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RAY AND WAVE OPTICS OF INTEGRABLE AND STOCHASTIC SYSTEMS

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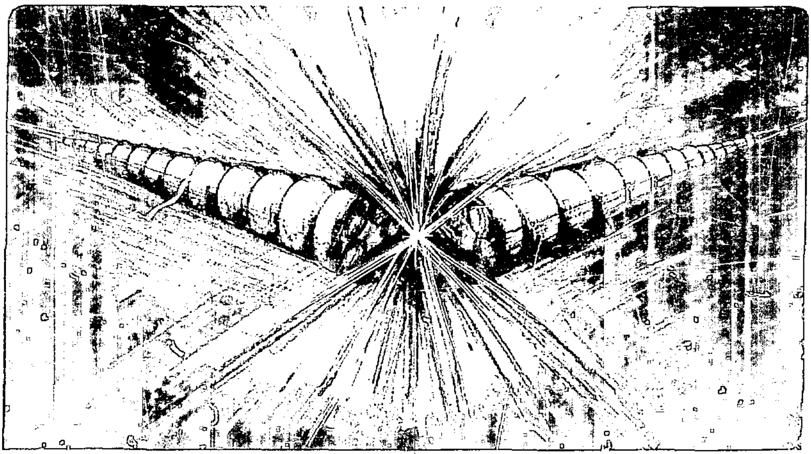

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Ray and Wave Optics of Integrable and Stochastic Systems

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Abstract

The generalization of WKB methods to more than one dimension is discussed in terms of the integrability or non-integrability of the geometrical optics (ray Hamiltonian) system derived in the short-wave approximation. In the two-dimensional case the ray trajectories are either regular or stochastic, and the qualitative differences between these types of motion are manifested in the characteristics of the spectra and eigenfunctions. We examine these for a model system which may be integrable or stochastic, depending on a single parameter.

The propagation of an electromagnetic or electrostatic wave in plasma can often be analyzed in terms of the WKB solution to the appropriate wave equation. This approximation requires the knowledge of the ray trajectories governed by the Hamiltonian set of equations

$$\frac{d\underline{k}}{dt} = -\frac{\partial\omega(\underline{k},\underline{x})}{\partial\underline{x}}, \quad \frac{d\underline{x}}{dt} = \frac{\partial\omega(\underline{k},\underline{x})}{\partial\underline{k}} \tag{1}$$

where $\omega(\underline{k},\underline{x})$ is the local dispersion relation. If the wave is trapped in a region of the plasma, one is led to the determination of the eigenfrequency spectrum and the normal modes. In order to extend the well-known WKB methods to nonseparable geometries of more than one dimension, the general relation between the trajectories of the classical (geometrical optics) system (1) and the states of the wave mechanical (physical optics) system must be understood. This is the regime of semiclassical mechanics, which has been studied extensively in the context of the correspondence between classical and quantum systems.[1]

The application of the WKB method to a time-independent, bound Hamiltonian system with N degrees of freedom is known as EBK quantization, and it relies on the structure of the classical phase space $(\underline{k},\underline{x})$. Quantization of integrable systems is well understood;[2] it is based on the fact that N analytic constants of the motion exist which constrain the motion to lie on N-dimensional tori in phase space. In the non-integrable case, a complete set of N invariants does not exist and EBK quantization fails. In the limiting case there is only one constant, the value of the frequency $\omega = \omega(\underline{k},\underline{x})$, so that the ray trajectory ergodically explores a 2N-1 dimensional frequency surface; only a few predictions[3-7] have been made regarding the qualitative aspects of the spectra and eigenfunctions of such systems. In the remainder of this paper we shall restrict ourselves to N=2 and then consider a model system which, depending on the value of a single parameter, is either integrable or non-integrable and ergodic. We indicate how the characteristics of the different types of classical (ray trajectory) motion are reflected in the features of the (wave) spectrum and eigenmodes.

