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NONLINEAR VISCOELASTIC ANALYSIS OF A CENTRALLY LOADED COLUMN

by

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Summary

A Volterra-Fréchet functional expansion is used to represent the stress-strain relation of a general nonlinear viscoelastic material. A procedure is given for the experimental determination of the material kernels appearing in the expansion. Application is made to the investigation of the stability of a centrally loaded nonlinear viscoelastic column.

By investigating the singularities of the integral equation governing the deflection of the column subjected to a small perturbation, general conditions for instantaneous stability are derived. Asymptotic stability of the column is also investigated, and by the use of a general Tauberian theorem, conditions which ensure long time stability of the column are obtained. Results based on a generalized version of the particular nonlinear viscoelastic stress-strain relations proposed by Leaderman and Rabotnov are also presented.

I - Introduction

Physical systems whose response depends on the previous history of the input to the system are common in various fields of physics and engineering. Phenomena governed by such systems were called "hereditary phenomena" by Volterra [1].* In the study of the mechanics of solids, materials which behave in such a manner are termed viscoelastic. The theory for the case in which the stress and strain histories of a viscoelastic material are related through a linear functional is now quite well developed, apparently having been formulated first by Boltzmann in 1874 [2].

The classical linear theory of viscoelasticity has proved very useful and has been employed to explain and predict many phenomena of interest. But there are frequent instances in which the linear theory fails, and in which a nonlinear hereditary theory is clearly required. Many specific nonlinear constitutive relations for various viscoelastic materials have been proposed, and these usually take the form of particular nonlinear differential or integral equations relating the stress with the strain [3,4,5]. Often, such relations, although appropriate for certain specific materials, fail to describe with sufficient accuracy the behavior of many important materials [6].

In such cases, it may prove convenient to deal with a constitutive equation sufficiently general to represent the behavior of a larger class of nonlinear viscoelastic materials. Volterra discussed such general laws, and considered them within the context of the general

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* References are listed at the end of the paper.

theory of functionals [1]. A convenient representation of general analytical functionals is the Volterra-Fréchet functional power series [1]. By means of this representation, the character of a general nonlinear viscoelastic material may be quantitatively described. Interest has been revived in the applications of such a representation to the construction of a general nonlinear theory of viscoelasticity [7], to the experimental determination of the constitutive relation for nonlinear viscoelastic materials [6,8], and to the solution of some simple boundary value problems of nonlinear viscoelasticity [9,10].

To gain some insight into the practical difficulties or advantages associated with the use of the Volterra-Fréchet expansion, a specific application is considered here. The general nonlinear viscoelastic theory mentioned above includes, besides physically nonlinear viscoelastic behavior, the possibility of large deformations of a three-dimensional continuum. Here, attention is restricted to a one-dimensional problem in which the strains are infinitesimal, but the viscoelastic material exhibits a general nonlinear relationship between stress and strain. Specifically, the stability of a centrally loaded bar is investigated, under the assumption of quasi-static behavior.

The behavior of a centrally loaded viscoelastic bar subjected to a small disturbance was first studied by Robotnov and Shesterikov [11]. In that paper, an equation of state for the material constitutive relation involving the stress, inelastic strain and inelastic strain rate is assumed to exist. Later, Rabotnov [12], using a particular nonlinear integral representation, studied the possibility of bifurcation of the solution of the equation governing the disturbed bar.

In 1962, Onat and Wang [13] discussed the stability of the bar assuming a linear, second order, constant coefficient, ordinary differential equation for the relationship between the increment of stress history and the induced increment of strain history.

In this paper, a more general constitutive law is considered. The first part of the paper deals with various representations of the general, one-dimensional, nonlinear viscoelastic stress-strain relation. Since the stability of the column is here tested by observing the behavior induced by a small lateral perturbation, then only the linearized relationship between the small increment of stress and the small increment of strain caused by the perturbation is required, and it is derived from the general nonlinear viscoelastic constitutive relation. This relationship is expressed by means of a linear integral equation whose kernel embodies the entire past history of stress or strain, and the entire complex of time dependent material functions of a general, one-dimensional, nonlinear viscoelastic medium. This indicates that the incremental response of the bar to a small perturbation can depend not only on spontaneous changes of the material properties (e.g., chemical hardening), but also on apparent changes induced by the previous stress or strain history. Similar linearized relationships are then obtained from less general, but more widely used, nonlinear viscoelastic stress-strain laws.

Based on these relations between the small increments of stress and strain, and on elementary beam theory, the stability of the centrally loaded bar is investigated. The possibility of an instability occurring at some finite (or zero) period of time after the application of the end load is considered. Finally, a "critical load" is found for the bar, such that the application of any central end load smaller than the "critical load" will never cause unbounded deflections of the bar.

II - Representation of nonlinear stress-strain relationship

In what follows attention will be restricted to classical infinitesimal deformation theory. We will be dealing with a non-linear viscoelastic material, in which the strain at any time may depend on the complete past stress history in a general way. It is customary [1] to express such a dependence by means of a non-linear functional

$$\epsilon(t) = \mathcal{F}\Big[\sigma(\tau)\Big]_{\tau=-\infty}^{\tau=t} \tag{1}$$

where attention has been restricted to the one-dimensional case. In the above equation, ϵ is the (small) strain, σ the stress, t the time and $\mathcal{F}$ a nonlinear functional.

Here, only materials for which neighbouring stress histories induce neighbouring strain histories will be considered. In such a case, the functional $\mathcal{F}$ will be continuous, and it is assumed that it may be represented by means of a Fréchet expansion [14]. Then the functional relationship given by equation (1) may be written

$$\epsilon(t) = \int_{-\infty}^{\infty} \sigma(\tau) f_1(t;\tau)\, d\tau +$$

$$+\frac{1}{2!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma(\tau_1)\,\sigma(\tau_2)\,f_2(t;\tau_1,\tau_2)\,d\tau_1\,d\tau_2+\cdots$$

$$\cdots+\frac{1}{n!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sigma(\tau_1)\,\sigma(\tau_2)\cdots\sigma(\tau_n)\,\cdot$$

$$\cdot\,f_n(t;\tau_1,\tau_2,\cdots,\tau_n)\,d\tau_1\,d\tau_2\cdots d\tau_n+\cdots \qquad (2)$$

where without any loss of generality the generalized functions f_n may be considered to be symmetrical with respect to their n arguments $\tau_1,\tau_2,\cdots,\tau_n$.

An alternative representation, which will be useful in the following developments, is given by

$$\epsilon(t)=\int_{-\infty}^{\infty}\dot{\sigma}(\tau)\,\epsilon^{(1)}(t;\tau)\,d\tau+$$

$$+\frac{1}{2!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dot{\sigma}(\tau_1)\,\dot{\sigma}(\tau_2)\,\epsilon^{(2)}(t;\tau_1,\tau_2)\,d\tau_1\,d\tau_2+\cdots$$

$$\cdots+\frac{1}{n!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\dot{\sigma}(\tau_1)\,\dot{\sigma}(\tau_2)\cdots\dot{\sigma}(\tau_n)\,\cdot$$

$$\cdot\,\epsilon^{(n)}(t;\tau_1,\tau_2,\cdots,\tau_n)\,d\tau_1\,d\tau_2\cdots d\tau_n+\cdots \qquad (3)$$

where the generalized function $\dot{\sigma}$ is the derivative of σ in the distribution sense. It is readily proved that the following relationships

$$f_n(t;\tau_1,\tau_2,\cdots,\tau_n) = (-1)^n \frac{\partial^n \epsilon^{(n)}(t;\tau_1,\tau_2,\cdots,\tau_n)}{\partial\tau_1\partial\tau_2\cdots\partial\tau_n} \qquad (4)$$

$$\epsilon^{(n)}(t;\tau_1,\tau_2,\cdots\tau_n) = \int_{\tau_1}^{\infty}\int_{\tau_2}^{\infty}\cdots\int_{\tau_n}^{\infty} f_n(t;\xi_1,\xi_2,\cdots\xi_n)\cdot$$

$$\cdot\, d\xi_1\, d\xi_2\cdots d\xi_n \qquad (5)$$

between the function $\epsilon^{(n)}$ and the generalized function f_n hold.

