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The Conformal Prescribed Scalar Curvature Problem on Orbifolds

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DEDICATION

To my parents Peixi Ju and Jing Yang

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ABSTRACT OF THE DISSERTATION

The Conformal Prescribed Scalar Curvature Problem on Orbifolds

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In this dissertation, we study the prescribed scalar curvature problem in a conformal class on orbifolds with isolated singularities. This problem is more subtle than the manifold case since the positive mass theorem does not hold for ALE metrics in general.

In Chapter 2, we give a positive mass theorem on AF orbifolds. Together with the negative mass ALE space example – LeBrun's metric, it shows that mass can be discontinuous on compact orbifolds. This further implies that the study of the Yamabe equation on orbifolds can be substantially different from the manifold case.

From Chapter 3 to Chapter 5, we prove the compactness and existence theorem for the orbifold prescribed scalar curvature problem, in dimension 4, which can also be generalized to higher dimensions. The proof is by carefully analyzing the Pohozaev Identity and the blow-up behavior of a sequence of solutions.

In Chapter 6, we provide another version of the existence theorem, by looking into the Yamabe energy for all functions, and also of a certain testing function. This existence theorem differs but also overlaps with the previous one, and they both have important use cases.

By applying our theorems to the families of Calabi metrics and LeBrun's metrics, we obtain lots of existence examples. Moreover, in Chapter 7, after playing some tricks on LeBrun's metrics, we get some non-existence examples, and thus establish the full theory of Leray-Schauder degree for radially symmetric solutions. Especially, we observe an interesting "wall" in the prescribed scalar curvature space such that the Leray-Schauder degree jumps by 1 upon crossing this "wall".

Chapter 1

Introduction

1.1 Background

To begin, we give the definition of a Riemannian orbifold.

Definition 1.1.1. We say that (M, g) is a Riemannian orbifold of dimension n if M is a smooth manifold of dimension n with a smooth Riemannian metric away from a finite singular set

$$\Sigma = \{(q_1, \dots, q_l)\}. \tag{1.1.1}$$

Near each singular point q_j, there exists a neighborhood U_j of q_j, a nontrivial finite subgroup $\Gamma_j \subset \mathrm{O}(n)$ acting freely on $\mathbb{R}^n \setminus \{0\}$ and a Γ_j-equivariant diffeomorphism

$$\varphi_j : \widetilde{U_j} \to B_{\sigma_j}(0), \tag{1.1.2}$$

where $\widetilde{U_j}$ is the completion (by adding a point $\tilde{q}_j$) of the universal cover of $U_j \setminus \{q_j\}$, and $B_{\sigma_j}(0)$ is a ball of radius σ_j about the origin in $\mathbb{R}^4$. Furthermore, $(\varphi_j)_* \pi_j^* g$ extends to a smooth Riemannian metric on $B_{\sigma_j}(0)$, where $\pi_j : \widetilde{U_j} \setminus \{\tilde{q}_j\} \to U_j \setminus \{q_j\}$ is a universal covering map.

Our convention is that if $l = 0$, then $\Sigma = \emptyset$, and (M, g) is a smooth Riemannian manifold. But if $l \geq 1$, then there must be nontrivial singular points. We next define the meaning of a $C^k(M)$ function on a Riemannian orbifold (M, g).

Definition 1.1.2. A function $f : M \to \mathbb{R}$ is of class $C^k(M)$ if $f : M \setminus \Sigma \to \mathbb{R}$ is of class $C^k(M \setminus \Sigma)$, and near each singularity, there exists a coordinate system φ_j such that the function $f \circ \pi_j \circ \varphi_j^{-1} : B_{\sigma_j}(0) \to \mathbb{R}$ is of class $C^k(B_{\sigma_j}(0))$. We can also define the spaces $C^\infty(M), C^{k,\alpha}(M), C^k_{loc}(M)$ in a similar fashion.

Note that since linear terms are never invariant under a nontrivial orbifold group, a C^1 function necessarily has a critical point at any nontrivial orbifold singularity, that is, $\Sigma \subset Crit(f) \equiv \{x \in M \mid \nabla_g f(x) = 0\}$.

Remark 1.1.3. In general, an orbifold can have higher-dimensional singular sets. So our definition is restrictive in that we only allow isolated quotient singularities.

Assume (M, g) is a compact Riemannian n-orbifold with positive scalar curvature $R_g > 0$. Let $K > 0$ be a positive C^2 function on M. We will study the following equation

$$-\Delta_g u + c(n) R_g u = K u^p \tag{1.1.3}$$

where $c(n) = \frac{n-2}{4(n-1)}$, $1 < p \leq \frac{n+2}{n-2}$ and Δ_g is the Laplacian operator associated with g. Let $L_g = \Delta_g - c(n) R_g$ denote the conformal Laplacian of the metric g. When $p = \frac{n+2}{n-2}$, the solution of equation (1.1.3) corresponds to the prescribed scalar curvature problem. That is, the metric given by $\tilde{g} = u^{\frac{4}{n-2}} g$ has scalar curvature $R_{\tilde{g}} = \frac{4(n-1)}{n-2} K$.

The Yamabe problem on manifolds is well-understood, and we refer to [7, 26, 42, 52] and references therein. The prescribed scalar curvature problem on S^n and other manifolds has been studied in many works; see for example [10, 15, 16, 30, 28, 32, 49]. Prescribed scalar curvature on manifolds is studied for example in [11, 33, 36]. More recently, the Yamabe problem on singular spaces has been of interest; see for example [2, 3, 4, 5, 14, 37, 38, 50, 51].

Analogous to the generalization from manifolds to orbifolds, there is the following generalization of asymptotically flat (AF) metrics.

Definition 1.1.4. A complete Riemannian orbifold (X^n, g) with finitely many singular points is called *asymptotically locally Euclidean* or ALE of order τ if it has finitely many ends and for each end there exists a finite subgroup $\Gamma \subset \mathrm{O}(n)$ acting freely on $\mathbb{R}^n \setminus \{0\}$ and a diffeomorphism $\psi : X \setminus K \to (\mathbb{R}^n \setminus \overline{B_R(0)})/\Gamma$ where K is a compact subset of X, and such that under this identification,

$$(\psi_* g)_{ij} = \delta_{ij} + O(\rho^{-\tau}), \tag{1.1.4}$$

$$\partial^k (\psi_* g)_{ij} = O(\rho^{-\tau-|k|}), \tag{1.1.5}$$

for any partial derivative of order $|k|$, as $\rho \to \infty$, where ρ is the distance to some fixed basepoint.

We will occasionally refer to an AF metric as an ALE metric, since AF is exactly the case of ALE with $\Gamma = \{e\}$. Next, we give the definition of the ADM mass on asymptotically locally Euclidean (ALE) orbifolds.

Definition 1.1.5. Given an n-dimensional ALE orbifold (X, g) with asymptotic coordinates $\{z^i\}$ and quotient group Γ near ∞, define the ADM mass as follows:

$$m(g) = \frac{|\Gamma|}{Vol(S^{n-1})} \lim_{r\to\infty} \int_{S_r/\Gamma} \sum_{i,j=1}^{n} (\partial_i g_{ij} - \partial_j g_{ii})(\partial_j \lrcorner dV_z). \tag{1.1.6}$$

where S_r/Γ is the hypersurface at $|z| = r$, and dV_z is the Euclidean volume element.

Remark 1.1.6. Bartnik proved that in the AF case, if $\tau > (n-2)/2$ then the mass is well-defined and independent of the choice of coordinates at infinity [9]. A similar argument shows that the same result holds in the ALE case.

Remark 1.1.7. Note that if $|\Gamma| = \{e\}$ is the trivial group, the formula above defines the mass for AF orbifold (X, g), which is consistent with [26, Definition 8.2]. Note also that if $\Gamma \neq \{e\}$, our coefficient differs from [21] due to the factor of $|\Gamma|$. Our convention has the advantage that it eliminates the need for writing extra factors of $|\Gamma|$ in several formulas.

For any Riemannian orbifold (M, g) with positive scalar curvature, we can construct a scalar-flat ALE orbifold $(X, \hat{g})$ by the following well-known procedure.

Definition 1.1.8. Take g-normal coordinates $\{x^i\}$ centered at point $\bar{x}$ and let $r = |x|$ denote the distance function. Let $\psi_{\bar{x}} > 0$ be the Green's function of L_g with leading term r^{2-n} near $\bar{x}$. Then $(X_{\bar{x}}, \hat{g}_{\bar{x}}) = (M \setminus \bar{x}, \psi_{\bar{x}}^{4/(n-2)} g)$ is a scalar flat ALE orbifold. We will refer to $(X_{\bar{x}}, \hat{g}_{\bar{x}})$ as the conformal blow-up of g at the point $\bar{x}$.

Of course, if $\bar{x}$ is a smooth point of M, then $(X_{\bar{x}}, \hat{g}_{\bar{x}})$ is an AF orbifold, but if $\bar{x}$ is a singular point, then $(X_{\bar{x}}, \hat{g}_{\bar{x}})$ is an ALE orbifold.

1.2 Positive Mass Theorem on AF orbifolds

Our first result is that the positive mass theorem does hold for AF orbifolds.

Theorem 1.2.1. *Let (X, g_X) be an asymptotically flat (AF) n-dimensional Riemannian orbifold with finitely many isolated singular points, with $R(g_X) \geq 0$, and which is of order $\tau_X > \frac{n-2}{2}$. Then* $\mathrm{mass}(g_X) \geq 0$. *and* $\mathrm{mass}(g_X) = 0$ *if and only if there are no nontrivial orbifold singularities, and (X, g_X) is isometric to $(\mathbb{R}^n, g_{Euc})$.*

This is proved in Section 2. The basic idea is to use a certain Green's function for the Laplacian based at the orbifold points, which we use to reduce to the positive mass theorem for manifolds with concave boundary due to Hirsch-Miao [22] using the fundamental work of Schoen-Yau [46, 47, 48].

Remark 1.2.2. The positive mass theorem holds on ALE spaces only with some extra assumptions; see for example [40]. In contrast, the positive mass theorem does not necessarily hold for arbitrary ALE metrics with nonnegative scalar curvature. The first examples were given in [24], and many more in [21]. Theorem 1.2.1 shows that the study of the Yamabe equation on orbifolds can be substantially different from the manifold case. In dimension 4, we can define a mass function $m : M \to \mathbb{R}$ by assigning the mass of the conformal blow-up at $\bar{x} \in M$. In the manifold case, this is a smooth function [20]. However, the above result shows that if (M, g) is an orbifold and the conformal blow-up has negative mass at $\bar{x} \in \Sigma$, then the mass function m is necessarily *discontinuous* at $\bar{x}$.

1.3 Existence and compactness results

There is a long history of existence and compactness results for the Yamabe problem, which is the case when $K = constant$. The fundamental idea for compactness was due to Rick Schoen [44, 45, 43]; compactness results for the Yamabe problem in low dimensions were then proved in [18, 23, 31, 29, 35]. As mentioned above, existence and compactness results for variable K have been studied in great detail on S^n and on manifolds. Our next result generalizes many of these results in dimension four to the case of a Riemannian orbifold.

Theorem 1.3.1. *Let (M, g) be a compact Riemannian 4-dimensional orbifold with positive scalar curvature. Let $K \in C^2(M)$ satisfy*

$$0 < \delta_1 \equiv \inf_M K, \tag{1.3.1}$$

$$0 < \delta_2 \equiv \inf_{\bar{x} \in Crit(K)} \left| m(\hat{g}_{\bar{x}}) + \frac{\Delta_g K(\bar{x})}{2K(\bar{x})} \right|. \tag{1.3.2}$$

Then there exists some constant C depending only on $M, g, \delta_1, \delta_2, \|K\|_{C^2(M)}$ such that

$$1/C \leq u \leq C \quad \text{and} \quad \|u\|_{C^{2,\alpha}(M)} \leq C \tag{1.3.3}$$

for all solutions u of (1.1.3) *with $p = 3$, where $0 < \alpha < 1$.*

Moreover, if

$$0 < \delta_3 \equiv \inf_{\bar{x} \in Crit(K)} \left(m(\hat{g}_{\bar{x}}) + \frac{\Delta_g K(\bar{x})}{2K(\bar{x})} \right) \tag{1.3.4}$$

then (1.3.3) *holds for all $1 < 1 + \varepsilon < p \leq 3$ where C in addition depends upon δ_3, ϵ. Consequently, in this case there exists a solution u of* (1.1.3) *with $p = 3$.*

Theorem 1.3.1 will be proved in Sections 3, 4, and 5. One of the main difficulties is due to the "discontinuity" of conformal normal coordinates at an orbifold point. In Section 5, we will show that if concentration of a sequence u_k happens at an orbifold point, then the local maxima of u_k must occur *exactly* at the orbifold point for k sufficiently large; see Condition 3.3.6 which is a generalization of the isolated simple blow-up condition. Once we prove that blow-up points satisfy Condition 3.3.6; we can then fix the coordinates to be centered at the orbifold point, and then there is a contribution only from the mass function at the blow-up point. Note: our argument does not need continuity of the mass function which, as pointed out above, is not true in general anyway.

Remark 1.3.2. We emphasize that under the assumption (1.3.2), we are only claiming compactness; we do not have the existence in this general case. The main reason is that the subcritical method only works under the stronger assumption (1.3.4). However, the compactness result does allow one to define the Leray-Schauder degree. We expect that by imposing extra assumptions on K, one should be able to prove this degree is non-zero in some cases where (1.3.4) does not hold, but we do not pursue this here. Results along this line in the manifold case can be found in [11, 28, 33, 36].

Remark 1.3.3. The similar compactness and existence results hold on higher dimensions if making more complicated assumptions on higher order derivatives of K. Basically, it is the process of carefully comparing the higher order terms of the Green function and the terms in the Pohozaev identity. Especially, in dimension $n \geq 6$, the mass term won't show up, but the Weyl tensor gets involved. Since here we care the most about the nature that mass could be negative on orbifolds, we are not going to dive into the compactness property in higher dimensions.

For completeness, we next give a (slightly informal) re-statement of Theorem 1.3.1 in the case that $K = constant$. By scaling, we can assume that $K \equiv 1$.

Corollary 1.3.4. *Let (M, g) be a compact Riemannian 4-dimensional orbifold with positive scalar curvature. In the case $K \equiv 1$, the set of solutions to* (1.1.3) *with $p = 3$ is compact as long as $\inf_{\bar{x} \in M} |m(\hat{g}_{\bar{x}})| > 0$. Moreover, if $\inf_{\bar{x} \in M} m(\hat{g}_{\bar{x}}) > 0$. then there exists a solution.*

Remark 1.3.5. In general, bubbling can only happen at the orbifold points $\bar{x} \in \Sigma$ satisfying $m(\hat{g}_{\bar{x}}) = 0$. This condition does not imply Ricci-flatness of the Green's function metric at $\bar{x}$, see the examples given in [21] and [6].

Next, we have another existence result in all dimensions which is proved using energy methods, and which some more general cases than under the assumption (1.3.4). Roughly, this says that if K is not too large, then we need assumption (1.3.4) to hold at only one orbifold point.

Theorem 1.3.6. *Let (M, g) be a compact Riemannian n-dimensional orbifold with singularities $\Sigma_\Gamma = \{(q_1, \Gamma_1), \cdots, (q_l, \Gamma_l)\}$ and positive scalar curvature. Let K be a positive*

C^2 function on M. Assume that

$$\max_{1\le i\le l}\{|\Gamma_i|^{\frac{2}{n-2}}K(q_i)\}\ge \sup K, \tag{1.3.5}$$

and

$$\begin{cases} m(\hat g_{q_{i_0}})+\frac{\Delta_g K(q_{i_0})}{2K(q_{i_0})}>0, & n=4,\\ \Delta_g K(q_{i_0})>0, & n\ge 5.\end{cases} \tag{1.3.6}$$

for some i_0 such that $|\Gamma_{i_0}|^{\frac{2}{n-2}}K(q_{i_0})=\max_{1\le i\le l}\{|\Gamma_i|^{\frac{2}{n-2}}K(q_i)\}$, where $\hat g_{q_{i_0}}$ is as in Definition 1.1.8. Then there exists a solution of equation (1.1.3) *with $p=3$.*

This will be proved in Section 6, which is closely related to [19, Theorem 2.1], [1, Theorem 3.1] and [34, Proposition 5.1]. The analogue on manifolds is that (1.3.6) has to hold on one maximum point of K, which is only possible in dimension $n=4$. Hence the theorem is quite special to the orbifold case in dimension $n\ge 5$.

In [27], they proved an existence result on S^n, with rotationally symmetric prescribed scalar curvatures. It is equivalent to the prescribed scalar curvature problem on the football orbifold metric obtained by taking a quotient on S^n by group $\mathbb{Z}/k\mathbb{Z}$. In this point of view, [27, Main Theorem 1.12] is a special case of our Theorem 1.3.6.

Furthermore, we have many other applications of Theorem 1.3.6, one is to the LeBrun's metric, which will be presented in Theorem 1.4.1.

Another one is to the Calabi metric. As introduced and studied in [39], the Calabi metric $g_{CAL(n)}$ is a Ricci-flat Kähler ALE metric on the total space X_n of the line bundle $\mathcal{O}(-n)\to\mathbb{P}^{n-1}$, which is $U(n)$-invariant. For $n=2$, $(X_2, g_{CAL(2)})$ is also known as the Eguchi Hanson metric, which is included in the family of LeBrun's metrics. When $n\ge 3$, $(X_n, g_{CAL(n)})$ is of real dimension $2n\ge 6$, where we may apply Theorem 1.3.6 in the following way.

Corollary 1.3.7. *For $n \geq 3$, take a $U(n)$-invariant conformal compactification $(\check{X}_n, \check{g}_{CAL(n)})$ of the Calabi metric $(X_n, g_{CAL(n)})$, such that the infinity of $g_{CAL(n)}$ is compactified to an orbifold point $\check{q}$, with quotient group $\mathbb{Z}/n\mathbb{Z}$. Let K be a positive C^2 function on $\check{X}_n$. Assume that*

$$n^{\frac{1}{n-1}} K(\check{q}) \geq \sup K, \tag{1.3.7}$$

$$\Delta_{g_{CAL(n)}} K(\check{q}) > 0, \tag{1.3.8}$$

then there exists a solution of equation (1.1.3) *with $g = \check{g}_{CAL(n)}$ and $p = \frac{n+1}{n-1}$.*

Remark 1.3.8. Furthermore, if assuming K is a $U(n)$-invariant positive function, a generalization of Theorem 1.3.1 to higher dimensions would imply the existence of the $U(n)$-invariant solution of equation (1.1.3) with $g = \check{g}_{CAL(n)}$ and $p = \frac{n+1}{n-1}$, with only requiring (1.3.8) to hold. The main reason is that a sequence of $U(n)$-invariant solutions can only possibly blow up at the orbifold point. We omit the detailed proof.

1.4 On LeBrun's metrics

In this subsection, we will use the previous results to analyze the prescribed scalar curvature problem on the family of orbifolds mentioned above: the LeBrun negative mass metrics on $\mathcal{O}_{\mathbb{P}^1}(-n)$. Note: these are 4-dimensional manfiolds, so in this subsection n will denote the first Chern class of the bundle, not the dimension. These metrics possess an isometric U(2) action. As we will see below, we have some results in the general case, but our most complete results are under a U(2) symmetry assumption.

In [24], LeBrun presented the first known example of a scalar-flat ALE metric of negative ADM mass. For any $n \in \mathbb{N}^*$, define

$$g_{LEB(n)} = \frac{1+\hat{r}^2}{n+\hat{r}^2} d\hat{r}^2 + (1+\hat{r}^2)\sigma_1^2 + (1+\hat{r}^2)\sigma_2^2 + \frac{\hat{r}^2(n+\hat{r}^2)}{1+\hat{r}^2}\sigma_3^2 \tag{1.4.1}$$

where $\hat{r} \in (0, \infty)$ is the radial coordinate, $\{\sigma_1, \sigma_2, \sigma_3\}$ is a left-invariant coframe on $S^3 = SU(2)$. Attach a $\mathbb{P}^1$ at $\hat{r} = 0$. After taking a quotient by $\Gamma_n = \mathbb{Z}/n\mathbb{Z}$, the metric extends smoothly over this $\mathbb{P}^1$. Furthermore, the metric is ALE with group action Γ_n near $\hat{r} = \infty$ and the resulting manifold is diffeomorphic to $\mathcal{O}_{\mathbb{P}^1}(-n)$.

The mass term defined in [24] differs from our Definition 1.1.5 by a scaling factor. With a modification of the mass computed in [24], we obtain

$$m(g_{LEB(n)}) = -2(n-2). \tag{1.4.2}$$

For any $n \in \mathbb{N}^*$, $g_{LEB(n)}$ is scalar-flat. Note that $g_{LEB(1)}$ is conformal to the Fubini-Study metric on $\mathbb{P}^2$, and is known as the Burns metric, which is AF and has positive mass. $g_{LEB(2)}$ is the Eguchi-Hanson metric, which is Ricci-flat ALE and has zero mass. For any $n \geq 3$, $g_{LEB(n)}$ has negative mass. More details and properties of LeBrun metric can be found in [24], [50] and [17].

We next choose an orbifold compactification by defining

$$\check{g}_{LEB(n)} = \frac{1}{(n+\hat{r}^2)^2} \cdot g_{LEB(n)}. \tag{1.4.3}$$

Then $(\check{\mathcal{O}}_{\mathbb{P}^1}(-n), \check{g}_{LEB(n)})$ is a compact orbifold with singular point $\check{q}$ at $\hat{r} = \infty$ with quotient group Γ_n. Its scalar curvature is computed to be

$$R_{\check{g}_{LEB(n)}} = \frac{24n(n+\hat{r}^2)}{1+\hat{r}^2} > 0. \tag{1.4.4}$$

In this case, Theorem 1.3.6 specializes to the following.

Theorem 1.4.1. *Consider the orbifold* $(\check{\mathcal{O}}_{\mathbb{P}^1}(-n), \check{g}_{LEB(n)})$ *for* $n \geq 2$*. Let* K *be a positive* C^2 *function such that*

$$\begin{aligned} &\sup K \leq nK(\check{q}) \\ &4(n-2)K(\check{q}) < \Delta_{\check{g}_{LEB(n)}} K(\check{q}). \end{aligned} \tag{1.4.5}$$

Then there exists a solution of equation (1.1.3) *with* $g = \check{g}_{LEB(n)}$ *and* $p = 3$*.*

Remark 1.4.2. For $n = 2$, the conditions are

$$\sup K \leq 2K(\check{q}), \quad 0 < \Delta_{\check{g}_{LEB(2)}} K(\check{q}). \tag{1.4.6}$$

In this case, from [50], we know that there is no solution with $K = constant$, which certainly does not contradict with the condition on the Laplacian at the orbifold point. However, Theorem 1.4.1 implies that any sufficiently small perturbation of a constant with $\Delta_{\check{g}_{LEB(2)}} K(\check{q}) > 0$ *does* admit a solution. So the case of $K = constant$ is right on the boundary of existence. Furthermore, if we take a sequence of functions K_k satisfying (1.4.6), but converging to a constant in C^2 norm, then this gives an example where bubbling *must* occur as $k \to \infty$.

Next, we turn to the U(2)-invariant problem. Define a function space $\mathcal{X}$

$$\mathcal{X} = \{K : \check{\mathcal{O}}_{\mathbb{P}^1}(-n) \to \mathbb{R} \mid K \text{ is smooth, U(2)-invariant, and } K > 0\}. \tag{1.4.7}$$

Define $s = 1/\hat{r}$ to be the inverted radial coordinate centered at the orbifold point $\check{q}$. Let $K = K(s)$ to be a function in the radial coordinate s. The smoothness condition on $K(s) \in \mathcal{X}$, implies that $K(s)$ is a smooth function of s, and the power expansion of $K(s)$ and $K(1/s)$ at $s = 0$ both only have even order terms.

Given any $K(s) \in \mathcal{X}$, we are asking whether there exists a solution $u(s) \in \mathcal{X}$ such that $u^2 g_{\check{g}_{LEB(n)}}$ is a compact orbifold with scalar curvature K, i.e.

$$\Delta_{\check{g}_{LEB(n)}} u - \frac{1}{6} R_{\check{g}_{LEB(n)}} u = -\frac{1}{6} K u^3. \tag{1.4.8}$$

Decompose $\mathcal{X}$ into three disjoint subsets

$$\begin{aligned}
\mathcal{X}_{n,+} &= \{K \in \mathcal{X} : K''(0)/K(0) > n-2\},\\
\mathcal{X}_{n,0} &= \{K \in \mathcal{X} : K''(0)/K(0) = n-2\},\\
\mathcal{X}_{n,-} &= \{K \in \mathcal{X} : K''(0)/K(0) < n-2\},
\end{aligned} \tag{1.4.9}$$

where $K''(0)$ and $K(0)$ are computed in the s-coordinate at the orbifold point $s = 0$.

Define a function set

$$\Omega_\Lambda = \{u \in \mathcal{X} : \|u\|_{C^{2,\alpha}(\check{\mathcal{O}}_{\mathbb{P}^1}(-n))} < \Lambda,\ u > \Lambda^{-1}\} \tag{1.4.10}$$

and a map

$$F_{p,K} : \bar{\Omega}_\Lambda \to C^{2,\alpha}(\check{\mathcal{O}}_{\mathbb{P}^1}(-n)) \ \text{ by } \ F_p(u) = u + \frac{1}{6} L^{-1}_{\check{g}_{LEB(n)}}(K u^p), \tag{1.4.11}$$

where $L_{\check{g}_{LEB(n)}} = \Delta_{\check{g}_{LEB(n)}} - \frac{1}{6} R_{\check{g}_{LEB(n)}}$. Similar to [42], [31], [29], [23], [35], we can define the U(2)-invariant Leray-Schauder degree of $F_{p,K}$ in the region Ω_Λ with respect to $0 \in C^{2,\alpha}(\check{\mathcal{O}}_{\mathbb{P}^1}(-n))$, which we denote by $\deg_{\mathrm{U}(2)}(F_{p,K}, \Omega_\Lambda, 0)$. Using Theorem 1.3.1 and some other special techniques, we present our last theorem, which gives a complete understanding of the U(2)-invariant Leray-Schauder degree.

