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Isotone maps on lattices

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Abstract. Let $\mathcal{L} = (L_i \mid i \in I)$ be a family of lattices in a nontrivial lattice variety $\mathbf{V}$, and let $\varphi_i \colon L_i \to M$, for $i \in I$, be isotone maps (not assumed to be lattice homomorphisms) to a common lattice M (not assumed to lie in $\mathbf{V}$). We show that the maps φ_i can be extended to an isotone map $\varphi \colon L \to M$, where $L = \mathrm{Free}_{\mathbf{V}}\, \mathcal{L}$ is the free product of the L_i in $\mathbf{V}$. This was known for $\mathbf{L} = \mathbf{V}$, the variety of all lattices.

The above free product L can be viewed as the free lattice in $\mathbf{V}$ on the partial lattice P formed by the disjoint union of the L_i. The analog of the above result does not, however, hold for the free lattice L on an arbitrary partial lattice P. We show that the only codomain lattices M for which that more general statement holds are the complete lattices. On the other hand, we prove the analog of our main result for a class of partial lattices P that are not-quite-disjoint unions of lattices.

We also obtain some results similar to our main one, but with the relationship **lattices : orders** replaced either by **semilattices : orders** or by **lattices : semilattices**.

Some open questions are noted.

1. Introduction

By Yu. I. Sorkin [15, Theorem 3], if $\mathcal{L} = (L_i \mid i \in I)$ is a family of lattices and $\varphi_i \colon L_i \to M$ are isotone maps of the lattices L_i into a lattice M, then there exists an isotone map φ from the free product $\mathrm{Free}\, \mathcal{L}$ of the L_i to M that extends all the φ_i. (There are some difficulties with Sorkin's original proof; but a simple, correct proof is given by G. Grätzer, H. Lakser and C. R. Platt in [11, §4].)

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Our main result, proved in Section 2, is a generalization of this fact, with $\mathrm{Free}\,\mathcal{L}$ replaced by $\mathrm{Free}_{\mathbf{V}}\,\mathcal{L}$, the free product of the L_i in any nontrivial variety $\mathbf{V}$ of lattices containing them — though not necessarily containing M. (In Section 3, we explore some variants of our proof of this result.)

We may regard $\mathrm{Free}_{\mathbf{V}}\,\mathcal{L}$ as the free lattice in $\mathbf{V}$ on the partial lattice P given by the disjoint union of the L_i. Does the analog of the above result hold for more general partial lattices P and their free lattices L? In Section 4 we find that the lattices M such that this statement holds for *all* partial lattices P are the complete lattices. On the other hand, we describe in Section 5 a class of partial lattices P, related to but distinct from the class considered in Section 2, for which the full analog of the result of that section holds.

Since *semilattices* lie between orders and lattices, it is plausible that statements similar to our main result should hold, either with "lattice" weakened to "semilattice", or with "lattice" unchanged but "isotone map" strengthened to "semilattice homomorphism". In Section 6 we shall find that the former statement is easy to prove. In that section and Section 7, we obtain several approximations to the latter statement; we do not know whether the full statement holds.

The reader familiar with the concepts of *quasi*variety and *pre*variety will find that the proofs given in this note for varieties of lattices in fact work for those more general classes. However, varieties are not sufficient for the result of Section 7, so we develop that in terms of prevarieties.

In Section 8 we note some open questions.

For general definitions and results in lattice theory, see [8] or [9].

For another context in which isotone maps among lattices have appeared in the literature—in this case, isotone sections of lattice homomorphisms—see R. Freese, J. Ježek, and J. B. Nation [6, pp. 99–107, 133] and R. Freese [5].

We are indebted to R. Freese for pointing us to those works, and for the related observation following Corollary 4 below; and to A. V. Kravchenko for an extensive correspondence regarding Sorkin's original argument.

2. Extending isotone maps to free product lattices

Let $\mathbf{V}$ be a nontrivial lattice variety, that is, a variety $\mathbf{V}$ of lattices having a member with more than one element. Let $\mathcal{L} = (L_i \mid i \in I)$ be a family of lattices in $\mathbf{V}$, and $L = \mathrm{Free}_{\mathbf{V}}\,\mathcal{L}$ their free product in $\mathbf{V}$. Finally, let $(\varphi_i \colon L_i \to M \mid i \in I)$ be a family of isotone maps into a lattice M, not assumed to lie in $\mathbf{V}$.

To show that the φ_i have a common extension to L, it suffices, by the universal property of L, to find some $L' \in \mathbf{V}$ such that each map φ_i factors $L_i \to L' \to M$, where the first map is a lattice homomorphism, and the second an isotone map not depending on i. So let us, for now, forget free products, and obtain such a lattice L'.

We first note (as remarked in [11, last paragraph] for $\mathbf{V} = \mathbf{L}$) that this is easy if M has a least element, or more generally, if its subsets $\varphi_i(L_i)$ have a common lower bound $e \in M$. In that case, we begin by enlarging all the L_i to lattices $\bar{L}_i = \{e_i\} + L_i$, where e_i is a new least element, and extend the φ_i to maps $\bar{\varphi}_i \colon \bar{L}_i \to M$ mapping e_i to e. Now let L' be the sublattice of $\prod(\bar{L}_i \mid i \in I)$ consisting of the elements $x = (x_i \mid i \in I)$ such that $x_i = e_i$ for all but finitely many i; and let us map each L_i into L' by the homomorphism sending $x \in L_i$ to the element having i-component x, and j-component e_j for all $j \neq i$. We now map L' to M using the isotone map ψ given by

$$\psi(x) = \bigvee(\, \bar{\varphi}_i(x_i) \mid i \in I\,) \qquad \text{for } x = (x_i \mid i \in I) \in L'. \tag{1}$$

This infinite join is defined because all but finitely many of the joinands are e; and it is easy to verify that for each i, the composite map $L_i \to L' \to M$ is the given isotone map φ_i, as required.

If, rather, the $\varphi_i(L_i)$ have a common *upper* bound, the dual construction is, of course, available.

In the absence of either sort of bound, we shall follow the same pattern of adjoining to the L_i elements e_i with a common image e in M (this time an arbitrary element of that lattice); but that construction takes a bit more work, as does the one analogous to the definition (1) of the isotone map $\psi : L' \to M$. The first of these steps is carried out in the following lemma (where L corresponds to the above L_i).

Lemma 1. *Let $\varphi\colon L \to M$ be any isotone map of lattices, and e any element of M. Then there exists a lattice extension $\bar{L}$ of L, and an isotone map $\bar{\varphi}\colon \bar{L} \to M$ extending φ, such that $e \in \bar{\varphi}(\bar{L})$. Moreover, $\bar{L}$ can be taken to lie in any nontrivial lattice variety $\mathbf{V}$ containing L.*

Proof. Let $\bar{L} = L \times \mathsf{C}_2 \times \mathsf{C}_2$, where C_2 is the 2-element lattice $\{0, 1\}$, and embed L in $\bar{L}$ by $x \mapsto (x, 0, 1)$. Define $\bar{\varphi}\colon \bar{L} \to M$ by

$$\begin{cases} \bar{\varphi}(x,0,1) &= \quad \varphi(x), \\ \bar{\varphi}(x,1,0) &= \quad e, \\ \bar{\varphi}(x,0,0) &= \quad \varphi(x) \wedge e, \\ \bar{\varphi}(x,1,1) &= \quad \varphi(x) \vee e. \end{cases} \tag{2}$$

It is easy to check that $\bar{\varphi}$ is isotone, and it clearly has e in its range. Since $\mathsf{C}_2 = \{0, 1\}$ belongs to every nontrivial variety of lattices, $\bar{L}$ will belong to any nontrivial variety $\mathbf{V}$ containing L. □

The next lemma gives the construction we will use to weld our I-tuple of isotone maps into one map.