It should be noted that for a real physical system, the response occurs after an excitation of the system, not prior to the excitation. This implies that

$$\epsilon^{(n)}(t;\tau_1,\tau_2,\cdots,\tau_n) \equiv f_n(t;\tau_1,\tau_2,\cdots,\tau_n) \equiv 0$$

$$\text{if any } \tau_i > t \;, \; i = 1,2,\cdots,n \qquad (6)$$

Then the upper limits of integration need not extend to ∞ , but may be set at t^+ in equation (2) and (5), and at t in equation (3). Moreover, if the system has been quiescent from $t = -\infty$

up to a certain instant t_o , at which the excitation $\sigma(t)$ is initiated, then the lower limit of integration need not extend from $-\infty$, but may be set at t_o and t_o^- in equations (2) and (3) respectively.

If the functional is differentiable to all orders, then the kernel $f_n(t;\tau_1,\tau_2,\cdots,\tau_n)$ represents the n^{th} functional derivative of $\mathcal{F}[y(t)]$ with respect to $y(t)$, evaluated at $y(t)\equiv 0$ [14]. Correspondingly, a typical term

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sigma(\tau_1)\sigma(\tau_2)\cdots\sigma(\tau_n)f_n(t;\tau_1,\tau_2,\cdots,\tau_n)\,d\tau_1 d\tau_2\cdots d\tau_n \tag{7}$$

appearing in equation (2) is then the n^{th} variation of the functional interpreted in the sense of Fréchet [14].

III - Approximate representation and evaluation of the kernels

Equation (2) or (3) serve to completely define a nonlinear system, provided all of the kernels in the expansion are known. In the usual case, the system is known only through the knowledge of the response to a finite number of inputs. From the responses to these inputs, one attempts to construct an approximate representation of the system. A convenient approximate representation appears to be a regular functional of degree n given by either of the following expressions:

$$\epsilon(t)\doteq\tilde{\mathcal{F}}\left[\sigma(\tau)\right]_{\tau=-\infty}^{\tau=t}=\int_{-\infty}^{\infty}\sigma(\tau)\tilde{f}_1(t;\tau)\,d\tau\;+$$

$$+\frac{1}{2!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma(\tau_1)\sigma(\tau_2)\tilde{f}_2(t;\tau_1,\tau_2)\,d\tau_1 d\tau_2+\cdots$$

$$+\frac{1}{n!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sigma(\tau_1)\sigma(\tau_2)\cdots\sigma(\tau_n)\tilde{f}_n(t;\tau_1,\tau_2,\cdots,\tau_n)\,d\tau_1\,d\tau_2\cdots d\tau_n \tag{8}$$

or

$$\epsilon(t) \doteq \tilde{\mathcal{F}}\Big[\sigma(\tau)\Big]_{\tau=-\infty}^{\tau=t} = \int_{-\infty}^{\infty}\dot{\sigma}(\tau)\,\tilde{\epsilon}^{(1)}(t;\tau)\,d\tau +$$

$$+\frac{1}{2!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dot{\sigma}(\tau_1)\dot{\sigma}(\tau_2)\,\tilde{\epsilon}^{(2)}(t;\tau_1,\tau_2)\,d\tau_1\,d\tau_2 + \cdots \tag{9}$$

$$+\frac{1}{n!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\dot{\sigma}(\tau_1)\dot{\sigma}(\tau_2)\cdots\dot{\sigma}(\tau_n)\,\tilde{\epsilon}^{(n)}(t;\tau_1,\tau_2,\cdots,\tau_n)\,d\tau_1\,d\tau_2\cdots d\tau_n$$

where $\tilde{\mathcal{F}}$, $\tilde{f}_n$ and $\tilde{\epsilon}^{(n)}$ are, in general, approximations of $\mathcal{F}$, f_n and $\epsilon^{(n)}$ respectively. It is noted that equations (4), (5), (6) and (7) hold for the approximate functional and kernels.

Kernels appearing in the approximate representation may be determined by a program of experiments in which a prescribed collection of inputs is applied and the resulting responses (outputs) are measured. A suitable collection of inputs may be represented by the n families of input histories

$$\sigma_i = \sigma(s_i, t) \quad , \qquad i = 1, 2, \cdots, n \tag{10}$$

where s_i is a parameter ranging, in general, over the interval $(-\infty, \infty)$.

Consider now the following differences of the n^{th} order system represented by $\tilde{\mathcal{F}}[\sigma]$

$$\Delta_1^{(i)}\tilde{\mathcal{F}} = \tilde{\mathcal{F}}\left[0+\sigma(s_i,\tau)\right] - \tilde{\mathcal{F}}\left[0\right] = \tilde{\mathcal{F}}\left[\sigma(s_i,\tau)\right]$$

$$\Delta_2^{(1,2)}\tilde{\mathcal{F}} = \Delta_1^{(2)}\Delta_1^{(1)}\tilde{\mathcal{F}} = \Delta_1^{(1)}\Delta_1^{(2)}\tilde{\mathcal{F}} =$$

$$= \tilde{\mathcal{F}}\left[\sigma(s_1,\tau)+\sigma(s_2,\tau)\right] - \tilde{\mathcal{F}}\left[\sigma(s_1,\tau)\right] - \tilde{\mathcal{F}}\left[\sigma(s_2,\tau)\right]$$

$$\vdots \tag{11}$$

$$\Delta_n^{(1,2,\cdots,n)}\tilde{\mathcal{F}} = \Delta_1^{(n)}\Delta_{n-1}^{(1,2,\cdots,n-1)}\tilde{\mathcal{F}} = \Delta_1^{(1)}\Delta_1^{(2)}\cdots\Delta_1^{(n)}\tilde{\mathcal{F}}$$

where $\Delta_n^{(1,2,\cdots,n)}$ is the n^{th} order difference operator. It should be noted that the differences are taken about the zero stress history, and it is assumed that $\tilde{\mathcal{F}}[0]$ (which represents the output to a zero input) is identically zero. Application of the n^{th} order difference operator to the representation of $\tilde{\mathcal{F}}$ given by equations (8) and (9) yields the following expressions:

$$\Delta_n^{(1,2,\cdots,n)}\tilde{\mathcal{F}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sigma(s_1,\tau_1)\sigma(s_2,\tau_2)\cdots\sigma(s_n,\tau_n)\cdot$$

$$\cdot\tilde{f}_n(t;\tau_1,\tau_2,\cdots,\tau_n)\,d\tau_1\,d\tau_2\cdots d\tau_n \tag{12a}$$

$$\Delta_n^{(1,2,\cdots,n)}\tilde{\mathcal{F}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\dot{\sigma}(s_1,\tau_1)\dot{\sigma}(s_2,\tau_2)\cdots\dot{\sigma}(s_n,\tau_n)\cdot$$

$$\cdot\tilde{\epsilon}^{(n)}(t;\tau_1,\tau_2,\cdots\tau_n)\,d\tau_1\,d\tau_2\cdots d\tau_n \tag{12b}$$