We consider the ray system (1), for the canonically conjugate two-dimensional variables $\underline{k}$ and $\underline{x}$, which has been derived in the short-wave (WKB-like) approximation from an underlying wave equation. We also assume that $\omega(\underline{k},\underline{x})$ is a real-valued function. If this set of equations represents an integrable system, the trajectories in the four-dimensional phase space are constrained to lie on two-dimensional tori, due to the existence of two analytic constant-valued functions of $\underline{k}$ and $\underline{x}$. These functions are the two invariant actions $\underline{I}$ and are the new momenta in a canonical transformation from $(\underline{k},\underline{x})$ to new variables, such that the new Hamiltonian $\omega = \omega(\underline{I})$ is independent of the new conjugate angles $\underline{\theta}$. The actions are determined by integrating around any two curves (not necessarily trajectories) that are deformable into the two independent closed circuits $\underline{\Gamma}$ on a particular torus

$$\underline{I} = \oint_{\underline{\Gamma}} \underline{k} \cdot d\underline{x} \tag{2}$$

Each torus in the continuum of tori is labelled by the values of the actions on that torus. If the system is non-integrable, then only the value of the Hamiltonian (the frequency) is constant and the motion covers a three dimensional frequency surface in phase space; the tori and the corresponding actions do not exist.

The method of EBK quantization stems from the requirement of the single-valuedness of the mode and consists of restricting the actions to discrete values

$$\underline{I}_{\underline{m}} = \underline{m} + \frac{1}{4}\underline{\alpha} \qquad [\underline{m} = 0,\ 1,\ 2\ldots] \tag{3}$$

where the Keller-Maslov indices $\underline{\alpha}$ are determined from the topology of the tori. The discrete set of actions represents a discrete set of tori which support the eigenfunctions, and the eigenfrequencies are now given by

$$\omega_{\underline{m}} = \omega(\underline{I}_{\underline{m}}) = \omega(\underline{m} + \frac{1}{4}\underline{\alpha}) \tag{4}$$

In the non-integrable case, there are no classical quantities analogous to the actions to which we can apply EBK quantization; one can simply label the modes $n = 0,\ 1,\ 2\ldots$ In contrast to integrable systems, where the classical analog of the eigenfunction is the family of orbits on the corresponding torus, there seems to be no such connection for non-integrable systems.

The EBK eigenfunction is similar to that in the one-dimensional WKB form

$$\psi_{\underline{m}}(x) = \sum_j A_j(\underline{x},\underline{I}_{\underline{m}})\, e^{iS_j(\underline{x},\underline{I}_{\underline{m}})} \tag{5}$$

Here, $\psi(x)$ is one component of an electromagnetic wave, for example, and is written as a sum over the branches of $\underline{k}$ at the point $\underline{x}$; $\underline{k}_j(\underline{x})$ is a finitely multi-valued vector function in integrable systems. This form is inappropriate for non-integrable systems in the sense that for these systems $\underline{k}$ is an infinitely multi-valued function of $\underline{x}$. Einstein[8] was the first to recognize this difficulty.

The phase in expression (5) is the indefinite action integral

$$S_j(\underline{x},\underline{I}_{\underline{m}}) = \int^{\underline{x}} \underline{k}_j(\underline{x},\underline{I}_{\underline{m}}) \cdot d\underline{x}$$

$$\underline{k}_j(\underline{x},\underline{I}_{\underline{m}}) = \frac{\partial S_j(\underline{x},\underline{I}_{\underline{m}})}{\partial \underline{x}} \tag{6}$$

and as such it is the generator of the canonical transformation $(\underline{k},\underline{x}) \to (\underline{I},\underline{\theta})$. The slowly varying amplitude is

$$A_j(\underline{x},\underline{I}_{\underline{m}}) = \left\{ \det \left[\frac{\partial^2 S_j(\underline{x},\underline{I})}{\partial \underline{x} \quad \partial \underline{I}} \right]_{\underline{I}_{\underline{m}}} \right\}^{1/2} \tag{7}$$