Theorem 1.4.3. *For any $n \in \mathbb{N}^*$, on $(\check{\mathcal{O}}_{\mathbb{P}^1}(-n), \check{g}_{LEB(n)})$, we have the following conclusions.*

(1) There exists C depending only on $(\check{\mathcal{O}}_{\mathbb{P}^1}(-n), \check{g}_{LEB(n)})$ and a small $\varepsilon > 0$ such that for all $p \in (1+\varepsilon, 3]$, for all $K \in \mathcal{X}_{n,+}$ and for all $\Lambda > C$, we have

$$\deg_{\mathrm{U}(2)}(F_{p,K}, \Omega_\Lambda, 0) = -1. \tag{1.4.12}$$

Consequently, equation (1.4.8) *admits at least one* U(2)*-invariant solution in $\mathcal{X}$.*

(2) There exists C depending only on $(\check{\mathcal{O}}_{\mathbb{P}^1}(-n), \check{g}_{LEB(n)})$ such that for all $K \in \mathcal{X}_{n,-}$ and for all $\Lambda > C$, we have

$$\deg_{\mathrm{U}(2)}(F_{3,K}, \Omega_\Lambda, 0) = 0. \tag{1.4.13}$$

Remark 1.4.4. We note that vanishing of the Leray-Schauder degree does not give any information regarding the existence of a solution. However, we can moreover show that there is no solution at all for a large class of functions in $\mathcal{X}_{n,-}$; see Theorem 7.1.6. In particular, there is no solution for $K = constant$ when $n \geq 2$. For $n = 2$ and $K = constant$, nonexistence of *any* solution (symmetric or non-symmetric) was proved in [50]. However, it is still an open question whether the case $K = constant$ possibly admits some non-symmetric solution when $n > 2$.

Remark 1.4.5. For any $n \in \mathbb{N}^*$, the set $\mathcal{X}_{n,0}$ can be viewed as a "wall" in the space of positive radial functions $\mathcal{X}$, and the Leray-Schauder degree jumps by 1 upon crossing this wall, which is a phenomenon observed in many other geometric PDE problems.

All of the above results regarding the LeBrun metrics are proved in Section 7.

Chapter 2

Properties of the mass

In this section, we will prove Theorem 1.2.1.

For simplicity, let us assume that there is exactly 1 orbifold point, which we denote as q, with orbifold group $\Gamma \subset \mathrm{O}(n)$ (the argument below easily generalizes to the case of multiple orbifold singularities). Let r denote a positive smooth function which is the Euclidean distance in the AF coordinate system, and near q is the distance to q.

We will first prove a lemma showing existence of a certain harmonic function on $X \setminus \{q\}$.

Lemma 2.0.1. *There exists a unique harmonic function $H : X \setminus \{q\} \to \mathbb{R}$ which satisfies $H > 1$ and admits the expansion*

$$H = \begin{cases} r^{2-n} + O(r^{4-n-\epsilon}) & \text{as } r \to 0 \\ 1 + Ar^{2-n} + O(r^{2-n-\epsilon}) & \text{as } r \to \infty \end{cases}, \tag{2.0.1}$$

for $\epsilon > 0$ sufficiently small, for some constant $A > 0$.

Proof. Let ϕ be the cutoff function

$$\phi(t) = \begin{cases} 1 & t \leq 1 \\ 0 & t \geq 2 \end{cases}, \tag{2.0.2}$$

and consider

$$h_0 = \phi(r/r_0) r^{2-n}, \tag{2.0.3}$$

for $r_0 > 0$ small. Since h_0 is harmonic with respect to the Euclidean metric near point q, by expanding Δ_g at the Euclidean metric, it is not hard to see that

$$\Delta_g h_0 = \begin{cases} O(r^{2-n}) & r \to 0 \\ 0 & r \geq 2r_0 \end{cases}. \tag{2.0.4}$$

Denote $X_q = X \setminus \{q\}$. The argument below uses weighted Hölder space theory; for background we refer to [9, 26]. Consider the doubly weighted Hölder space $C^{k,\alpha}_{\delta_0,\delta_\infty}(X_q)$, which satisfies if $u \in C^{k,\alpha}_{\delta_0,\delta_\infty}(X_q)$ then

$$u = \begin{cases} O(r^{\delta_0}) & r \to 0 \\ O(r^{\delta_\infty}) & r \to \infty \end{cases}. \tag{2.0.5}$$

For any $\epsilon > 0$, from (2.0.4), we have

$$\Delta h_0 \in C^{k-2,\alpha}_{2-n-\epsilon,-n+\epsilon}. \tag{2.0.6}$$

Consider the operator

$$\Delta_g : C^{k,\alpha}_{4-n-\epsilon,2-n+\epsilon}(X_q) \to C^{k-2,\alpha}_{2-n-\epsilon,-n+\epsilon}(X_q). \tag{2.0.7}$$

The adjoint operator has domain $(C^{k-2,\alpha}_{2-n-\epsilon,-n+\epsilon}(X_q))^*$, and kernel elements lie in the doubly weighted space $C^{k,\alpha}_{-2+\epsilon,-\epsilon}(X_q)$. The removable singularity theorem then says that a kernel element u in this space extends to X, and then $u \in C^{k,\alpha}_{-\epsilon}(X)$ (the weighted space on X with only a weight $\delta_\infty = -\epsilon$ at infinity), so $u \equiv 0$. Thus for $\epsilon > 0$ sufficiently small, the operator in (2.0.7) is surjective, and we can solve for $\Delta h_\epsilon = \Delta_g h_0$, with $h_\epsilon \in C^{k,\alpha}_{4-n-\epsilon,2-n+\epsilon}(X_q)$.

The function $h \equiv h_0 - h_\epsilon$ satisfies $\Delta_g h = 0$, and by the existence of a harmonic expansion near ∞, it admits the expansion.

$$h = \begin{cases} r^{2-n} + O(r^{4-n-\epsilon}) & r \to 0 \\ Ar^{2-n} + O(r^{2-n-\epsilon}) & r \to \infty \end{cases}, \tag{2.0.8}$$

for some constant A. We then define $H = 1 + h$, which is harmonic. We have that $\lim_{r\to\infty} H = 1$, and $\lim_{r\to 0} H = +\infty$. If H were not strictly larger than 1, then it would have an interior minimum. The strong maximum principle would then imply that H is constant, which is impossible. So $H > 1$, which clearly implies that $A > 0$. Obviously, H is unique. □

2.1 Proof of Theorem 1.2.1

For any constant $\delta > 0$, we define

$$H_\delta = \delta H + (1 - \delta) \tag{2.1.1}$$

Then H_δ satisfies $\Delta_g H_\delta = 0$, $H_\delta > 1$, and admits the expansion

$$H_\delta = \begin{cases} \delta r^{2-n} + O(r^{4-n-\epsilon}) & r \to 0 \\ 1 + \delta A r^{2-n} + O(r^{2-n-\epsilon}) & r \to \infty \end{cases}, \tag{2.1.2}$$

for $\epsilon > 0$ sufficiently small, for the fixed constant A from Lemma 2.0.1.

Next, we consider the metric $(X_q, g_\delta) = (X \setminus \{q\}, H_\delta^{\frac{4}{n-2}} g_X)$. Near $r \sim \infty$, g_δ has a single AF end of order $\min\{\tau_X, n-2\}$. Since q is an orbifold point, near q, g_δ has a single ALE end of order $\tau = 2 - \epsilon$. To see this, choose Riemannian normal coordinates $\{x^i\}$ for g_X around q, then we have the expansions

$$g_X = dx^2 + O(|x|^2) \tag{2.1.3}$$

$$H_\delta = \delta |x|^{2-n} + O(|x|^{4-n-\epsilon}), \tag{2.1.4}$$

which yield the expansion

$$g_\delta = H_\delta^{\frac{4}{n-2}} g_X = \delta^{\frac{4}{n-2}} |x|^{-4} (1 + O(|x|^{2-\epsilon}))^{\frac{4}{n-2}} (dx^2 + O(|x|^2)), \tag{2.1.5}$$

as $|x| \to 0$. Next, define coordinates y by

$$y = \delta^{\frac{2}{n-2}} \frac{x}{|x|^2}. \tag{2.1.6}$$

A computation then shows that

$$g_\delta = dy^2 + O(|y|^{-2+\epsilon}) \tag{2.1.7}$$

as $|y| \to \infty$, so g_δ is indeed ALE of order $\tau = 2 - \epsilon$. Note also that the scalar curvature of g_δ is given by

$$R(g_\delta) = c(n)^{-1} H_\delta^{-\frac{n+2}{n-2}} (-\Delta_g H_\delta + c(n) R_g H_\delta) = H_\delta^{-\frac{4}{n-2}} R_g \geq 0. \tag{2.1.8}$$

Given $\delta > 0$, we can choose a very large distance sphere Σ_δ in the ALE end of g_δ which is strictly concave with respect to the normal pointing to the AF end. Let X_{Σ_δ} be the manifold with boundary obtained by removing the ALE end outside of Σ, which is a manifold with strictly concave boundary with a single AF end. From [22, Theorem 1.5 and Remark 1.7], we conclude that

$$m(X_{\Sigma_\delta}, g_\delta) \geq 0. \tag{2.1.9}$$

But an easy computation shows that

$$m(X_{\Sigma_\delta}, g_\delta) = m(X, g_X) + b(n)\delta A, \tag{2.1.10}$$

where $b(n) > 0$ is a dimensional constant. Since this is true for any constant $\delta > 0$, and A is a fixed constant, we conclude that

$$m(X, g_X) \geq 0. \tag{2.1.11}$$

Note that if $m(X, g_X) = 0$, then we cannot conclude that $m(X_{\Sigma_\delta}, g_\delta) = 0$, since $A > 0$. Therefore we cannot directly use the equality case in [22, Theorem 1.5]. So to finish the proof, if $\text{mass}(X, g_X) = 0$ then we instead argue as in [26, Lemma 10.7] to conclude that g_X is Ricci-flat (this argument is valid in our orbifold setting). Since g_X is asymptotically flat, we have asymptotic equality in Bishop's volume inequality (which holds for orbifolds; see [12]). This implies that g_X is flat, which clearly implies that there can be no nontrivial orbifold singularities, and (X, g_X) is isometric to $(\mathbb{R}^n, g_{Euc})$.

2.2 Some remarks on the dimension

If the dimension n is odd, we have the following.

Proposition 2.2.1. *Let (X, g) is a compact Riemannian orbifold with isolated singularities and odd-dimensional. Then any nontrivial orbifold point must have $\Gamma = \mathbb{Z}/2\mathbb{Z}$. Furthermore, (X, g) is a good orbifold. That is, there is a $\mathbb{Z}/2\mathbb{Z}$ action on a compact* *manifold $\tilde{X}$ with finitely many fixed points such that $X = \tilde{X}/(\mathbb{Z}/2\mathbb{Z})$. Letting $\pi : \tilde{X} \to X$ denote the quotient mapping, then $\pi^* g$ is a smooth Riemannian metric on $\tilde{X}$.*

Proof. For n odd, any element $A \in \mathrm{O}(n)$ must have ± 1 as an eigenvalue, so the only possibility for a nontrivial orbifold point is $\Gamma = \mathbb{Z}/2\mathbb{Z}$. Near any singular point q, a small distance sphere is homeomorphic to $\mathbb{RP}^{n-1}$, which is non-orientable if n is odd. So if there is any nontrivial orbifold point, then X contains a non-orientable 2-sided hypersurface, which implies that $X \setminus \Sigma$ is non-orientable, where Σ is the finite set of singular points. Let $\pi : X' \to X \setminus \Sigma$ denote the orientable double cover. Consider $\tilde{X} = X' \cup \{\tilde{q}_1, \dots, \tilde{q}_j\}$ where $\tilde{q}_j$ are points, and extend $\pi : \tilde{X} \to X$ by letting $\pi(\tilde{q}_j) = q_j$. We extend to $\mathbb{Z}/2\mathbb{Z}$ action to $\tilde{X}$ with fixed points at $\tilde{q}_j$, and endow $\tilde{X}$ with the quotient topology. It is then straightforward to show that $\tilde{X}$ is a smooth manifold and $\pi^* g$ extends as a smooth Riemannian metric to $\tilde{X}$. □

Corollary 2.2.2. *Let (X, g) be a compact Riemannian orbifold with isolated singularities and odd-dimensional. Then the mass function $m : X \to \mathbb{R}$ satisfies $m > 0$ everywhere, unless (X, g) is conformal to (S^n, g_{round}) or a "football" metric $S^n/(\mathbb{Z}/2\mathbb{Z})$ with exactly 2 singular points.*

Proof. At any smooth point of X, the mass of the Green's function metric is positive by Theorem 1.2.1. If the mass at a smooth point were zero, then the Green's function metric would be Euclidean space, which would imply that (X, g) is conformal to (S^n, g_{round}). At a singular point q, by uniqueness of the Green's function metric, the Green's function metric at q must be the $\mathbb{Z}/2\mathbb{Z}$ quotient of the Green's function of $(\tilde{X}, \pi^* g)$ at $\tilde{q}$. Since $\tilde{X}$ is a manifold, by the usual positive mass theorem, the Green's function metric upstairs must have non-negative mass, so the Green's function downstairs must also. If the mass at an orbifold point was 0, then the Green's function metric upstairs would have to be Euclidean space. This implies that the Green's function metric downstairs is a $\mathbb{Z}/2\mathbb{Z}$-

quotient of Euclidean space, which implies that (X, g) is conformal to a "football" metric $S^n/(\mathbb{Z}/2\mathbb{Z})$. $\square$

The above remarks show that the odd-dimensional case of the orbifold Yamabe problem is equivalent to $\mathbb{Z}/2\mathbb{Z}$-equivariant Yamabe problem on the manifold $\tilde{X}$. The even-dimensional case is more interesting: there are many examples of "bad" orbifolds which are not quotients of manifolds. Furthermore, in dimension 4, there are examples of ALE spaces which have negative mass [24, 21]. So we note the following corollary of Theorem 1.2.1.

Corollary 2.2.3. *If (X, g) is a 4-dimensional Riemannian orbifold with negative mass at an orbifold point q, then the mass function $m : X \to \mathbb{R}$ is necessarily discontinuous at q.*

2.3 The mass and the term A_P in dimension 4

By [26, Lemma 6.4], if dimension $n = 3, 4, 5$ or M is conformally flat in a neighborhood of a point P, and g is the conformal normal metric at P, that is, there exist g-conformal normal coordinates $\{x^i\}$ centered at P, then ψ_P as defined in Definition 1.1.8 has the following expansion near P:

$$\psi_P = r^{2-n} + A_P + O(r), \tag{2.3.1}$$

where the constant A_P is called the regular term corresponding to the conformal blow-up of metric g at point P.

In dimension 4, we compute the relation between the mass and the term A_P as following.

Proposition 2.3.1. *Let (M, g_1) be a 4-dimensional Riemannian compact orbifold and P be any point on M. Then there exists ϕ such that*

$$\phi(P) = 1, \quad \nabla\phi(P) = 0, \tag{2.3.2}$$

and

$$g_2 = \phi^2 g_1 \tag{2.3.3}$$

is the conformal normal metric with conformal normal coordinates centered at P.

Let A_P be the regular term corresponding to the conformal blow-up of g_2 at point P. Let $\hat{g}_1$ be the conformal blow-up of g_1 at point P. Then we have

$$m(\hat{g}_1) = 12A_P. \tag{2.3.4}$$

Proof. We start with a 4-dimensional compact Riemannian orbifold (M, g_1). Without loss of generality, assume P is a singular point on M accosiated with a quotient group Γ. The case that P being a smooth point follows similarly by letting $\Gamma = \{e\}$ and replacing "ALE" by "AF" in the following arguments.

Let $\{x^i\}$ be the normal coordinates centered at P. Let $s = |x|$ be the distance function. Then for small s,

$$g_1 = (\delta_{ij} + O(s^2))dx^i dx^j. \tag{2.3.5}$$

By [26, Theorem 5.1], there exists ϕ such that $g_2 = \phi^2 g_1$ is the conformal normal metric with conformal normal coordinates $\{y^i\}$ centered at P. Note that ϕ is a globally defined function on M, but the conformal normal coordinates $\{y^i\}$ is locally defined in a small neighborhood near P. Moreover, by [26, Theorem 5.6], after applying a dilation and a translation to the coordinates $\{y^i\}$, we may assume $\phi(P) = 1$ and $\nabla\phi(P) = 0$, in other words, for small s,

$$\phi = 1 + O(s^2). \tag{2.3.6}$$

Let $r = |y|$ be the distance function on metric g_2. For small r,

$$g_2 = (\delta_{ij} + O(r^2))dy^i dy^j, \tag{2.3.7}$$

and

$$r(q) = s(q) + O(s(q)^3) \tag{2.3.8}$$

for any point q near P.

By Definition 1.1.8, there exists a Green function ψ_2 with expansion

$$\psi_2 = r^{-2} + A_P + O(r) \tag{2.3.9}$$

on g_2 such that

$$\hat{g}_2 = \psi_2^2 g_2 \tag{2.3.10}$$

is the conformal blow-up of g_2 at point P, which is a scalar-flat ALE manifold. Let $\hat{y}^i = r^{-2}y^i$ be the inverted conformal normal coordinates in a neighborhood of P and let

$\hat{r} = |\hat{y}| = r^{-1}$ be the distance function near ∞. We have

$$
\begin{aligned}
\hat{g}_2 =& (r^{-2} + A + O(r))^2 g_2 \\
=& (r^{-2} + A + O(r))^2 (\delta_{ij} + O(r^2)) dy^i dy^j \\
=& (r^{-2} + A + O(r))^2 (\delta_{ij} + O(r^2)) r^4 d\hat{y}^i d\hat{y}^j \\
=& (\delta_{ij} + O(r^2)) d\hat{y}^i d\hat{y}^j \\
=& (\delta_{ij} + O(\hat{r}^{-2})) d\hat{y}^i d\hat{y}^j
\end{aligned}
\tag{2.3.11}
$$

Hence the coordinates $\{\hat{y}^i\}$ is the asymptotic coordinates on $\hat{g}_2$. By [26, Section 9], the ADM mass of metric g_2 is $m(\hat{g}_2) = 4(n-1)A_P$, where n is the dimension. Especially, in dimension 4,

$$
m(\hat{g}_2) = 12A_P. \tag{2.3.12}
$$

Furthermore, on metric g_1, let ψ_1 be the Green function with leading term $1/s^2$ near $s = 0$, then for small s, ψ_1 has expansion

$$
\psi_1 = s^{-2} + O(\log(s)) \tag{2.3.13}
$$

and

$$
\hat{g}_1 = \psi_1^2 g_1 \tag{2.3.14}
$$

is the conformal blow-up of g_1 at point P, which is also a scalar-flat ALE manifold. Let $\hat{x}^i = s^{-2}x^i$ be the inverted normal coordinates in a neighborhood of P and let $\hat{s} = |\hat{x}| = s^{-1}$ be the distance function near ∞. Similar to above, we have

$$
\hat{g}_1 = \big(\delta_{ij} + O(\log(\hat{s})\hat{s}^{-2})\big) d\hat{x}^i d\hat{x}^j \tag{2.3.15}
$$

Hence $\{\hat{x}^i\}$ is the asymptotic coordinates on $\hat{g}_1$.

We have four metrics in the conformal class of g_1, shown in the following diagram:

$$\begin{array}{ccc} \hat{g}_1 & & \hat{g}_2 \\ {\scriptstyle \cdot\psi_1^2}\uparrow & & {\scriptstyle \cdot\psi_2^2}\uparrow \\ g_1 & \xrightarrow{\cdot\phi^2} & g_2 \end{array}$$

Chasing the diagram, we get

$$\hat{g}_2 = (\psi_1^{-1}\phi\psi_2)^2\hat{g}_1. \tag{2.3.16}$$

Denote $w = \psi_1^{-1}\phi\psi_2$.

Note that near infinity on the ALE metric $\hat{g}_1$,

$$\begin{aligned} \psi_1^{-1} &= \left(s^{-2} + O(\log(s))\right)^{-1} = s^2 + O(\log(s)s^4) = \hat{s}^{-2} + O(\log(\hat{s})\hat{s}^{-4}), \\ \phi &= 1 + O(s^2) = 1 + O(\hat{s}^{-2}), \\ \psi_2 &= r^{-2} + A + O(r) = O(s^{-2}) + O(1) = O(\hat{s}^2) + O(1). \end{aligned} \tag{2.3.17}$$

Thus

$$w = 1 + O(\log(\hat{s})\hat{s}^{-2}) \quad \text{as } \hat{s} \to \infty \tag{2.3.18}$$

On the other hand, $\hat{g}_1$ and $\hat{g}_2$ are both scalar-flat, so w is a bounded harmonic function on $(X_P, \hat{g}_1)$. By the Liouville Theorem, w is a constant function. By (2.3.18), we know $w \equiv 1$.

Hence $\hat{g}_1 = \hat{g}_2$ and

$$m(\hat{g}_1) = m(\hat{g}_2) = 12A_P. \tag{2.3.19}$$

□

Remark 2.3.2. This result can be generalized to dimension $n = 3, 5$, or the case M is conformally flat near P. However, since we only need the result for dimension 4, we omit the statement and proof for other dimensions.

Chapter 3

Compactness preliminaries

Remarkable work in analyzing the blow-up points of equation (1.1.3) has been done in [35], [31], [29], [23]. In this section, we are going to quote some of their definitions and local results on manifolds, which also appear to hold on orbifolds by modification of proofs.

In the following context, we will write $\mathbb{R}^n/\Gamma$ or $B_r(\bar{x})/\Gamma$. If $\Gamma = \{e\}$ is the trivial group, it just denotes the Euclidean space or a smooth ball; if $\Gamma \neq \{e\}$ is a finite nontrivial group in $O(n)$, it denotes the Euclidean cone or a quotient of a ball centered at a singular point $\bar{x}$.

3.1 Conformal scalar curvature equation

Instead of dealing with equation (1.1.3), we will study the following conformal scalar curvature equation.

Let $\Omega \subset \mathbb{R}^n/\Gamma$ be an open neighborhood of the origin, and suppose g is a Riemannian metric in Ω. Suppose also f is a positive C^1 function defined in Ω.

Consider a positive C^2 function u satisfying

$$L_g u + K f^{-\tau} u^p = 0 \tag{3.1.1}$$

where K is a positive C^2 function, $1 < p \leq \frac{n+2}{n-2}$ and $\tau = \frac{n+2}{n-2} - p$.

This equation has two good properties.

It is scale invariant. Let u be a solution to equation (3.1.1). For any $s > 0$, define the rescaled solution $v(y) = s^{\frac{2}{p-1}} u(sy)$. Then $L_h v + \tilde{K}\tilde{f}^{-\tau} v^p = 0$, where $\tilde{K}(y) = K(sy)$, $\tilde{f}(y) = f(sy)$ and the components in metric h in normal coordinates are given by $h_{ij}(y) = g_{ij}(sy)$. Note that v satisfies an equation of the same type as equation (3.1.1).

It is also conformally invariant. Suppose $\tilde{g} = \phi^{\frac{4}{n-2}} g$ is a metric conformal to g and let u be a solution to equation (3.1.1). Then $\phi^{-1}u$ satisfies $L_{\tilde{g}}(\phi^{-1}u) + K(\phi f)^{-\tau}(\phi^{-1}u)^p = 0$, which is again an equation of the same type.

Thanks to the above two properties, which allow us to take a rescaling of the coordinates and conformally map g to some conformal normal metric in the later context, without changing the type of equation (3.1.1).

3.2 Pohozaev Identity

Suppose $u : B_\sigma(0)/\Gamma \to \mathbb{R}$ is a positive C^2 solution to the equation

$$a^{ij}(x)\partial_{ij}u + b^i(x)\partial_i u + c(x)u + K(x)u^p = 0, \tag{3.2.1}$$

where $p \neq -1, K \in C^1$ and a^{ij}, b^i, c are continuous functions, $1 \leq i, j \leq n$.

Define

$$P(r,u) = \int_{\{|x|=r\}/\Gamma} \Big(\frac{n-2}{2}u\frac{\partial u}{\partial \sigma} - \frac{r}{2}|\nabla u|^2 + r\Big|\frac{\partial u}{\partial r}\Big|^2\Big) d\sigma(r) \tag{3.2.2}$$

whenever $0 < r < \sigma$. Then, we have the following lemma.