An explicit expression for the p^{th} difference of ${}^{\sigma}\tilde{\mathcal{F}}$ in terms of p outputs based on p families of input histories is given by

$$\Delta_p^{(1,2,\cdots,p)}\,{}^{\sigma}\tilde{\mathcal{F}} = \sum_{k=1}^{p}(-1)^{p-k}\,S_k^{(p)} \tag{13}$$

where

$$S_1^{(p)} = {}^{\sigma}\tilde{\mathcal{F}}[\sigma_1] + {}^{\sigma}\tilde{\mathcal{F}}[\sigma_2] + \cdots + {}^{\sigma}\tilde{\mathcal{F}}[\sigma_3]$$

$$\begin{aligned} S_2^{(p)} = {}^{\sigma}\tilde{\mathcal{F}}[\sigma_1+\sigma_2] + {}^{\sigma}\tilde{\mathcal{F}}[\sigma_1+\sigma_3] + \cdots + {}^{\sigma}\tilde{\mathcal{F}}[\sigma_1+\sigma_p] &+ \\ + {}^{\sigma}\tilde{\mathcal{F}}[\sigma_2+\sigma_3] + \cdots + {}^{\sigma}\tilde{\mathcal{F}}[\sigma_2+\sigma_p] &+ \\ + \cdots\cdots\cdots &+ \\ + {}^{\sigma}\tilde{\mathcal{F}}[\sigma_{p-1}+\sigma_p] & \end{aligned}$$

$$\vdots$$

$$S_p^{(p)} = {}^{\sigma}\tilde{\mathcal{F}}[\sigma_1+\sigma_2+\cdots+\sigma_p] \tag{14}$$

$S_k^{(p)}$ is the sum of the outputs generated by the $\binom{p}{k}$ combinations of k inputs chosen from the p input families. The actual computation of all the terms $S_k^{(p)}$, and consequently of the p^{th} difference of ${}^{\sigma}\tilde{\mathcal{F}}$, can be performed once the general response ${}^{\sigma}\tilde{\mathcal{F}}[\sigma_1+\sigma_2+\cdots+\sigma_n]$ is known, for then all the "lower order" responses of the form ${}^{\sigma}\tilde{\mathcal{F}}[\sigma_1+\sigma_2+\cdots+\sigma_k]$, $k \le n$, are known, as they are just special cases of the general response.

If the computation of the n^{th} difference is thought of as being performed for all values of the parameters ς_i, then equations (12a) and (12b) represent n multiple integral transformations of the kernels $\tilde{f}_n$ and $\tilde{\epsilon}^{(n)}$ respectively. Provided the inverse transformation exists, equations (12a) and (12b) furnish a way for the computation of $\tilde{f}_n$ or $\tilde{\epsilon}^{(n)}$.

The $(n-1)^{th}$ order kernels may be obtained by means of a similar procedure, considering the $(n-1)^{th}$ difference of the following $(n-1)^{th}$ order system

$$\tilde{\mathcal{F}}[\sigma(\tau)] - \frac{1}{n!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sigma(\tau_1)\sigma(\tau_2)\cdots\sigma(\tau_n)\cdot$$

$$\cdot\tilde{f}_n(t;\tau_1,\tau_2,\cdots,\tau_n)\,d\tau_1\,d\tau_2\cdots d\tau_n \qquad (15)$$

In general, the following recurrence formula applies for the determination of the p^{th} order kernel

$$\Delta_p^{(1,2,\cdots,p)}\Bigg[\tilde{\mathcal{F}}[\sigma(\tau)] - \sum_{i=p+1}^{n}\frac{1}{i!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sigma(\tau_1)\sigma(\tau_2)\cdots\sigma(\tau_i)\cdot$$

$$\cdot\tilde{f}_i(t;\tau_1,\tau_2,\cdots,\tau_i)\,d\tau_1\,d\tau_2\cdots d\tau_i\Bigg] =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sigma(s_1,\tau_1)\,\sigma(s_2,\tau_2)\cdots\sigma(s_p,\tau_p)\cdot$$

$$\cdot\tilde{f}_p(t;\tau_1,\tau_2,\cdots,\tau_p)\,d\tau_1\,d\tau_2\cdots d\tau_p \qquad (16)$$

where it should be kept in mind that the p^{th} difference has to be performed around the stress history $\sigma \equiv 0$. Similar equations can be obtained for the determination of $\tilde{\epsilon}^{(p)}$, if required.

From a practical point of view, the inversion of the n^{th} multiple integral transformation of the kernels may be considerably simplified when the n input families are taken to be unit step functions

$$\sigma(s_i, t) = H(t - s_i) \tag{17}$$

where the parameter s_i generating the family of functions σ_i, now represents the time of the initiation of the step function. (For second and third order systems, such a family of inputs has been suggested by Dong [9].) Consider, for instance, the case of the n^{th} order kernel. When equation (17) is substituted into equations (12), they reduce to

$$\Delta_n^{(1,2,\cdots,n)} \tilde{\sigma}_f = \int_{s_1}^{\infty}\int_{s_2}^{\infty}\cdots\int_{s_n}^{\infty} \tilde{f}_n(t; \tau_1, \tau_2; \cdots, \tau_n)\, d\tau_1\, d\tau_2 \cdots d\tau_n \tag{18a}$$

$$\Delta_n^{(1,2,\cdots,n)} \tilde{\sigma}_f = \tilde{\epsilon}^{(n)}(t; \tau_1, \tau_2, \cdots, \tau_n) \tag{18b}$$

where it is emphasized that in this case the left hand members represent the n^{th} difference of the system response based on unit step function inputs. If the kernel $\tilde{f}_n$ is desired, it may be calculated taking into account equation (4).

IV - Instantaneous elastic response

In this section, discontinuities in the strain history as induced by discontinuities in the stress history will be investigated. Only discontinuities of the first kind, i.e. finite jumps, will be considered.

Suppose the material is submitted to a stress history $\sigma(t)$ continuous around a point t_1 . If at t_1 a step function $\sigma_0 H(t-t_1)$ is superimposed on the function $\sigma(t)$, the following limit

$$\Delta \epsilon(t_1) = \lim_{t \to t_1^+} \mathcal{F}\left[\sigma(\tau) + \sigma_0 H(\tau - t_1)\right]_{\tau=-\infty}^{\tau=t} - \lim_{t \to t_1^-} \mathcal{F}\left[\sigma(\tau)\right]_{\tau=-\infty}^{\tau=t} \quad (19)$$

will be defined as the "instantaneous elastic response" induced by the step function.

If the expansion of the nonlinear functional $\mathcal{F}$ given by equation (3) is taken into account, it is not difficult to prove that the limit indicated in equation (19) is given by

$$\Delta \epsilon(t_1) = \sum_{n=1}^{\infty} \frac{\sigma_0^n}{n!} \epsilon^{(n)}(t_1^+; t_1, t_1, \cdots, t_1) + \quad (20)$$

$$+ \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{i=1}^{n-1} \sigma_0^i \binom{n}{i} \int_{-\infty}^{t_1} \int_{-\infty}^{t_1} \cdots \int_{-\infty}^{t_1} \dot{\sigma}(\tau_1) \dot{\sigma}(\tau_2) \cdots \dot{\sigma}(\tau_{n-i}) \cdot$$

$$\cdot \epsilon^{(n)}(t_1^+; \tau_1, \tau_2, \cdots, \tau_{n-i}, t_1, t_1 \cdots, t_1) d\tau_1 d\tau_2 \cdots d\tau_{n-i}$$

In this equation, the first sum represents the instantaneous elastic response which would occur in any aging nonlinear elastic material in the absence of hereditary effects. The second double sum is the contribution to the immediate elastic response due to hereditary effects, and in general depends on the complete past stress history. This type of behavior is not at all exhibited by linear viscoelastic materials. Real materials do in fact behave in such a way, to a certain extent, due to the change in properties induced by the previous inelastic strain which has occurred. The nonlinear representation considered here thus offers a variety of ways of analytically including hereditary dependent instantaneous response.