Lemma 2. *Let M be a lattice, e any element of M, and I a nonempty set. Let M' be the sublattice of M^I consisting of those elements f such that $f(i) = e$*

for all but finitely many $i \in I$*. Then there exists a map* $\psi\colon M' \to M$ *such that*

$$\psi \text{ is isotone} \tag{3}$$

and

For every $i \in I$*, and every* $f \in M'$ *satisfying* $f(j) = e$ *for all* $j \neq i$*, we have* $\psi(f) = f(i)$*.* (4)

Proof. For $f \in M'$, define

$$\psi(f) = \begin{cases} \bigwedge(\, f(i) \mid i \in I\,) & \text{if } f(i) \leq e \text{ for all } i \in I, \\ \bigvee(\, f(i) \mid i \in I,\ f(i) \not\leq e\,) & \text{otherwise.} \end{cases} \tag{5}$$

These meets and joins are defined because for each $f \in M'$, there are only finitely many distinct values $f(i)$.

It is easy to see that ψ satisfies (4). To obtain (3), observe that

$$\text{For } f \leq g \text{ in } M', \text{ we have } \{i \mid f(i) \not\leq e\} \ \subseteq\ \{i \mid g(i) \not\leq e\}. \tag{6}$$

Hence given $f \leq g$, there are three possibilities: Either the definitions of $\psi(f)$ and $\psi(g)$ both fall under the first case of (5), or they both fall under the second, or that of $\psi(f)$ falls under the first and that of $\psi(g)$ under the second.

If both fall under the first case, then $\psi(f) \leq \psi(g)$ because the meet operation of M is isotone.

If both fall under the second, the same conclusion follows using the fact that the join operation is isotone, together with the fact that bringing in more joinands, as can happen in view of (6), yields a join greater than or equal to what we would get without those additional terms.

Finally, if the evaluation of $\psi(f)$ falls under the first case and that of $\psi(g)$ under the second, we may choose, in view of the latter fact, an i such that $g(i) \not\leq e$. Then

$$\psi(f)\ \leq\ f(i)\ \leq\ g(i)\ \leq\ \psi(g), \tag{7}$$

completing the proof of (3). □

We can now fill in the proof of our main theorem.

Theorem 3. *Let* $\mathbf{V}$ *be a nontrivial variety of lattices,* $\mathcal{L} = (L_i \mid i \in I)$ *a family of lattices in* $\mathbf{V}$*, and* $(\varphi_i\colon L_i \to M \mid i \in I)$ *a family of isotone maps from the* L_i *to a lattice* M *not necessarily in* $\mathbf{V}$*. Then there exists an isotone map* $\varphi\colon \mathrm{Free}_{\mathbf{V}}\,\mathcal{L} \to M$ *whose restriction to each* $L_i \subseteq \mathrm{Free}_{\mathbf{V}}\,\mathcal{L}$ *is* φ_i*.*

Proof. Choose any element $e \in M$, and extend each φ_i as in Lemma 1 to a map $\bar{\varphi}_i\colon \bar{L}_i \to M$ on a lattice extension $\bar{L}_i \supseteq L_i$ in $\mathbf{V}$, so that some $e_i \in \bar{L}_i$ is mapped by $\bar{\varphi}_i$ to $e \in M$. Now map each L_i into $\prod(\bar{L}_i \mid i \in I)$ by sending every element $x \in L_i$ to the element having i-th coordinate x, and j-th coordinate e_j for all $j \neq i$. These maps are lattice homomorphisms, hence together they induce a homomorphism $\mathrm{Free}_{\mathbf{V}}\,\mathcal{L} \to \prod(\bar{L}_i \mid i \in I)$. Moreover, this map has range in the sublattice L' of elements whose j-coordinates are e_j for almost all j, since the image of each L_i lies in that sublattice.

Mapping $\prod \bar{L}_i$ to M^I by the isotone map $\prod \bar{\varphi}_i$, we see that the above sublattice $L' \subseteq \prod \bar{L}_i$ is carried into the sublattice $M' \subseteq M^I$ of Lemma 2. Bringing in the isotone map $f\colon M' \to M$ of that lemma, we get our desired isotone map φ as the composite $\mathrm{Free}_{\mathbf{V}}\,\mathcal{L} \to L' \to M' \to M$. It follows from (4) that the restriction of φ to each L_i is φ_i. □

We note a curious consequence of the fact that the M of Theorem 3 need not lie in $\mathbf{V}$.

Corollary 4. *Let $\mathcal{L} = (L_i \mid i \in I)$ be a family of lattices in a nontrivial lattice variety $\mathbf{V}$, let $\mathrm{Free}_{\mathbf{V}}\,\mathcal{L}$ be their free product in $\mathbf{V}$, and let $\mathrm{Free}\,\mathcal{L}$ be their free product in the variety $\mathbf{L}$ of all lattices. Then there exists an isotone map $\mathrm{Free}_{\mathbf{V}}\,\mathcal{L} \to \mathrm{Free}\,\mathcal{L}$ which acts as the identity on each L_i.*

In particular, for any nontrivial lattice variety $\mathbf{V}$ and any set X, there exists an isotone map $\mathrm{Free}_{\mathbf{V}}(X) \to \mathrm{Free}(X)$ (where these denote the free lattices on the set X in $\mathbf{V}$ and in $\mathbf{L}$ respectively), which acts as the identity map on X.

Proof. For the first statement, apply Theorem 3 to the inclusions of the L_i in $\mathrm{Free}\,\mathcal{L}$. The second is the case of the first where all L_i are one-element lattices. □

The existence of a map as in the last sentence of the above corollary also follows from results in the literature, which in fact show that it can be taken to be a set-theoretic right inverse to the natural lattice homomorphism $\mathrm{Free}(X) \to \mathrm{Free}_{\mathbf{V}}(X)$. Namely, it is shown by R. Freese [4] that free lattices in any lattice variety satisfy an elegant order-theoretic property called "finite separability", and it is shown in R. Freese and J. B. Nation [5] (where finite separability was called "condition (4)"; for a rewrite of that development, using the present name, see R. Freese, J. Ježek, and J. B. Nation [6, Lemma 5.6]) that this property allows one to construct isotone right inverses to surjective homomorphisms.

It would be interesting to know whether the isotone map of the first paragraph of the above corollary can similarly be taken to be a right inverse to the natural homomorphism $\mathrm{Free}\,\mathcal{L} \to \mathrm{Free}_{\mathbf{V}}\,\mathcal{L}$. The lattice $\mathrm{Free}_{\mathbf{V}}\,\mathcal{L}$ will not be finitely separable if the L_i are not; but perhaps it has some corresponding relative property.

3. Digression: sketches of some alternate proofs of Theorem 3

The definition (5) of the isotone map ψ used in the proof of Lemma 2 is clearly asymmetric in the meet and join operations.

We sketch below a variant proof of Theorem 3 which uses a function that is symmetric in these operations — but lacks instead (when $|I| > 2$) the symmetry in the family of lattices L_i which the proof given above clearly has.

We shall then show that one cannot have it both ways: a map of the required sort having both sorts of symmetry does not, in general, exist. However, we show that we *can* get such a map if M lies in the given variety $\mathbf{V}$.

This section will be sketchier than the rest of the paper. In particular, we will be informal about our two sorts of symmetry; though in the next-to-last paragraph, we will indicate how to make these considerations precise.

Our new proof of Theorem 3 starts with a generalization of the construction of Lemma 1. Namely, suppose we are given isotone maps of *two* lattices into a common lattice, $\varphi_i \colon L_i \to M$ for $i = 0, 1$. Let

$$L' \ = \ L_0 \times L_1 \times \mathsf{C}_2 \times \mathsf{C}_2. \tag{8}$$

Then taking any $e_0 \in L_0$, $e_1 \in L_1$, we can embed our two lattices in L' by the homomorphisms

$$\begin{aligned} &L_0 \to L' \quad \text{acting by} \quad x \mapsto (x, e_1, 1, 0),\\ &L_1 \to L' \quad \text{acting by} \quad y \mapsto (e_0, y, 0, 1). \end{aligned} \tag{9}$$

Now define the isotone map $\varphi' \colon L' \to M$ by

$$\begin{cases} \varphi'(x, y, 1, 0) & = \quad \varphi_0(x),\\ \varphi'(x, y, 0, 1) & = \quad \varphi_1(y),\\ \varphi'(x, y, 0, 0) & = \quad \varphi_0(x) \wedge \varphi_1(y),\\ \varphi'(x, y, 1, 1) & = \quad \varphi_0(x) \vee \varphi_1(y). \end{cases} \tag{10}$$

Clearly, φ' acts on the embedded images of the L_i by the φ_i; and as before, since C_2 belongs to every nontrivial variety of lattices, L' belongs to any nontrivial variety $\mathbf{V}$ containing the L_i.