Lemma 3.2.1 ([35] Lemma 2.1)**.** *$\forall 0 < r < \sigma$,*

$$\begin{aligned} P(r,u) = & - \int_{\{|x|\le r\}/\Gamma} \Big(x\cdot\nabla u + \frac{n-2}{2}u\Big)((a^{ij}-\delta^{ij})\partial_{ij}u + b^i\partial_i u)dx \\ & + \int_{\{|x|\le r\}/\Gamma} \Big(\frac{1}{2}x\cdot\nabla c + c\Big)u^2 dx - \frac{r}{2}\int_{\{|x|=r\}/\Gamma} cu^2 d\sigma(r) \\ & + \frac{1}{p+1}\int_{\{|x|\le r\}/\Gamma} (x\cdot\nabla K(x))u^{p+1}dx \\ & + \Big(\frac{n}{p+1} - \frac{n-2}{2}\Big)\int_{\{|x|\le r\}/\Gamma} K(x)u^{p+1}dx \\ & - \int_{\{|x|=r\}/\Gamma} \frac{1}{p+1}K(x)ru^{p+1}d\sigma(r). \end{aligned} \tag{3.2.3}$$

3.3 Isolated and isolated simple blow-up points

Let $\Omega = B_\sigma(\bar{x})/\Gamma$ be (a quotient of) an open ball centered at point $\bar{x}$. Suppose $\{g_k\}$ is a sequence of Riemannian metrics in Ω converging, in the C^2_{loc} topology, to a metric g. Let R_k denote the scalar curvature of g_k and R_g denote the scalar curvature of the limit metric g.

Suppose $\{f_k\}$ is a sequence of positive C^1 functions converging in the C^1_{loc} topology to a positive function f. Also suppose $\{K_k\}$ is a sequence of positive C^2 functions converging in the C^2_{loc} topology to a positive function K_∞.

Consider a sequence of positive C^2 functions u_k satisfying

$$L_{g_k}u_k + K_k f_k^{-\tau_k}u_k^{p_k} = 0 \text{ in } \Omega \tag{3.3.1}$$

where $1+\epsilon_0 < p_k \leq \frac{n+2}{n-2}$ for some $\epsilon_0 > 0$ and $\tau_k = \frac{n+2}{n-2} - p_k$.

Definition 3.3.1. Suppose u_k is a sequence of positive functions satisfying equation (3.3.1).

If $\Gamma = \{e\}$, define $\bar{x}$ to be an isolated blow-up point for u_k if there exists a sequence $x_k \in \Omega$, converging to $\bar{x}$, so that:

1) x_k is a local maximum point of u_k;

2) $M_k := u_k(x_k) \to \infty$ as $k \to \infty$;

3) there exist $r, C > 0$ such that $u_k(x) \leq C d_{g_k}(x, x_k)^{-\frac{2}{p_k-1}}$ for every $x \in B_r(x_k) \subset \Omega$. Here, $B_r(x_k)$ denotes the geodesic ball of radius r, centered at x_k, with respect to the metric g_k.

If $\Gamma \neq \{e\}$, let $\pi : B_\sigma(\bar{x}) \to B_\sigma(\bar{x})/\Gamma$ be the projection map. We say that $\bar{x}$ is an isolated blow-up point for u_k if $\pi^*(\bar{x}) = \bar{x}$ is an isolated blow-up point for $\pi^*(u_k)$ in the lifting-up space.

Define

$$U_c(y) = \Big(\frac{n(n-2)}{c}\Big)^{\frac{n-2}{4}} (1+|y|^2)^{\frac{2-n}{2}} \tag{3.3.2}$$

on $\mathbb{R}^n/\Gamma$, where c is some positive constant. It is not hard to check that

$$\Delta U_c(y) + c U_c^{\frac{n+2}{n-2}}(y) = 0. \tag{3.3.3}$$

Especially, in dimension $n = 4$,

$$U_c(y) = \Big(\frac{8}{c}\Big)^{\frac{1}{2}} (1+|y|^2)^{-1} \tag{3.3.4}$$

Remark 3.3.2. From now on, for each k, assume that we work in g_k-normal coordinates $\{x^i\}$ centered at point x_k. Then, we will simply write $u_k(x)$ instead of $u_k(\exp_{x_k}(x))$ and $|x|$ instead of $d_{g_k}(x, x_k)$.

Moreover, by [26, Theorem 5.1], there exists a conformal factor ϕ_k such that $\tilde{g}_k = \phi_k^{\frac{4}{n-2}} g_k$ is the conformal normal metric with conformal normal coordinates $\{\tilde{x}^i\}$ centered at x_k. After the conformal change, u_k becomes $\tilde{u}_k = \phi_k^{-1} u_k$. By the property that equation (3.1.1) is conformally invariant as stated in Section 3.1, u_k and $\tilde{u}_k$ satisfies the same type of conformal scalar curvature equation (3.1.1). Hence we may assume g_k is already the conformal normal metric and $\{x^i\}$ is already the conformal normal coordiantes centered at x_k.

Take change of variables

$$y = M_k^{\frac{p_k-1}{2}} x \tag{3.3.5}$$

and define the rescaled metric and functions

$$\begin{aligned} &(h_k)_{ij}(y) = (g_k)_{ij}(M_k^{-\frac{p_k-1}{2}} y),\ v_k(y) = M_k^{-1} u_k(M_k^{-\frac{p_k-1}{2}} y), \\ &\tilde{f}_k(y) = f_k(M_k^{-\frac{p_k-1}{2}} y),\ \tilde{K}_k(y) = K_k(M_k^{-\frac{p_k-1}{2}} y) \text{ and } \tilde{R}_k(y) = R_k(M_k^{-\frac{p_k-1}{2}} y). \end{aligned} \tag{3.3.6}$$

for $|y| < M_k^{\frac{p_k-1}{2}} r$, where r is as in Definition 3.3.1.

Using the property that equation (3.1.1) is rescale invariant as stated in Section 3.1, the rescaled functions satisfy

$$L_{h_k} v_k + \tilde{K}_k \tilde{f}_k^{-\tau_k} v_k^{p_k} = 0 \tag{3.3.7}$$

The following property holds for an isolated blow-up point.

Proposition 3.3.3 ([35] Proposition 4.3). *Assume u_k is a sequence of positive functions satisfying equation* (3.3.1) *and $x_k \to \bar{x}$ is an isolated blow-up point. Moreover, if $\Gamma \neq \{e\}$, we require $x_k = \bar{x}$ for all large k's. Assume $p_k \to \frac{n+2}{n-2}$, then there exist $R'_k \to \infty$ and $\epsilon_k \to 0$, such that after passing to a subsequence,*

$$\left\| v_k(y) - U_{K_\infty(\bar{x})}(y) \right\|_{C^2(B_{R'_k}(0))/\Gamma} \leq \epsilon_k, \tag{3.3.8}$$

and

$$\frac{R'_k}{\log(M_k)} \to 0, \tag{3.3.9}$$

as $k \to \infty$.

Proof. The case $K_k = constant$ is proved in [35, Proposition 4.3].

For variable K_k, because $K_k \to K_\infty$ in the C^0_{loc} norm, $\tilde{K}_k \to K_\infty(\bar{x})$ in the C^0_{loc} norm. By equation (3.3.7) and the proof of [35, Proposition 4.3], after passing to a subsequence, $v_k \to v > 0$ in C^2_{loc} norm, with

$$\begin{cases} \Delta v(y) + K_\infty(\bar{x}) v(y)^p = 0, & y \in \mathbb{R}^n/\Gamma \\ v(0) = 1, \quad \nabla v(0) = 0, \end{cases} \tag{3.3.10}$$

where $p = \lim_{k\to\infty} p_k$ and Δ denotes the Euclidean Laplacian. By [13], there must be $p = \frac{n+2}{n-2}$ and $v(y) = U_{K_\infty(\bar{x})}(y)$. Thus the Proposition is proved. □

Remark 3.3.4. Under the same assumptions as Proposition 3.3.3, we also have

$$\left\| v_k(y) - U_{K_k(x_k)}(y) \right\|_{C^2(B_{R'_k}(0))/\Gamma} \leq \epsilon_k, \tag{3.3.11}$$

simply because $U_{K_k(x_k)} \to U_{K_\infty(\bar{x})}$ uniformly in $\mathbb{R}^n/\Gamma$.

Definition 3.3.5. Suppose u_k is a sequence of positive functions satisfying equation (3.3.1) and $x_k \to \bar{x}$ is an isolated blow-up point.

If $\Gamma = \{e\}$, define

$$\bar{u}_k(r) = \frac{1}{Vol(S^{n-1})r^{n-1}} \int_{\partial B_r(x_k)} u_k d\sigma(r), \tag{3.3.12}$$

where we are using g_k-normal coordinates and integrating with respect to the Euclidean volume form.

We say $x_k \to \bar{x}$ is an isolated simple blow-up point if there exists a real number $0 < \rho < r$ such that the functions

$$\hat{u}_k(r) = r^{\frac{2}{p_k - 1}} \bar{u}_k(r) \tag{3.3.13}$$

have exactly one critical point in the interval $(0, \rho)$, for k large.

If $\Gamma \neq \{e\}$, let $\pi : B_\sigma(\bar{x}) \to B_\sigma(\bar{x})/\Gamma$ be the projection map. We say that $\bar{x}$ is an isolated simple blow-up point for u_k if $\pi^*(\bar{x})$ is an isolated simple blow-up point for $\pi^*(u_k)$ in the lifting-up space.

Next, we give the special blow-up condition that we will work on.

Condition 3.3.6. Assume u_k is a sequence of positive functions satisfying equation (3.3.1) and $x_k \to \bar{x}$ is an isolated simple blow-up point. Moreover, if $\Gamma \neq \{e\}$, we require $x_k = \bar{x}$ for all k's.

Remark 3.3.7. In Condition 3.3.6, there are two reasons why we require $x_k = \bar{x}$ for all k's in the singular point case $\Gamma \neq \{e\}$:

a) By assuming so, for each k, the geodesic ball centered at x_k is always $B_r(\bar{x})/\Gamma$. Then, when we later analyze some local integral and let $k \to \infty$, we will not run into the case that integrals over smooth balls converge to an integral over a quotient of a smooth ball.

b) Eventually, we will prove in Corollary 5.1.4 that if blow-up occurs at a singular point, then $x_k = \bar{x}$ for all sufficiently large k's, which implies Condition 3.3.6 is actually a necessary condition for any blow-up sequence.

From now on, assume we are working in dimension $n = 4$. In the following context, we will use C to denote various constants which only depend on the limit metric g, $\inf K_\infty$, $\|K_\infty\|_{C^2}$ and possibly the chosen small radius ρ_1, δ and σ. The dependency is implied in the proof.

Fix $\delta > 0$, and define

$$\lambda_k = (2-\delta)\frac{p_k - 1}{2} - 1. \tag{3.3.14}$$

Proposition 3.3.8 ([29]). *Assuming Condition 3.3.6, for sufficiently small $\delta > 0$, there exists constants $0 < \rho_1 < \rho$ and $C > 0$ such that*

$$\begin{aligned} M_k^{\lambda_k} u_k(x) &\le C|x|^{-2+\delta} \\ M_k^{\lambda_k} |\nabla u_k(x)| &\le C|x|^{-3+\delta} \\ M_k^{\lambda_k} |\nabla^2 u_k(x)| &\le C|x|^{-4+\delta} \end{aligned} \tag{3.3.15}$$

for every x satisfying

$$R_k' M_k^{-\frac{p_k-1}{2}} \le |x| \le \rho_1. \tag{3.3.16}$$

As a consequence, it implies

$$\begin{aligned} v_k(y) &\le CM_k^{\delta\frac{p_k-1}{2}}(1+|y|)^{-2} \\ |\nabla v_k(y)| &\le CM_k^{\delta\frac{p_k-1}{2}}(1+|y|)^{-3} \\ |\nabla^2 v_k(y)| &\le CM_k^{\delta\frac{p_k-1}{2}}(1+|y|)^{-4} \end{aligned} \tag{3.3.17}$$

for every y satisfying

$$|y| \le \rho_1 M_k^{\frac{p_k-1}{2}}. \tag{3.3.18}$$

Proof. The proof is the same as [29, Lemma 7.3]. That proof was for $n = 3$, but directly generalizes to higher dimensions. □

Proposition 3.3.9. *Assuming Condition 3.3.6, then there exists a constant C such that*

$$|\nabla K_k(x_k)| \le C(M_k^{-2+2\delta}), \tag{3.3.19}$$

as $k \to \infty$. Consequently,

$$|\nabla K_\infty(\bar{x})| = 0. \tag{3.3.20}$$

Proof. The proof is very similar to [29, Lemma 7.8], we provide here only a brief outline.

Recall that $\{x^i\}$ is the g_k-normal coordinates centered at point x_k. For some fixed positive small σ, let η be a smooth cutoff function such that $\eta(x) = 1$ for $|x| \le \sigma/2$ and $\eta(x) = 0$ for $|x| \ge \sigma$. Multipying equation (3.3.1) by $\eta(\partial u_k/\partial x^j)$, integrating by parts on $\{|x| \le \sigma\}/\Gamma$, we get

$$\begin{aligned} &\int_{\{|x|\le\sigma\}/\Gamma} \frac{\partial K_k}{\partial x^j}\eta f_k^{-\tau_k} u_k^{p_k+1}dx \\ \le& C\int_{\{|x|\le\sigma\}/\Gamma}\Big(|\nabla\eta|\cdot|\nabla u_k|^2 + \Big|\frac{\partial(\eta R_k)}{\partial x^j}\Big|u_k^2 + \Big|\frac{\partial(\eta f_k^{-\tau_k})}{\partial x^j}\Big|K_k u_k^{p_k+1}\Big)dx \end{aligned} \tag{3.3.21}$$

Then, using Proposition 3.3.3 in the ball $|x| \leq R'_k M_k^{-\frac{p_k-1}{2}}$ and Proposition 3.3.8 in the annuli $R'_k M_k^{-\frac{p_k-1}{2}} \leq |x| \leq \sigma$, together with the assumption that K_k converges to K_∞ in the C^0_{loc} norm, for large k's we have

$$\int_{\{|x|\leq\sigma\}/\Gamma} \frac{\partial K_k}{\partial x^j} u_k^{p_k+1} dx \leq C_1 M_k^{-2+2\delta}, \tag{3.3.22}$$

where C_1 depends on $\inf K_\infty$, $\|K_\infty\|_{C^0}$ and σ.

Next, consider power expansion

$$\frac{\partial K_k}{\partial x^j}(x) = \frac{\partial K_k}{\partial x^j}(0) + \frac{\partial^2 K_k}{\partial x^j \partial x^i}(0) \cdot x^i + O(|x|^2), \tag{3.3.23}$$

so

$$\Big|\frac{\partial K_k}{\partial x^j}(0)\Big| \leq \Big|\frac{\partial K_k}{\partial x^j}(x)\Big| + C_2|x| \tag{3.3.24}$$

where C_2 depends on $\|K_\infty\|_{C^2}$. Multiplying it by $u_k^{p_k+1}$, integrating over $\{|x| \leq \sigma\}/\Gamma$ and using inequality (3.3.22), we get

$$\Big|\frac{\partial K_k}{\partial x^j}(0)\Big| \int_{\{|x|\leq\sigma\}/\Gamma} u_k(x)^{p_k+1} dx \leq C\Big(M_k^{-2+2\delta} + \int_{\{|x|\leq\sigma\}/\Gamma} |x| u_k(x)^{p_k+1} dx\Big) \tag{3.3.25}$$

The integral on the left limits to the volume of the bubble which is a finite constant, the integral on the right can be estimated similarly as above using Proposition 3.3.3 and Proposition 3.3.8. Therefore, we've proved

$$|\nabla K_k(x_k)| \leq C(M_k^{-2+2\delta}) \tag{3.3.26}$$

where C is a constant depending on $\inf K_\infty$, $\|K_\infty\|_{C^2}$, g and σ. □

Proposition 3.3.10 ([35] Proposition 4.5)**.** *Assuming Condition 3.3.6, then there exists a constant* $C > 0$ *and* $0 < \rho_1 < \rho$ *such that*

$$M_k u_k(x) \leq C d_{g_k}(x, x_k)^{-2} \tag{3.3.27}$$

for x *satisfying* $d_{g_k}(x, x_k) \leq \rho_1$.

Proof. The proof is very similar to the proof of [35, Proposition 4.5]. That proof was assuming $K_k = constant$. For variable K_k, every step in the proof remains valid, except for Marques' Claim 2, which says that there exists $C > 0$ such that

$$\tau_k \leq C M_k^{-2+2\delta+o(1)} log(M_k) \quad \text{as} \quad k \to \infty. \tag{3.3.28}$$

This estimate does not hold in our setting, however, a modification of his arguments shows that there exists a constant $C > 0$ such that

$$\tau_k \leq C M_k^{-2+6\delta+o(1)} \quad \text{as} \quad k \to \infty. \tag{3.3.29}$$

To verify this, note that when K_k is a variable function, there will be an extra term on the left hand side of [35, inequality (4.18)]. That extra term is

$$\begin{aligned} &\frac{1}{p_k+1}\Big| \int_{\{|x| \leq \frac{\rho_1}{2}\}/\Gamma} (\nabla K_k(x) \cdot x) u_k(x)^{p_k+1} dx \Big| \\ \leq &\frac{1}{p_k+1} M_k^{3-p_k} \int_{\{|y| \leq \frac{\rho_1}{2} M_k^{\frac{p_k-1}{2}}\}/\Gamma} |\nabla_x K_k(M_k^{-\frac{p_k-1}{2}} y)| \cdot |M_k^{-\frac{p_k-1}{2}} y| v_k(y)^{p_k+1} dy \end{aligned} \tag{3.3.30}$$

For small ρ_1 and large k, when $|x| \leq \frac{\rho_1}{2}$, by power expansion and Proposition 3.3.9, we have

$$|\nabla K_k(x)| \leq |\nabla K_k(x_k)| + C|x| \leq C(M_k^{-2+2\delta} + |x|). \tag{3.3.31}$$

Then for $|y| \leq \frac{\rho_1}{2} M_k^{\frac{p_k-1}{2}}$ and large k,

$$
\begin{aligned}
|\nabla_x K_k(M_k^{-\frac{p_k-1}{2}} y)| \cdot |M_k^{-\frac{p_k-1}{2}} y| \leq & C(M_k^{-2-\frac{p_k-1}{2}+2\delta}|y| + M_k^{-p_k+1}|y|^2) \\
\leq & C(\rho_1 M_k^{-2+2\delta} + M_k^{-p_k+1})|y|^2 \\
\leq & C M_k^{-2+2\delta}|y|^2.
\end{aligned}
\tag{3.3.32}
$$

On the other hand, by Proposition 3.3.8, we know

$$
v_k(y)^{p_k+1} \leq C M_k^{\delta\frac{p_k^2-1}{2}}(1+|y|)^{-2(p_k+1)} \tag{3.3.33}
$$

Therefore,

$$
\begin{aligned}
& M_k^{3-p_k} \int_{\{|y| \leq \frac{\rho_1}{2} M_k^{\frac{p_k-1}{2}}\}/\Gamma} |\nabla_x K_k(M_k^{-\frac{p_k-1}{2}} y)| \cdot |M_k^{-\frac{p_k-1}{2}} y| v_k(y)^{p_k+1} dy \\
\leq & C M_k^{3-p_k} M_k^{-2+2\delta} M_k^{\delta\frac{p_k^2-1}{2}} \int_{\{|y| \leq \frac{\rho_1}{2} M_k^{\frac{p_k-1}{2}}\}/\Gamma} (1+|y|)^{-2(p_k+1)}|y|^2 dy \\
\leq & C M_k^{-2+6\delta+o(1)} \int_{\mathbb{R}^4/\Gamma} (1+|y|)^{-8+o(1)}|y|^2 dy \\
\leq & C M_k^{-2+6\delta+o(1)} \quad \text{as } k \to \infty.
\end{aligned}
\tag{3.3.34}
$$

Then (3.3.29) is proved following the rest proof in Claim 2 in [35, Proposition 4.5]. □

Corollary 3.3.11 ([35] Corollary 4.6)**.** *Assuming Condition 3.3.6, after maybe passing to a subsequence, we have*

$$
M_k u_k \to h_{\bar{x}} \text{ in } C^2_{loc}((B_r(\bar{x}) \setminus \{\bar{x}\})/\Gamma) \tag{3.3.35}
$$

where M_k is as defined in Definition 3.3.1 and $h_{\bar{x}} = aG(\cdot, \bar{x})$ is a constant multiple of the standard Green function, i.e. $L_g(G(\cdot, \bar{x})) = \delta_{\bar{x}}$ is the Dirac delta function at point $\bar{x}$. (Here, g stands for the limit metric.)

Proof. The proof of Marques remains valid for variable K_k. □

Then, we have the following:

Proposition 3.3.12. *Assuming Condition 3.3.6, then*

$$\tau_k \leq C M_k^{-2} \tag{3.3.36}$$

and there exists $\delta > 0$ such that

$$|v_k(y) - U_{K_k(x_k)}(y)| \leq C M_k^{-2} \tag{3.3.37}$$

$$|\nabla(v_k - U_{K_k(x_k)})(y)| \leq C M_k^{-2}(1+|y|)^{-1} \tag{3.3.38}$$

$$|\nabla^2(v_k - U_{K_k(x_k)})(y)| \leq C M_k^{-2}(1+|y|)^{-2} \tag{3.3.39}$$

for $|y| \leq \delta M_i^{\frac{p_i-1}{2}}$.

Proof. Proof for $K_k = \textit{constant}$ is by [35, Chapter 5]. We need to verify that Marques' proof is valid for variable K_k.

When K_k is a variable function instead of a fixed constant, every U_0 in [35, Chapter 5] has to be replaced by $U_{K_k(x_k)}$, then [35, equation (5.1)] will become

$$\begin{aligned} & Q_k(y) \\ =& \Lambda_k^{-1}\Big\{c(n) M_k^{-(p_k-1)} R_{g_k}\Big(M_k^{-\frac{p_k-1}{2}} y\Big) U_{K_k(x_k)}(y) + M_k^{-(1+N)\frac{p_k-1}{2}} O(|y|^N)|y|(1+|y|^2)^{-2} \\ & + \Big(K_k(x_k) U^3_{K_k(x_k)} - \tilde{K}_k(y) \tilde{f}_k^{-\tau_k} U^{p_k}_{K_k(x_k)}\Big)\Big\} \end{aligned} \tag{3.3.40}$$

where $\tilde{K}_k(y) = K_k(M_k^{-\frac{p_k-1}{2}} y)$ and $\tilde{f}_k(y) = f_k(M_k^{-\frac{p_k-1}{2}} y)$. Only the last term in (3.3.40) differs from the last term in [35, equation (5.1)]. Denote it by

$$\mathbb{B}_k = K_k(x_k) U^3_{K_k(x_k)} - \tilde{K}_k(y) \tilde{f}_k^{-\tau_k} U^{p_k}_{K_k(x_k)} \tag{3.3.41}$$

To analyze it, by power expansion we get

$$
\begin{aligned}
|\tilde{K}_k(y) - K_k(x_k)| =& |K_k(M_k^{-\frac{p_k-1}{2}} y) - K_k(0)| \\
\leq& |\nabla K_k(0)| M_k^{-\frac{p_k-1}{2}} |y| + C M_k^{-(p_k-1)} |y|^2.
\end{aligned}
\tag{3.3.42}
$$

By Proposition 3.3.8, $|\nabla K_k(0)| = |\nabla K_k(x_k)| \leq C M_k^{-2+2\delta}$, hence

$$
\begin{aligned}
|\tilde{K}_k(y) - K_k(x_k)| \leq& C\Big(M_k^{-2-\frac{p_k-1}{2}+2\delta}|y| + M_k^{-(p_k-1)}|y|^2\Big) \\
\leq& C\Big(M_k^{-3+2\delta+o(1)} + M_k^{-2+\tau_k}|y|\Big)|y|
\end{aligned}
\tag{3.3.43}
$$

Then

$$
\begin{aligned}
\mathbb{B}_k \leq& K_k(x_k) U^3_{K_k(x_k)}(1 - (\tilde{f}_k U^3_{K_k(x_k)})^{-\tau_k}) + |\tilde{K}_k(y) - K_k(x_k)| \tilde{f}_k^{-\tau_k} U^{p_k}_{K_k(x_k)} \\
\leq& C\Big(\tau_k \log(\tilde{f}_k U_{K_k(x_k)})(1+|y|^2)^{-3} + (M_k^{-3+2\delta+o(1)} + M_k^{-2+\tau_k}|y|)|y|(1+|y|^2)^{-2}\Big).
\end{aligned}
\tag{3.3.44}
$$

Suppose

$$
\Lambda_k^{-1} \max\{M_k^{-2}, \tau_k\} \to 0 \quad \text{as } k \to \infty. \tag{3.3.45}
$$

Note that (3.3.29) implies $\lim_{k\to\infty} M_k^{\tau_k} = 1$, so $\lim_{k\to\infty} \Lambda_k^{-1}\mathbb{B}_k \to 0$. Thus $|Q_k| \to 0$ as $k \to 0$ and [35, Lemma 5.1] can be proved following the rest of his proof. Also, [35, Lemma 5.3] remains valid with a similar modification to the term $\tilde{Q}_k$ in his proof. Therefore, our proposition is proved for variable K_k. □

Chapter 4

Local blow-up analysis

4.1 Application of Pohozaev Identity

Assuming Condition 3.3.6, recall that g_k is the conformal normal metric with conformal normal coordinates $\{x^i\}$ centered at point x_k. By [26, Chapter 5] and the explicit computations in [31, Chapter 2], we have the following for each k:

At coordinate $x = 0$ (the point x_k), the scalar curvature satisfies

$$R_k(0) = 0, (R_k)_{,i}(0) = 0. \tag{4.1.1}$$

Locally in a neighborhood of $x = 0$, we have

$$(g_k)_{ij} = \delta_{ij} + O(r^2) \quad \text{and} \quad \det(g_k) = 1 + O(r^N) \tag{4.1.2}$$

for some $N \geq 5$.