Examination of the quasi-classical eigenfunction reveals a feature reminiscent of WKB functions: the divergence of the mode at classical turning points. The multi-dimensional generalization of a turning point is termed a caustic which is a surface on which the new angle variables are rapidly varying functions of the old position variables so that

$$A^2(\underline{x},\underline{I}_{\underline{m}}) = \det \left[\frac{\partial^2 S}{\partial \underline{x} \ \partial \underline{I}} \right]_{\underline{I}_{\underline{m}}} = \det \left[\frac{\partial \underline{\theta}}{\partial \underline{x}} \right]_{\underline{m}} \to \infty \tag{8}$$

Naturally, the divergence signals the fact that the assumed form of the EBK solution (5) is not applicable near the caustic, but one nonetheless expects a peak in the amplitude of the mode in those regions. However, the existence of caustics is a property only of integrable systems; modes of non-integrable systems will not exhibit this enhanced "turning point" structure.

The wave mechanical analog of the classical Liouville phase-space density is the Wigner function, defined as

$$W_{\underline{m}}(\underline{k},\underline{x}) = \int d^2 s \ \psi_{\underline{m}}(\underline{x} - \tfrac{1}{2}\ \underline{s}) \ \psi^*_{\underline{m}}(\underline{x} + \tfrac{1}{2}\ \underline{s}) e^{-i\underline{k}\cdot\underline{s}} \tag{9}$$

which produces a local wave vector spectrum for each value of $\underline{x}$. Substitution of (5) into (9) yields, after averaging over many oscillations of the wave function, the result for integrable systems[3]

$$W_{\underline{m}}(\underline{k},\underline{x}) = (2\pi)^{-2} \ \delta^2(\underline{I}(\underline{k},\underline{x}) - \underline{I}_{\underline{m}}) \tag{10}$$

The delta function on the torus is in agreement with the classical analogy and implies that the Wigner function exhibits a discrete anisotropic distribution of local wave vectors at a point $\underline{x}$ since $\underline{k}(\underline{x};\underline{I}_{\underline{m}})$ is finitely multi-valued. For non-integrable systems, where the Liouville density ultimately spreads over the entire frequency surface, one has[9]

$$W_n(\underline{k},\underline{x}) \propto \delta(\omega(\underline{k},\underline{x}) - \omega_n), \tag{11}$$

which is an isotropic local wave vector spectrum.

Features of the *eigenfrequency spectrum* may also be deduced by considering the classical orbits. The eigenvalue density

$$n(\omega) = \sum_{\underline{m}} \delta(\omega-\omega_{\underline{m}}); \qquad \omega_{\underline{m}} \equiv \omega(\underline{I}_{\underline{m}}) \tag{12}$$

for integrable systems may be transformed, using the Poisson summation formula, to

$$n(\omega) = \sum_{\underline{M}} e^{-i\pi\underline{\alpha}\cdot\underline{M}/2} \int d^2I\ \delta(\omega-\omega(\underline{I}))\ e^{2\pi i\underline{M}\cdot\underline{I}} \tag{13}$$

where we have used (4). Evaluation of the integral by the stationary phase method leads to the result that each term $\underline{M}$ in the sum is dominated by the contribution from the torus $\underline{I}$ which contains closed periodic orbits with classical frequencies in the *commensurable ratio*

$$\Omega_1/\Omega_2 = M_1/M_2, \qquad \underline{\Omega} \equiv \frac{d\underline{\theta}}{dt} = \frac{\partial\omega(\underline{I})}{\partial\underline{I}} \tag{14}$$

Therefore, periodicities in the spectrum of integrable systems (regular spectra) are due to the existence of closed orbits.[5] Further investigation of (13) also predicts a generic clustering of the eigenvalues $\omega_{\underline{m}}$ characterized by an exponential distribution for the probability density $P(\Delta\omega)$ *of finding a separation* $\Delta\omega$ between neighboring eigenfrequencies

$$P(\Delta\omega) \sim e^{-\alpha\Delta\omega} \tag{15}$$

The preceding analysis is inapplicable to non-integrable systems in which closed orbits are unstable. Predictions have been made, however, regarding the characteristics of the (irregular) spectra in this case. In contrast to the clustering found in the regular spectrum, Zaslavskii[7] and Berry expect level "repulsion", or a more even spacing:

$$P(\Delta\omega) \to 0 \quad \text{as} \quad \Delta\omega \to 0, \tag{16}$$

the maximum of $P(\Delta\omega)$ occurring at a finite value of $\Delta\omega$. This is reminiscent of the behavior of $P(\Delta\omega)$ for a random matrix.[10]