This, in fact, gives us Theorem 3 for $|I| = 2$ by a construction symmetric both in meet and join, and in our family of lattices.

Now suppose more generally that we have lattices $L_i \in \mathbf{V}$ and isotone maps $\varphi_i \colon L_i \to M$ indexed by an arbitrary set I. Assuming without loss of generality that I is an ordinal, we shall construct the desired $L' \in \mathbf{V}$ and isotone map $\varphi' \colon L' \to M$ by a recursive transfinite iteration of the above construction. It is the recursion that will lose us our symmetry in the L_i, via the arbitrary choice of an identification of I with an ordinal, i.e., of a well-ordering on I.

To describe the recursion, let $1 < k \leq I$, and assume that we have constructed lattices $L'_{(j)}$ for all $1 \leq j < k$, which satisfy

$$L_0 \ = \ L'_{(1)} \ \subseteq \ L'_{(2)} \ \subseteq \ \cdots \ \subseteq \ L'_{(j)} \ \subseteq \ \cdots, \tag{11}$$

together with lattice embeddings $L_i \to L'_{(j)}$ for $i < j$, and isotone maps $L'_{(j)} \to M$, and that these form a coherent system, in the sense that for $i < j < j'$, the composite $L_i \to L'_{(j)} \subseteq L'_{(j')}$ is the embedding $L_i \to L'_{(j')}$, and the composite $L'_{(j)} \subseteq L'_{(j')} \to M$ is the isotone map $L'_{(j)} \to M$; and, finally, such that for every $i < j$, the composite $L_i \to L'_{(j)} \to M$ is the given isotone map φ_i.

If k is a successor ordinal, $k = j+1$, we apply the $|I| = 2$ case of our construction, described in (8)-(10), to the pair of lattices $L'_{(j)}$ and L_j and their isotone maps to M, calling the resulting lattice

$$L'_{(j+1)} \;=\; L'_{(j)} \times L_j \times \mathsf{C}_2 \times \mathsf{C}_2, \tag{12}$$

and identifying $L'_{(j)}$ with its image therein under the first map of (9). If, on the other hand, k is a limit ordinal, we let $L'_{(k)}$ be the union of the $L'_{(j)}$ over all $j < k$. In each case, the asserted properties are immediate. Thus, we can carry our construction up to $k = I$, the resulting lattice $L'_{(I)}$ being our desired L'.

What are the consequences of the different kinds of symmetry of the construction of the preceding section and the one just sketched?

Because the former was symmetric in the L_i, we can deduce, for instance, that in the final statement of Corollary 4, if X is finite, then the isotone map $\mathrm{Free}_{\mathbf{V}}(X) \to \mathrm{Free}(X)$ can be taken to respect the actions of the symmetric group $\mathrm{Sym}(X)$ on these two lattices. (Why assume X to be finite? So that in applying Lemma 1, we can choose an $e \in M = \mathrm{Free}(X)$ invariant under that group action, say the join of the given generators. Alternatively, without this finiteness assumption, if we choose any $x_0 \in X$ and perform our construction with $e = x_0$, we can get our map to respect the action of $\mathrm{Sym}(X - \{x_0\})$.)

On the other hand, using our new construction we can deduce that if $\mathbf{V}$ is closed under taking dual lattices, we can, instead, in that same final statement of Corollary 4, take the isotone map $\mathrm{Free}_{\mathbf{V}}(X) \to \mathrm{Free}(X)$ to respect the anti-automorphisms of the domain and codomain that fix the free generators but interchange meet and join. (Again, we have to decide what to use for our distinguished elements e_0, e_1 at each application of (9). In this case, we may, at each such step, take e_0 to be any of the preceding generators, while for e_1 we have no choice but to use the generator we are adjoining.)

Let us now show that for $|I| = 3$, we cannot get a construction with both sorts of symmetry. If we could, then letting $\mathbf{D}$ denote the variety of distributive lattices, we could get an isotone map φ: $\mathrm{Free}_{\mathbf{D}}(3) \to \mathrm{Free}(3)$ respecting all permutations of the generators, and also respecting the anti-automorphisms that interchange meets and joins.

Now $\mathrm{Free}_{\mathbf{D}}(3)$ has an element invariant under all these symmetries; namely, writing its three generators a, b, c, the element

$$(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \;=\; (a \vee b) \wedge (b \vee c) \wedge (c \vee a). \tag{13}$$

On the other hand, for any set X, the only elements of $\mathrm{Free}(X)$ that can be invariant under an anti-automorphism are the given free generators; this follows from the fact that every element of $\mathrm{Free}(X)$ other than those generators is either meet-reducible or join-reducible, but never both. (Cf. [9, condition (W) on p. 477, and Corollary 534(iii)].) So Free(3) has no element with both sorts of symmetry to which one could send the element given by (13).

Finally, let us show that we *can* get both sorts of symmetry if the lattice M lies in $\mathbf{V}$. (Of course, this restriction makes it impossible to use the result to prove a version of Corollary 4.) We record in the next lemma the raw construction used in the proof. Though that lemma requires M to lie in $\mathbf{V}$, it does not require the same of the L_i. But it is easy to see that if we add the assumption that the L_i lie in $\mathbf{V}$, the lattice L' obtained will lie there as well, hence the construction will induce, as in Theorem 3, an isotone map $\varphi\colon \mathrm{Free}_{\mathbf{V}}\,\mathcal{L} \to M$ acting as φ_i on each L_i. Moreover, the construction clearly has all the asserted symmetries. (We remark that the factor $\mathrm{Free}_{\mathbf{V}}(I)$ in the construction reduces, when $|I| = 2$, to the lattice $\mathsf{C}_2 \times \mathsf{C}_2$ of (8). So one could say it was the fact that all nontrivial lattice varieties have the same 2-generator free lattice that allowed us to get the doubly symmetric construction in that two-lattice case with no added restriction on M.)

Lemma 5. *Let $\mathcal{L} = (L_i \mid i \in I)$ be a family of lattices, let M be a lattice, and for each $i \in I$, let $\varphi_i\colon L_i \to M$ be an isotone map.*

Let $\mathbf{V}$ be a lattice variety containing M, and in the free lattice $\mathrm{Free}_{\mathbf{V}}(I)$*, let the i-th generator be denoted g_i for each i. Let*

$$L' \;=\; \textstyle\prod(L_i \mid i \in I) \;\times\; \mathrm{Free}_{\mathbf{V}}(I). \tag{14}$$

Suppose we choose, for each $i \in I$, a lattice homomorphism $\xi_i\colon L_i \to L'$ which takes every $x \in L_i$ to an element whose i-th coordinate is x and whose coordinate in $\mathrm{Free}_{\mathbf{V}}(I)$ *is the generator g_i. (For instance, such a family of homomorphisms ξ_i can be determined by fixing an element e_j in each L_j, and letting $\xi_i(x)$ have, in addition to the two coordinates just specified, j-th coordinate e_j for all $j \in I - \{i\}$.)*

Finally, let $\varphi'\colon L' \to M$ be the function taking each pair (x, w), where

$$x \;=\; (x_i \mid i \in I) \in \textstyle\prod(L_i \mid i \in I) \quad \textit{and} \quad w \in \mathrm{Free}_{\mathbf{V}}(I), \tag{15}$$

to $\bar{w}(\varphi_i(x_i) \mid i \in I)$, where $\bar{w}\colon M^I \to M$ is the operation of evaluating the lattice expression $w \in \mathrm{Free}_{\mathbf{V}}(I)$ at I-tuples of elements of M.