Write

$$\Delta_{g_k} = \frac{1}{\sqrt{\det(g_k)}}\partial_i(\sqrt{\det(g_k)}(g_k)^{ij}\partial_j) = \Delta_0 + (b_k)_i\partial_i + (d_k)_{ij}\partial_{ij} \tag{4.1.3}$$

where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{ij} = \frac{\partial^2}{\partial x^i \partial x^j}$ and $\Delta_0 = \sum_{i=1}^n \frac{\partial^2}{(\partial x^i)^2}$, then

$$(b_k)_i = O(r^2) \quad \text{and} \quad (d_k)_{ij} = g^{ij} - \delta_{ij} = O(r^2) \tag{4.1.4}$$

Recall the change of variables, rescaled metric and functions defined in (3.3.5) and (3.3.6), write

$$\Delta_{h_k} = \bar{\Delta}_0 + (\bar{b}_k)_i\bar{\partial}_i + (\bar{d}_k)_{ij}\bar{\partial}_{ij} \tag{4.1.5}$$

where

$$\begin{aligned}
&(\bar{b}_k)_i(y) = M_k^{-\frac{p_k-1}{2}}(b_k)_i(M_k^{-\frac{p_k-1}{2}}y), \quad (\bar{d}_k)_{ij}(y) = (d_k)_{ij}(M_k^{-\frac{p_k-1}{2}}y), \\
&\bar{\partial}_i = \frac{\partial}{\partial y^i}, \quad \bar{\partial}_{ij} = \frac{\partial^2}{\partial y^i \partial y^j} \quad \text{and} \quad \bar{\Delta}_0 = \sum_{i=1}^n \frac{\partial^2}{(\partial y^i)^2}.
\end{aligned} \tag{4.1.6}$$

It follows

$$|(\bar{b}_k)_i(y)| = M_k^{-\frac{3(p_k-1)}{2}}O(|y|)^2, \quad |(\bar{d}_k)_{ij}(y)| = M_k^{-(p_k-1)}O(|y|)^2 \tag{4.1.7}$$

Rewrite equation (3.3.1) as

$$((d_k)_{ij} + \delta_{ij})\partial_{ij}u_k + (b_k)_i\partial_i u_k - \frac{1}{6}R_k u_k + K_k f_k^{-\tau_k}u_k^{p_k} = 0, \tag{4.1.8}$$

For some small σ, apply Lemma (3.2.1) to the above equation in $\{|x| \le \sigma\}/\Gamma$, we get

$$\begin{aligned}P(\sigma, u_k) = &- \int_{\{|x|\le\sigma\}/\Gamma} \Big(x \cdot \nabla_x u_k + u_k\Big)((d_k)_{ij}\partial_{ij}u_k + (b_k)_i\partial_i u_k) \\ &- \frac{1}{12}\int_{\{|x|\le\sigma\}/\Gamma}(x\cdot\nabla_x R_k + 2R_k)u_k^2 \\ &+ \frac{\sigma}{12}\int_{\{|x|=\sigma\}/\Gamma} R_k u_k^2 + \frac{1}{p_k+1}\int_{\{|x|\le\sigma\}/\Gamma}(x\cdot\nabla_x(K_k f_k^{-\tau_k}))u_k^{p_k+1} \\ &+ \frac{\tau_k}{p_k+1}\int_{\{|x|\le\sigma\}/\Gamma} K_k f_k^{-\tau_k} u_k^{p_k+1} - \frac{\sigma}{p_k+1}\int_{\{|x|=\sigma\}/\Gamma} K_k f_k^{-\tau_k}u_k^{p_k+1} \\ = &\int_{\{|x|=\sigma\}/\Gamma}\Big(u_k\frac{\partial u_k}{\partial\nu} - \frac{\sigma}{2}|\nabla u_k|^2 + \sigma\Big|\frac{\partial u_k}{\partial \nu}\Big|^2\Big)\end{aligned} \tag{4.1.9}$$

Denote $R_k' = M_k^{\frac{p_k-1}{2}}\sigma$. Define the following

$$\begin{aligned}I_{k,1} = &- \frac{\sigma}{p_k+1}\int_{\{|x|=\sigma\}/\Gamma} K_k f_k^{-\tau_k}u_k^{p_k+1}d\sigma(x) \\ = &- \frac{\sigma}{p_k+1}M_k^{\frac{5-p_k}{2}}\int_{\{|y|=R_k'\}/\Gamma}\tilde{K}_k\tilde{f}_k^{-\tau_k}v_k^{p_k+1}d\sigma(y)\end{aligned} \tag{4.1.10}$$

$$\begin{aligned}I_{k,2} = &\int_{\{|x|\le\sigma\}/\Gamma}(-(b_k)_i\partial_i u_k - (d_k)_{ij}\partial_{ij}u_k)(\nabla_x u_k\cdot x + u_k)dx \\ = &M_k^{\tau_k}\int_{\{|y|\le R_k'\}/\Gamma}(-(\bar{b}_k)_i\bar{\partial}_i v_k - (\bar{d}_k)_{ij}\bar{\partial}_{ij}v_k)(y\cdot\nabla_y v_k + v_k)dy\end{aligned} \tag{4.1.11}$$

$$\begin{aligned}I_{k,3} = &\frac{\sigma}{12}\int_{\{|x|=\sigma\}/\Gamma} R_k u_k^2 d\sigma(x) \\ = &\frac{\sigma}{12}M_k^{\frac{7-3p_k}{2}}\int_{\{|y|=R_k'\}/\Gamma}\tilde{R}_k v_k^2 d\sigma(y)\end{aligned} \tag{4.1.12}$$

$$\begin{aligned}I_{k,4} = &- \frac{1}{12}\int_{\{|x|\le\sigma\}/\Gamma}(x\cdot\nabla_x R_k + 2R_k)u_k^2 dx \\ = &- \frac{1}{12}M_k^{4-2p_k}\int_{\{|y|\le R_k'\}/\Gamma}(y\cdot\nabla_y\tilde{R}_k + 2\tilde{R}_k)v_k^2 dy\end{aligned} \tag{4.1.13}$$

$$I_{k,5} = \frac{1}{p_k+1}\int_{\{|x|\le\sigma\}/\Gamma}(x\cdot\nabla_x(K_k f_k^{-\tau_k}))u_k^{p_k+1}dx = \frac{1}{p_k+1}M_k^{\tau_k}\int_{\{|y|\le R'_k\}/\Gamma}(y\cdot\nabla_y(\tilde{K}_k\tilde{f}_k^{-\tau_k}))v_k^{p_k+1}dy \tag{4.1.14}$$

$$I_{k,6} = \frac{\tau_k}{p_k+1}\int_{\{|x|\le\sigma\}/\Gamma}K_k f_k^{-\tau_k}u_k^{p_k+1}dx = \frac{\tau_k}{p_k+1}M_k^{\tau_k}\int_{\{|y|\le R'_k\}/\Gamma}\tilde{K}_k\tilde{f}_k^{-\tau_k}v_k^{p_k+1}dy \tag{4.1.15}$$

$$I_{k,7} = \int_{\{|x|=\sigma\}/\Gamma}\Big\{\Big(\Big|\frac{\partial u_k}{\partial\nu_x}\Big|^2-\frac{1}{2}|\nabla_x u_k|^2\Big)\sigma+u_k\frac{\partial u_k}{\partial\nu_x}\Big\}d\sigma(x) = M_k^{\tau_k}\int_{\{|y|=R'_k\}/\Gamma}\Big\{\Big(\Big|\frac{\partial v_k}{\partial\nu_y}\Big|^2-\frac{1}{2}|\nabla_y v_k|^2\Big)R'_k+v_k\frac{\partial v_k}{\partial\nu_y}\Big\}d\sigma(y) \tag{4.1.16}$$

Then the Pohozaev identity becomes

$$I_{k,7} = \sum_{j=1}^{6} I_{k,j}. \tag{4.1.17}$$

4.2 Main Estimations

By [13], it is sufficient to consider the blow-up case when $p_k \to 3$, consequently $\tau_k = 3 - p_k \to 0$. Then we know $\lim_{k\to\infty} M_k^{\tau_k} = 1$ from (3.3.29). We can estimate terms in the Pohozaev identity through the following lemmas. We will use C, C_1 to denote various positive constants independent of k and σ. We will omit $dx, dy, d\sigma(x)$ and $d\sigma(y)$ terms in integrals, since the integral domains will explain themselves well.

Lemma 4.2.1. *For small $\sigma > 0$,*

$$\lim_{k\to\infty} M_k^2 I_{k,1} = 0 \tag{4.2.1}$$

Proof. Using (3.3.1),

$$I_{k,1} = \frac{\sigma}{p_k+1}\int_{\{|x|=\sigma\}/\Gamma}(L_{g_k}u_k)\cdot u_k. \tag{4.2.2}$$

By Corollary 3.3.11, $M_k u_k \to h$ in the C^2_{loc} norm, hence for small σ,

$$\begin{aligned}\lim_{k\to\infty} M_k^2 I_{k,1} =& \lim_{k\to\infty}\frac{\sigma}{p_k+1}\int_{\{|x|=\sigma\}/\Gamma}(L_{g_k}(M_k u_k))\cdot(M_k u_k)\\ =&\frac{\sigma}{p_k+1}\int_{\{|x|=\sigma\}/\Gamma}(L_g h)\cdot h\\ =&0.\end{aligned} \tag{4.2.3}$$

□

Lemma 4.2.2. *For $\sigma \le 1$, there exists some constant $C > 0$ such that*

$$\limsup_{k\to\infty} M_k^2 |I_{k,2}| \le C\sigma^2 \tag{4.2.4}$$

Proof. Because $U_{K_k(x_k)}(y)$ is a radially symmetric function and $\{y^i\}$ is a rescale of the conformal normal coordinates, we know

$$(\Delta_{h_k} - \bar{\Delta}_0)U_{K_k(x_k)} = ((\bar{b}_k)_i\bar{\partial}_i + (\bar{d}_k)_{ij}\bar{\partial}_{ij})U_{K_k(x_k)} \equiv 0 \tag{4.2.5}$$

Then, we have

$$\begin{aligned}M_k^2|I_{k,2}| \le& M_k^{2+\tau_k}\int_{\{|y|\le\sigma M_k^{\frac{p_k-1}{2}}\}/\Gamma}|((\bar{b}_k)_i\partial_i + (\bar{d}_k)_{ij}\partial_{ij})v_k|\cdot|\nabla_y v_k\cdot y + v_k|\\ =&M_k^{2+\tau_k}\int_{\{|y|\le\sigma M_k^{\frac{p_k-1}{2}}\}/\Gamma}|((\bar{b}_k)_i\partial_i + (\bar{d}_k)_{ij}\partial_{ij})(v_k - U_{K_k(x_k)})|\cdot|\nabla_y v_k\cdot y + v_k|\end{aligned} \tag{4.2.6}$$

For $\sigma \leq 1$, we have $|y| \leq \sigma M_k^{\frac{p_k-1}{2}} \leq M_k^{\frac{p_k-1}{2}}$, hence $M_k^{-1} \leq M_k^{-\frac{p_k-1}{2}} \leq C(1+|y|)^{-1}$ for large k. By (4.1.7) and Proposition 3.3.12, for large k, we have

$$\begin{aligned}
&|((\bar{b}_k)_i(y)\bar{\partial}_i + (\bar{d}_k)_{ij}(y)\bar{\partial}_{ij})(v_k(y) - U_{K_k(x_k)}(y))| \\
\leq& C M_k^{-2} M_k^{-(p_k-1)}|y|^2(1+|y|)^{-1}(M_k^{-\frac{p_k-1}{2}} + (1+|y|)^{-1}) \\
\leq& C M_k^{-1-p_k}|y|^2(1+|y|)^{-2} \\
\leq& C M_k^{-1-p_k}
\end{aligned} \tag{4.2.7}$$

and

$$\begin{aligned}
|y \cdot \nabla_y v_k(y) + v_k(y)| \leq& |\nabla_y U_{K_k(x_k)}| \cdot |y| + |\nabla_y (v_k - U)| \cdot |y| + |v_k - U| + U \\
\leq& C\Big(|y|^2(1+|y|^2)^{-2} + M_k^{-2}(1+|y|)^{-1}|y| + M_k^{-2} + (1+|y|^2)^{-1}\Big) \\
\leq& C\Big((1+|y|^2)^{-1} + M_k^{-2}\Big) \\
\leq& C(1+|y|^2)^{-1}.
\end{aligned} \tag{4.2.8}$$

Thus, for large k,

$$\begin{aligned}
M_k^2|I_{k,2}| \leq& C M_k^{2+\tau_k} M_k^{-1-p_k} \int_{\{|y| \leq \sigma M_k^{\frac{p_k-1}{2}}\}/\Gamma} (1+|y|^2)^{-1} \\
\leq& C M_k^{1-p_k+\tau_k} \int_0^{\sigma M_k^{\frac{p_k-1}{2}}} r^3(1+r^2)^{-1} dr \\
\leq& C M_k^{1-p_k+\tau_k}((\sigma M_k^{\frac{p_k-1}{2}})^2 + C_1) \\
=& C M_k^{\tau_k}\sigma^2 + CC_1 M_k^{1-p_k+\tau_k} \\
\leq& C\sigma^2 \quad \text{as } k \to \infty.
\end{aligned} \tag{4.2.9}$$

□

Lemma 4.2.3. *There exist constants $C > 0$ and $0 < \delta < 1$ such that when $\sigma < \delta$,*

$$\limsup_{k \to \infty} M_k^2 |I_{k,3}| \leq C\sigma^2 \tag{4.2.10}$$

Proof. Due to (4.1.1), there exist $C > 0$ and $0 < \delta < 1$ such that when $\sigma < \delta$, on $|x| = \sigma$, by power expansion,

$$|R_k(x)| \leq C|x|^2 = C\sigma^2 \tag{4.2.11}$$

Hence $|\tilde{R}_k(y)| \leq C\sigma^2$ on $|y| = \sigma M_k^{\frac{p_k-1}{2}}$.

On the other hand, on $|y| = \sigma M_k^{\frac{p_k-1}{2}} \leq M_k^{\frac{p_k-1}{2}}$, $M_k^{-1} \leq M_k^{-\frac{p_k-1}{2}} \leq |y|^{-1}$. By Proposition 3.3.12,

$$\begin{aligned} v_k^2(y) \leq & [|v_k(y) - U_{K_k(x_k)}(y)| + U_{K_k(x_k)}(y)]^2 \\ \leq & C[M_k^{-2} + (1+|y|^2)^{-1}]^2 \\ \leq & C|y|^{-4} \end{aligned} \tag{4.2.12}$$

Then, we have the estimate

$$\begin{aligned} M_k^2|I_{k,3}| \leq & \frac{\sigma}{12} M_k^{\frac{11-3p_k}{2}} \int_{\{|y|=\sigma M_k^{\frac{p_k-1}{2}}\}/\Gamma} |\tilde{R}_k| v_k^2 \\ \leq & C\sigma^3 M_k^{\frac{11-3p_k}{2}} \int_{\{|y|=\sigma M_k^{\frac{p_k-1}{2}}\}/\Gamma} |y|^{-4} \\ \leq & C\sigma^3 M_k^{\frac{11-3p_k}{2}} (\sigma M_k^{\frac{p_k-1}{2}})^{-1} \\ = & C\sigma^2 M_k^{2\tau_k} \\ \leq & C\sigma^2 \quad \text{as } k \to \infty. \end{aligned} \tag{4.2.13}$$

□

Lemma 4.2.4. *There exist constants $C > 0$ and $0 < \delta < 1$ such that when $\sigma < \delta$,*

$$\limsup_{k\to\infty} M_k^2|I_{k,4}| \leq C\sigma^2 \tag{4.2.14}$$

Proof. By (4.1.1), there exist $C > 0$ and $0 < \delta < 1$ such that when $\sigma < \delta$, in the ball $|x| \leq \sigma$, by power expansion, we get

$$|R_k(x)| \leq C|x|^2, \quad |\nabla_x R_k(x)| \leq |\nabla_x R_k(0)| + C|x| \leq C|x|. \tag{4.2.15}$$

The second inequality implies

$$|x \cdot \nabla_x R_k(x)| \leq |x| \cdot |\nabla_x R_k(x)| \leq C|x|^2. \tag{4.2.16}$$

It follows that in the ball $|y| \leq \sigma M_k^{\frac{p_k-1}{2}}$,

$$\begin{aligned} &|\tilde{R}_k(y)| \leq C M_k^{-(p_k-1)}|y|^2, \\ &|y \cdot \nabla_y \tilde{R}_k(y)| = |(M_k^{-\frac{p_k-1}{2}} y) \cdot \nabla_x R_k(M_k^{-\frac{p_k-1}{2}} y)| \leq C M_k^{-(p_k-1)}|y|^2. \end{aligned} \tag{4.2.17}$$

On the other hand, similarly to (4.2.12), for $|y| \leq \sigma M_k^{\frac{p_k-1}{2}}$ and large k,

$$v_k^2(y) \leq C(1+|y|^2)^{-2} \tag{4.2.18}$$

Then, for large k, we have the estimate

$$\begin{aligned} M_k^2|I_{k,4}| \leq& C M_k^{6-2p_k} \int_{\{|y| \leq \sigma M_k^{\frac{p_k-1}{2}}\}/\Gamma} (|y \cdot \nabla_y \tilde{R}_k| + 2|\tilde{R}_k|) v_k^2 \\ \leq& C M_k^{6-2p_k} M_k^{-(p_k-1)} \int_{\{|y| \leq \sigma M_k^{\frac{p_k-1}{2}}\}/\Gamma} |y|^2 (1+|y|^2)^{-2} \\ \leq& C M_k^{7-3p_k} \int_0^{\sigma M_k^{\frac{p_k-1}{2}}} r^5 (1+r^2)^{-2} dr \\ \leq& C M_k^{7-3p_k} ((\sigma M_k^{\frac{p_k-1}{2}})^2 + C_1) \\ =& C\sigma^2 M_k^{2\tau_k} + C C_1 M_k^{7-3p_k} \\ \leq& C\sigma^2 \quad \text{as } k \to \infty. \end{aligned} \tag{4.2.19}$$

□

Next, let's estimate the most important term $M_k^2 I_{k,5}$.

Lemma 4.2.5. *There exists a constant* $0 < \delta < 1$ *such that when* $\sigma < \delta$*,*

$$\lim_{k\to\infty} M_k^2 I_{k,5} = \frac{2\Delta_x K_\infty(\bar{x}) \cdot Vol(S^3)}{3|\Gamma| K_\infty(\bar{x})^2} \tag{4.2.20}$$

Proof. Firstly, note that

$$\begin{aligned}
&\lim_{k\to\infty} f_k^{-\tau_k}(x) = 1, \\
&\lim_{k\to\infty} |\nabla_x f_k^{-\tau_k}(x)| = \lim_{k\to\infty} -\tau_k f_k^{-\tau_k - 1}(x)|\nabla_x f_k(x)| = 0
\end{aligned} \tag{4.2.21}$$

uniformly for $|x| \leq \sigma$. It follows

$$\begin{aligned}
\lim_{k\to\infty} M_k^2 I_{k,5} =& \frac{1}{p_k+1} M_k^2 \int_{\{|x|\leq\sigma\}/\Gamma} (x \cdot \nabla_x (K_k f_k^{-\tau_k})) u_k^{p_k+1} \\
=& \frac{1}{p_k+1} M_k^2 \int_{\{|x|\leq\sigma\}/\Gamma} (x \cdot \nabla_x K_k) u_k^{p_k+1} \\
=& \frac{1}{p_k+1} M_k^{2+\tau_k} \int_{\{|y|\leq\sigma M_k^{\frac{p_k-1}{2}}\}/\Gamma} (y \cdot \nabla_y \tilde{K}_k) v_k^{p_k+1}.
\end{aligned} \tag{4.2.22}$$

There exists a constant $0 < \delta < 1$ such that when $\sigma < \delta$, by power expansion and Proposition 3.3.9, we have

$$\begin{aligned}
K_k(x) =& K_k(0) + (K_k)_{,i}(0) x^i + \frac{1}{2}(K_k)_{,ij}(0) x^i x^j + O(|x|^3) \\
=& K_k(0) + \frac{1}{2}(K_k)_{,ij}(0) x^i x^j + O(M_k^{-2+2\delta}|x|) + O(|x|^3)
\end{aligned} \tag{4.2.23}$$

for $|x| \leq \sigma$, where $(K_k)_{,ij}(0)$ denotes the second order partial derivatives of K_k in coordinates $\{x^i\}$ at the point x_k. It's not hard to verify that

$$\begin{aligned}
&x \cdot \nabla_x [K_k(0) + O(M_k^{-2+2\delta}|x|) + O(|x|^3)] = O(M_k^{-2+2\delta}|x|) + O(|x|^3). \\
&x \cdot \nabla_x \Big(\frac{1}{2}(K_k)_{,ij}(0) x^i x^j\Big) = (K_k)_{,ij}(0) x^i x^j.
\end{aligned} \tag{4.2.24}$$

Thus

$$x \cdot \nabla_x K_k(x) = (K_k)_{,ij}(0)x^i x^j + O(M_k^{-2+2\delta}|x|) + O(|x|^3), \tag{4.2.25}$$

which implies

$$y \cdot \nabla_y \tilde{K}_k(y) = M_k^{-(p_k-1)}(K_k)_{,ij}(0)y^i y^j + O(M_k^{-2-\frac{p_k-1}{2}+2\delta}|y|) + O(M_k^{-3\frac{p_k-1}{2}}|y|^3). \tag{4.2.26}$$

Hence

$$|y \cdot \nabla_y \tilde{K}_k(y)| \leq CM_k^{-(p_k-1)}(1+|y|)^2 \tag{4.2.27}$$

and

$$|y \cdot \nabla_y \tilde{K}_k(y) - M_k^{-(p_k-1)}(K_k)_{,ij}(0)y^i y^j| \leq CM_k^{-3\frac{p_k-1}{2}}(1+|y|)^3. \tag{4.2.28}$$

for $|y| \leq \sigma M_k^{\frac{p_k-1}{2}}$ and large k.

On the other hand, by power serious and Proposition 3.3.12,

$$\begin{aligned} |v_k^{p_k+1}(y) - U_{K_k(x_k)}^{p_k+1}(y)| \leq& C \cdot (p_k+1)U_{K_k(x_k)}^{p_k}(y)|v_k(y) - U_{K_k(x_k)}(y)| \\ \leq& C(1+|y|^2)^{-p_k}M_k^{-2} \end{aligned} \tag{4.2.29}$$

Together with estimate (4.2.27), for large k, we have

$$\begin{aligned}
&M_k^{2+\tau_k}\int_{\{|y|\le\sigma M_k^{\frac{p_k-1}{2}}\}/\Gamma}(y\cdot\nabla_y\tilde{K}_k)|v_k^{p_k+1}-U_{K_k(x_k)}^{p_k+1}|\\
\le& CM_k^{2+\tau_k}M_k^{-(p_k-1)}M_k^{-2}\int_{\{|y|\le\sigma M_k^{\frac{p_k-1}{2}}\}/\Gamma}(1+|y|)^2(1+|y|^2)^{-p_k}\\
\le& CM_k^{-(p_k-1)+\tau_k}\int_0^{\sigma M_k^{\frac{p_k-1}{2}}} r^3(1+r)^2(1+r^2)^{-3+o(1)}dr\\
\le& CM_k^{-(p_k-1)+\tau_k}(\sigma M_k^{\frac{p_k-1}{2}}+C_1)\\
\le& CM_k^{-\frac{p_k-1}{2}+\tau_k}\to 0 \quad\text{as } k\to\infty.
\end{aligned}\tag{4.2.30}$$

Therefore,

$$\lim_{k\to\infty}M_k^2 I_{k,5}=\lim_{k\to\infty}\frac{1}{p_k+1}M_k^{2+\tau_k}\int_{\{|y|\le\sigma M_k^{\frac{p_k-1}{2}}\}/\Gamma}(y\cdot\nabla_y\tilde{K}_k)U_{K_k(x_k)}^{p_k+1}\tag{4.2.31}$$

Using estimate (4.2.28), for large k, we have

$$\begin{aligned}
&M_k^{2+\tau_k}\int_{\{|y|\le\sigma M_k^{\frac{p_k-1}{2}}\}/\Gamma}|y\cdot\nabla_y\tilde{K}_k(y)-M_k^{-(p_k-1)}(K_k)_{,ij}(0)y^iy^j|\cdot U_{K_k(x_k)}^{p_k+1}\\
\le& CM_k^{2+\tau_k}M_k^{-3\frac{p_k-1}{2}}\int_{\{|y|\le\sigma M_k^{\frac{p_k-1}{2}}\}/\Gamma}(1+|y|)^3(1+|y|^2)^{-4+o(1)}\\
\le& CM_k^{2-3\frac{p_k-1}{2}+\tau_k}\int_0^{\sigma M_k^{\frac{p_k-1}{2}}} r^3(1+r)^3(1+r^2)^{-4+o(1)}dr\\
\le& CM_k^{2-3\frac{p_k-1}{2}+\tau_k}((\sigma M_k^{\frac{p_k-1}{2}})^{-1+o(1)}+C_1)\to 0 \quad\text{as } k\to 0.
\end{aligned}\tag{4.2.32}$$

Therefore,

$$\lim_{k\to\infty}M_k^2 I_{k,5}=\lim_{k\to\infty}\frac{1}{p_k+1}M_k^{2\tau_k}\int_{\{|y|\le\sigma M_k^{\frac{p_k-1}{2}}\}/\Gamma}(K_k)_{,ij}(0)y^iy^jU_{K_k(x_k)}^{p_k+1}\tag{4.2.33}$$

Since $U_{K_k(x_k)}$ is a radially symmetric function,

$$\int_{\{|y|\le\sigma M_k^{\frac{p_k-1}{2}}\}/\Gamma}y^iy^jU_{K_k(x_k)}=\begin{cases}0, & i\ne j,\\ \frac{1}{4}\int_{\{|y|\le\sigma M_k^{\frac{p_k-1}{2}}\}/\Gamma}|y|^2U_{K_k(x_k)}, & i=j.\end{cases}\tag{4.2.34}$$

Thus we get

$$\lim_{k\to\infty} M_k^2 I_{k,5} = \lim_{k\to\infty} \frac{1}{p_k+1} M_k^{2\tau_k} \int_{\{|y|\le\sigma M_k^{\frac{p_k-1}{2}}\}/\Gamma} \frac{1}{4}\Delta_x K_k(0)\cdot|y|^2 U_{K_k(x_k)}^{p_k+1} \tag{4.2.35}$$

where $\Delta_x K_k(0) = \Delta_x K_k(x_k)$ is the Laplacian of K_k in coordinates $\{x^i\}$ at point x_k. Because $\Delta_x K_k(x_k) \to \Delta_x K_\infty(\bar{x})$ and $U_{K_k(x_k)}^{p_k+1} \to U_{K_\infty(\bar{x})}^4$ uniformly on $\mathbb{R}^4/\Gamma$, we have

$$\begin{aligned}
\lim_{k\to\infty} M_k^2 I_{k,5} =& \frac{1}{16}\int_{y\in\mathbb{R}^4/\Gamma} \Delta_x K_\infty(\bar{x})\cdot|y|^2\Big(\frac{8}{K_\infty(\bar{x})}\Big)^2(1+|y|^2)^{-4} \\
=& \frac{4\Delta_x K_\infty(\bar{x})\cdot Vol(S^3)}{|\Gamma|K_\infty(\bar{x})^2}\int_0^\infty r^5(1+r^2)^{-4}dr \\
=& \frac{2\Delta_x K_\infty(\bar{x})\cdot Vol(S^3)}{3|\Gamma|K_\infty(\bar{x})^2}
\end{aligned} \tag{4.2.36}$$

□

Then we achieve the following proposition.