In order to demonstrate these *ideas, we consider* the Helmholtz equation $(\nabla^2 + k^2)\,\psi(\underline{x}) = 0$ within a two-dimensional stadium boundary adjusted by the parameter $\gamma \equiv a/R$ (a = half length of straight side, R = radius of semicircle, constant area = π) and on which $\psi = 0$. The ray approximation produces trajectories governed by $\omega = \omega(\underline{k},\underline{x}) = k^2$ with specular reflection at the walls: a classical billiard system which is integrable for $\gamma = 0$ (the circle) and non-integrable for all $\gamma > 0$ (the stadium).[11]

Figures 1 and 2 are examples of trajectories for $\gamma = 0$ and 1 respectively. Orbits in the integrable circle conserve angular momentum I_θ, so that the motion is within an annular region, the inner edge of which is a caustic. The radial action I_r may also be determined so that one can write $\omega = \omega(I_r, I_\theta)$. This is impossible for the non-integrable stadium, where the only constant is the value of $\omega = k^2$; the motion covers the entire configuration space (as well as all orientations of $\underline{k}$), and no caustics exist.

Since the circle is a separable geometry, the wave equation may be solved by one dimensional WKB methods, the resulting quantization of the actions yielding eigenfrequencies and modes which are indeed asymptotic to the exact solutions. An example of the exact Bessel function structure (Fig. 3) exhibits a regular nodal pattern and a peak in the amplitude in the region of the classical caustic.

We have used a method due to Riddell and Lepore[12] to solve the Helmholtz equation numerically, and have concentrated on high mode numbers, which should correspond to the geometrical optics limit. A typical mode for the case $\gamma = 1$ is shown in Fig. 4; it is of odd-odd parity so that there are nodal lines along the x and y axis which bound the quadrant of the stadium. In marked contrast to circular modes, this eigenfunction exhibits a quite irregular nodal structure (with no nodal crossings) and an apparently isotropic distribution of local wave vector orientation. The amplitude is fairly constant throughout the region indicating the absence of caustics. These observations demonstrate the spreading of the Wigner function over the entire frequency surface in phase space, in accordance with Eq. (11).

A segment of the eigenfrequency spectrum for odd-odd parity modes in both the circular and stadium cases is shown in Fig. 5. The eigenvalues for $\gamma = 0$ clearly exhibit clustering, and there is a tendency toward periodicity in the asymptotic spectrum of Bessel function zeroes. The eigenvalues for $\gamma = 1$ are more evenly spaced, as has been conjectured. The statistics of neighboring eigenvalue spacings has been studied[13] and the predictions of Zaslavskii and Berry seem to be verified.

We have also studied the γ-dependence of these modes and eigenfrequencies, and have found a wide range of sensitivity. For small values of γ, modes corresponding to low angular momentum ("bouncing ball" modes) are extremely sensitive, as are their eigenvalues (Figures 6 and 7); high angular momentum ("whispering gallery") modes are very insensitive. In the regime near $\gamma = 1$ such classification of modes is not possible and all modes appear to be relatively insensitive to changes in γ.

The change in the low angular momentum (m=2) mode in Fig. 7 may be interpreted as follows: as γ increases, so does the probability of finding the classical billard bouncing between the straight sides of the stadium, corresponding to the least unstable family of trajectories for $\gamma > 0$. Indeed, this almost one-dimensional motion may be quantized in the usual fashion with

$$L(\gamma) = \frac{n\pi}{L(\gamma)} \qquad n = 1,2.... \tag{17}$$

with $L(\gamma) = 2R(\gamma)$, since the semicircle radius is γ-dependent if the stadium area is constant. When (17) is linearized for small γ, good agreement is obtained with the slope of the m=2 curve in Fig. 6, with a value of n that corresponds to the actual number of radial modes of the given eigenfunction.