Then φ' is an isotone map, and for each $i \in I$, $\varphi'\xi_i = \varphi_i$.

Sketch of proof. The final equation is clear. To show that φ' is isotone, let $(x, w) \leq (x', w')$ in L'. Then we claim that $\varphi'(x, w) \leq \varphi'(x', w) \leq \varphi'(x', w')$; in other words, that $\bar{w}(\varphi_i(x_i)) \leq \bar{w}(\varphi_i(x'_i)) \leq \bar{w}'(\varphi_i(x'_i))$. The first inequality holds because $\bar{w}$, a derived lattice operation, is isotone; the second, because evaluation at $(\varphi_i(x'_i))$, a lattice homomorphism $\mathrm{Free}_{\mathbf{V}}(I) \to M$, is isotone. □

We remark that the isotone map φ' of the above lemma is not, in general, a lattice homomorphism. (For instance, let $I = \{0, 1\}$, let $L_0 = L_1 = M = \mathsf{C}_2$, let the $\varphi_i\colon L_i \to M$ be the identity map, and let $\mathbf{V}$ be any nontrivial lattice variety. Denoting the generators of $\mathrm{Free}_{\mathbf{V}}(I)$ by g_0 and g_1, we note that

$$((0,1), g_1 \wedge g_2) \vee ((1,0), g_1 \wedge g_2) \;=\; ((1,1), g_1 \wedge g_2) \quad \text{in } L'. \tag{16}$$

The map φ' takes each of the joinands on the left to 0, but the term on the right to 1.)

We have discussed symmetries of our construction informally above, leaving it to the reader to see that a construction with a certain sort of symmetry would imply corresponding properties of the maps constructed. These considerations can, of course, be formalized. If we describe our constructions as functors on appropriate categories of systems of lattices, isotone maps, and distinguished elements, then constructions with various sorts of symmetry allow us to strengthen the conclusion of Theorem 3 to say that we have functors respecting certain additional structure on those categories. We shall not go into these details here, however.

We turn now to some variants of our main result.

4. When does the same result hold for the inclusion of a general partial lattice P in its free lattice L?

If the lattice M of Theorem 3 happens to be a *complete* lattice, the conclusion of that theorem follows from a much more general fact: Any isotone map from an order P into a complete lattice can be extended to any order extension Q of P. In other words, in the category of orders, complete lattices are *injective* with respect to inclusions of orders.

The inclusions of orders are not, up to isomorphism, the only monomorphisms in the category of orders and isotone maps. B. Banaschewski and G. Bruns [1] characterize the inclusions category-theoretically among the monomorphisms, calling them the *strict* monomorphisms, and they formulate the above result as the statement that every complete lattice (in their terminology, every complete partially ordered set) is a "strict injective"; to which they also prove the converse [1, Proposition 1, (i) $\Longleftrightarrow$ (ii)].

Theorem 3 can thus be looked at as saying that if P is the disjoint union of a family of lattices L_i belonging to a variety $\mathbf{V}$, regarded as a partial lattice, then the inclusion of P in its free lattice $L = \mathrm{Free}_{\mathbf{V}}\, P$ behaves a little better than a general inclusion of orders, in that isotone maps of P to arbitrary lattices, and not only to complete lattices, can be extended to L.

In contrast, we shall see below that the inclusion of a general partial lattice P in its free lattice $\mathrm{Free}\, P$ behaves no better in this way than do arbitrary extensions of orders, at least insofar as isotone maps to lattices are concerned. (For the concepts of a partial lattice and of the free lattice on such an object, see [8, Section I.5], [9, Sections I.5.4-I.5.5].)

We begin with the building blocks from which the "test cases" showing this will be put together.

Lemma 6. *Let B be a boolean lattice with > 2 elements. Then as an extension of $B - \{0, 1\}$, the lattice* $\mathrm{Free}(B - \{0, 1\})$ *is isomorphic to B.*

Proof. Let $P = B - \{0,1\}$. The only joins that P is missing are those that in B yield 1; likewise, the only missing meets are those that yield 0. We shall show that all pairs of elements which had join 1 in B give equal joins in any lattice L into which we map P by a homomorphism of partial lattices. By symmetry, the dual statement holds for 0 and meets. Hence the free lattice on P just restores these two elements, i.e., it is naturally isomorphic to B.

So suppose we are given a map of P into a lattice L, which preserves the meets and joins of P. By abuse of notation, we shall use the same symbols for elements of P and their images in L.

Let us first consider any two elements $a, b \in P$ which are distinct in P from each other and from each other's complements, and compare the joins $a \vee a^{\mathrm{c}}$ and $b \vee b^{\mathrm{c}}$ in L (writing $(\ )^{\mathrm{c}}$ for complementation in B).

Note that in B, we have

$$a \;=\; (a \wedge b) \vee (a \wedge b^{\mathrm{c}}) \quad \text{and} \quad a^{\mathrm{c}} \;=\; (a^{\mathrm{c}} \wedge b) \vee (a^{\mathrm{c}} \wedge b^{\mathrm{c}}). \tag{17}$$

If the four meets appearing in these two expressions are all nonzero, then they belong to P, and the relations (17) hold there, and hence in L. In this situation, if we expand $a \vee a^{\mathrm{c}}$ in L using these two formulas, we can rearrange the result as $((a \wedge b) \vee (a^{\mathrm{c}} \wedge b)) \vee ((a \wedge b^{\mathrm{c}}) \vee (a^{\mathrm{c}} \wedge b^{\mathrm{c}}))$, which (by (17) with a and b interchanged) simplifies to $b \vee b^{\mathrm{c}}$, giving the desired equality. On the other hand, if any of the four pairwise meets of a and a^{c} with b and b^{c} is zero, this can, under the assumptions made in the preceding paragraph, be true only of one such meet; say $a \wedge b = 0$. Then we can repeat the above computation, everywhere omitting "$(a \wedge b) \vee$". (Thus, we have a version of (17) with the first equation simplified to $a = a \wedge b^{\mathrm{c}}$, and the second unchanged.) With this slight modification, our computation still works, and again gives $a \vee a^{\mathrm{c}} = b \vee b^{\mathrm{c}}$.

So let us write i for the common value, for all $a \in P$, of $a \vee a^{\mathrm{c}} \in L$. We now consider two elements $a, b \in P$ which are not assumed to be complements, but whose join in B is 1. This relation implies that $b \geq a^{\mathrm{c}}$; note that both these terms lie in P. Hence in L we have $a \vee b \;\geq\; a \vee a^{\mathrm{c}} \;=\; i$, while the reverse inequality holds because $a \leq i,\ b \leq i$. Thus, $a \vee b = i$, completing the proof that all pairs of elements having join 1 in B have the same join, namely i, in L. □

Let us now consider, independent of the above result, the same inclusion $B - \{0,1\} \subseteq B$ in the context of isotone maps.

Lemma 7. *Let M be a lattice, X a nonempty subset of M, and B the free Boolean lattice on X. Then there exists an isotone map $\varphi\colon B - \{0,1\} \to M$ with the property that*

The pairs of elements $y, z \in M$ such that φ can be extended to an isotone map $\bar{\varphi}\colon B \to M$ taking 0 to y and 1 to z, are precisely those for which y is a lower bound, and z an upper bound, for X in M. (18)

Proof. Let B have free generators g_x for $x \in X$. For every $a \in B - \{0,1\}$, let $\varphi(a)$ be the join in M of all elements of the form

$$x_1 \wedge \cdots \wedge x_n \tag{19}$$

where $n \geq 1$, and $x_1, \ldots, x_n$ are distinct elements of X such that for some choice of $\varepsilon_1, \ldots, \varepsilon_n \in \{1, \mathrm{c}\}$, we have

$$a \ \geq \ g_{x_1}^{\varepsilon_1} \wedge \cdots \wedge g_{x_n}^{\varepsilon_n}. \tag{20}$$

Here for $b \in B$, b^ε denotes b if $\varepsilon = 1$, and b^{c} if $\varepsilon = \mathrm{c}$.