In particular, it is possible to define a time dependent tangent modulus at time t_1

$$1/E(t_1) = \lim_{\sigma_0 \to 0} \frac{\Delta\epsilon(t_1)}{\sigma_0} \tag{21}$$

In terms of equation (20), the tangent modulus takes the form

$$1/E(t_1) = \epsilon^{(1)}(t_1^+; t_1) + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{-\infty}^{t_1}\int_{-\infty}^{t_1}\cdots\int_{-\infty}^{t_1} \dot{\sigma}(\tau_1)\,\dot{\sigma}(\tau_2)\cdots$$

$$\cdots\dot{\sigma}(\tau_k)\,\epsilon^{(k+1)}(t_1^+; \tau_1, \tau_2, \cdots, \tau_k, t_1)\,d\tau_1\,d\tau_2\cdots d\tau_k \tag{22}$$

Then, for a very small increment of stress, $\delta\sigma_0$, occurring at t_1, the following equation holds

$$\delta\epsilon(t_1) = \delta\sigma_0 / E(t_1) \tag{23}$$

which may be considered as the linearized form of equation (20).

V - Special nonlinear stress-strain relationships

In various studies of nonlinear viscoelastic behavior, other more well-known stress-strain relationships have been used. In 1943, Leaderman [4] introduced a functional of the type

$$\epsilon(t) = \frac{\sigma(t)}{E} + \int_{-\infty}^{t} F[\sigma(\tau)]\, f(t-\tau)\, d\tau \tag{24}$$

where F represents a prescribed nonlinear function. Afterwards, Arutiunian used this type of functional in a form more suitable for the representation of the behavior of aging materials [15].

Rabotnov, in 1948 [5], introduced the functional

$$\varphi[\epsilon(t)] = \sigma(t) + \int_{-\infty}^{t} \sigma(\tau)\, f(t-\tau)\, d\tau \tag{25}$$

where φ is a given nonlinear function.

A simple generalization of the two preceding laws is

$$\varphi[\epsilon(t)] = F_1[t, \sigma(t)] + \int_{-\infty}^{t} F_2[\tau, \sigma(\tau)]\, f(t;\tau)\, d\tau \tag{26}$$

where φ , F_1 and F_2 are given functions.

It is obvious, of course, that equations (24), (25) and (26) are less general than the comprehensive form given by Volterra in terms of equation (2).

VI - First variation of the nonlinear functional

In the analysis of the problem under consideration, we shall be concerned with small variations in the strain history and associated variations of the stress history which result from a small perturbation about an equilibrium state. In what follows, $\delta\epsilon(t)$ and $\delta\sigma(t)$ will be used to denote such a small variation of strain and stress history respectively.

The relationship between $\delta\epsilon(t)$ and $\delta\sigma(t)$ is given by a linear functional which may be represented either by

$$\delta\epsilon(t) = \int_{-\infty}^{\infty} \delta\dot{\sigma}(\tau)\, K(t;\tau)\, d\tau \tag{27}$$

or by

$$\delta\epsilon(t) = -\int_{-\infty}^{\infty} \delta\sigma(\tau)\, \frac{\partial K(t;\tau)}{\partial\tau}\, d\tau \tag{28}$$

The kernel appearing in equation (27) may be obtained by performing the first variation of equation (3). Then, the following expression results

$$K(t;\tau) = \epsilon^{(1)}(t;\tau) + \sum_{i=1}^{\infty} \frac{1}{i} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \dot{\sigma}(\tau_1)\,\dot{\sigma}(\tau_2)\cdots$$

$$\cdots\dot{\sigma}(\tau_i)\,\epsilon^{(i+1)}(t;\tau_1,\tau_2,\cdots,\tau_i,\tau)\,d\tau_1\,d\tau_2\cdots d\tau_i \tag{29}$$

where $\dot{\sigma}(t)$ represents the time derivative of the equilibrium stress state, about which small perturbations are being taken.

For the more particular form of the nonlinear functional given by equation (26), the kernel $\dfrac{\partial K(t;\tau)}{\partial\tau}$ of equation (28) is given by

$$-\frac{\partial K(t;\tau)}{\partial\tau} = \Big\{ \frac{\partial}{\partial\sigma} F_2[\tau,\sigma(\tau)]\, f(t;\tau) +$$

$$+\ \delta(t-\tau)\frac{\partial}{\partial\sigma} F_1[t,\sigma(t)] \Big\} \Big/ \varphi'[\epsilon(t)] \tag{30}$$

where $\delta(t)$ is the delta function and φ' is the derivative of φ. The expressions for $\frac{\partial K}{\partial \tau}$ associated with equations (24) and (25) may be derived from equation (30) by suitable specializations.

If the material is assumed to be quiescent before a certain time t_0 and if $\sigma(t) = \sigma_0 H(t - t_0)$, then equations (29) and (30) reduce to

$$K(t;\tau) = \epsilon^{(1)}(t;\tau) + \sum_{i=1}^{\infty} \frac{\sigma_0^{\,i}}{i!} \epsilon^{(i+1)}(t; t_0, t_0, \cdots, t_0, \tau) \tag{31}$$

$$-\frac{\partial K(t;\tau)}{\partial \tau} = \left\{ \frac{\partial}{\partial \sigma_0} F_2[\tau, \sigma_0] f(t;\tau) + \right.$$

$$\left. + \delta(t - t_0) \frac{\partial}{\partial \sigma_0} F_1[t, \sigma_0] \right\} \Big/ \varphi'[\epsilon(t)] \tag{32}$$

VII - Integro-differential equations of a centrally loaded bar

Consider a straight column of length L subjected to compressive concentric end loads $P(t)$, which in general may vary with time. Moreover, it will be assumed that the bar is submitted to a prescribed imposed state of stress (e.g., a prestress) independent of the axial load $P(t)$. For the sake of simplicity, it is considered here that the imposed stress $\tilde{\sigma}(t)$, although variable with time, is uniform over the cross section A of the bar. Under these assumptions, the strain history will be given by the nonlinear functional

$$\epsilon(t) = \underset{\tau=-\infty}{\overset{\tau=t}{\mathcal{F}}} \left[\tilde{\sigma}(\tau) + \frac{P(\tau)}{A} \right] \tag{33}$$

Within the scope of this paper, instantaneous and asymptotic stability of the bar will be studied. The bar will be said to be instantaneously stable at a time t_1, if when it is subjected to a small perturbation initiated at that time, it does not instantaneously buckle. If under a stress history $\sigma = \tilde{\sigma} + \frac{P}{A}$, the bar is instantaneously stable for all times, and if when subjected to a small perturbation at any time t in the interval $t_1 \leq t \leq \infty$ the deflection remains bounded forever, then the bar is said to be asymptotically stable.