In conclusion, we have demonstrated how properties of ray trajectories are reflected in the structure of the corresponding modes and eigenvalues. The integrability or non-integrability of the ray system is manifested by major qualitative differences in the eigenfunctions and spectra obtained.

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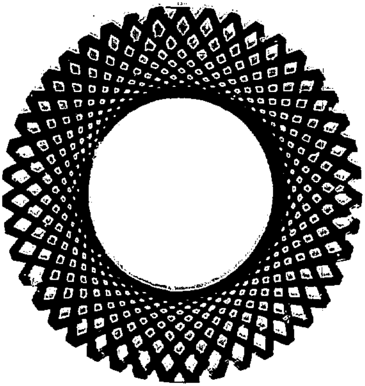


Figure 1

Single trajectory inside a circle ($\gamma = 0$). Note interior caustic which is an envelope of the trajectory.

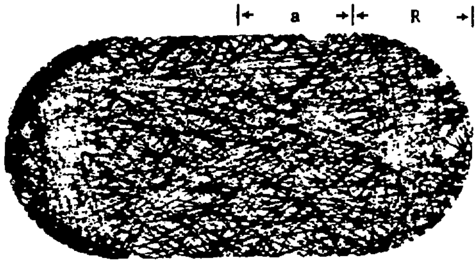


Figure 2

Single trajectory in a stadium ($\gamma = 1$); a = half length of straight side, R = radius of semicircle.

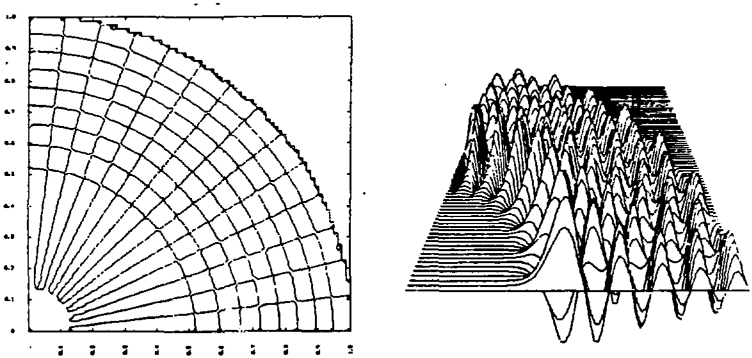


Figure 3

Left: Regular nodal structure of $J_{28}(kr)\sin 28\theta$, with eigenvalue $k_{28,9} = 65.38142$. Non-crossing of nodal lines is due to computer-graphics. Right: Amplitude in perspective, looking along positive y axis.

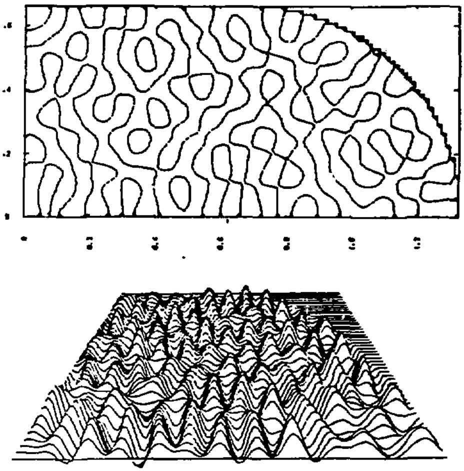
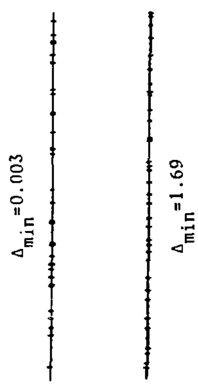


Figure 4

Top: Irregular nodal structure of eigenfunction for $\gamma = 1$, with eigenvalue k=65.036. Closer analysis reveals no nodal line crossings.

Bottom: Amplitude in perspective indicates uniform amplitude throughout interior of stadium.