Because $a \neq 0$, the set of instances of (20) is nonempty, hence so is the set of joinands (19). This set is in general infinite; however, if we take the least subset $X_0 \subseteq X$ such that a is in the Boolean sublattice generated by the g_x with $x \in X_0$, then X_0 is finite and (since $a \neq 0, 1$) nonempty; and we find that the irredundant relations (20) (those relations (20) from which no meetand can be dropped) involve only terms g_x^ε with $x \in X_0$. Thus, each expression (19) in our description of $\varphi(a)$ is majorized by one that arises from one of these finitely many irredundant relations (20); so the join describing $\varphi(a)$ is effectively a finite join, and so exists in M.

It is not hard to see from our definition that φ is isotone, and that for all $x \in X$, $\varphi(g_x) = \varphi(g_x^{\mathrm{c}}) = x$.

Suppose now that we have an extension $\bar{\varphi}\colon B \to M$ of this isotone map φ. Then for every $x \in X$, $\bar{\varphi}(1) \geq \varphi(g_x) = x$, so $\bar{\varphi}(1)$ is an upper bound of X. Conversely, any upper bound for X in M will majorize all elements (19), and hence all joins of such elements, hence will indeed be an acceptable choice for a value of $\bar{\varphi}(1)$ making $\bar{\varphi}$ isotone. Though our construction of φ is not symmetric in $\vee$ and $\wedge$, the duals of these observations are easily seen to hold, so the choices for $\bar{\varphi}(0)$ are, likewise, the lower bounds of X. □

Note that (18) above can be summarized as saying that

The upper and lower bounds in M of $\varphi(B - \{0,1\})$ are the same as the upper and lower bounds in M of X. (21)

In our next result, for any two partial lattices P and Q, we will denote by $P + Q$ the disjoint union of P and Q, made a partial lattice using the partial meet and join operations of P and Q, together with the further meet and join relations corresponding to the condition that every element of P be majorized by every element of Q (namely, $p \wedge q = p$ and $p \vee q = q$ for all $p \in P,\ q \in Q$). It is not hard to see that

$$\mathrm{Free}(P + Q) \ \cong \ \mathrm{Free}\, P + \mathrm{Free}\, Q. \tag{22}$$

Theorem 8. *Let M be a lattice. Then the following conditions are equivalent.*

M is complete. (23)

Any isotone map from a partial lattice P to M can be extended to an isotone map $\mathrm{Free}\, P \to M$. (24)

For B a free Boolean lattice on a nonempty set, any isotone map $B_1 - \{0,1\} \to M$ can be extended to an isotone map $B \to M$; and for B_1, B_2 any two free Boolean lattices on nonempty sets, any isotone map $(B_1 - \{0,1\}) + (B_2 - \{0,1\}) \to M$ can be extended to an isotone map $B_1 + B_2 \to M$. (25)

Proof. (23) $\Longrightarrow$ (24) is a case of [1, Proposition 1, (i) $\Longrightarrow$ (ii)], which says that every complete lattice is injective with respect to inclusions of orders. In view of Lemma 6 and (22), the implication (24) $\Longrightarrow$ (25) is clear.

To complete the argument, assume (25).

Calling on the first statement of (25), together with the case $X = M$ of the preceding lemma, we see that M must have a greatest and a least element.

Now take any nonempty subset $X_1 \subseteq M$, let X_2 be the set of its upper bounds (which is nonempty, since M has a greatest element), let B_1 be the free Boolean lattice on X_1, and let B_2 be the free Boolean lattice on X_2. Map $B_1 - \{0,1\}$ and $B_2 - \{0,1\}$ into M by maps φ_1, φ_2 satisfying (18) with respect to X_1 and X_2, respectively. By the equivalence of (18) and (21), $\varphi_1(B_1 - \{0,1\})$ is majorized by all upper bounds of X_1, i.e., by all elements of X_2, hence (again using that equivalence) by all elements of $\varphi_2(B_2 - \{0,1\})$; so φ_1 and φ_2 together constitute an isotone map $\varphi\colon (B_1 - \{0,1\}) + (B_2 - \{0,1\}) \to M$. Extending this to the free lattice $B_1 + B_2$ on that partial lattice, we see that the image of the 1 of B_1 (and likewise that of the 0 of B_2) will be both an upper bound of X_1 and a lower bound of X_2, hence must be a least upper bound of X_1. So M is upper semicomplete.

By symmetry (or by the known fact that in a lattice with 0 and 1, upper semicompleteness and lower semicompleteness are equivalent), M is also lower semicomplete, establishing (23). □

5. Lattices amalgamated over convex retracts

The results of the preceding section show that Theorem 3, looked at as a property of the inclusion of a certain kind of partial lattice P in $\mathrm{Free}_{\mathbf{V}}\, P$, does not go over to the inclusion of a general partial lattice P in its free lattice. Can we describe other interesting partial lattices P for which it does?

In proving Theorem 3, after reducing to the case where the given lattices contained elements e_i that mapped to the same element of M, we effectively proved that the free lattice on the union of those lattices with amalgamation of the e_i had the desired extension property. The next theorem will slightly generalize this result, replacing the singletons $\{e_i\}$ with any family of isomorphic sublattices that are both retracts of the L_i, and convex therein. We will need the following observation.

Lemma 9. *Let M be a lattice, and ϱ a lattice-theoretic retraction of M to a convex sublattice. Then if an element $x \in M$ is majorized by some element of $\varrho(M)$, then it is majorized by $\varrho(x)$.*

Proof. Say $x \leq r \in \varrho(M)$. Applying ϱ to this relation, and taking the join of the original relation with the resulting one, we get $x \vee \varrho(x) \leq r$. Hence $x \vee \varrho(x)$ lies in the interval between $\varrho(x)$ and r, so as $\varrho(M)$ is assumed convex, $x \vee \varrho(x) \in \varrho(M)$. This means that $x \vee \varrho(x)$ is fixed under the idempotent lattice homomorphism ϱ; but its image under that map is $\varrho(x) \vee \varrho(x) = \varrho(x)$. Thus, $x \vee \varrho(x) = \varrho(x)$, which is equivalent to the desired conclusion $x \leq \varrho(x)$. □

In the above lemma, the assumption that ϱ is a lattice homomorphism could have been weakened to say that it is a join-semilattice homomorphism. We have stated it as above for conceptual simplicity, and because in the proof of the next result, the maps ϱ_i must be lattice homomorphisms anyway.

Theorem 10. *Let $(L_i \mid i \in I)$ be a family of lattices which are disjoint except for a common sublattice K, which is convex in each L_i, and is a retract of each L_i via a lattice-theoretic retraction $\varrho_i \colon L_i \to K$.*

Let P denote the partial lattice given by the union of the L_i with amalgamation of the common sublattice K, and let $L = \mathrm{Free}_{\mathbf{V}}\, P$, where $\mathbf{V}$ is any variety containing all the L_i.

Then for any lattice M (not necessarily belonging to $\mathbf{V}$) given with isotone maps $\varphi_i \colon L_i \to M$ agreeing on K, there exists an isotone map $\varphi \colon L \to M$ extending all the φ_i.

In other words, every isotone map $P \to M$ extends to L.

Proof. Let us assume that I does not contain the symbol 0, and use 0 to index the factor K in $K \times (\prod L_i \mid i \in I)$. Now let L' denote the sublattice of that direct product consisting of those elements f such that $f(i) = f(0)$ for almost all i, and $\varrho_i(f(i)) = f(0)$ for all i. Then we can map each L_i into L' by sending $x \in L_i$ to the element having i-th coordinate x, and having $\varrho_i(x)$ for all other coordinates (including the 0-th coordinate). These maps are lattice homomorphisms (this is where we need the ϱ_i to be lattice homomorphisms and not just join-semilattice homomorphisms), which agree on K; hence they extend to a lattice homomorphism $L \to L'$.