Proposition 4.2.6. *Assuming Condition 3.3.6, we have the following inequality*

$$\lim_{\sigma\to 0} P(\sigma, h_{\bar{x}}) \ge \frac{2\Delta_g K_\infty(\bar{x})\cdot Vol(S^3)}{3|\Gamma|K_\infty(\bar{x})^2}. \tag{4.2.37}$$

Moreover, if assuming $\tau_k = 0$ for all k, we get the equality

$$\lim_{\sigma\to 0} P(\sigma, h_{\bar{x}}) = \frac{2\Delta_g K_\infty(\bar{x})\cdot Vol(S^3)}{3|\Gamma|K_\infty(\bar{x})^2}. \tag{4.2.38}$$

Here, Δ_g is the Laplacian with respect to the limit conformal normal metric g.

Proof. By Lemma 4.2.1 – Lemma 4.2.5 and equation (4.1.17),

$$\lim_{\sigma\to 0}\limsup_{k\to\infty} M_k^2 I_{k,7} = \frac{2\Delta_g K_\infty(\bar{x})\cdot Vol(S^3)}{3|\Gamma|K_\infty(\bar{x})^2} + \lim_{\sigma\to 0}\limsup_{k\to\infty} M_k^2 I_{k,6}. \tag{4.2.39}$$

By Corollary 3.3.11,

$$\begin{aligned}&\limsup_{k\to\infty} M_k^2 I_{k,7}\\ =&\limsup_{k\to\infty}\int_{\{|x|=\sigma\}/\Gamma}\Big\{\Big(\Big|\frac{\partial(M_k u_k)}{\partial\nu_x}\Big|^2-\frac{1}{2}|\nabla_x(M_k u_k)|^2\Big)\sigma+(M_k u_k)\frac{\partial(M_k u_k)}{\partial\nu_x}\Big\}\\ =&P(\sigma,h_{\bar{x}})\end{aligned} \tag{4.2.40}$$

On the other hand, recall that

$$I_{k,6}=\frac{\tau_k}{p_k+1}\int_{\{|x|\le\sigma\}/\Gamma}K_k f_k^{-\tau_k}u_k^{p_k+1}. \tag{4.2.41}$$

It's clear that $I_{k,6}\ge 0$, espcially $I_{k,6}=0$ if $\tau_k=0$. Our proposition is proved. □

4.3 Green Function

Assuming Condition 3.3.6, we will further study the Green function. Corollary 3.3.11 tells us that $M_k u_k \to h_{\bar{x}} = aG(\cdot,\bar{x})$ in $C^2_{loc}((B_r(\bar{x})-\{\bar{x}\})/\Gamma)$. We are going to determine the constant a and the regular part of $G(\cdot,\bar{x})$ evaluated at $\bar{x}$.

$G(\cdot,\bar{x})$ is the standard Green function for $L_g=\Delta_g-\frac{1}{6}R_g$ at point $\bar{x}$, where g is the limit metric, i.e. for any C^2 function ϕ and small $\sigma>0$,

$$\int_{\{|x|\le\sigma\}/\Gamma} G(L_g\phi)dV_g=\phi(\bar{x}) \tag{4.3.1}$$

Hence

$$\lim_{k\to\infty}\int_{\{|x|\le\sigma\}/\Gamma}(M_k u_k)(L_{g_k}\phi)dV_{g_k}=a\phi(\bar{x}) \tag{4.3.2}$$

Lemma 4.3.1. *The constant a in Corollary 3.3.11 is*

$$a = -\frac{4\sqrt{2}Vol(S^3)}{|\Gamma|\sqrt{K_\infty(\bar{x})}} \tag{4.3.3}$$

Proof. Recall the coordinates setting in Remark 3.3.2 and the change of variables in (3.3.5), (3.3.6). For any C^2 function ϕ, define $\tilde{\phi}(y) = \phi(M_k^{-\frac{p_k-1}{2}} y)$. Integrating by parts, we get

$$\begin{aligned}
&\int_{\{|x|\le\sigma\}/\Gamma} (M_k u_k)(L_{g_k}\phi) \\
=&\int_{\{|x|\le\sigma\}/\Gamma} \Big(\Delta_x - \frac{1}{6}R_k\Big)(M_k u_k)\cdot\phi \\
=&M_k^{-2(p_k-1)} \int_{\{|y|\le\sigma M_k^{\frac{p_k-1}{2}}\}/\Gamma} \Big(M_k^{p_k-1}\Delta_y - \frac{1}{6}\tilde{R}\Big)(M_k^2 v_k)\cdot\tilde{\phi} \\
=&\Big(M_k^{\tau_k}\int_{\{|y|\le\sigma M_k^{\frac{p_k-1}{2}}\}/\Gamma} (\Delta_y v_k)\cdot\tilde{\phi}\Big) - \Big(\frac{1}{6}M_k^{4-2p_k}\int_{\{|y|\le M_k^{\frac{p_k-1}{2}}\sigma\}/\Gamma} \tilde{R}_k v_k\tilde{\phi}\Big)
\end{aligned} \tag{4.3.4}$$

Since $\lim_{k\to\infty} M_k^{4-2p_k} = 0$, the second integral vanishes when $k\to\infty$. Thus

$$\begin{aligned}
a\phi(\bar{x}) &= \lim_{\sigma\to 0}\lim_{k\to\infty}\int_{\{|x|\le\sigma\}/\Gamma} (M_k u_k)(L_{g_k}\phi) \\
&= \lim_{\sigma\to 0}\lim_{k\to\infty}\int_{\{|y|\le M_k^{\frac{2}{n-2}}\sigma\}/\Gamma} (\Delta_y v_k)\cdot\phi(M_k^{-\frac{p_k-1}{2}} y) \\
&=\Big(\int_{\mathbb{R}^n/\Gamma} \Delta_y U_{K_\infty(\bar{x})}(y)dy\Big)\cdot\phi(\bar{x})
\end{aligned} \tag{4.3.5}$$

where the last equality is by Proposition 3.3.3. Therefore the constant a is

$$\begin{aligned} a &= \int_{\mathbb{R}^n/\Gamma} \Delta_y U_{K_\infty(\bar{x})}(y)dy \\ &= -\int_{\mathbb{R}^n/\Gamma} K_\infty(\bar{x}) U^3_{K_\infty(\bar{x})}(y)dy \\ &= -\frac{16\sqrt{2}}{\sqrt{K_\infty(\bar{x})}} \int_{\mathbb{R}^n/\Gamma} (1+|y|^2)^{-3}dy \\ &= -\frac{16\sqrt{2}Vol(S^3)}{|\Gamma|\sqrt{K_\infty(\bar{x})}} \int_0^\infty r^3(1+r^2)^{-3}dr \\ &= -\frac{4\sqrt{2}Vol(S^3)}{|\Gamma|\sqrt{K_\infty(\bar{x})}} \end{aligned} \tag{4.3.6}$$

□

On the other hand, recall that we assume for each k, g_k is the conformal normal metric with conformal normal coordinates centered at point x_k, hence the limit metric g is the conformal normal metric with conformal normal coordinates centered at point $\bar{x}$. Thus we have the following:

Proposition 4.3.2. *In g-conformal normal coordinates $\{x^i\}$ centered at $\bar{x}$, $G(\cdot, \bar{x})$ has the following expansion:*

$$G(\cdot, \bar{x}) = b\psi_{\bar{x}} = b[r^{-2} + A_{\bar{x}} + O(r)] \tag{4.3.7}$$

where $r = |x|$, $\psi_{\bar{x}}$ and $A_{\bar{x}}$ are defined in Definition 1.1.8, and

$$b = -\frac{|\Gamma|}{2Vol(S^3)} \tag{4.3.8}$$

Proof. The proof is given by [26, Definition 6.2, Lemma 6.4]. Note that our notation is different from [26]. Our point $\bar{x}$ is their point P; our operator L_g is equal to $-1/6$ multiplying with their box operator □; our $G(\cdot, \bar{x})$ is equal to -6 multiplying with their Γ_P, our $\psi_{\bar{x}}$ is their G. And the Γ in our equation (4.3.8) is the quotient group near point

$\bar{x}$.

□

Then we can relate the Pohozaev identity with the constant term $A_{\bar{x}}$.

Lemma 4.3.3.

$$\lim_{\sigma\to 0} P(\sigma, h_{\bar{x}}) = -\frac{16Vol(S^3)}{|\Gamma|K_\infty(\bar{x})} \cdot A_{\bar{x}} \tag{4.3.9}$$

Proof. We will write G instead of $G(\cdot, \bar{x})$. Using $h_{\bar{x}} = aG$, we get

$$P(\sigma, h_{\bar{x}}) = a^2 \int_{\{|x|=\sigma\}/\Gamma} \Big\{ \Big(\Big|\frac{\partial G}{\partial \nu}\Big|^2 - \frac{1}{2}|\nabla G|^2 \Big)\sigma + G\frac{\partial G}{\partial \nu_x} \Big\} \tag{4.3.10}$$

By Proposition 4.3.2, on $|x| = \sigma$ for small σ,

$$\begin{aligned} &G = b[\sigma^{-2} + A_{\bar{x}} + O(\sigma)], \\ &\Big|\frac{\partial G}{\partial \nu}\Big| = b[-2\sigma^{-3} + O(1)], \\ &|\nabla G| = b[-2\sigma^{-3} + O(1)]. \end{aligned} \tag{4.3.11}$$

Then we get

$$\begin{aligned} &\Big(\Big|\frac{\partial G}{\partial \nu}\Big|^2 - \frac{1}{2}|\nabla G|^2 \Big)\sigma + G\frac{\partial G}{\partial \nu_x} \\ =&b^2\Big(\frac{\sigma}{2} \cdot [-2\sigma^{-3} + O(1)]^2 + [\sigma^{-2} + A_{\bar{x}} + O(\sigma)] \cdot [-2\sigma^{-3} + O(1)] \Big) \\ =&b^2\Big(\frac{\sigma}{2} \cdot [4\sigma^{-6} + O(\sigma^{-3})] - 2\sigma^{-5} - 2A_{\bar{x}}\sigma^{-3} + O(\sigma^{-2}) \Big) \\ =& - 2b^2 A_{\bar{x}}\sigma^{-3} + O(\sigma^{-2}) \end{aligned} \tag{4.3.12}$$

Hence

$$\begin{aligned} P(\sigma, h_{\bar{x}}) =&a^2 \int_{\{|x|=\sigma\}/\Gamma} \Big(- 2b^2 A_{\bar{x}}\sigma^{-3} + O(\sigma^{-2}) \Big) \\ =& - \frac{2a^2b^2 A_{\bar{x}} \cdot Vol(S^3)}{|\Gamma|} + O(\sigma). \end{aligned} \tag{4.3.13}$$

Let $\sigma \to 0$, using Lemma 4.3.1 and Proposition 4.3.2, we have

$$\begin{aligned}\lim_{\sigma\to 0} P(\sigma, h_{\bar{x}}) =& -\frac{2a^2b^2 A_{\bar{x}} \cdot Vol(S^3)}{|\Gamma|} \\ =&(-2)\cdot \frac{Vol(S^3)}{|\Gamma|} \cdot \frac{32Vol(S^3)^2}{|\Gamma|^2 K_\infty(\bar{x})} \cdot \frac{|\Gamma|^2}{4Vol(S^3)^2} \cdot A_{\bar{x}} \\ =& -\frac{16Vol(S^3)}{|\Gamma| K_\infty(\bar{x})} \cdot A_{\bar{x}}\end{aligned} \tag{4.3.14}$$

□

Proposition 4.3.4. *Assuming Condition 3.3.6, we have the following inequality*

$$A_{\bar{x}} \leq -\frac{\Delta_g K_\infty(\bar{x})}{24K_\infty(\bar{x})} \tag{4.3.15}$$

Moreover, if assuming $\tau_k = 0$ for all k, we get the equality

$$A_{\bar{x}} = -\frac{\Delta_g K_\infty(\bar{x})}{24K_\infty(\bar{x})} \tag{4.3.16}$$

Here, Δ_g is the Laplacian with respect to the limit conformal normal metric g.

Proof. Assuming Condition 3.3.6, by Proposition 4.2.6, we have

$$\lim_{\sigma\to 0} P(\sigma, h_{\bar{x}}) \geq \frac{2\Delta_g K_\infty(\bar{x}) \cdot Vol(S^3)}{3|\Gamma| K_\infty(\bar{x})^2}. \tag{4.3.17}$$

By Lemma 4.3.4, we have

$$\lim_{\sigma\to 0} P(\sigma, h_{\bar{x}}) = -\frac{16Vol(S^3)}{|\Gamma| K_\infty(\bar{x})} \cdot A_{\bar{x}}. \tag{4.3.18}$$

Hence

$$A_{\bar{x}} \leq -\frac{2\Delta_g K_\infty(\bar{x}) \cdot Vol(S^3)}{3|\Gamma| K_\infty(\bar{x})^2} \cdot \frac{|\Gamma| K_\infty(\bar{x})}{16Vol(S^3)} = -\frac{\Delta_g K_\infty(\bar{x})}{24K_\infty(\bar{x})} \tag{4.3.19}$$

The case that $\tau_k = 0$ for all k follows similarly. $\square$

Using the result in Section 2.3, we can remove the assumption "conformal normal metric".

Proposition 4.3.5. *Assuming Condition 3.3.6, denote* $g = \lim_{k\to\infty} g_k$ *to be the limit metric, but not assuming* g_k, g *are conformal normal metrics.*

Let

$$\hat{g}_{\bar{x}} = \psi_{\bar{x}}^2 g \tag{4.3.20}$$

be the conformal blow-up of g *at the point* $\bar{x}$*, as in Definition 1.1.8.*

We have the following inequality

$$m(\hat{g}_{\bar{x}}) \leq -\frac{\Delta_g K_\infty(\bar{x})}{2K_\infty(\bar{x})}. \tag{4.3.21}$$

Moreover, if assuming $\tau_k = 0$ *for all* k*, we get the equality*

$$m(\hat{g}_{\bar{x}}) = -\frac{\Delta_g K_\infty(\bar{x})}{2K_\infty(\bar{x})}. \tag{4.3.22}$$

Proof. For each k, let r_k denote the g_k-distance function from the point x_k. Assume

$$\tilde{g}_k = \phi_k^2 g_k \tag{4.3.23}$$

is the conformal normal metric with conformal normal coordinates $\{\tilde{x}^i\}$ centered at point x_k. By [26, Theorem 5.6], after applying a dilation and a translation to the coordinates $\{\tilde{x}^i\}$, we may assume $\phi_k(x_k) = 1$ and $\nabla\phi_k(x_k) = 0$, in other words, for small r_k,

$$\phi_k = 1 + O(r_k^2). \tag{4.3.24}$$

When $k \to 0$, we have that

$$\tilde{g} = \phi^2 g \tag{4.3.25}$$

where $g, \tilde{g}, \phi$ are the limit of $g_k, \tilde{g}_k, \phi_k$ as $k \to \infty$. Moreover,

$$\phi = 1 + O(r^2) \tag{4.3.26}$$

where r is the g-distance function from the point $\bar{x}$.

Recall the conformal transformation law of Laplacian for $\tilde{g} = e^{2\varphi} g$ is

$$\Delta_{\tilde{g}} = e^{-2\varphi}\Delta_g + (n-2)e^{-2\varphi} g^{ij} \frac{\partial \varphi}{\partial x_j} \frac{\partial}{\partial x_i}. \tag{4.3.27}$$

Thus, for any $f \in C^2$, we have

$$\Delta_{\tilde{g}} f(\bar{x}) = \Delta_g f(\bar{x}). \tag{4.3.28}$$

Let $A_{\bar{x}}$ be the regular part corresponding to conformal blow-up of $\tilde{g}$ at point $\bar{x}$. Let $\hat{g}_{\bar{x}}$ be the conformal blow-up of g at point $\bar{x}$. By Proposition 2.3.1,

$$m(\hat{g}_{\bar{x}}) = 12 A_{\bar{x}}. \tag{4.3.29}$$

Therefore, we know

$$\begin{aligned} m(\hat{g}_{\bar{x}}) \le -\frac{\Delta_g K_\infty(\bar{x})}{2K_\infty(\bar{x})} \quad &\Leftrightarrow \quad A_{\bar{x}} \le -\frac{\Delta_{\tilde{g}} K_\infty(\bar{x})}{24K_\infty(\bar{x})}, \\ m(\hat{g}_{\bar{x}}) = -\frac{\Delta_g K_\infty(\bar{x})}{2K_\infty(\bar{x})} \quad &\Leftrightarrow \quad A_{\bar{x}} = -\frac{\Delta_{\tilde{g}} K_\infty(\bar{x})}{24K_\infty(\bar{x})}, \end{aligned} \tag{4.3.30}$$

which implies our proposition is equivalent to Proposition 4.3.4. □

Chapter 5

Blow-up points

5.1 Blow-up points must be isolated and simple

In [29], they proved that on a 3-dimensional compact manifold, all blow-up points for equation (3.3.1) must be isolated simple blow-up points. The same result in higher dimensions is proved by [23], [31], but only for constant prescribed scalar curvature. Here, we will modify their proofs to show that the same result holds on 4-dimensional compact orbifolds, for a sequence of variable prescribed scalar curvatures.

Let (M, g) be a compact Riemannian 4-dimensional orbifold with singularities

$$\Sigma_\Gamma = \{(q_1, \Gamma_1), \cdots, (q_l, \Gamma_l)\} \tag{5.1.1}$$

and positive scalar curvature R_g. Assume u is a positive C^2 solution of equation (3.1.1) on M, where K is a positive C^2 function and f is a positive C^1 function.

For any point $\bar{x} \in M$, define $\Omega_{\bar{x},\sigma}$ in the following:

a) if $\bar{x}$ is a smooth point, define $\Omega_{\bar{x},\sigma} = B_\sigma(\bar{x})$ for some $\sigma > 0$ such that its closure $\bar{\Omega}_{\bar{x},\sigma}$ doesn't include any singular point. In other words, $d_g(\bar{x}, \{q_1, \cdots, q_l\}) > \sigma$;

b) if $\bar{x} = q_j$ for some $1 \le j \le l$, choose $\sigma = \sigma_j$ where σ_j is as defined in Definition 1.1.1. Then the neighborhood of $\bar{x}$ is a quotient ball $B_\sigma(\bar{x})/\Gamma$. Define $\Omega_{\bar{x},\sigma} = \pi_j^*(B_\sigma(\bar{x})/\Gamma)$ to be the lifting-up space. Denote the lifting-up functions and metric still by u, f, K and g.

Firstly, let's introduce a lemma from [29].

Lemma 5.1.1 ([29] Lemma 5.1)**.** *Let $(M, g), u, K, f, \Omega_{\bar{x},\sigma}$ be as defined above. Given any small $\varepsilon > 0$ and large $R' > 1$, there exists a large positive C_0, depending only on $M, g, \|f\|_{C^1(M)}$, $\inf_M K, \|K\|_{C^2(M)}, \varepsilon$ and R' such that for any compact $S \subset \bar{\Omega}_{\bar{x},\sigma}$, if u satisfies*

$$\max_{x \in \bar{\Omega}_{\bar{x},\sigma} \setminus S} d_g(x, S)^{\frac{2}{p-1}} u(x) \ge C_0, \tag{5.1.2}$$

then we have $p > 3 - \varepsilon$ and for some local maximum point of u in $\Omega_{\bar{x},\sigma} \setminus S$, denoted as x_0,

$$\left\| u(x_0)^{-1} u\Big(\exp_{x_0} \left(u(x_0)^{-\frac{p-1}{2}} x \right)\Big) - U_{K(x_0)}(x) \right\|_{C^2(|x|<2R')} < \varepsilon, \tag{5.1.3}$$

where $d_g(x, S)$ denotes the distance of y to S, and $d_g(x, S) = 1$ if $S = \emptyset$.

Especially, the constant C_0 does not depend on the choice of σ and $\bar{x}$.

Proof. The case that K be a positive constant and $\Omega_{\bar{x},\sigma}$ be replaced by a compact 3-dimensional manifold M is proved in [29, Lemma 5.1]. As mentioned in [29, Section 7], it can be generalized to the case K be a variable function. With a straightforward modification, it can also be generalized to higher dimensions.

Moreover, note that the statement and proof in [29, Lemma 5.1] is just a local argument in $M \setminus S$. Thus his proof remains valid in our case by choosing his S to be any compact subset of M such that $M \setminus S \subset B_\sigma(\bar{x})$, and considering the lifting-up of $M \setminus S$ if $\bar{x}$ is a singular point. $\square$

Using Lemma 5.1.1, we can prove the following:

Proposition 5.1.2 ([29] Proposition 5.1). *Let $(M,g), u, K, f, \Omega_{\bar{x},\sigma}$ be as defined in the beginning of this section. Given small $\varepsilon > 0$ and large R', there exist some positive constants C_0 and C_1 depending on $M, g, \|f\|_{C^1(M)}, \inf_M K, \|K\|_{C^2(M)}, \varepsilon$ and R' such that if*

$$\max_{\Omega_{\bar{x},\sigma}} u > C_0, \tag{5.1.4}$$

then there exists some integer $N = N(u) \geq 1$ and N local maximum points of u denoted as $\{x_1, \cdots, x_N\} \subset \Omega_{\bar{x},\sigma}$, such that:
1) $3 - \varepsilon < p \leq 3$,
2) $\overline{B_{r_i}(x_i)} \cap \overline{B_{r_j}(x_j)} = \emptyset$ for $i \neq j$, where $r_j = R' u(x_j)^{-\frac{p-1}{2}}$,
and for each j,

$$\left\| u(x_j)^{-1} u\Big(\exp_{x_j} \Big(u(x_j)^{-\frac{p-1}{2}} x \Big) \Big) - U_{K(x_j)}(x) \right\|_{C^2(|x|<2R')} < \varepsilon, \tag{5.1.5}$$

3) $d_g(x_i, x_j)^{\frac{2}{p-1}} u(x_j) \geq C_0$ for $j > i$, while $d_g(x, \{x_1, \cdots, x_N\})^{\frac{2}{p-1}} u(x) \leq C_1$ for all $x \in \Omega_{\bar{x},\sigma}$.

Proof. The case that K be a positive constant and $\Omega_{\bar{x},\sigma}$ be replaced by a compact 3-dimensional manifold M is proved in [29, Proposition 5.1].

Briefly, that proof was completed by an induction process as following: first apply [29, Lemma 5.1] with $S = \emptyset$ to get the first local maximum point of u, denoted by x_1;

assuming we have already got local maximum points $\{x_1, \cdots, x_l\}$, apply [29, Lemma 5.1] with $S = \cup_{j=1}^{l} \overline{B_{r_j}(x_j)}$ to get the next local maximum point. If the condition (5.1.2) is not satisfied in any step, the process stops. This process must stop after a finite number of times because each time $\int_{B_{r_j}(x_j)} |\nabla u|^2$ is greater than some positive universal constant and their sum $\sum_j \int_{B_{r_j}(x_j)} |\nabla u|^2 \leq \int_M |\nabla u|^2$ which is finite. It's not hard to verify the set $\{x_1, \cdots, x_N\}$ constructed by the process satisfies all the above properties.

In our case, we can continue the same process by inductively applying our Lemma 5.1.1 with S as mentioned above. Everything follows the same way and our proposition is proved. □

Then we can rule out bubble accumulation.

Proposition 5.1.3. *Let $(M, g), u, K, f, \Omega_{\bar{x},\sigma}$ be as defined in the beginning of this section. Let $\varepsilon, R', C_0, C_1$ and $\{x_1, \cdots, x_N\}$ be as defined in Proposition 5.1.2. If ε is sufficiently small and R' is sufficiently large, then there exists a positive constant $\bar{C}$ which only depends on $M, g, \|f\|_{C^1(M)}, \inf_M K, \|K\|_{C^2(M)}, \varepsilon$ and R' such that if*

$$\max_{\Omega_{\bar{x},\sigma}} u \geq C_0, \tag{5.1.6}$$

then

$$d_g(x_j, x_l) \geq \bar{C} \tag{5.1.7}$$

for all $j \neq l$.