In order to check stability in either of the senses considered above, it is necessary to construct the equation of the deflected bar when it is subjected to a small perturbation causing lateral deflection. Consider, for instance, that a small lateral perturbation is initiated at time t_1, inducing a small perturbation in the stress, $\delta\sigma(t)$, with $\delta\sigma(t_1) \neq 0$. Then, if due account is taken of the singularity of the function $\delta\dot{\sigma}$ around t_1, equation (27) may be written

$$\delta\epsilon(t) = K(t;t_1)\,\delta\sigma(t_1) + \int_{t_1^+}^{\infty} \delta\dot{\sigma}(\tau)\,K(t;\tau)\,d\tau \qquad (34)$$

which, after an integration by parts, yields

$$\delta\epsilon(t) = K(t;t)\,\delta\sigma(t) - \int_{t_1}^{\infty} \delta\sigma(t)\,\frac{\partial}{\partial\tau}K(t;\tau)\,d\tau \qquad (35)$$

where the delta function contribution associated with the term $\frac{\partial}{\partial\tau}K(t;\tau)$ has been separated and explicitly included as the first term on the right hand side of equation (35), so that the term $\frac{\partial}{\partial\tau}K(t;\tau)$ has at most a step discontinuity at $t = \tau$. From the above equation, and assuming the usual hypotheses of elementary beam theory, the relationship between the curvature of the bar and the bending moment, M, may be derived as

$$\mu(x,t)I = M(x,t)K(t;t) - \int_{t_1}^{\infty} M(x,\tau)\frac{\partial}{\partial\tau}K(t;\tau)d\tau \qquad (36)$$

where I is the area moment of inertia of the column cross section about its centroidal axis. The bending moment M is related to the column deflection w and the small perturbing moment δM_0 by

$$M(x,t) = \delta M_0(x,t) + P(t)w(x,t) \qquad (37)$$

Approximating the curvature by

$$\mu(x,t) = -\frac{\partial^2}{\partial x^2}w(x,t) \qquad (38)$$

and eliminating μ and M in equation (36) by using equations (37) and (38), the following integro-differential equation is obtained

$$-\alpha(x)I_0\frac{\partial^2 w}{\partial x^2} = P(t)w(x,t)K(t;t) -$$

$$-\int_{t_1}^{\infty} P(\tau)w(x,\tau)\frac{\partial}{\partial\tau}K(t;\tau)d\tau +$$

$$+\delta M_0K(t;t) - \int_{t_1}^{\infty}\delta M_0(x,\tau)\frac{\partial}{\partial\tau}K(t;\tau)d\tau \qquad (39)$$

where I_o is a reference moment of inertia defined by

$$I(x) = \alpha(x) I_o \tag{40}$$

so that the cross-section of the bar is allowed to vary with x .

To solve equation (39), the following expansions are used

$$w(x,t) = \sum_{i=1}^{\infty} b_i(t)\, \varphi_i(x) \tag{41}$$

$$\delta M_o(x,t) = \sum_{i=1}^{\infty} a_i(t)\, \varphi_i(x) \tag{42}$$

where $\varphi_i(x)$ are the eigenfunctions of the differential equations

$$\frac{d^2}{dx^2}\left[\alpha(x) \frac{d^2 \varphi_i}{dx^2} + k_i \varphi_i\right] = 0 \tag{43}$$

and associated boundary conditions at the ends of the bar.
It is simple to prove that φ_i' and φ_i'' are orthogonal functions in the following sense

$$\int_0^L \varphi_i' \varphi_j' \, dx = 0 \tag{44}$$

$$\int_0^L \alpha(x)\, \varphi_i'' \varphi_j'' \, dx = 0 \tag{45}$$

$$\text{if} \quad i \neq j$$

Then, if the expressions for w and M_o given by equations (41) and (42) are substituted into equation (39), and equations (43), (44) and (45) are taken into account, the following integral equations result for the coefficients $b_i(t)$

$$\left[K(t;t)\,P(t) - I_o k_i\right] b_i(t) - \int_{t_1}^{\infty} P(\tau)\, b_i(\tau) \cdot$$

$$\cdot \frac{\partial K(t;\tau)}{\partial \tau}\, d\tau = g_i(t, t_1) \tag{46}$$

where

$$g_i(t,t_1) = -\left[a_i(t)\,K(t;t) - \int_{t_1}^{\infty} a_i(\tau)\, \frac{\partial K(t;\tau)}{\partial \tau}\, d\tau\right] \tag{47}$$

It follows from comparison of equations (22) and (29) that

$$K(t;t) = {}^{1}/{E_T(t)} \tag{48}$$

Then equation (46) may be rewritten

$$K(t;t)\left[P(t) - P_T^{\,i}(t)\right] b_i(t) - \int_{t_1}^{t} P(\tau)\, b_i(\tau) \cdot$$

$$\cdot \frac{\partial K(t;\tau)}{\partial \tau}\, d\tau = g_i(t,t_1) \tag{49}$$

where

$$P_T^{(i)}(t) = k_i\, E_T(t)\, I_o \tag{50}$$

is the i^{th} tangent modulus buckling load of the column at time t .

VIII - Conditions of stability

It follows from equation (41) that the stability of the bar depends on the behavior of $b_i(t)$. Hence the problem reduces to the study of the stability of the integral equations (49). If the eigen values k_i of equation (43) are assumed to be ordered, so that

$$P_T^{(1)} < P_T^{(2)} < \cdot \cdot \cdot < P_T^{(i)} < \cdot \cdot \cdot ,$$

then it will be sufficient to investigate the stability of the function $b_1(t)$. In what follows, the sub-index 1 will be understood, and not written explicitly.

Assuming conditions of piecewise continuity for the functions $K(t;t)$, $P(t)$, $P_T(t)$ and $g(t,t_1)$, the function $b(t)$ will be a piecewise continuous function in the interval $t_1 \leq t < t^*$ provided $P(t) < P_T(t)$. If at t^*, $P(t^*) = P_T(t^*)$, then the solution of the integral equation may exhibit a singular behavior at t^* . In order to investigate the behavior around t^*, equation (49) (with $i = 1$) will be rewritten in the following form

$$\left[P(t) - P_T(t)\right] b(t) - \int_{t^*}^{t} \frac{P(\tau)}{K(t;t)} \frac{\partial K(t;\tau)}{\partial \tau} b(\tau) d\tau =$$

$$= \frac{g(t,t_1)}{K(t;t)} + \int_{t_1}^{t^*} \frac{P(\tau)}{K(t;t)} \frac{\partial K(t;\tau)}{\partial \tau} b(\tau) d\tau \qquad (51)$$

where the right hand member is a well-behaved function.

Performing an expansion of $b(t)$ around t^* of the form

$$b(t) = \sum_{i=0}^{\infty} c_i \xi^{i+r} \quad ; \quad \xi = t - t^* \tag{52}$$

and considering that the kernel of the integral equation (51) and the coefficient of $b(t)$ in the same equation are regular at t^*, the following indicial equation on r is obtained

$$1 + r = \frac{P(t^*)}{\dot{P}(t^*) - \dot{P}_T(t^*)} \cdot \frac{1}{K(t^*;t^*)} \cdot \left. \frac{\partial K(t;\tau)}{\partial \tau} \right|_{t=\tau=t^*} \tag{53}$$

where the superposed dot indicates differentiation and the subscript at the right of the vertical bar indicates that the function $\frac{\partial K}{\partial \tau}$ must be evaluated at $t = \tau = t^*$.

Since the axial load is essentially positive, and $P < P_T$ for $t < t^*$, then $P(t^*) > 0$ and $\left[\dot{P}(t^*) - \dot{P}_T(t^*) \right] > 0$.
Moreover, if attention is restricted to those cases for which

$$K(t^*;t^*) > 0 \quad ; \quad \left. \frac{\partial K(t;\tau)}{\partial \tau} \right|_{t=\tau=t^*} < 0 \tag{54}$$

then r in equation (53) will be negative, and therefore, as seen from equation (52), $b(t) \to \infty$ as $t \to t^*$. This result indicates that if a time t^* exists such that $P(t^*) = P_T(t^*)$, then instability will occur at that time. It is worthwhile noting that $E_T(t)$, and consequently $P_T(t)$, does not depend on the perturbation. This is a natural consequence of having considered only a small perturbation, for then higher order powers of $\delta\sigma_o$ in equation (22) may be neglected. Hence t^* will not depend on the perturbation and, more specifically, on the time t_1 at which it was initiated.