We shall now map L' isotonely to M using the idea of Lemma 2. Namely, given $f \in L'$, we define

$$\psi(f) \;=\; \begin{cases} \bigwedge(\, \varphi_i(f(i)) \mid i \in I\,) & \text{if for all } i \in I, f(i) \leq f(0), \\ \bigvee(\, \varphi_i(f(i)) \mid i \in I,\ f(i) \not\leq f(0)\,) & \text{otherwise.} \end{cases} \tag{26}$$

These are defined because for each f, all but finitely many $i \in I$ have $\varphi_i(f(i))$ equal to the common image of $f(0)$ in M under the φ_j. (Recall that $f(0) \in K$, and all φ_j agree on K.) We now claim that

$$\psi \text{ is isotone,} \tag{27}$$

and

$$\text{for every } i \in I, \text{ and every } f \in L' \text{ such that } f(j) = f(0) \text{ for all } j \neq i, \text{ we have } \psi(f) = \varphi_i(f(i)). \tag{28}$$

Assertion (28) is clear. The proof of (27) is exactly like that of the corresponding statement, (3), in the proof of Lemma 2, once we know the analog of (6), namely

$$\text{for } f \leq g \text{ in } L', \text{ we have } \{i \mid f(i) \not\leq f(0)\} \;\subseteq\; \{i \mid g(i) \not\leq g(0)\}. \tag{29}$$

To prove (29), consider any i not lying in the right-hand side. Then

$$f(i) \;\leq\; g(i) \;\leq\; g(0) \in \varrho(M), \tag{30}$$

so by Lemma 9, $f(i) \leq \varrho(f(i)) = f(0)$, showing that i also fails to lie in the left-hand set.

Composing ψ with the map $L \to L'$ of the first paragraph of this proof, we get our desired isotone map $L \to M$. □

(If we think of the constant e of Lemma 2 as "sea level", then the $f(0)$ of the above proof brings in "tides".)

We remark that though in the free-lattice-with-amalgamation L of the above proof, K is necessarily a retract, since it was a retract in each of the L_i, it does not follow similarly that K is convex in L. To see this, let us first note an example of a lattice L' having a sublattice K which is convex and a retract in each of two sublattices L_0 and L_1 containing K, but is not convex in the sublattice that these generate. Let L' be the lattice of all subspaces of a 3-dimensional vector space V over any field, let $K = \{\{0\}, a\}$ where a is a 2-dimensional subspace of V, and let each L_i be the sublattice generated by a and a 1-dimensional subspace b_i not contained in a, with $b_0 \neq b_1$. Then the stated hypotheses are satisfied, but $0 < (b_0 \vee b_1) \wedge a < a$, so K is not convex in the lattice generated by L_1 and L_2. It easily follows that in the free product L of L_0 and L_1 with amalgamation of $K = \{0, 1\}$, we likewise have $0 < (b_0 \vee b_1) \wedge a < a$ with the middle term not in K.

6. Semilattice variants—two easy results

In our main theorem, free products of a family of lattices L_i, whose normal role is to admit a lattice homomorphism extending a given family of lattice homomorphisms on the L_i, were made to do the same for isotone maps (homomorphisms of orders). One might expect it to be easier to get similar results if the gap between lattices and orders is replaced by one of the smaller gaps between lattices and semilattices, or between semilattices and orders.

For the latter case, the result is indeed easy; it is only for parallelism with our other results that we dignify it with the title of theorem.

Theorem 11. *Let $(L_i \mid i \in I)$ be a family of join-semilattices, and $\varphi_i \colon L_i \to M$ a family of isotone maps from the L_i to a common join-semilattice. Let L denote the free product of the L_i as join-semilattices. Then there exists an isotone map $\varphi \colon L \to M$ whose restrictions to the $L_i \subseteq L$ are the φ_i.*

Proof. The general element $x \in L$ is a formal join $x_{i_1} \vee \cdots \vee x_{i_n}$ of elements $x_{i_m} \in L_{i_m}$, where $i_1, \ldots, i_n$ are a finite nonempty family of distinct indices in I. If we send each such x to $\varphi_{i_1}(x_{i_1}) \vee \cdots \vee \varphi_{i_n}(x_{i_n})$, this is easily seen to have the desired properties. □

There was no analog, in the above result, to the $\mathbf{V}$ of Theorem 3, since the variety of semilattices has no proper nontrivial subvarieties.

On the other hand, if we wish to get an analog of Theorem 3 with the L_i and M again lattices, but for semilattice homomorphisms, rather than isotone maps, we may again start with lattices L_i in an arbitrary lattice variety $\mathbf{V}$. For this situation the authors have not been able to prove the full analog of Theorem 3. The difficulty with adapting our proofs of that theorem is that the map ψ of Lemma 2, though isotone, does not respect joins; nor do the variant constructions of Section 3.

The map of Lemma 2 does, however, respect joins when the e_i are least elements in the $\bar{L}_i$. In that case, the composite $L' \to M' \to M$ reduces to the map (1) in our sketch of the "easy case" of Theorem 3, and we find that if the φ_i are join-semilattice homomorphisms, that composite will also be one. Hence we get

Proposition 12. *Let $\mathbf{V}$ be a nontrivial variety of lattices, $\mathcal{L} = (L_i \mid i \in I)$ a family of lattices in $\mathbf{V}$, $L = \mathrm{Free}_{\mathbf{V}}\,\mathcal{L}$, and $\varphi_i \colon L_i \to M$ a family of join-semilattice homomorphisms from the L_i to a common lattice, not necessarily belonging to $\mathbf{V}$.*

If the image-sets $\varphi_i(L_i)$ have a common lower bound $e \in M$, then there exists a join-semilattice homomorphism $\varphi \colon L \to M$ whose restrictions to the $L_i \subseteq L$ are the φ_i. In particular, this is so if M has a least element, or if I is finite and every L_i has a least element. □

One could modify this result in the spirit of Theorem 10, assuming that each L_i has a retraction ϱ_i to a common *ideal* K on which the φ_i agree.

In another direction, the condition in Proposition 12 that there exist a common lower bound e in M to all the $\varphi_i(L_i)$ can be weakened slightly (for I infinite) to say that M contains a chain C such that every $\varphi_i(L_i)$ is bounded below by some member of C. Let us sketch the argument that gets this, by transfinite induction, from the statement as given. First, by passing to a subchain, assume without loss of generality that C is dually well-ordered. Then apply Proposition 12, first, to those L_i such that $\varphi_i(L_i)$ is bounded below by the top element, c_0, of C, concluding that those φ_i can be factored through some lattice $L'_{(0)}$ in $\mathbf{V}$. Then go to the next member, c_1, of C, and combine $L'_{(0)}$ with all the L_i that are bounded below by c_1 but not by c_0, factoring these together through a lattice $L'_{(1)} \in \mathbf{V}$; and so on. As in the discussion following (12), we take the union of the preceding steps whenever we hit a limit ordinal.

7. Semilattice variants—a harder result

What if we have nothing like the lower-bound condition of Proposition 12?

For free products taken in the variety $\mathbf{L}$ of all lattices, the analog of that proposition, without the lower bound condition, is obtained in [11, middle of p. 239, "We note finally ..."]. Indeed, the map f used in [11] to prove Theorem 3 for $L = \operatorname{Free}\mathcal{L}$ has the property that $f(x \vee y) = f(x) \vee f(y)$ *except* possibly when x and y are bounded below by elements $x_{(i)}$, $y_{(i)} \in L_i$ for some i, and $\varphi_i(x_{(i)} \vee y_{(i)}) > \varphi_i(x_{(i)}) \vee \varphi_i(y_{(i)})$. (Cf. [11, p. 238, (ii)].) If the φ_i are join-semilattice homomorphisms, that strict inequality never occurs, so φ is also a join-semilattice homomorphism.

If $\mathbf{V}$ is a nontrivial variety of lattices containing the L_i, we do not know whether the corresponding result holds for the free product of the L_i in $\mathbf{V}$, but we shall show below that we can get such a result for their free product in the larger class $\mathbf{D} \circ \mathbf{V}$ (definition recalled in (31) and (32) below). Our construction will be similar in broad outline to those used in preceding sections, but the intermediate lattice L', rather than being a subdirect product, will be a certain lattice of downsets in a direct product.