Proof. The proof is similar to [23, Proposition 8.3] and [29, Proposition 5.2].

We will prove it by contradiction. Suppose that such a constant $\bar{C}$ does not exist, then there exist sequences $p_k \to p \in (3-\varepsilon, 3]$ and $\{u_k\}$ with $\max_{\Omega_{\bar{x},\sigma}} u_k \geq C_0$ and

$$\lim_{k\to\infty} \min_{j\neq l} d_g(x_j(u_k), x_l(u_k)) = 0. \tag{5.1.8}$$

Without loss of generality, assume that

$$\delta_k = d_g(x_1(u_k), x_2(u_k)) = \min_{j\neq l} d_g(x_j(u_k), x_l(u_k)) \to 0 \text{ as } k \to \infty. \tag{5.1.9}$$

For each k, take normal coordinates $\{x^i\}$ centered at point $x_1(u_k)$ and consider change of variables $y = \delta_k^{-1} x$. Rescale u_k by

$$v_k(y) = \delta_k^{\frac{2}{p_k-1}} u_k(\delta_k y), \quad \forall |y| < \delta_k^{-1}. \tag{5.1.10}$$

Then v_k satisfies

$$L_{h_k} v_k + \tilde{K}_k \tilde{f}_k^{-\tau_k} v_k^{p_k} = 0 \tag{5.1.11}$$

where $(h_k)_{ij}(y) = g_{ij}(\delta_k y)$, $\tilde{K}_k(y) = K(\delta_k y)$ and $\tilde{f}_k(y) = f(\delta_k y)$.

If $x_j(u_k) \in B_{\sqrt{\sigma_k}(x_1)}$, denote $y_j(u_k) = \delta_k^{-1} x_j(u_k)$ to be the y-coordinate of point $x_j(u_k)$. By the proof of [23, Proposition 8.3], we have $y_1(u_k) = 0$, $y_2(u_k) \to \bar{y}_2$ with $|\bar{y}_2| = 1$, $\{0, \bar{y}_2\}$ are isolated simple blow-up points for $\{v_k\}$, and

$$v_k(0) v_k(y) \to a_1(|y|^{-2} + b_1 + O(y)) \text{ in } C^2_{loc}(\mathbb{R}^4 - S'), \tag{5.1.12}$$

where S' denotes the set of blow-up points for $\{v_k\}$ and $a_1, b_1 > 0$ are some positive constants.

On the other hand,

$$\begin{aligned}
&\tilde{K}_k(y) = K(\delta_k y) \to K(0) \text{ in } C^0_{loc} \text{ norm,} \\
&|\nabla_y \tilde{K}_k(y)| = \delta_k |\nabla_x K(\delta_k y)| \to 0 \text{ in } C^0 \text{ norm,} \\
&|\nabla^2_y \tilde{K}_k(y)| = \delta_k^2 |\nabla^2_x K(\delta_k y)| \to 0 \text{ in } C^0 \text{ norm.}
\end{aligned} \tag{5.1.13}$$

Hence $\tilde{K}_k$ converges to the constant $K(0)$ in the C^2_{loc} norm, where $K(0)$ by definition is the K value at the limit point of $x_1(u_k)$ as $k \to \infty$, possibly by passing to a subsequence.

Apply Proposition 4.3.4 to the blow-up sequence $\{v_k\}$, we get

$$b_1 \leq 0, \tag{5.1.14}$$

which contradicts against $b_1 > 0$.

Therefore our proposition is proved. □

Corollary 5.1.4. *Let (M, g) be a compact Riemannian 4-dimensional orbifold with positive scalar curvature. Suppose $\{f_k\}$ is a sequence of positive C^1 functions converging in the C^1_{loc} topology to a positive function f. Also suppose $\{K_k\}$ is a sequence of positive C^2 functions converging in the C^2_{loc} topology to a positive function K_∞.*

Let $\{u_k\}$ be a sequence of positive solutions of equation (3.3.1) *on M with $g_k = g$ and $\max_M u_k \to \infty$. Then $p_k \to 3$ and the set of blow-up points is finite and consists only of isolated blow-up points.*

Moreover, if blow-up occurs at a singular point, i.e. $x_k \to \bar{x}$ and $u_k(x_k) \to \infty$ where $\bar{x}$ is a singular point, then there exists an integer $N \in \mathbb{N}^*$ such that for any $k > N$, $x_k = \bar{x}$.

Proof. By the assumption of f_k and K_k, there exists a constant C_2 such that for large k,

$$\begin{aligned}
&\|f_k\|_{C^1(M)} \leq C_2\|f\|_{C^1(M)}, \\
&\inf_M K_k \geq C_2 \inf_M K_\infty, \qquad (5.1.15) \\
&\|K_k\|_{C^2(M)} \leq C_2\|K_\infty\|_{C^2(M)}.
\end{aligned}$$

By proposition 5.1.3, in each $\Omega_{x,\sigma}$, blow-up points must be isolated blow-up points and the number $N(u_k)$ as defined in Proposition 5.1.2 must have an uniformly upper bound, otherwise, there cannot exist a constant $\bar{C}$ such that $d_g(x_i(u_k), x_j(u_k)) \geq \bar{C}$ for all $i \neq j$ and k.

A compact orbifold can be covered by finitely many sufficiently small open balls, which gives us finitely many $\Omega_{x,\sigma}$ as defined in the beginning of this section. Therefore the set of blow-up points on M is finite and consists only of isolated blow-up points.

Moreover, if blow-up occurs at a singular point, i.e. $x_k \to \bar{x}$ and $u_k(x_k) \to \infty$ where $\bar{x}$ is a singular point associated with a nontrivial quotient group Γ. Suppose there exists a subsequence, still denoted by (u_k, x_k), such that $x_k \neq \bar{x}$ for any k.

Consider the lifting-up space $\Omega_{\bar{x},\sigma}$ as defined in the beginning of this section. Let $\tilde{x}_k^{(1)}, \cdots, \tilde{x}_k^{(|\Gamma|)}$ denote the lifting-up points of x_k. In the lifting-up space, it's clear that

$$d_{\tilde{g}}(\tilde{x}_k^{(1)}, \tilde{x}_k^{(2)}) \to 0 \quad \text{as } k \to \infty, \qquad (5.1.16)$$

which contradicts against Proposition 5.1.3. Thus our corollary is proved. □

Given Corollary 5.1.4, we are able to conclude the following.

Proposition 5.1.5. *Let $(M, g), f_k, K_k, g_k$ be as defined in Corollary 5.1.4. Assume u_k is a sequence of positive functions satisfying equation* (3.3.1) *and $x_k \to \bar{x}$ is an isolated blow-up point. Then $\bar{x}$ is an isolated simple blow-up point for $\{u_k\}$.*

Proof. The proof is similar to [23, Lemma 8.2] and [29, Proposition 4.1].

Firstly, by Corollary 5.1.4, without loss of generality, we may assume $x_k = \bar{x}$ for all k if $\bar{x}$ is a singular point.

Suppose $\bar{x}$ is an isolated blow-up point, but not an isolated simple blow-up point. Let $x = \{x^i\}$ be normal coordinates centered at x_k and define the rescaled function

$$v_k(y) = \tau_k^{\frac{2}{p_k-1}} u_k(\tau_k y), \quad \forall |y| < \tau_k^{-1}. \tag{5.1.17}$$

Then v_k satisfies

$$L_{h_k} v_k + \tilde{K}_k \tilde{f}_k^{-\tau_k} v_k^{p_k} = 0 \tag{5.1.18}$$

where $(h_k)_{ij}(y) = (g_k)_{ij}(\tau_k y)$, $\tilde{K}_k(y) = K_k(\tau_k y)$ and $\tilde{f}_k(y) = f_k(\tau_k y)$. By the proof of [23, Lemma 8.2], the origin $y = 0$ is an isolated simple blow-up point for $\{v_k\}$ and

$$v_k(0) v_k(y) \to h(y) = a_2(|y|^{-2} + b_2) \ \text{ in } C^2_{loc}((\mathbb{R}^4 - \{0\})/\Gamma), \tag{5.1.19}$$

where $a_2 = b_2 = 1$. Here, we put "$/\Gamma$" like what we did in previous section, in the sense that: $\Gamma = \{e\}$ if $\bar{x}$ is a smooth point; Γ is the quotient group if $\bar{x}$ is a singular point.

On the other hand, because K_k converges to K_∞ in the C^2_{loc} norm, we know

$$\begin{aligned}
&\tilde{K}_k(y) = K_k(\tau_k y) \to K_\infty(\bar{x}) \text{ in } C^0_{loc} \text{ norm},\\
&|\nabla_y \tilde{K}_k(y)| = \tau_k |\nabla_x K_k(\tau_k y)| \to 0 \text{ in } C^0_{loc} \text{ norm},\\
&|\nabla^2_y \tilde{K}_k(y)| = \tau_k^2 |\nabla^2_x K_k(\tau_k y)| \to 0 \text{ in } C^0_{loc} \text{ norm}.
\end{aligned} \tag{5.1.20}$$

Hence $\tilde{K}_k$ converges to the constant $K_\infty(\bar{x})$ in the C^2_{loc} norm.

Apply Proposition 4.3.4 to the blow-up sequence $\{v_k\}$, we get

$$b_2 \leq 0, \tag{5.1.21}$$

which contradicts against $b_2 = 1$.

Therefore $\bar{x}$ is an isolated simple blow-up points for $\{u_k\}$. □

Corollary 5.1.6. *Let $(M, g), f_k, K_k, g_k$ be as defined in Corollary 5.1.4. Assume u_k is a sequence of positive functions satisfying equation* (3.3.1)*, then our Condition 3.3.6 is a necessary condition for any blow-up point.*

5.2 Proof of Theorem 1.3.1

To prove the upper bound $u \leq C$ under assumption (1.3.4) in Theorem 1.3.1, suppose the contrary, then there exist $p_k \to 3$ and $\{u_k\}$ satiesfying

$$L_g u_k + K_k u_k^{p_k} = 0 \tag{5.2.1}$$

with $\max_M u_k \to \infty$ and $\{K_k\}$ be a sequence of positive C^2 functions converging in the C^2_{loc} topology to a positive function K_∞. Let x_k denote the point where u_k obtains maximum, after possibly passing to a subsequence, we may assume $x_k \to \bar{x}$ is a blow-up point. By Corollary 5.1.6, the sequence (u_k, x_k) satisfies Condition 3.3.6, where g_k in equation (3.3.1) is the metric conformal to g with conformal normal coordinates centered at x_k. By Proposition 4.3.5, we know

$$m(\hat{g}_{\bar{x}}) \leq -\frac{\Delta_g K_\infty(\bar{x})}{2K_\infty(\bar{x})}, \tag{5.2.2}$$

which contradicts against assumption (1.3.4) in Theorem 1.3.1. Therefore we know

$$u \leq C \tag{5.2.3}$$

for u and C as stated in Theorem 1.3.1. By the Harnack inequality[1] and the standard elliptic estimate similar to [23], [29], [35], we also get

$$u \geq 1/C \quad \text{and} \quad \|u\|_{C^{2,\alpha}(M)} \leq C. \tag{5.2.4}$$

Furthermore, because for $p < 3$, there always exists subcritical solution for any K. Take a sequence of subcritical solutions and let $p \to 3$, due to (1.3.3), they limit to a critical solution for $p = 3$. Thus the second part of Theorem 1.3.1 is proved.

The first part of Theorem 1.3.1 can be proved similarly, by fixing $p_k = 3$ in the above proof. □

[1]Assume u obtains $\sup_M u$ at point P, then $\Delta u(P) < 0$. By (1.1.3), $u(P) \geq \sqrt{R_g(P)/(6K(P))} \geq c_0$. By Harnack inequality, $\inf_M u \geq (1/c_1)u(P) \geq c_0/c_1 \geq 1/C$, for sufficiently large C.

Chapter 6

Energy approach

6.1 An energy bound

Let (M, g) be a compact Riemannian n-orbifold with singularities $\Sigma_\Gamma = \{(q_1, \Gamma_1), \cdots, (q_l, \Gamma_l)\}$ and positive scalar curvature R. Let K be a positive C^2 function on M. In this section, we will study equation (1.1.3) using an energy approach.

Consider the energy functional

$$E(u) = \int_M \Big(|\nabla u|^2 + \frac{n-2}{4(n-1)} R u^2\Big) dVol_g \tag{6.1.1}$$

for $u \in W^{1,2}(M)$. Let $\mathcal{C}_{p,K}(M) \subset W^{1,2}(M)$ be the constraint set

$$\mathcal{C}_{p,K}(M) = \Big\{u \in H_1(M) : \int_M K|u|^{p+1} dVol_g = 1\Big\}. \tag{6.1.2}$$

The condition that K be positive guarantees $\mathcal{C}_{p,K} \neq \emptyset$. Define

$$E(p,K) = \inf_{u \in \mathcal{C}_{p,K}} E(u) \tag{6.1.3}$$

It's a well-known result that there exists minimizing function $u \in \mathcal{C}_{p,K}$ such that $E(u) = E(p,K)$ for $p < \frac{n+2}{n-2}$, which satisfies

$$Lu + E(p,K)Ku^p = 0. \tag{6.1.4}$$

Let $Q(S^n)$ be the Sobolev quotient of S^n, i.e.

$$Q(S^n) = \inf_{\phi \in H_1(S^n)} \frac{\int_{S^n} \left(|\nabla \phi|^2 + \frac{n(n-2)}{4}\phi^2\right) dv}{\left(\int_{S^n} \phi^{\frac{2n}{n-2}} dv\right)^{\frac{n-2}{n}}} = \frac{n(n-2)}{4} Vol(S^n)^{2/n}. \tag{6.1.5}$$

The following theorem generalizes [19, Proposition 1.1] to our orbifold setting. This also generalizes [1, Theorem 3.1] for the case $K = \mathit{constant}$ to the case of variable K.

Theorem 6.1.1. *Let (M,g) be a compact Riemannian n-orbifold with singularities $\Sigma_\Gamma = \{(q_1,\Gamma_1),\cdots,(q_l,\Gamma_l)\}$ and positive scalar curvature. Let K be a positive C^2 function on M. Define the modified maximum value of K*

$$B_K := \max\Big\{ \sup_{x \in M}\{K(x)\}, \max_{1 \le i \le l}\{|\Gamma_i|^{\frac{2}{n-2}} K(q_i)\}\Big\}. \tag{6.1.6}$$

Then we have the following inequality

$$(B_K)^{\frac{n-2}{n}} E\Big(\frac{n+2}{n-2}, K\Big) \le Q(S^n). \tag{6.1.7}$$

Furthermore, if

$$(B_K)^{\frac{n-2}{n}} E\Big(\frac{n+2}{n-2}, K\Big) < Q(S^n), \tag{6.1.8}$$

then there exists a positive smooth function u of (1.1.3) *with $p = \frac{n+2}{n-2}$ such that u lies in $\mathcal{C}_{\frac{n+2}{n-2},K}$ and achieves the minimum energy $E(\frac{n+2}{n-2}, K)$ in this set. In other words, there exists an orbifold metric $\tilde{g} \in [g]_{orb}$ such that $\tilde{g}$ has scalar curvature K.*

Proof. We first prove the inequality (6.1.7). We need to modify proof in [26] and [19] to put a test function on the orbifold.

Let's first assume B_K is obtained at some singular point, without loss of generality assuming that $|\Gamma_1|^{\frac{2}{n-2}} K(q_1) = B_K$. Define

$$u_\varepsilon(x) = \Big(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\Big)^{(n-2)/2} \tag{6.1.9}$$

to be a function in the Euclidean cone $\mathbb{R}^n/\Gamma_1$. For any fixed $\delta > 0$, choose a smooth radial cut-off function $0 \le \eta \le 1$ in $\mathbb{R}^n/\Gamma_1$ such that $\eta \equiv 1$ on $B_\delta(0)/\Gamma_1$ and $\eta \equiv 0$ on $(\mathbb{R}^n - B_{2\delta}(0))/\Gamma_1$. Let $\tilde{u}_\varepsilon$ and $\tilde{\eta}$ be the lifting up function of u_ε and η in $\mathbb{R}^n$.

On M, take the normal coordinates $\{x^i\}$ centered at q_1. Let $\phi = \eta u_\varepsilon$ in $\{x^i\}$ in a neighborhood of q_1 and extended by zero to a smooth function on M. Let $\tilde{\phi} = \tilde{\eta}\tilde{u}_\varepsilon$ in $B_{2\delta}(0)$. Note that ϕ is supported in $B_{2\delta}(0)/\Gamma_1$, so

$$\begin{aligned} E(\phi) &= \int_M \Big(|\nabla\phi|^2 + \frac{n-2}{4(n-1)}R\phi^2\Big)dV_g \\ &= \int_{B_{2\delta(0)}/\Gamma_1} \Big(|\nabla\phi|^2 + \frac{n-2}{4(n-1)}R\phi^2\Big)dV_g \\ &= |\Gamma_1|^{-1} \cdot \int_{B_{2\delta}(0)} \Big(|\nabla\tilde{\phi}|^2 + \frac{n-2}{4(n-1)}\tilde{R}\tilde{\phi}^2\Big)dV_{\tilde{g}} \end{aligned} \tag{6.1.10}$$

By the proof of [26, Lemma 3.4], we have

$$E(\phi) \le |\Gamma_1|^{-1}(1 + C\delta)(Q(S^n) + C\varepsilon)\Big(\int_{B_{2\delta}(0)} \tilde{\phi}^{\frac{2n}{n-2}} dV_{\tilde{g}}\Big)^{\frac{n-2}{n}} \tag{6.1.11}$$

$$= |\Gamma_1|^{-\frac{2}{n}}(1 + C\delta)(Q(S^n) + C\varepsilon)\Big(\int_{B_{2\delta}(0)/\Gamma_1} \phi^{\frac{2n}{n-2}} dV_g\Big)^{\frac{n-2}{n}} \tag{6.1.12}$$

where C is some constant independent of δ and ε, $\tilde{R}$ and $\tilde{g}$ are scalar curvature and metric in the lifting up space.

Consider the power expansion of $|\Gamma_1|^{\frac{2}{n-2}}K(x)$ near the center point q_1, we have

$$B_K \le |\Gamma_1|^{\frac{2}{n-2}}K(x) + C|x| \tag{6.1.13}$$

for some constant C. Then

$$\int_{B_{2\delta}(0)/\Gamma_1} \phi^{\frac{2n}{n-2}} dV_g \tag{6.1.14}$$
$$= B_K^{-1}\int_{B_{2\delta}(0)/\Gamma_1} B_K \phi^{\frac{2n}{n-2}} dV_g \tag{6.1.15}$$
$$\le B_K^{-1}|\Gamma_1|^{\frac{2}{n-2}}\int_{B_{2\delta}(0)/\Gamma_1} K(x)\phi(x)^{\frac{2n}{n-2}} dV_g + CB_K^{-1}\int_{B_{2\delta}(0)/\Gamma_1} |x|u_\varepsilon(x)^{\frac{2n}{n-2}} dV_g \tag{6.1.16}$$

By fundamental calculus,

$$\int_{B_{2\delta}(0)/\Gamma_1} |x|u_\varepsilon^{\frac{2n}{n-2}} dx = \frac{Vol(S^{n-1})}{|\Gamma_1|}\varepsilon\int_0^{2\delta/\varepsilon}\Big(\frac{r}{1+r^2}\Big)^n dr \le C\varepsilon. \tag{6.1.17}$$

Combining (6.1.12), (6.1.16) and (6.1.17), we get

$$B_K^{\frac{n-2}{n}} E(\phi) \le (1+C\delta)(Q(S^n)+C\varepsilon)\Big[\Big(\int_{B_{2\delta}(0)/\Gamma_1} K\phi^{\frac{2n}{n-2}} dV_g\Big)^{\frac{n-2}{n}} + C\varepsilon\Big] \tag{6.1.18}$$

Take a rescaling

$$\bar{\phi} = \phi \cdot \Big(\int_{B_{2\delta}(0)/\Gamma_1} K\phi^{\frac{2n}{n-2}} dV_g\Big)^{-\frac{n-2}{2n}} \tag{6.1.19}$$

Then $\bar{\phi} \in \mathcal{C}_{\frac{n+2}{n-2},K}$ and

$$B_K^{\frac{n-2}{n}} E(\bar{\phi}) \le (1+C\delta)(Q(S^n)+C\varepsilon)(1+C\varepsilon) \tag{6.1.20}$$

By choosing δ and ε small, (6.1.7) is proved.

On the other hand, if B_K is obtained at some smooth point instead of a singular point, i.e. $K(x) = B_K$ for some smooth point $x \in M$, following the same proof as above without lifting everything upstairs, or in other words, with treating $\Gamma_1 = \{e\}$ to be the trivial group, it's not hard to see (6.1.7) holds as well.

We next show the strict inequality in (6.1.8) implies the existence. The proof is a modification of [1, Theorem 3.1].

If we've proved the case that M has only one singularity, the more general cases can be proved by induction on the number of singularity points. Hence we may assume M has only one singularity $\Sigma_\Gamma = \{(q, \Gamma)\}$. Let $X = M - \{q\}$.

First, note that

$$E(\frac{n+2}{n-2}, K) = \inf_{u \in C_c^\infty(X) \cap \mathcal{C}_{\frac{n+2}{n-2}, K}} E(u) \tag{6.1.21}$$

Let B_ρ be the open geodesic ball centered at q of radius ρ. Set

$$Y_k := \inf_{u \in C_c^\infty(X - \overline{B_{1/k}}) \cap \mathcal{C}_{\frac{n+2}{n-2}, K}} E(u) \tag{6.1.22}$$

for $i \in \mathbb{N}$. It follows

$$Y_k > Y_{k+1} > Y_{k+2} > \cdots \tag{6.1.23}$$

$$\lim_{k \to \infty} Y_k = \inf_{u \in C_c^\infty(X) \cap \mathcal{C}_{\frac{n+2}{n-2}, K}} E(u) = E(\frac{n+2}{n-2}, K) \tag{6.1.24}$$

By (6.1.8), there exists a large integer k_0 such that

$$(B_K)^{\frac{n-2}{n}} Y_k < \frac{n(n-2)}{4} Vol(S^n)^{2/n} \text{ for any } k \geq k_0. \tag{6.1.25}$$

Due to (6.1.6),

$$\Big(\max_{x \in X - \overline{B_{1/k}}} K(x) \Big)^{\frac{n-2}{n}} Y_k < \frac{n(n-2)}{4} Vol(S^n)^{2/n} \text{ for any } k \geq k_0. \tag{6.1.26}$$

Note that on the manifold with boundary $(N, \partial N) = (X - B_{1/k}, \partial B_{1/k})$, when we apply integration by parts to any function in $C_c^\infty(X - \overline{B_{1/k}})$, the boundary term always vanishes. As a result, the variational method used in [19, Proposition 1.1] can be applied here. Thus, (6.1.26) implies that for each $k \geq k_0$, there exists a non-negative $E_{(X-B_{1/k}, g)}$-minimizer $u_k \in C^\infty(X - B_{1/k}) \cap \mathcal{C}_{\frac{n+2}{n-2}, K}(X - B_{1/k})$ such that,

$$E_{(X-B_{1/k}, g)}(u_k) = Y_k, \quad \int_{X - B_{1/k}} K u_k^{\frac{2n}{n-2}} dVol_g = 1 \tag{6.1.27}$$

$$u_k = 0 \text{ on } \partial B_{1/k}, \quad u_k > 0 \text{ in } X - \overline{B_{1/k}}. \tag{6.1.28}$$

Denote the zero extension of each u_k to M by also the same symbol u_k.

Suppose the sequence $\{u_k\}$ has a uniform C^0-bound, i.e. there exists a constant $L > 0$ such that

$$\|u_k\|_{C^0(M)} \leq L \text{ for } k \geq k_0, \tag{6.1.29}$$

then there exists a non-negative $E_{(M,g)}$-minimizer $u \in W^{1,2}(M)$ with $||u||_{C^0(M)} \leq L$ such that

$$u_k \to u \text{ weakly in } W^{1,2}(M), \quad u_k \to u \text{ strongly in } L^2(M). \tag{6.1.30}$$

By (6.1.27), Lebesgue's bounded convergence theorem and the above uniform C^0-estimate, we have

$$\int_M K u^{\frac{2n}{n-2}} dVol_g = 1. \tag{6.1.31}$$

By this equation and the fact that $\{u_k\}$ is a $E_{(M,g)}$-minimizing sequence, we know

$$u_k \to u \text{ strongly in } W^{1,2}(M). \tag{6.1.32}$$

Under the C^0-bound assumption $\|u\|_{C^0(M)} \le L$, applying the standard elliptic estimate, we obtain that $u \in C^\infty(M)$. By [8], the maximum principle implies that $u > 0$ everywhere on M. Thus we get an orbifold metric $\tilde{g} = u^{4/(n-2)} \cdot g \in [g]_{orb}$ such that $\tilde{g}$ has K as its scalar curvature.

To complete the proof, it is sufficient to show a uniform C^0-estimate for the sequence $\{u_k\}$. For each k, take the absolute maximum point $x_k \in X$ of u_k and set $M_k := u_k(x_k)$. Taking a subsequence if necessary, there exists a point $\bar{x} \in M$ such that

$$\lim_{k\to\infty} x_k = \bar{x}. \tag{6.1.33}$$

Suppose that there isn't a uniform C^0-estimate for $\{u_k\}$, that is $\lim_{k\to\infty} M_k = \infty$. There will be two cases.