These results establish that <u>when $P(t)$ approaches $P_T(t)$, then the bar reaches a state of instantaneous instability</u>.

<u>Asymptotic stability</u>

Hereafter the bar will be said to be asymptotically stable if-- for any small perturbation initiated at any time--the deflection remains bounded for $t \to \infty$. It is immediately recognized that asymptotic stability implies instantaneous stability, but the inverse is not true. Therefore, the investigation of asymptotic stability reduces to the investigation of the boundedness of the function $b(t)$ with $P(t) < P_T(t)$.

For convenience, consider equation (49) with $i = 1$ written in the following form

$$\int_{t_1}^{\infty} Q(t,\tau)\, b(\tau)\, d\tau = q(t,t_1) \tag{55}$$

where

$$Q(t,\tau) = K(t;t)\left[P(t) - P_T(t)\right]\delta(t-\tau) - P(\tau)\frac{\partial K(t;\tau)}{\partial \tau} \tag{56}$$

Now, the boundedness of the function $b(t)$ may be studied by means of a Tauberian theorem [16] (for "imperfect" kernels) which affirms that if $q(t,t_1)$ is bounded and if there exists a function $q(t-\tau)$ so that

$$\int_{-\infty}^{\infty} q(\tau) e^{-\nu\tau} d\tau \neq 0 \; ; \quad \operatorname{Re} \nu > 0 \tag{57}$$

and that approximates the kernel $Q(t,\tau)$ in the sense that

$$\lim_{t \to \infty} \int_{-\infty}^{t} | Q(t,\tau) - q(t-\tau) | e^{-\eta(t-\tau)} d\tau = 0 \; ; \begin{cases} \text{for some} \\ \eta > 0 \end{cases} \tag{58}$$

then $b(t)$ is bounded over the whole interval $(-\infty, \infty)$. It should be noted that for the real physical systems under consideration here, the response does not precede the input, so that $q(t)$ will be understood to be identically zero for $t < 0$.

It is apparent that once a function $q(t-\tau)$ satisfying equation (58) is found, then equation (57) serves to furnish a sufficient condition for the boundedness of $b(t)$. To find an appropriate function $q(t-\tau)$ fulfilling the requirements of equations (57) and (58), it is necessary to possess some essential knowledge of the properties of the material. When the behavior of the material is assumed to be given by any of the expansions expressed by equations (2) or (3), then some restrictions need to be imposed on the kernels $\epsilon^{(i)}$ and the stress history in order to construct the function $q(t-\tau)$.

In what follows, a case will be considered in which rather general behavior (likely to occur in practical applications) of the material functions, $\epsilon^{(i)}$, and the stress history, $\sigma(t)$, is assumed. The possibility of investigation of asymptotic stability using still less restrictive conditions is not excluded.

The imposed stress history $\tilde{\sigma}(t)$ and the applied load $P(t)$ will be assumed to converge asymptotically towards finite limits $\tilde{\sigma}_\infty$ and P_∞ as $t \to \infty$, so that

$$\lim_{t\to\infty} \sigma(t) = \lim_{t\to\infty}\left[\tilde{\sigma}(t) + \frac{P(t)}{A}\right] = \sigma_\infty \tag{59}$$

Only materials exhibiting bounded creep will be considered. This implies

$$\lim_{t,\tau_i\to\infty} \epsilon^{(n)}(t;\tau_1,\tau_2,\cdots,\tau_n) < \infty \; ; \; \begin{cases} i = 1,2,\cdots,n \\ n = 1,2,\cdots \end{cases} \tag{60}$$

Moreover, it will be assumed that the material ages asymptotically -- that is, after a long period of time the material properties will be time invariant. This implies that for large values of the variables t and τ_i, the function $\epsilon^{(n)}$ will tend asymptotically to a limit function

$$\epsilon^{(n)}(t;\tau_1,\tau_2,\cdots,\tau_n) \longrightarrow \epsilon^{(n)}_\infty(t-\tau_1,t-\tau_2,\cdots,t-\tau_n) \tag{61}$$

for large values of t and τ_i.

When conditions expressed by equations (59), (60) and (61) are fulfilled, then equation (58) will be satisfied by constructing the function $g(t-\tau)$ as follows

$$g(t-\tau) = K_\infty(0)\left[P_\infty - P_{T\infty}\right]\delta(t-\tau) - P_\infty \frac{\partial K_\infty(t-\tau)}{\partial \tau} \tag{62}$$

where

$$K_\infty(t-\tau) = \epsilon_\infty^{(1)}(t-\tau) + \sum_{i=1}^{\infty} \frac{\alpha_\infty^i}{i!}\, \epsilon_\infty^{i+1}(\infty, \infty, \cdots, \infty, t-\tau) \tag{63}$$

and

$$P_{T_\infty} = I_0 k_1 / K_\infty(0) \tag{64}$$

Substituting the value of $q(t-\tau)$ given by equation (62) in equation (57), the following equation is obtained

$$\int_{-\infty}^{\infty} q(\tau) e^{-\nu\tau} d\tau = K_\infty(0)\left[P_\infty - P_{T_\infty}\right] + $$

$$+ P_\infty \int_{-\infty}^{\infty} \dot{K}_\infty(\tau) e^{-\nu\tau} d\tau \neq 0 \qquad Re\, \nu > 0 \tag{65}$$

where $\dot{K}_\infty$ is the derivative of K_∞.

Now, since conditions expressed by equation (54) are assumed to be fulfilled, then $K_\infty(0)$ and $\dot{K}_\infty(\tau)$ will be positive. Moreover, recalling that $P < P_T$, then $P_\infty < P_{T_\infty}$. Hence equation (63) will be satisfied if

$$\int_{-\infty}^{\infty} \dot{K}_\infty(\tau) d\tau = K_\infty(\infty) - K_\infty(0) < K_\infty(0) \frac{P_{T_\infty} - P_\infty}{P_\infty}$$

using the fact that $K_\infty(t) \equiv 0$ in the interval $-\infty \leq t < 0$.

This equation may be written in the more convenient form

$$P_\infty < I_0 k_1 / K_\infty(\infty) = P_{T\infty} \frac{K_\infty(0)}{K_\infty(\infty)} \tag{66}$$

It must be observed that the boundedness of $K_\infty(0)$, $K_\infty(\infty)$ and consequently of $P_{T\infty}$, follow from equations (60), (61) and (63).

<u>The above result establishes the fact that--when conditions expressed by equations (54), (59), (60) and (61) are fulfilled--the bar will be asymptotically stable provided equation (66) is satisfied.</u>

Recalling the definition of the tangent modulus given in section IV, and taking into account equations (29) and (48), a convenient physical meaning may be given to equation (66). In fact $1/K_\infty(0)$ represents the tangent modulus of the indefinitely aged material submitted to a previous stress history $\sigma(t)$. On the other hand, $K_\infty(\infty)$ is the total asymptotic increment of strain per unit of increment of stress obtained when a constant increment of stress (small compared with the value of $\sigma(t)$) is applied to the indefinitely aged material which was submitted to a previous stress history $\sigma(t)$. Both quantities may then be obtained experimentally by submitting a specimen of the material to the stress history $\sigma(t)$ and--after a large period of time--the initial and "relaxed" modulus is measured by applying a small increment of stress. From the previous discussion it follows that the initial modulus obtained from the test will be $1/K_\infty(0)$ while the relaxed modulus will be $1/K_\infty(\infty)$.