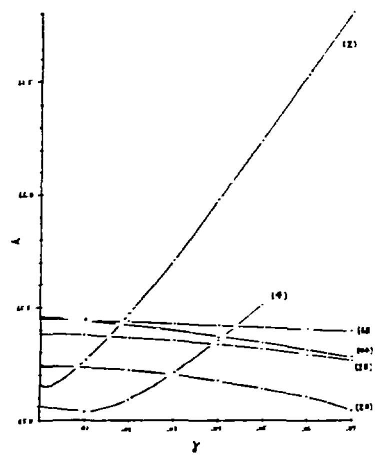


Figure 5

Left: Segment of eigenvalue spectrum for $\gamma=0$. Note clustering, with minimum separation indicated.
Right: Same range of spectrum with $\gamma=1$, exhibiting a more even spacing of eigenvalues.

Figure 6

Variation of eigenvalues with γ. Number in parentheses refers to angular momentum of a level. The large curvature and reversal of direction of the m=2 and m=4 eigenvalues is not predicted by Rayleigh perturbation theory. Only eigenvalues with $65.0 < k < 65.5$ at $\gamma=0$ are plotted.

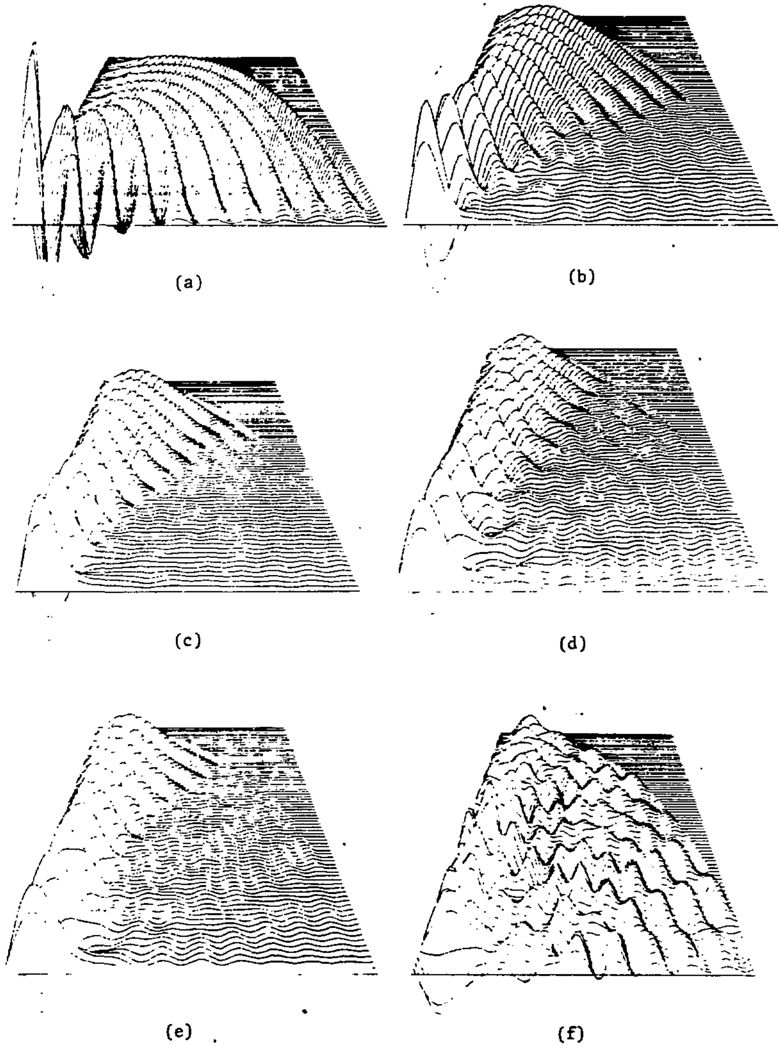


Figure 7

Sensitive variation of the $m = 2$ eigenfunction of Figure 6: (a) $\gamma = 0$, $J_2(kr)\sin 2\theta$, $k=65.159$; (b) $\gamma = .01$; (c) $\gamma = .02$; (d) $\gamma = .03$; (e) $\gamma = .04$; (f) $\gamma = .05$.