Case 1. $\bar{x} \neq q$: Similarly to Chapter 3, for each k, let $\{x^i\}$ be a normal coordinate system in a small ball $B_\sigma(x_k)$ centered at each x_k. Take change of variables $y = M_k^{\frac{2}{n-2}} x$ and define the rescaled function $v_k(y) = M_k^{-1} \cdot u_k(M_k^{-\frac{2}{n-2}} y)$. Since all u_k's satisfy equation (6.1.4) with $p = \frac{n+2}{n-2}$, by Proposition (3.3.3),

$$v_k \to v \text{ in } C^2_{loc}(\mathbb{R}^n) \tag{6.1.34}$$

where $v = U_{E(\frac{n+2}{n-2},K)K(\bar{x})}$ is the bubble on $\mathbb{R}^n$, as defined in (3.3.2).

Hence, v satisfies equation

$$-\Delta_0 v = E\Big(\frac{n+2}{n-2}, K\Big) K(\bar{x}) v^{\frac{n+2}{n-2}} \text{ on } \mathbb{R}^n, \tag{6.1.35}$$

where Δ_0 denotes the Euclidean Laplacian.

We know that

$$\int_{\mathbb{R}^n} K(\bar{x}) v(y)^{\frac{2n}{n-2}} dy = \lim_{r\to\infty} \int_{|y|\le r} K(\bar{x}) v(y)^{\frac{2n}{n-2}} dy. \tag{6.1.36}$$

And, for each $r > 0$, by (6.1.34) and change of variables, we have

$$\begin{aligned}
\int_{|y|\le r} K(\bar{x}) v(y)^{\frac{2n}{n-2}} dy &= \lim_{k\to\infty} \int_{|y|\le r} K(M_k^{-\frac{2}{n-2}} y) v_k(y)^{\frac{2n}{n-2}} dy \\
&= \lim_{k\to\infty} \int_{|x|\le M_k^{-\frac{2}{n-2}} r} K(x) u_k(x)^{\frac{2n}{n-2}} dx \\
&\le \lim_{k\to\infty} \int_M K u_k^{\frac{2n}{n-2}} dVol_g \\
&= 1.
\end{aligned} \tag{6.1.37}$$

It implies

$$\int_{\mathbb{R}^n} v^{\frac{2n}{n-2}} \le K(\bar{x})^{-1}. \tag{6.1.38}$$

Multiplying v with (6.1.35) and integrating by parts, we get

$$\int_{\mathbb{R}^n} |\nabla v|^2 = E\Big(\frac{n+2}{n-2}, K\Big) K(\bar{x}) \cdot \int_{\mathbb{R}^n} v^{\frac{2n}{n-2}} \tag{6.1.39}$$

By (6.1.38),

$$\frac{\int_{\mathbb{R}^n}|\nabla v|^2}{\left(\int_{\mathbb{R}^n} v^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}} = E\Big(\frac{n+2}{n-2}, K\Big)K(\bar{x})\cdot\Big(\int_{\mathbb{R}^n} v^{\frac{2n}{n-2}}\Big)^{\frac{2}{n}} \leq E\Big(\frac{n+2}{n-2}, K\Big)K(\bar{x})^{\frac{n-2}{n}}. \tag{6.1.40}$$

The left hand side of the above inequality is the Yamabe functional on $\mathbb{R}^n$, which is $\geq Q(S^n)$. Hence

$$E\Big(\frac{n+2}{n-2}, K\Big)K(\bar{x})^{\frac{n-2}{n}} \geq Q(S^n). \tag{6.1.41}$$

Note that $\bar{x}$ is a smooth point on M, so $K(\bar{x}) \leq B_K$, which leads to a contradiction against (6.1.8).

Case 2: $\bar{x} = q$: blow-up occurs at the singular point. In this case, we consider the universal cover of a small neighborhood around q. Let $\{\tilde{x}_k\}$ be a sequence of lifting-up points of $\{x_k\}$ in the same branch of the lifting-up space. Let $\tilde{u}_k$, $\tilde{K}$ denote the lifting-up function of u_k, $\tilde{K}$, respectively. In the lifting-up space, for each k, let $\{\tilde{x}^i\}$ be a normal coordinate system in a small ball $B_\sigma(\tilde{x}_k)$ centered at each $\tilde{x}_k$. Take change of variables $\tilde{y} = M_k^{\frac{2}{n-2}}\tilde{x}$ and define the rescaled function $\tilde{v}_k(\tilde{y}) = M_k^{-1}\cdot\tilde{u}_k(M_k^{-\frac{2}{n-2}}\tilde{y})$. Similar to Case 1, we know

$$\tilde{v}_k \to \tilde{v} \text{ in } C^2_{loc}(\mathbb{R}^n) \tag{6.1.42}$$

where $\tilde{v} = U_{E(\frac{n+2}{n-2},K)K(q)}$ is on $\mathbb{R}^n$ and $\tilde{v}$ satisfies equation

$$-\Delta_0\tilde{v} = E\Big(\frac{n+2}{n-2}, K\Big)K(q)\tilde{v}^{\frac{n+2}{n-2}} \text{ on } \mathbb{R}^n. \tag{6.1.43}$$

We know that

$$\int_{\mathbb{R}^n} K(q)\tilde{v}(y)^{\frac{2n}{n-2}}dy = \lim_{r\to\infty}\int_{|\tilde{y}|\le r} K(q)\tilde{v}(\tilde{y})^{\frac{2n}{n-2}}d\tilde{y}. \tag{6.1.44}$$

And, for each $r > 0$, by (6.1.42) and change of variables, we have

$$\begin{aligned}\int_{|\tilde{y}|\le r} K(q)\tilde{v}(\tilde{y})^{\frac{2n}{n-2}}d\tilde{y} =& \lim_{k\to\infty}\int_{|\tilde{y}|\le r}\tilde{K}(M_k^{-\frac{2}{n-2}}\tilde{y})\tilde{v}_k(\tilde{y})^{\frac{2n}{n-2}}d\tilde{y}\\ =& \lim_{k\to\infty}\int_{|\tilde{x}|\le M_k^{-\frac{2}{n-2}}r}\tilde{K}(\tilde{x})\tilde{u}_k(\tilde{x})^{\frac{2n}{n-2}}d\tilde{x}\\ \le& |\Gamma|\cdot\lim_{k\to\infty}\int_M K u_k^{\frac{2n}{n-2}}dVol_g\\ =& |\Gamma|.\end{aligned} \tag{6.1.45}$$

It implies

$$\int_{\mathbb{R}^n}\tilde{v}^{\frac{2n}{n-2}} \le |\Gamma|K(q)^{-1}. \tag{6.1.46}$$

Multiplying $\tilde{v}$ with (6.1.43) and integrating by parts, we get

$$\int_{\mathbb{R}^n}|\nabla\tilde{v}|^2 = E\Big(\frac{n+2}{n-2},K\Big)K(q)\cdot\int_{\mathbb{R}^n}\tilde{v}^{\frac{2n}{n-2}} \tag{6.1.47}$$

By (6.1.46),

$$\frac{\int_{\mathbb{R}^n}|\nabla\tilde{v}|^2}{(\int_{\mathbb{R}^n}\tilde{v}^{\frac{2n}{n-2}})^{\frac{n-2}{n}}} = E\Big(\frac{n+2}{n-2},K\Big)K(q)\cdot\Big(\int_{\mathbb{R}^n}\tilde{v}^{\frac{2n}{n-2}}\Big)^{\frac{2}{n}} \le E\Big(\frac{n+2}{n-2},K\Big)[|\Gamma|^{\frac{2}{n-2}}K(q)]^{\frac{n-2}{n}}. \tag{6.1.48}$$

The left hand side of the above inequality is the Yamabe functional on $\mathbb{R}^n$, which is $\ge Q(S^n)$. Hence

$$E\Big(\frac{n+2}{n-2},K\Big)[|\Gamma|^{\frac{2}{n-2}}K(q)]^{\frac{n-2}{n}} \ge Q(S^n). \tag{6.1.49}$$

Note that q is a singular point on M, so $|\Gamma|^{\frac{2}{n-2}}K(q) \leq B_K$. Again, it contradicts (6.1.8).

$\square$

6.2 Proof of Theorem 1.3.6

The proof will be based on our Theorem 6.1.1 and [34, Proposition 5.1].

Under the assumption of Theorem 1.3.6, let's simply denote the singular point (q_{i_0}, Γ_{i_0}) by (q, Γ). Take a conformal mapping $g_q = u_q^{\frac{4}{n-2}} g$ such that g_q is the conformal normal metric centered at point q. Consider the Green function with expansion

$$G_q = \frac{1}{4n(n-1)Vol(S^{n-1})}(r_q^{2-n} + H_q) \tag{6.2.1}$$

where r_q is the geodesic distance from q based on metric g_q and H_q is the higher order term. Take a family of test functions

$$\varphi_{q,\lambda} = u_q \cdot \left(\frac{\lambda}{1 + \lambda^2 (r_q^{2-n} + H_q)^{\frac{2}{2-n}}} \right)^{\frac{n-2}{2}} \tag{6.2.2}$$

where $\lambda > 0$.

Similarly to [34], for any $u \in C^2(M, g)$, define the functional

$$J(u) = \frac{\int_M (\frac{4(n-1)}{n-2} |\nabla u|^2 + R_g u^2 d) Vol_g}{(\int_M K u^{\frac{2n}{n-2}} dVol_g)^{\frac{n-2}{n}}}. \tag{6.2.3}$$

By [34, Proposition 5.1], we have the estimation

$$J(\varphi_{q,\lambda}) = \frac{\hat{c}_0}{|\Gamma|^{\frac{2}{n}} K(q)^{\frac{n-2}{n}}} \left(1 - \hat{c}_2 \frac{\Delta K(q)}{K(q)\lambda^2} - \hat{d}_1 \begin{pmatrix} \frac{H_q + O(\frac{\log\lambda}{\lambda^2})}{\lambda^2} & \text{for } n = 4 \\ \frac{H_q}{\lambda^3} & \text{for } n = 5 \\ \frac{W_q \log\lambda}{\lambda^4} & \text{for } n = 6 \\ 0 & \text{for } n \geq 7 \end{pmatrix}\right) \tag{6.2.4}$$

with positive constants $\hat{c}_0$, $\hat{c}_2$ and $\hat{d}_1$ up to error $O(1/\lambda^4)$ for large λ. Here we get an extra term $|\Gamma|^{\frac{2}{n}}$ in the leading factor of the above estimation, which comes from estimating integrals in functional J by lifting up to the universal cover near the orbifold point q.

According to [34], the constants[1] are

$$\begin{aligned} \hat{c}_0 &= 4n(n-1)\left(\int_{\mathbb{R}^n} \frac{dx}{(1+r^2)^n}\right)^{\frac{2}{n}}, \\ \hat{c}_2 &= \frac{\int_{\mathbb{R}^n} \frac{r^2 dx}{(1+r^2)^n}}{2n \int_{\mathbb{R}^n} \frac{dx}{(1+r^2)^n}}, \\ \hat{d}_1 &= \frac{2n \int_{\mathbb{R}^n} \frac{r^n dx}{(1+r^2)^{n+1}}}{(n-2) \int_{\mathbb{R}^n} \frac{dx}{(1+r^2)^n}}. \end{aligned} \tag{6.2.5}$$

Especially, in dimension 4,

$$\hat{c}_2 = \frac{1}{4} \quad \text{and} \quad \hat{d}_1 = 6. \tag{6.2.6}$$

Then, it's clear that under assumption

$$\begin{cases} H_q + \frac{\Delta_g K(q)}{24K(q)} > 0, & \text{for } n = 4, \\ \Delta_g K(q) > 0, & \text{for } n \geq 5, \end{cases} \tag{6.2.7}$$

[1] The constant factor $\frac{2n}{n-2}$ was missing when the authors computed $\hat{d}_1$ in [34, Proposition 5.1]. We've corrected the mistake here.

by choosing sufficiently large λ, we can get

$$J(\varphi_{q,\lambda}) < \frac{\hat{c}_0}{|\Gamma|^{\frac{2}{n}} K(q)^{\frac{n-2}{n}}}. \tag{6.2.8}$$

Because the functional $J(u)$ is invariant under scalar multiple of u, we may assume $\varphi_{q,\lambda}$ is already normalized such that $\varphi_{q,\lambda} \in \mathcal{C}_{\frac{n+2}{n-2},K}(M)$, in which case

$$E(\varphi_{q,\lambda}) = \frac{4(n-1)}{n-2} J(\varphi_{q,\lambda}) < \frac{\frac{4(n-1)}{n-2}\hat{c}_0}{|\Gamma|^{\frac{2}{n}} K(q)^{\frac{n-2}{n}}}, \tag{6.2.9}$$

where the funcional E and the function class $\mathcal{C}_{p,K}(M)$ are defined in (6.1.1) and (6.1.2).

Also note that the constant $\frac{4(n-1)}{n-2}\hat{c}_0 = Q(S^n)$ is the Sobolev quotient of S^n. Hence the above inequality implies

$$Q(S^n) > E(\varphi_{q,\lambda})|\Gamma|^{\frac{2}{n}} K(q)^{\frac{n-2}{n}} \geq E\Big(\frac{n+2}{n-2}, K\Big) \cdot [|\Gamma|^{\frac{2}{n-2}} K(q)]^{\frac{n-2}{n}} \tag{6.2.10}$$

By the assumption of theorem 1.3.6, $|\Gamma|^{\frac{2}{n-2}} K(q) = B_K$, hence

$$(B_K)^{\frac{n-2}{n}} E\Big(\frac{n+2}{n-2}, K\Big) < Q(S^n). \tag{6.2.11}$$

By Theorem 6.1.1, there exists a solution of equation (1.1.3). $\square$

Chapter 7

On LeBrun's metrics

7.1 Radial symmetric case analysis

Recall from Section 1, for any $n \in \mathbb{N}^*$, on $\mathcal{O}_{\mathbb{P}^1}(-n)$ we have the LeBrun metric

$$g_{LEB(n)} = \frac{1+\hat{r}^2}{n+\hat{r}^2} d\hat{r}^2 + (1+\hat{r}^2)\sigma_1^2 + (1+\hat{r}^2)\sigma_2^2 + \frac{\hat{r}^2(n+\hat{r}^2)}{1+\hat{r}^2}\sigma_3^2, \tag{7.1.1}$$

where $\mathcal{O}_{\mathbb{P}^1}(-n)$ is the complex line bundle over $\mathbb{CP}^1$, which is a 4-dimensional manfiold. And n denotes the first Chern class of the bundle, but not the dimension.

Its mass is computed to be

$$m(g_{LEB(n)}) = -2(n-2). \tag{7.1.2}$$

Then we take the conformal compactification

$$\check{g}_{LEB(n)} = \frac{1}{(n+\hat{r}^2)^2} \cdot g_{LEB(n)} \tag{7.1.3}$$

such that $(\check{\mathcal{O}}_{\mathbb{P}^1}(-n), \check{g}_{LEB(n)})$ is a compact orbifold with singular point $\check{q}$ at $\hat{r} = \infty$ with quotient group $\Gamma_n = \mathbb{Z}/n\mathbb{Z}$. Its scalar curvature is computed to be

$$R_{\check{g}_{LEB(n)}} = \frac{24n(n + \hat{r}^2)}{1 + \hat{r}^2} > 0. \tag{7.1.4}$$

Define

$$s = 1/\hat{r} \tag{7.1.5}$$

to be the inverted radial coordinate centered at the orbifold point $\check{q}$. Then

$$\check{g}_{LEB(n)} = \frac{1 + s^2}{(1 + ns^2)^3} ds^2 + \frac{s^2(s^2 + 1)}{(ns^2 + 1)^2} \Big[\sigma_1^2 + \sigma_2^2 + \frac{(1 + ns^2)}{(1 + s^2)^2} \sigma_3^2\Big] \tag{7.1.6}$$

$$= (1 + O(s^2)) \cdot [ds^2 + s^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)] \ \text{ for small } s. \tag{7.1.7}$$

Observe that $g_{\mathbb{R}^4/\Gamma_n} = ds^2 + s^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$ is the standard metric for Euclidean cone. Hence we can choose normal coordinate $\{x^i\}$ centered at $\check{q}$ such that $s = |x|$. Then

$$g_{LEB(n)} = (n + \hat{r}^2)^2 \cdot \check{g}_{LEB(n)} = (s^{-2} + n)^2 \cdot \check{g}_{LEB(n)}. \tag{7.1.8}$$

It implies $s^{-2} + n$ is the Green function on $\check{g}_{LEB(n)}$ with leading term s^{-2}, and $g_{LEB(n)}$ is the corresponding Green function metric. Thus $g_{LEB(n)}$ is the conformal blow-up of $\check{g}_{LEB(n)}$ at the point $\check{q}$, as in Definition 1.1.8.

Moreover, let

$$\Delta_{\check{g}_{LEB(n)}} u - \frac{1}{6} R_{\check{g}_{LEB(n)}} u = -\frac{1}{6} K u^p \tag{7.1.9}$$

to be the variational equation to (1.4.8), where $1 < p \leq 3$.

Then we have the following.

Proposition 7.1.1. *Let* $\mathcal{X}, \mathcal{X}_{n,+}, \mathcal{X}_{n,0}, \mathcal{X}_{n,-}$ *be as defined in* (1.4.7) *and* (1.4.9).
On $(\check{\mathcal{O}}_{\mathbb{P}^1}(-n), \check{g}_{LEB(n)})$*, for any* $K \in \mathcal{X}$*, there exists some constant* C *depending only on* $\check{\mathcal{O}}_{\mathbb{P}^1}(-n), \check{g}_{LEB(n)}, \inf K$ *and* $\|K\|_{C^2}$ *such that if*

$$K \in \mathcal{X}_{n,+} \cup \mathcal{X}_{n,-}, \tag{7.1.10}$$

then

$$1/C \leq u \leq C \quad \text{and} \quad \|u\|_{C^{2,\alpha}} \leq C \tag{7.1.11}$$

for all $U(2)$*-invariant solution* $u \in \mathcal{X}$ *of* (7.1.9) *with* $p = 3$*, where* $0 < \alpha < 1$.

Moreover, if

$$K \in \mathcal{X}_{n,+}, \tag{7.1.12}$$

then (7.1.11) *holds for all* $1 < 1 + \varepsilon < p \leq 3$. *Consequently, in this case there exists a solution* u *of* (7.1.9) *with* $p = 3$.

Proof. Assume $\{u_k\} \subset \mathcal{X}$ is a family of U(2)-invariant solutions of (7.1.9) with corresponding exponent $p_k \to 3$ as $k \to \infty$.

Assume u_k blows up at a point $\bar{x}$ when $k \to \infty$. If $\bar{x}$ is in the set $\{s = s_0\}$ for some $s_0 > 0$, because u_k is U(2)-invariant, u_k blows up on the entire set $\{s = s_0\}$, which is either a hypersurface if $s_0 < \infty$ or the $\mathbb{CP}^1$ component if $s_0 = \infty$. However, either case is impossible because Proposition 5.1.3 implies blow-up points are isolated. Therefore, it is only possible for u_k to blow up at the orbifold point $\check{q}$.

Near $s = 0$, K has power expansion

$$K(s) = K(0) + \frac{1}{2}K''(0)s^2 + O(s^4) \tag{7.1.13}$$

By (7.1.7) and (4.3.27),

$$\Delta_{\check{g}_{LEB(n)}} K(0) = \Delta_{g_{\mathbb{R}^4/\Gamma_n}} K(0) = K''(0) + \frac{3}{s} K'(s)\Big|_{s=0} = 4K''(0). \tag{7.1.14}$$

It's clear that

$$\begin{aligned} K \in \mathcal{X}_{n,+} &\Leftrightarrow m(g_{LEB(n)}) > -\frac{\Delta_{\check{g}_{LEB(n)}} K(\check{q})}{2K(\check{q})} \\ K \in \mathcal{X}_{n,0} &\Leftrightarrow m(g_{LEB(n)}) = -\frac{\Delta_{\check{g}_{LEB(n)}} K(\check{q})}{2K(\check{q})} \\ K \in \mathcal{X}_{n,-} &\Leftrightarrow m(g_{LEB(n)}) < -\frac{\Delta_{\check{g}_{LEB(n)}} K(\check{q})}{2K(\check{q})} \end{aligned} \tag{7.1.15}$$

By the proof and statement of Theorem 1.3.1, this proposition is proved. □

Now, for each n, let's define another radial coordinate

$$t(\hat{r}) = \log\left(\frac{n + \hat{r}^2}{\hat{r}^2}\right), \quad \Rightarrow \quad \hat{r}(t) = \sqrt{\frac{n}{e^t - 1}} \tag{7.1.16}$$

Hence

$$t(s) = \log(ns^2 + 1) \quad \text{and} \quad s(t) = \sqrt{\frac{e^t - 1}{n}}. \tag{7.1.17}$$

Note that we've defined three radial coordinates $\hat{r}, s, t \in [0, \infty]$. It's not hard to see that $1/\hat{r}$, s and t are monotonic to each other.

By basic computation, on the ALE manifold $\mathcal{O}_{\mathbb{P}^1}(-n)$, we have

$$\Delta_{g_{LEB(n)}} = \frac{n + 3\hat{r}^2}{\hat{r}(1 + \hat{r}^2)} \frac{\partial}{\partial \hat{r}} + \frac{n + \hat{r}^2}{1 + \hat{r}^2} \frac{\partial^2}{\partial \hat{r}^2} \tag{7.1.18}$$

$$= \frac{4(1 - e^{-t})^3}{e^{-t}(1 + (n-1)e^{-t})} \frac{\partial^2}{\partial t^2}. \tag{7.1.19}$$

Note the case that $u(s)$ satisfies equation (1.4.8) is equivalent to that $u^2 g_{\breve{g}_{LEB(n)}}$ has scalar curvature $K(s)$, which is equivalent to that $v^2 g_{g_{LEB(n)}}$ has scalar curvature $K(s)$ where

$$v(s) = \frac{u(s)}{n + 1/s^2}, \tag{7.1.20}$$

Furthermore, it is equivalent to

$$\Delta_{g_{LEB(n)}} v = -\frac{1}{6} K v^3 \tag{7.1.21}$$

since $g_{LEB(n)}$ is scalar-flat.

Define a function space

$$\mathcal{Y} = \left\{ \frac{f(s)}{n + 1/s^2} : \forall f(s) \in \mathcal{X} \right\}. \tag{7.1.22}$$

Thus there is an obvious bijection between the solution set of equation (1.4.8) in $\mathcal{X}$ and the solution set of equation (7.1.21) in $\mathcal{Y}$.

Proposition 7.1.2. *On* $(\mathcal{O}_{\mathbb{P}^1}(-2), g_{LEB(2)})$, *assume* $K(s) \in \mathcal{X}$ *satisfies*

$$K'(s) \le 0 \quad \forall s \ge 0, \tag{7.1.23}$$

then there is no solution in $\mathcal{Y}$ *for equation* (7.1.21) *with* $n = 2$.

Proof. Suppose $v(s) \in \mathcal{Y}$ solves equation (7.1.21) with $n = 2$, which implies $\breve{g} = v^2 g_{LEB(2)}$ is a compact orbifold metric with scalar curvature $K(s)$ and its orbifold point $\breve{q}$ is at $s = 0$. Denote the orbifold of compactification by $\breve{M}$.

Define

$$\hat{v}(\hat{r}) = v(1/\hat{r}), \quad \hat{K}(\hat{r}) = K(1/\hat{r}). \tag{7.1.24}$$

Then in $\hat{r}$-coordinate,

$$\frac{n+3\hat{r}^2}{\hat{r}(1+\hat{r}^2)}\hat{v}'(\hat{r})+\frac{n+\hat{r}^2}{1+\hat{r}^2}\hat{v}''(\hat{r})=-\frac{1}{6}\hat{K}(\hat{r})\hat{v}(\hat{r})^3. \tag{7.1.25}$$

Denote the left hand side of the above equation by LHS, note that

$$\text{LHS}\leq 0 \quad \text{and} \quad \text{LHS}=\frac{[(n\hat{r}+\hat{r}^3)\hat{v}'(\hat{r})]'}{\hat{r}(1+\hat{r}^2)}, \tag{7.1.26}$$

It follows

$$[(n\hat{r}+\hat{r}^3)\hat{v}'(\hat{r})]'\leq 0 \tag{7.1.27}$$

which implies

$$(n\hat{r}+\hat{r}^3)\hat{v}'(\hat{r})\leq \hat{v}'(0). \tag{7.1.28}$$

Since v is smooth on the whole manifold, $\hat{v}'(0)=0$. Thus

$$\hat{u}'(\hat{r})\leq 0, \quad \forall \hat{r}\geq 0. \tag{7.1.29}$$

It implies

$$v'(s)\geq 0, \quad \forall s\geq 0. \tag{7.1.30}$$

Similar to the proof of [50, Theorem 1.3], Let E denote the traceless Ricci tensor, by the transformation formula and the fact that $g_{LEB(2)}$ is Ricci-flat, we have

$$E_{\breve{g}}=v^{-1}(-2\nabla^2 v+(\Delta v/2)\breve{g}) \tag{7.1.31}$$

where ∇ and Δ are respect to metric $\check{g}$ in the above equation and the following context in this proof.