This result establishes the fact that complete knowledge of the kernels $\epsilon^{(i)}$ is not per se necessary in order to determine $K_\infty(0)$ and $K_\infty(\infty)$. Thus, if a specified stress history is given, only one experiment is necessary in order to establish the physical parameters required in equation (66) for the determination of the asymptotic stability of the bar under consideration.

At this stage it should be noted that--as would be expected--the condition for asymptotic stability is independent of the kind of lateral perturbation applied to the bar. The application of the general Tauberian theorem mentioned above did not introduce any restriction (except boundedness) on the lateral perturbation, but from a more physical point of view it should be kept in mind that the perturbation has to be chosen small enough so as to induce small deflections compatible with the linearization of the corresponding equations. A complete discussion of the determination of the asymptotic lateral deflections induced by a small given perturbation is beyond the scope of this paper. However, it is noted that if only nonaging materials are considered, then the asymptotic deflection will depend on the asymptotic value of the perturbing force, and not on the previous history of that force. Hence it can be concluded that if the perturbing force ceases for all times after any finite time, the deflection will tend asymptotically to zero, provided the condition for asymptotic stability is fulfilled. This is not the case for an aging material for which, in general, a certain deflection of the bar will remain even if the perturbation ceases at a finite time. The evaluation of the final deflection for a certain given perturbation may be performed by using equation (49). An upper estimation of the asymptotic behavior of a similar integral equation may be found in reference [17].

A practical question which can arise is the determination of the maximum constant load below which asymptotic stability is assured. This problem reduces to finding the critical load, P_{CR} , which satisfies the following equation

$$P_{CR} = P_{T\infty} \frac{K_\infty(0)}{K_\infty(\infty)} \tag{67}$$

obtained from the inequality given by equation (66).

It is apparent that since $P_{T\infty}$, $K_\infty(0)$ and $K_\infty(\infty)$ will depend in general on P_{CR} , then equation (67) will be a nonlinear equation in P_{CR} . In most cases, a trial and error procedure is a suitable method to be used for the evaluation of P_{CR} , particularly when analytical expressions for $K_\infty(0)$ and $K_\infty(\infty)$ are not available. It would then be necessary to experimentally determine the dependence of $K_\infty(0)$ and $K_\infty(\infty)$ on the stress level. In order to do this recall, as pointed out above, that for any stress history, only one experiment is necessary to measure $K_\infty(0)$ and $K_\infty(\infty)$. Then, performing a series of such experiments for various constant axial end loads it is possible to determine the dependence of $K_\infty(0)$, $K_\infty(\infty)$ and $P_{T\infty}$ on the value of the axial end load. On the basis of such data the value of P_{CR} may be obtained from equation (67) by a trial and error procedure.

IX - Stability for special nonlinear materials

It is of interest to investigate the possibilities of instantaneous and asymptotic instability when the material constitutive equation is represented by the particular law given by equation (26).

In the previous discussion for the general law, it was concluded that instantaneous instability occurs when

$$P(t) - P_T^{(1)}(t) = 0 \tag{68}$$

where $P_T^{(1)}(t)$ is the minimum tangent modulus buckling load defined by equation (50). Taking into account equations (30) and (48), and after some calculations, the following expression is found for $P_T^{(1)}(t)$

$$P_T^{(1)}(t) = k_1 I_o \left\{ \frac{\partial F_1[t, \sigma(t)]}{\partial \sigma(t)} \Big/ \frac{\partial \varphi[\epsilon(t)]}{\partial \epsilon(t)} \right\} \tag{69}$$

It is noticed that the function F_2 involved in the constitutive equation (26) does not appear in equation (69). This means that the function F_2 , which completely embodies the nonlinear properties for a pure Leaderman material (equation (24)), is not responsible for any change in the tangent modulus, and consequently it will not have any influence on instantaneous instability.

It is apparent that instantaneous instability can occur if, and only if, equation (68) possesses a real, positive bounded solution. In order to discuss the possibilities of such a solution for different--rather simple--materials, the following cases are discussed:

1. $\frac{\partial F_1}{\partial \sigma}$ and $\frac{\partial \varphi}{\partial \epsilon}$ are constants. Then $P_T^{(1)}$ is a constant, and a solution of equation (68) will exist only if $P(t)$ increases so as to reach the value $P_T^{(1)}$. Otherwise, if P is constant, and less than $P_T^{(1)}$, instantaneous instability cannot occur.

This is the case of a pure Leaderman material for which the instantaneous modulus of elasticity (E in equation (24)) remains constant.

2. $\frac{\partial \varphi}{\partial \epsilon}$ is independent of ϵ , and $\frac{\partial F_1}{\partial \sigma}$ is independent of σ but time-dependent. Then a solution of equation (68) may exist, even when P is constant. This is the case of a Leaderman material with nonlinear elastic response whose instantaneous modulus E varies with time. If the change in the instantaneous modulus is due only to chemical hardening--a behavior exhibited by many real materials submitted to moderate levels of stress--then the instantaneous modulus will increase with time, and instantaneous instability may occur only if $P(t)$ increases sufficiently rapidly.

3. P and $\frac{\partial F_1}{\partial \sigma}$ are constants. Then, a solution may exist, if $\frac{\partial \varphi}{\partial \epsilon}$ increases so that $P_T^{(i)}(t)$ given by equation (69) decreases to the value of P . This case was previously discussed in the literature [12], implicitly assuming $\frac{\partial F_2}{\partial \sigma}$ constant. As was already mentioned, the function F_2 does not have any effect on instantaneous instability.

In order to investigate asymptotic stability, some assumption regarding the behavior, for large values of t , of the material functions appearing in equation (26) will be made. It will be assumed that the functions F_1 and F_2 , as well as their derivatives $\frac{\partial F_1}{\partial \sigma}$ and $\frac{\partial F_2}{\partial \sigma}$, are bounded and tend asymptotically towards finite limits

$$\lim_{t \to \infty} F_1[t, \sigma(t)] = F_{1\infty}(\sigma_\infty) < \infty \tag{70a}$$

$$\lim_{t \to \infty} F_2[t, \sigma(t)] = F_{2\infty}(\sigma_\infty) < \infty \tag{70b}$$

$$\lim_{t \to \infty} \frac{\partial F_1}{\partial \sigma} = F'_{1\infty}(\sigma_\infty) < \infty \tag{70c}$$

$$\lim_{t \to \infty} \frac{\partial F_2}{\partial \sigma} = F'_{2\infty}(\sigma_\infty) < \infty \tag{70d}$$

in which it is implicitly assumed that the limit σ_∞ given by equation (59) exists.