We recall the definition:

If $\mathbf{K}_1$ and $\mathbf{K}_2$ are classes of lattices, then the class of those lattices L which admit homomorphisms $\varepsilon \colon L \to L_2$ such that $L_2 \in \mathbf{K}_2$, and such that the inverse image of every element of L_2 lies in $\mathbf{K}_1$, is denoted $\mathbf{K}_1 \circ \mathbf{K}_2$. (31)

The class $\mathbf{K}_1 \circ \mathbf{K}_2$ is often called the *product* of the classes $\mathbf{K}_1$ and $\mathbf{K}_2$, but we will not use that name here, to avoid confusion with direct products and free products of lattices.

If $\mathbf{K}_1$ and $\mathbf{K}_2$ are varieties, the class $\mathbf{K}_1 \circ \mathbf{K}_2$ need not be a variety; but as noted in [13], if $\mathbf{K}_1$ and $\mathbf{K}_2$ are prevarieties or quasivarieties (classes closed under taking direct products and sublattices; respectively, under taking direct products, ultraproducts, and sublattices), then $\mathbf{K}_1 \circ \mathbf{K}_2$ will also be a prevariety, respectively a quasivariety. In particular, if $\mathbf{K}_1$ and $\mathbf{K}_2$ are varieties, $\mathbf{K}_1 \circ \mathbf{K}_2$ is, at least, a quasivariety.

We also recall the standard notation:

The variety of distributive lattices is denoted $\mathbf{D}$. (32)

Now suppose $(L_i \mid i \in I)$ is a family of lattices. To begin the construction of the lattice L' that we shall use in proving our final result, let us adjoin to each L_i a new top element, 1_i, form the direct product $\prod(L_i + \{1_i\})$, and define the subset

$$P \;=\; \{f \in \textstyle\prod(L_i + \{1_i\}) \mid \{i \mid f(i) \neq 1_i\} \text{ is finite but nonempty}\}. \tag{33}$$

The condition that $\{i \mid f(i) \neq 1_i\}$ be nonempty means that we are excluding the top element of $\prod(L_i + \{1_i\})$; hence if $|I| > 1$, P is not a lattice, though it is a lower semilattice.

For each $i \in I$, let us define a map $\theta_i \colon L_i \to P$ by

$$\theta_i(x)(i) = x, \qquad \theta_i(x)(j) = 1_j \quad \text{for } j \neq i. \tag{34}$$

We see that every element of $p \in P$ has a representation

$$p \;=\; \theta_{i_1}(x_1) \wedge \cdots \wedge \theta_{i_n}(x_n) \quad \text{with } n > 0, \tag{35}$$

unique up to order of terms, where $i_1, \dots, i_n$ are distinct elements of I, and $x_m \in L_{i_m}$.

We now let

$$L' \;= \text{the set of all nonempty finitely generated downsets } F \subseteq P \text{ such that} \tag{36}$$

$$\text{for all } i \in I \text{ and } x,\, y \in L_i, \text{ if } \theta_i(x),\, \theta_i(y) \in F, \text{ then } \theta_i(x \vee y) \in F. \tag{37}$$

Thus

Each element $F \in L'$ is the union of the principal downsets $\downarrow(\theta_{i_1}(x_1) \wedge \ldots \wedge \theta_{i_n}(x_n))$ determined by its finitely many maximal elements $\theta_{i_1}(x_1) \wedge \ldots \wedge \theta_{i_n}(x_n)$. Moreover, for each i, F can have at most one such maximal element of the form $\theta_i(x)$ (i.e., with $n = 1$, and with the one meetand arising from L_i). (38)

The last sentence above follows from (37): Given distinct $\theta_i(x)$, $\theta_i(y) \in F$, we also have $\theta_i(x \vee y) \in F$, so $\theta_i(x)$ and $\theta_i(y)$ cannot both be maximal in F.

Let us now prove

Lemma 13. *Let $(L_i \mid i \in I)$ be a family of lattices, and let L' be constructed as in* (33)–(37) *above. Then*

L', partially ordered by inclusion, is a lattice. (39)

For each $i \in I$, the map $\xi_i \colon L_i \to L'$ defined by $\xi_i(x) = \downarrow\theta_i(x)$ is a lattice homomorphism. (40)

If $\mathbf{V}$ is any prevariety containing all the L_i, then $L' \in \mathbf{D} \circ \mathbf{V}$. (41)

Proof. In verifying (39), the only points that need a moment's thought are (i) that the intersection $F \cap G$ of two sets as in (38) remains nonempty and finitely generated; but indeed, in any meet-semilattice, the intersection of two nonempty finitely generated downsets $\bigcup \downarrow p_i$ and $\bigcup \downarrow q_j$ is the nonempty finitely generated downset $\bigcup \downarrow(p_i \wedge q_j)$; (ii) that the closure operation of (37) cannot produce the element $(1_i)_{i \in I} \notin P$; this follows from the fact that each L_i is closed under joins in $L_i + \{1_i\}$; and (iii) that repeated application of that operation when we form a join $F \vee G$ cannot lead to a violation of finite generation as a downset. This is clear once we observe that in constructing $F \vee G$ from $F \cup G$, it is enough to apply the closure operation of (37) to pairs consisting of one of the finitely many maximal elements of F and one of the finitely many maximal elements of G (and then close as a downset).

Statement (40) is easily checked. (Here (37) guarantees that ξ_i respects joins — that is the point of that condition.)

To show (41), let us now adjoin to each L_i a bottom element 0_i, and define maps $\pi_i \colon L' \to \{0_i\} + L_i$ as follows. For $F \in L'$,

$$\begin{array}{l}\text{If there are elements } x \in L_i \text{ such that } \theta_i(x) \in F, \text{ let } \pi_i(F) \text{ be the largest such } x \text{ (cf. second sentence of (38)).} \\ \text{If there are no such } x, \text{ let } \pi_i(F) \ = \ 0_i.\end{array} \tag{42}$$

In view of (37), each π_i is a homomorphism; hence together they give us a homomorphism $\pi \colon L' \to \prod(\{0_i\} + L_i \mid i \in I) \in \mathbf{V}$.

We claim that the inverse image under π of each $f \in \prod(\{0_i\} + L_i \mid i \in I)$ is distributive. Indeed, when we take the join of two elements F, $G \in \pi^{-1}(f)$, we see that for each i, the sets F and G agree in what elements $\theta_i(x)$ they contain, hence there is no occasion for enlarging $F \cup G$ via (37). So $F \vee G = F \cup G$. We always have $F \wedge G = F \cap G$ in L'; hence $\pi^{-1}(f)$ is a lattice of subsets of P under unions and intersections, hence it is distributive. Thus, $L' \in \mathbf{D} \circ \mathbf{V}$, as claimed. □

Now suppose that for each $i \in I$ we are given an upper semilattice homomorphism $\varphi_i \colon L_i \to M$, for a fixed lattice M. We define $\psi \colon L' \to M$ by

$$\psi(F) \ = \ \bigvee(\, \varphi_{i_1}(x_1) \wedge \ldots \wedge \varphi_{i_n}(x_n) \ \mid \ \theta_{i_1}(x_1) \wedge \ldots \wedge \theta_{i_n}(x_n) \in F\,). \tag{43}$$

This is formally an infinite join; but it is clearly equivalent to the corresponding join over the finitely many maximal elements of F, hence is defined.

We claim that

$$\psi \text{ is a join-semilattice homomorphism.} \tag{44}$$

To see this, note that if we temporarily extend the definition (43) to arbitrary finitely generated downsets F, not necessarily satisfying (37), then we have

$$\psi(F \cup G) \ = \ \psi(F) \vee \psi(G). \tag{45}$$

Now for F, $G \in L'$, the element $F \vee G$ is obtained by bringing into $F \cup G$ elements $\theta_i(x \vee y)$ where $\theta_i(x) \in F$ and $\theta_i(y) \in G$ (and the elements they majorize). In this situation, the join defining $\psi(F \cup G)$ already contains joinands $\varphi_i(x)$ and $\varphi_i(y)$, resulting from the presence of $\theta_i(x)$ and $\theta_i(y)$ in F and G, hence its value in M already majorizes $\varphi_i(x) \vee \varphi_i(y) = \varphi_i(x \vee y)$. So bringing $\theta_i(x \vee y)$ into $F \cup G$ does not increase its image under ψ, establishing (44).