Similar to the case of general nonlinear behavior previously treated, only materials with bounded creep will be considered, so that

$$\lim_{t \to \infty} \int_\xi^t f(t;\tau)\, d\tau < \infty \qquad -\infty \leq \xi \leq \infty \tag{71}$$

As in the general case, it is further assumed that

$$f(t;\tau) \longrightarrow f_\infty(t-\tau) \tag{72}$$

for large values of t and τ. When all these requirements are fulfilled, it is not difficult to establish, by means of a similar mathematical treatment as was performed for the general case, that the condition for asymptotic stability is given by

$$P_\infty < I_0 k_1 / \bar{K}(\infty) = \bar{P}_\infty \frac{\bar{K}_\infty(0)}{\bar{K}_\infty(\infty)} \tag{73}$$

where

$$\overline{P}_{T\infty} = I_0 k_1 / \overline{K}_\infty(0) \tag{74}$$

Equation (73) is similar to equation (66) but $\overline{K}_\infty(0)$ and $\overline{K}_\infty(\infty)$ are now obtained from the asymptotic value of $K(t;\tau)$ given by

$$\overline{K}_\infty(t) = \left[F'_{1\infty}(\sigma_\infty) + F'_{2\infty}(\sigma_\infty)\int_0^t f_\infty(\tau)\,d\tau\right] \Big/ \varphi'(\epsilon_\infty) \tag{75}$$

so that

$$\overline{K}_\infty(0) = F'_{1\infty}(\sigma_\infty) / \varphi'(\epsilon_\infty) \tag{76}$$

$$\overline{K}_\infty(\infty) = \left[F'_{1\infty}(\sigma_\infty) + F'_{2\infty}(\sigma_\infty)\int_0^\infty f_\infty(\tau)\,d\tau\right] \Big/ \varphi'(\epsilon_\infty) \tag{77}$$

It should be noted that the derivative of φ appearing in equations (75), (76) and (77) must be performed at $\epsilon_\infty = \lim_{t\to\infty} \epsilon(t)$, that is to say at the asymptotic value of the strain which occurred in the bar under the considered stress history. The value of ϵ_∞ is given by

$$\epsilon_\infty = \varphi^{-1}\left\{\lim_{t\to\infty}\left[F_1[t,\sigma(t)] + \int_{-\infty}^t F_2[\tau,\sigma(\tau)]\,f(t;\tau)\,d\tau\right]\right\} \tag{78}$$

where φ^{-1} is the inverse function of φ. In equations (73), (74), (76) and (77), as should be expected, the quantities $\overline{K}_\infty(0)$ and $\overline{K}_\infty(\infty)$ have the same physical interpretation as was pointed out for

$K_\infty(0)$ and $K_\infty(\infty)$ respectively in the discussion following equation (66).

A more suitable form of equation (73) may be obtained by utilizing the values of $\bar{K}_\infty(0)$ and $\bar{K}_\infty(\infty)$ given by equation (76) and (77). Then, the condition for asymptotic stability may be written

$$P_\infty < \bar{P}_{T\infty} \Big/ \left\{1 + \frac{F'_{2\infty}(\sigma_\infty^o)}{F'_{1\infty}(\sigma_\infty^o)} \int_0^\infty f_\infty(\tau)\, d\tau\right\} \tag{79}$$

It is seen--contrary to what was found for the case of instantaneous stability--that the function F_2 (or more precisely its derivative) appears in the condition of asymptotic stability.

For the particular case in which $\varphi(\epsilon)$, $F_1[t, \sigma(t)]$ and $F_2[t, \sigma(t)]$ are linear functions of ϵ and σ respectively, it is not difficult to show that equation (79) reduces to

$$P_\infty < P_{E\infty} \Big/ \left\{1 + E_\infty \int_0^\infty f_\infty(\tau)\, d\tau\right\} \tag{80}$$

where $P_{E\infty}$ is the Euler load of the bar based on the instantaneous modulus of elasticity, E_∞, of the completely aged material. This result was previously obtained in reference [17].

X - Concluding Remarks

It has been shown that a functional power series of the Volterra-Fréchet type may be a useful phenomenological approach to represent the behavior of general nonlinear viscoelastic materials. The kernels (material properties) occurring in the series may be obtained from a program of experiments which measures the response of the material to certain families of inputs. The order of the functional polynomial and the selection of the type of inputs to be used, depends to a large extent on the degree of accuracy desired and on the possibility of devising an effective procedure for inverting the integral transformations given by equation (12).

The analysis presented in this paper was based on the creep-type constitutive equations (2) and (3) which give the strain as a functional of the stress. If certain conditions are imposed on the kernels of first order appearing in equations (2) and (3), and if the kernels of higher order are bounded, then it could be shown that inverses of equations (2) and (3) exist. These inverses, which give the stress as a functional of strain, would represent the constitutive relation in the form of a relaxation law and would have the same form as equations (2) and (3). Thus, to determine directly the kernels appearing in the functional expansion of the relaxation form of the stress-strain relation, a set of experiments, similar to those previously discussed in conjunction with the creep form of the constitutive relation, could be programmed using n families of strain inputs rather than stress inputs.

The analysis of some of the problems considered here did not require having complete knowledge of the kernels for all values of their arguments. For example, asymptotic stability was found to depend on the asymptotic form of the nonlinear functional relating stress and strain. The experimental determination of this asymptotic form is considerably simpler than the experimental determination of the kernels for all values of their arguments.

However, it should be pointed out that if interest is centered on the calculation of the transient response of a general nonlinear viscoelastic material subjected to arbitrary excitation, then the use of functional expansions could prove very complicated and tedious. As soon as the number of terms required in the series representation of the functional exceeds three or so, then the testing program required to obtain a numerical evaluation of the kernels obviously becomes quite extensive. Furthermore, when such a representation of the constitutive law is incorporated into theories for the determination of transient stress and deformation of deformable solids, then the boundary value problems which result will, in general, be extremely difficult to solve.

To overcome difficulties of this nature, other kinds of representations of the nonlinear viscoelastic constitutive relation may prove more convenient. Different approaches are being investigated in fields where nonlinear hereditary phenomena occur. In the field of systems theory, a method which is still undergoing investigation, and which in the future may prove useful in applications to the mechanics of deformable solids, is the "state space" approach [18]. To a large extent, interest in that method is motivated by a conscious effort to avoid the difficulties associated with the use of series representations of nonlinear functionals.

The specific application treated in this paper was an investigation of the stability of a centrally loaded column. The column was assumed to be made of a general nonlinear viscoelastic material, and the constitutive relation of the material was represented by means of a Volterra-Fréchet expansion of a general nonlinear analytic functional. A small lateral perturbation of the bar was introduced in order to check stability. Subsequent boundedness of the deflection of the slightly disturbed bar was used as the basic criterion of stability. Attention was restricted to small disturbances in order that suitable linearizations could be made. A more general investigation which allowed for the introduction of large perturbations would be considerably more complicated and difficult than the analysis presented here.

The equation governing the bar subjected to a small lateral perturbation is derived from an appropriate linearization of the curvature expression and of the nonlinear constitutive equation, resulting in a linear integral equation whose kernel, in general, depends not only on the material functions (kernels of the nonlinear expansion) but also on the complete stress history of the bar. The stress history is assumed to be known, depending in general on the time behavior of the externally applied axial end load and a possible state of initial imposed stress as may occur due to an axial restraint of the bar.

The condition for instantaneous instability was derived from the singular behavior exhibited by the integral equation of the slightly disturbed bar, in the neighborhood of the critical time. It was shown that instantaneous instability occurs if the applied load reaches the value of the tangent modulus buckling load based on the tangent modulus defined in Section IV.

In the investigation of asymptotic stability, a very powerful and general Tauberian theorem established by Pitt was used. To shorten the analysis presented here, some relatively mild requirements were imposed on the asymptotic behavior of the stress history and material functions;

namely, that these quantities possess finite asymptotic limits. If this requirement is changed to the still weaker requirement that the stress history and material functions have a suitably bounded behavior for all times, then it is possible to extend the criterion of asymptotic stability to cover this case.

Finally, a nonlinear stress-strain law more restricted in nature than the general nonlinear viscoelastic relation, and which includes as special cases, Leaderman's and Rabotnov's laws of nonlinear viscoelastic behavior, was considered in the stability investigation. This particular form of the constitutive equation contains--even if in a limited manner--many of the important and interesting properties exhibited by a large variety of real materials.

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