Finally, comparing the definition (40) of the ξ_i and the definition (43) of ψ, we see that

$$\text{For all } i \in I, \quad \varphi_i \ = \ \psi\, \xi_i. \tag{46}$$

Now given $\mathbf{V}$ as in (41), let $\mathcal{L} = (L_i \mid i \in I)$ and $L = \mathrm{Free}_{\mathbf{D} \circ \mathbf{V}}\, \mathcal{L}$. Then the lattice homomorphisms $\xi_i \colon L_i \to L'$ are equivalent to a single homomorphism $\xi \colon L \to L'$; and we see that by taking $\varphi = \psi\, \xi \colon L \to L' \to M$, we get our desired result:

Theorem 14. *Let $\mathcal{L} = (L_i \mid i \in I)$ be a family of lattices, and $\varphi_i \colon L_i \to M$ a family of join-semilattice homomorphisms from the L_i to a common lattice. Suppose all L_i lie in some prevariety $\mathbf{V}$ of lattices, and let $L = \mathrm{Free}_{\mathbf{D}\circ\mathbf{V}}\,\mathcal{L}$. Then there exists a join-semilattice homomorphism $\varphi \colon L \to M$ whose restrictions to the L_i are the φ_i.* □

Let us show now by example that the lattice L' constructed in the above proof may fail to lie in $\mathbf{V}$ itself. We start with two distributive lattices, namely, the one-element lattice $L_0 = \{e\}$, and the four-element lattice L_1 generated by two elements a and b. Let us use bar notation for the images of these generators under the embeddings $\xi_i \colon L_i \to L'$, so that $\bar{e} = \xi_0(e) = {\downarrow}(e, 1_1)$, $\bar{a} = \xi_1(a) = {\downarrow}(1_0, a)$, $\bar{b} = \xi_1(b) = {\downarrow}(1_0, b)$. We claim that in L',

$$\bar{e} \wedge (\bar{a} \vee \bar{b}) \;\neq\; (\bar{e} \wedge \bar{a}) \vee (\bar{e} \wedge \bar{b}). \tag{47}$$

Indeed, one finds that the left-hand side of (47) is the principal down-set ${\downarrow}(e, a \vee b)$, while the right-hand side is $({\downarrow}(e, a)) \cup ({\downarrow}(e, b))$, a nonprincipal down-set. Hence L' is not distributive. It is not even modular: one can similarly verify that a copy of N_5 is given by the elements

$$(\bar{e}\wedge\bar{a})\vee(\bar{e}\wedge\bar{b})\vee(\bar{a}\wedge\bar{b}),\quad \bar{a}\vee(\bar{e}\wedge\bar{b}),\quad \bar{a}\vee(\bar{e}\wedge(\bar{a}\vee\bar{b})),\quad \bar{a}\vee\bar{b},\quad (\bar{e}\wedge\bar{a})\vee\bar{b}. \tag{48}$$

Evidence suggesting that the task of extending semilattice homomorphisms from a family of lattices in $\mathbf{V}$ to their free product in $\mathbf{V}$ is likely to be harder than the corresponding task for isotone maps is [2, Theorem 1] = [12, Theorem 2.8], which says that the injective objects in the category of meet-semilattices are the *frames*, i.e., the complete lattices satisfying the join-infinite distributive identity. (This result is generalized in [16, Theorem 3.1].) Thus, dually, the injective join-semilattices are the complete lattices satisfying the meet-infinite distributive identity; in particular, they are distributive; so Theorem 14 does not "almost" follow from a general injectivity statement, as Theorem 3 did.

(While on the topic of injective objects, what are the injectives in the variety of lattices? It is shown in [1, next-to-last paragraph] that the only one is the trivial lattice. This is generalized in [3] to any nontrivial variety $\mathbf{V}$ of lattices other than the variety of distributive lattices, and in [14], with a very quick proof, to any class of lattices containing a 3-element chain and a nondistributive lattice.)

8. Questions

The example following Theorem 14 does not mean that there is no way to factor a family of maps as in that theorem through the free product of the L_i in $\mathbf{V}$; only that the construction by which we have proved that theorem doesn't lead to such a factorization. Indeed, for that particular pair of lattices, one does have such a factorization, by the final clause of Proposition 12. So we ask

Question 1. *For* $\mathbf{V}$ *a general nontrivial variety of lattices, can one prove the full analog of Theorem 3 with join-semilattice homomorphisms in place of isotone maps* (*i.e., a result like Theorem 14 with* $\mathbf{V}$ *in place of* $\mathbf{D} \circ \mathbf{V}$*; equivalently, a result like Proposition 12 without the assumptions on lower bounds*)*?*

If that result is not true in general, is it true if M *also belongs to the given variety* $\mathbf{V}$*?*

A counterexample to either version of the above question would probably have to be fairly complicated, in view of Proposition 12.

In a different direction, note that in our main result, Theorem 3, the assumption that M had a lattice structure did not come into the statement, except to make the concept of isotone map meaningful, for which a structure of order would have sufficed; though the lattice structure was used in the proof. The same observation applies to many of our other results. This suggests a family of questions.

Question 2. *For each of Lemmas 1 and 2 and Theorems 3, 8, and 10, does the same conclusion hold for a significantly wider class of orders* M *than the underlying orders of lattices?*

Likewise, for Proposition 12 and Theorem 14, does the same conclusion hold for a significantly wider class of join-semilattices M *than the underlying join-semilattices of lattices?*

When we showed in Section 3 that our main result could not be proved by a construction with "too much symmetry", we called on the fact that in a free lattice in the variety $\mathbf{L}$ of all lattices, no element is doubly reducible (both a proper meet and a proper join; see sentence following display (13)). Lattices (not necessarily free) with the latter property were considered in [7]. We do not know the answer to

Question 3. *Are there any nontrivial proper subvarieties* $\mathbf{V}$ *of* $\mathbf{L}$ *such that in every free lattice* $\mathrm{Free}_{\mathbf{V}}(X)$*, no element is doubly reducible?*

We also record the question arising at the end of Section 2.

Question 4. *In the first paragraph of Corollary 4, can the isotone map*

$$\mathrm{Free}_{\mathbf{V}}\,\mathcal{L} \to \mathrm{Free}\,\mathcal{L}$$

be taken to be a section (*left inverse*) *to the natural lattice homomorphism* $\mathrm{Free}\,\mathcal{L} \to \mathrm{Free}_{\mathbf{V}}\,\mathcal{L}$*?*

9. An afterthought

Readers familiar with the study of free products in the variety $\mathbf{L}$ of all lattices will be aware of the interplay in that theory between the full forms of lattice terms, and "upper and lower covers" of those terms in the lattices L_i

[11, Definition 2], [8, p. 356, Definition 3], [9, Definition 521]. Looking back at the present paper, the authors find that our basic idea can be thought of as that, when constructing isotone maps, one may ignore altogether the full forms of the words, and consider only (something like) their "covers".

This has the advantage of preserving identities satisfied by the L_i, and of making the construction fairly transparent.

On the other hand, it has required us to introduce the elements e_i to serve when "covers" are not available, and also to use the asymmetric definition (5), where for free products in $\mathbf{L}$, one can use formulas that depend (symmetrically) on whether the full form of a reduced lattice word is a meet or a join [11, Definition 2, and p. 238, conditions (ii) and (iii)]. In a general lattice variety $\mathbf{V}$, there is no such meet-join dichotomy for elements of a free product, as shown for $\mathbf{D}$ by (13). (Question 3 above essentially asks whether such a dichotomy can hold in *any* lattice variety other than $\mathbf{L}$.)

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