Using the argument of [41], integrating on $\check{M}$, we get

$$\int_{\check{M}} v|E_{\check{g}}|^2 dVol_{\check{g}} = \int_{\check{M}} vE^{ij}_{\check{g}}\Big\{v^{-1}(-2\nabla^2 v + (\Delta v/2)\check{g})_{ij}\Big\}dVol_{\check{g}} \tag{7.1.32}$$

$$= -2\int_{\check{M}} E^{ij}_{\check{g}}(\nabla^2 v)_{ij} dVol_{\check{g}} \tag{7.1.33}$$

$$= -2\lim_{\varepsilon\to 0}\int_{\check{M}\setminus B_\varepsilon(\check{P})} E^{ij}_{\check{g}}(\nabla^2 v)_{ij} dVol_{\check{g}} \tag{7.1.34}$$

where $B_\varepsilon(\check{q})$ denotes the geodesic ball $\{q \in \check{M}, s(q) \leq \varepsilon\}$.

Integrating by parts again, we get

$$\int_{\check{M}} v|E_{\check{g}}|^2 dVol_{\check{g}} = -2\lim_{\varepsilon\to 0}\Big(\int_{\partial B_\varepsilon(\check{P})} E^{ij}_{\check{g}}(\nabla v)_i \nu_j d\sigma_{\check{g}} - \int_{\check{M}\setminus B_\varepsilon(\check{P})} \nabla_j E^{ij}_{\check{g}} \cdot (\nabla v)_i dVol_{\check{g}}\Big) \tag{7.1.35}$$

By (7.1.20), it's clear that $v(s) = O(s^2)$ near $s = 0$. Also because the curvature and volume term for $\check{g}$ is bounded near $s = 0$, the first integral term on the right hand side limits to zero as $\varepsilon \to 0$.

For the second integral term on the right hand side, since $v(s)$ is radial symmetric, the only nonvanishing component of $(\nabla v)_i$ is $(\nabla v)_s = v'(s)$ along the $\partial/\partial s$ direction. On the other hand, by the Bianchi identity,

$$\nabla_j E^{ij} = \nabla_j(Ric^{ij} - \frac{1}{4}R\cdot g^{ij}) = \frac{1}{2}\nabla^i R - \frac{1}{4}\nabla^i R = \frac{1}{4}\nabla^i R \tag{7.1.36}$$

for any metric g and its Ricci tensor Ric and scalar curvature R.

Since $K(s)$ is radial symmetric, the only nonvanishing component of $(\nabla K)^i$ is $(\nabla K)^s = K'(s)$ along the ds direction. Hence

$$\nabla_j E_{\breve{g}}^{ij} = (\nabla K)^s = K'(s). \tag{7.1.37}$$

It follows

$$\int_{\breve{M}} v|E_{\breve{g}}|^2 dVol_{\breve{g}} = \lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty} C(s)K'(s)v'(s)ds \tag{7.1.38}$$

where $C(s)$ is a positive function in s, depending on the volume term of $\breve{g}$ at each s.

By (7.1.23) and (7.1.30), we know

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty} C(s)K'(s)v'(s)ds \leq 0, \tag{7.1.39}$$

hence

$$\int_{\breve{M}} v|E_{\breve{g}}|^2 dVol_{\breve{g}} \leq 0. \tag{7.1.40}$$

It follows $E_{\breve{g}} = 0$. By (7.1.37), we know $K'(s) = 0$, so that $K = constant$. By [50, Theorem 1.3], such solution v does not exist. □

By playing with the t-coordinate between different n's, we can get the following.

Proposition 7.1.3. *Given any* $n_1, n_2 \in \mathbb{N}^*$, *on* $(\mathcal{O}_{\mathbb{P}^1}(-n_1), g_{LEB(n_1)})$, *for some* $K_1(s) \in \mathcal{X}$, *if there exists* $v_1(s) \in \mathcal{Y}$ *such that*

$$\Delta_{g_{LEB(n_1)}} v_1 = -\frac{1}{6}K_1 v_1^3, \tag{7.1.41}$$

then on $(\mathcal{O}_{\mathbb{P}^1}(-n_2), g_{LEB(n_2)})$*, there exists* $v_2(s) \in \mathcal{Y}$ *such that*

$$\Delta_{g_{LEB(n_2)}} v_2 = -\frac{1}{6} K_2 v_2^3 \tag{7.1.42}$$

where

$$v_2(s) = v_1\Big(\sqrt{\frac{n_2}{n_1}} s\Big) \quad \text{and} \quad K_2(s) = \frac{n_1 + n_2 s^2}{n_2 + n_2 s^2} K_1\Big(\sqrt{\frac{n_2}{n_1}} s\Big). \tag{7.1.43}$$

Proof. We will prove it using the view of t-coordinate.

For $v_1(s), K_1(s)$ as given above, define

$$\bar{v}_1(t) = v_1\Big(\sqrt{\frac{e^t - 1}{\alpha}}\Big), \quad \bar{K}_1(t) = K_1\Big(\sqrt{\frac{e^t - 1}{\alpha}}\Big) \tag{7.1.44}$$

By (7.1.17) and (7.1.19) with n replaced by n_1, we know (7.1.41) is equivalent to the following equation

$$\bar{v}_1''(t) = -\bar{K}_1(t) \cdot \frac{e^{-t}(1 + (n_1 - 1)e^{-t})}{24(1 - e^{-t})^3} \bar{v}_1(t)^3. \tag{7.1.45}$$

It implies

$$\bar{v}_1''(t) = -\frac{(1 + (n_1 - 1)e^{-t})\bar{K}_1(t)}{1 + (n_2 - 1)e^{-t}} \cdot \frac{e^{-t}(1 + (n_2 - 1)e^{-t})}{24(1 - e^{-t})^3} \bar{v}_1(t)^3. \tag{7.1.46}$$

Define

$$\bar{K}_2(t) = \frac{(1 + (n_1 - 1)e^{-t})\bar{K}_1(t)}{1 + (n_2 - 1)e^{-t}}, \tag{7.1.47}$$

then (7.1.46) becomes

$$\bar{v}_1''(t) = -\bar{K}_2(t) \cdot \frac{e^{-t}(1 + (n_2 - 1)e^{-t})}{24(1 - e^{-t})^3} \bar{v}_1(t)^3. \tag{7.1.48}$$

The above equation can be viewed as the conformal mapping equation on $(\mathcal{O}_{\mathbb{P}^1}(-n_2), g_{LEB(n_2)})$ in t-coordinate. More explicitly, define

$$v_2(s) = \bar{v}_1(\log(n_2 s^2 + 1)), \quad K_2(s) = \bar{K}_2(\log(n_2 s^2 + 1)). \tag{7.1.49}$$

By (7.1.17) and (7.1.19), it's not hard to see that (7.1.48) is equivalent to (7.1.42).

It remains to compute out $v_2(s)$ and $K_2(s)$ in terms of v_1 and K_1. By direct computation,

$$v_2(s) = \bar{v}_1(\log(n_2 s^2 + 1)) = v_1\Big(\sqrt{\frac{n_2}{n_1}} s\Big) \tag{7.1.50}$$

and

$$K_2(s) = \bar{K}_2(\log(n_2 s^2 + 1)) = \frac{n_1 + n_2 s^2}{n_2 + n_2 s^2} K_1\Big(\sqrt{\frac{n_2}{n_1}} s\Big). \tag{7.1.51}$$

□

Corollary 7.1.4. *Given any $n_1, n_2 \in \mathbb{N}^*$, on $(\mathcal{O}_{\mathbb{P}^1}(-n_1), g_{LEB(n_1)})$, for some $K_1(s) \in \mathcal{X}$, if there doesn't exist any $v_1(s) \in \mathcal{Y}$ such that*

$$\Delta_{g_{LEB(n_1)}} v_1 = -\frac{1}{6} K_1 v_1^3, \tag{7.1.52}$$

then on $(\mathcal{O}_{\mathbb{P}^1}(-n_2), g_{LEB(n_2)})$, there doesn't exist any $v_2(s) \in \mathcal{Y}$ such that

$$\Delta_{g_{LEB(n_2)}} v_2 = -\frac{1}{6} K_2 v_2^3 \tag{7.1.53}$$

where

$$K_2(s) = \frac{n_1 + n_2 s^2}{n_2 + n_2 s^2} K_1\Big(\sqrt{\frac{n_2}{n_1}} s\Big). \tag{7.1.54}$$

Proof. This corollary can be proved by switching n_1 and n_2 in the contrapositive statement of Proposition 7.1.3. □

Corollary 7.1.5. *Using the setting and notations in Proposition 7.1.3, we have*

$$K_1(s) \in \mathcal{X}_{n_1,+} \;\Leftrightarrow\; K_2(s) \in \mathcal{X}_{n_2,+}, \tag{7.1.55}$$

$$K_1(s) \in \mathcal{X}_{n_1,0} \;\Leftrightarrow\; K_2(s) \in \mathcal{X}_{n_2,0}, \tag{7.1.56}$$

$$K_1(s) \in \mathcal{X}_{n_1,-} \;\Leftrightarrow\; K_2(s) \in \mathcal{X}_{n_2,-}. \tag{7.1.57}$$

Proof. Assume $K_1(s) \in \mathcal{X}_{n_1,+}$, i.e.

$$\frac{K_1''(s)}{K_1(s)}\Big|_{s=0} > n_1 - 2. \tag{7.1.58}$$

Then, by direction computation,

$$\begin{aligned}
\frac{K_2''(s)}{K_2(s)}\Big|_{s=0} &= \frac{\frac{d^2}{ds^2}\left(\frac{n_1+n_2s^2}{n_2+n_2s^2}K_1(\sqrt{\frac{n_2}{n_1}}s)\right)}{\frac{n_1+n_2s^2}{n_2+n_2s^2}K_1(\sqrt{\frac{n_2}{n_1}}s)}\Bigg|_{s=0} && (7.1.59)\\
&= \frac{2n_2}{n_1} - 2 + \frac{n_1}{n_2}\cdot\frac{K_1''(s)}{K_1(s)}\Big|_{s=0} && (7.1.60)\\
&> \frac{2n_2}{n_1} - 2 + \frac{n_2}{n_1}\cdot(n_1-2) && (7.1.61)\\
&= n_2 - 2. && (7.1.62)
\end{aligned}$$

Hence $K_2(s) \in \mathcal{X}_{n_2,+}$. All the other directions follow similarly. □

Using the above, we can now proof the following non-existence and uniqueness results.

Theorem 7.1.6. *For any $n \in \mathbb{N}^*$, on $(\check{\mathcal{O}}_{\mathbb{P}^1}(-n), \check{g}_{LEB(n)})$, we have the following conclusions.*

(1) There is no solution in $\mathcal{X}$ *for equation* (1.4.8) *with*

$$K = K_{n,-}(s) = \frac{2+ns^2}{n+ns^2} K_{2,-}(s), \tag{7.1.63}$$

where $K_{2,-}(s) \in \mathcal{X}$ *is any monotonically decreasing function in* s*. Moreover, we know*

$$K = K_{n,-}(s) \in \mathcal{X}_{n,-} \text{ unless } K_{2,-} \equiv constant. \tag{7.1.64}$$

In particular, for $n \geq 2$*, there is no radial symmetric solution for the Yamabe Problem* $K = constant$*.*

(2) $u \equiv 1$ *is the unique solution in* $\mathcal{X}$ *for* (1.4.8) *with*

$$K = K_{n,+}(s) = \frac{24n(ns^2+1)}{s^2+1}. \tag{7.1.65}$$

Moreover, $K_{n,+} \in \mathcal{X}_{n,+}$*.*

Remark 7.1.7. The nonexistence of a radial solution for $K = constant$ was previously claimed in [50, Section 5.2], but the argument there for $n > 2$ was based on an ODE argument using an incorrect formula for the LeBrun metric, so was not valid. The proof here corrects that oversight, and also has the advantage of proving nonexistence for a larger class of functions. The other results in [50], in particular for $n = 2$, are not affected by this.

Proof of Theorem 7.1.6. By Proposition 7.1.2, for $K_{2,-}(s) \in \mathcal{X}$ satisfying

$$K'_{2,-}(s) \leq 0 \quad \forall s \geq 0, \tag{7.1.66}$$

there is no solution in $\mathcal{Y}$ for equation

$$\Delta_{g_{LEB(2)}} v = -\frac{1}{6} K_{2,-} \cdot v^3, \tag{7.1.67}$$

Using Corollary 7.1.4 with $n_1 = 2$, $K_1 = K_{2,-}$ and $n_2 = n$, it implies that for any $n \in \mathbb{N}^*$, there is no solution in $\mathcal{Y}$ for equation

$$\Delta_{g_{LEB(n)}} v = -\frac{1}{6} K_{n,-} \cdot v^3, \tag{7.1.68}$$

where

$$K_{n,-}(s) = \frac{2 + ns^2}{n + ns^2} K_{2,-}(s). \tag{7.1.69}$$

As a consequence, there is no solution in $\mathcal{X}$ for equation (1.4.8) with $K = K_{n,-}$. Moreover, by Corollary 7.1.5, since $K_{2,-} \in \mathcal{X}_{2,-}$ unless $K_{2,-} = constant$, we get $K_{n,-} \in \mathcal{X}_{n,-}$ unless $K_{2,-} = constant$. This proves (1) in Theorem 7.1.6. In particular, by choosing

$$K_{2,-}(s) = \frac{n + ns^2}{2 + ns^2}, \tag{7.1.70}$$

it proves that for $n \geq 2$, there is no U(2)-invariant solution for the Yamabe Problem $K = constant$.

On the other hand, note that $(\mathcal{O}_{\mathbb{P}^1}(-1), g_{LEB(1)})$ is the Fubini-Study metric, where the Yamabe problem for compactification has a unique solution. In other words, there exists a unique solution in $v(s) \in \mathcal{Y}$ for equation

$$\Delta_{g_{LEB(2)}} v = -\frac{1}{6} C_0 \cdot v^3, \tag{7.1.71}$$

where C_0 is a constant. Furthermore, we can write out $v(s)$ explicitly

$$v(s) = \sqrt{\frac{24}{C_0} \frac{s^2}{s^2 + 1}}. \tag{7.1.72}$$

Using Proposition 7.1.3 with $n_1 = 1$, $K_1 = C_0$ and $n_2 = n$, it implies that for any $n \in \mathbb{N}^*$, there exists a unique solution in $v_n(s) \in \mathcal{Y}$ for equation

$$\Delta_{g_{LEB(n)}} v_n = -\frac{1}{6} K_{n,+} \cdot v_n^3, \tag{7.1.73}$$

where

$$v_n(s) = \sqrt{\frac{24}{C_0}} \frac{ns^2}{ns^2+1} \quad \text{and} \quad K_{n,+}(s) = \frac{1+ns^2}{n+ns^2} C_0. \tag{7.1.74}$$

As a consequence, there exists a unique solution $u_n(s) \in \mathcal{X}$ for equation

$$\Delta_{\breve{g}_{LEB(n)}} u_n = -\frac{1}{6} K_{n,+} \cdot u_n^3, \tag{7.1.75}$$

where

$$u_n(s) = \sqrt{\frac{24}{C_0}} n. \tag{7.1.76}$$

Note that this is a constant rescale of $\breve{g}_{LEB(n)}$ itself. Moreover, by Corollary 7.1.5, since $C_0 \in \mathcal{X}_{1,+}$, we get $K_{n,+} \in \mathcal{X}_{n,+}$. By choosing $C_0 = 24n^2$, it proves (2) in Theorem 7.1.6. □

7.2 Proof of Theorem 1.4.3

By Proposition 7.1.1, $\deg(F_{p,K}, \Omega_\Lambda, 0)$ is equal to a constant under either assumption (1) or (2) in Theorem 1.4.3. By the non-existence results in (1) of Theorem 7.1.6, we know that for any $n \in \mathbb{N}^*$,

$$\deg(F_{3,K_{n,-}}, \Omega_\Lambda, 0) = 0, \tag{7.2.1}$$

which proves (2) of Theorem 1.4.3. Part (1) of Theorem 1.4.3 is proved by a standard subcritical degree counting argument; see [45]. Alternatively, this can be shown using the uniqueness in part (2) of Theorem 7.1.6 (but this requires an extra argument to show that solution is non-degenerate, which we omit). □

7.3 Hyperbolic ansatz

In this last section, we will introduce the hyperbolic ansatz on LeBrun's metric and compute its Yamabe invariant in some S^1-invariant function space.

Denote the metric for 3-dimensional hyperbolic space $\mathcal{H}^3 = \mathbb{R}^2 \times \mathbb{R}_+$ by

$$h_{\mathcal{H}^3} = \frac{dx^2 + dy^2 + dz^2}{z^2} = \frac{dr^2 + r^2 d\tau^2 + dz^2}{z^2}, \tag{7.3.1}$$

where (r, τ) is the polar coordinate for $x, y-$plane.

The distance function from the point $(0, 0, z_0)$ is

$$\rho = \cosh^{-1}\Big(\frac{r^2 + z^2 + z_0^2}{2zz_0}\Big) \;\Rightarrow\; \coth\rho = \frac{r^2 + z^2 + z_0^2}{\sqrt{(r^2 + z^2 + z_0^2)^2 - 4z^2 z_0^2}}. \tag{7.3.2}$$

The fundamental solution for $\Delta_{\mathcal{H}^3}$ at the point $(0, 0, z_0)$ is

$$\Gamma_{(0,0,z_0)} = \frac{1}{2}(\coth\rho - 1) = \frac{1}{e^{2\rho} - 1}. \tag{7.3.3}$$

For any $n \in \mathbb{N}^*$, define

$$V_n = 1 + n \cdot \Gamma_{(0,0,z_0)} = 1 + \frac{n}{2}\Big(\frac{r^2 + z^2 + z_0^2}{\sqrt{(r^2 + z^2 + z_0^2)^2 - 4z^2 z_0^2}} - 1\Big). \tag{7.3.4}$$

By [25, Proposition 2], using the hyperbolic ansatz, LeBrun's metric can be viewed as a circular bundle over $\mathcal{H}^3$:

$$g_{LEB(n)} = z^2(V_n \cdot h_{\mathcal{H}^3} + V_n^{-1} \cdot \omega_n \odot \omega_n), \tag{7.3.5}$$

where the connection 1-form ω_n is given by $d\omega_n = *_{\mathcal{H}^3} dV_n$. Here $*$ is the Hodge star operator on $\mathcal{H}^3$.

To compute ω_n, note that V_n is a function only in r and z, we may assume

$$dV_n = V_{n,r}dr + V_{n,z}dz, \tag{7.3.6}$$

where $V_{n,r}$ and $V_{n,z}$ are the r, z partial derivatives of V_n.

Hence

$$dV_n \wedge *_{\mathcal{H}^3} dV_n = \langle dV_n, dV_n \rangle \frac{dr}{z} \wedge \frac{r d\tau}{z} \wedge \frac{dz}{z} \tag{7.3.7}$$

$$= \frac{r}{z}(V_{n,r}^2 + V_{n,z}^2) dr \wedge d\tau \wedge dz. \tag{7.3.8}$$

Then, it's not hard to verify

$$*_{\mathcal{H}^3} dV_n = \frac{r}{z}(-V_{n,r}dz + V_{n,z}dr) \wedge d\tau. \tag{7.3.9}$$

It implies that $\omega_n = d\theta + f_n(r,z)d\tau$ where $d\theta$ is the angular chart for fibers and the function f_n satisfies

$$f_{n,z} = -\frac{r}{z}V_{n,r}, \quad f_{n,r} = \frac{r}{z}V_{n,z}, \tag{7.3.10}$$

where $f_{n,r}$ and $f_{n,z}$ are the r, z partial derivatives of f_n.

By solving the differential equations (7.3.10), we get

$$f_n(r,z) = \frac{n}{2}\frac{r^2+z^2-z_0^2}{\sqrt{(r^2+z^2+z_0^2)^2-4z^2z_0^2}} + const. \tag{7.3.11}$$

To determine the constant here, observing that

$$\lim_{r\to 0}\frac{r^2+z^2-z_0^2}{\sqrt{(r^2+z^2+z_0^2)^2-4z^2z_0^2}} = \begin{cases} -1, & 0<z<z_0 \\ 1, & z>z_0 \end{cases} \tag{7.3.12}$$

Also, because $f_n(r,z)$ is the coefficient to $d\tau$, it requires

$$\lim_{r\to 0} f_n(r,z) = 0. \tag{7.3.13}$$

So, we may choose an open subset

$$U = \mathcal{H}^3 \setminus \{(0,0,z)|z>z_0\}, \tag{7.3.14}$$

and on U, we have

$$f_n(r,z) = \frac{n}{2}\Big(\frac{r^2+z^2-z_0^2}{\sqrt{(r^2+z^2+z_0^2)^2-4z^2z_0^2}} + 1\Big). \tag{7.3.15}$$

Now, we can write out LeBrun's metric explicitly.

$$\begin{aligned} g_{LEB(n)} =& z^2(V_n h_{\mathcal{H}^3} + V_n^{-1}\omega_n^2) \\ =& V_n dr^2 + V_n r^2 d\tau^2 + V_n dz^2 + z^2 V_n^{-1}(d\theta + f_n d\tau)^2 \\ =& V_n dr^2 + (V_n r^2 + z^2 V_n^{-1} f_n^2)d\tau^2 + V_n dz^2 + z^2 V_n^{-1} d\theta^2 \\ & + z^2 V_n^{-1} f_n (d\theta\otimes d\tau + d\tau\otimes d\theta). \end{aligned} \tag{7.3.16}$$

In coordinates chart (r, τ, z, θ),

$$(g_{LEB(n)})_{ij} = \begin{pmatrix} V_n & 0 & 0 & 0 \\ 0 & V_n r^2 + z^2 V_n^{-1} f_n^2 & 0 & z^2 V_n^{-1} f_n \\ 0 & 0 & V_n & 0 \\ 0 & z^2 V_n^{-1} f_n & 0 & z^2 V_n^{-1} \end{pmatrix} \tag{7.3.17}$$

By computing its determinant, we know the volume term is

$$dVol_{g_{LEB(n)}} = rzV_n dr \wedge d\tau \wedge dz \wedge d\theta \triangleq rzV_n \; dVol_0. \tag{7.3.18}$$

Moreover, its inverse matrix is

$$(g_{LEB(n)})^{ij} = \begin{pmatrix} V_n^{-1} & 0 & 0 & 0 \\ 0 & V_n^{-1} r^{-2} & 0 & -V_n^{-1} r^{-2} f_n \\ 0 & 0 & V_n^{-1} & 0 \\ 0 & -V_n^{-1} r^{-2} f_n & 0 & V_n z^{-2} + V_n^{-1} r^{-2} f_n^2 \end{pmatrix} \tag{7.3.19}$$

Similar to what we have done in Section 6, define the Yamabe functional on $(\mathcal{O}(-n), g_{LEB(n)})$:

$$Q_n(u) = \frac{\int_{\mathcal{O}(-n)} |\nabla u|^2 \; dVol_{g_{LEB(n)}}}{(\int_{\mathcal{O}(-n)} u^4 \; dVol_{g_{LEB(n)}})^{\frac{1}{2}}} \tag{7.3.20}$$

LeBrun's metric is scalar-flat, so there is no scalar curvature term in the Yamabe functional.

Based on the above computations, we may write the numerator and denominator as

$$\begin{aligned}
&\int_{\mathcal{O}(-n)} |\nabla u|^2 dVol_{g_{LEB(n)}} \\
&= \int g^{ij}\partial_i u \partial_j u \cdot rzV_n \, dVol_0 \\
&= \int \Big[rz(u_r^2 + r_z^2) + zr^{-1}u_\tau^2 + (V_n^2 rz^{-1} + f_n^2 zr^{-1})u_\theta^2 - 2f_n zr^{-1}u_\tau u_\theta\Big] dVol_0 \\
&= \int \Big[rz(u_r^2 + r_z^2) + V_n^2 rz^{-1}u_\theta^2 + zr^{-1}(f_n u_\theta - u_\tau)^2\Big] dVol_0
\end{aligned} \tag{7.3.21}$$

and

$$\Big(\int_{\mathcal{O}(-n)} u^4 \, dVol_{g_{LEB(n)}}\Big)^{1/2} = \Big(\int u^4 rzV_n \, dVol_0\Big)^{1/2} \tag{7.3.22}$$

Theorem 7.3.1. *For any $u \in C^2(\mathcal{O}_{\mathbb{P}^1}(-n))$, if u satisfies both the following conditions:*
1) The conformal mapping $u^2 g_{LEB(n)}$ compactifies $\mathcal{O}_{\mathbb{P}^1}(-n)$ to a compact orbifold with an orbifold point $(\check{q}, \mathbb{Z}/n\mathbb{Z})$ attached at infinity;
2) For each r, z, the function u is invariant either in τ or in θ;
then,

$$Q_n(u) > \frac{Q(S^4)}{\sqrt{n}}. \tag{7.3.23}$$

where $Q(S^4)$ is defined as in (6.1.5). *Furthermore, it implies there is no such u that solves the Yamabe equation and also minimizes the Yamabe energy.*

Proof. By (7.3.4), it's clear that $V_n > 0$ and for any $n > 2$,

$$V_2 \le V_n < \frac{n}{2} \cdot V_2. \tag{7.3.24}$$

By (7.3.15), for any $n > 2$, we know $f_n^2 \ge f_2^2$.

For a function u satisfying conditions of the theorem, if u is invariant in τ, then $u_\tau = 0$, hence

$$
\begin{aligned}
&rz(u_r^2 + r_z^2) + V_n^2 rz^{-1} u_\theta^2 + zr^{-1}(f_n u_\theta - u_\tau)^2 \\
=&rz(u_r^2 + r_z^2) + V_n^2 rz^{-1} u_\theta^2 + zr^{-1} f_n^2 u_\theta^2 \\
\geq&rz(u_r^2 + r_z^2) + V_2^2 rz^{-1} u_\theta^2 + zr^{-1} f_2^2 u_\theta^2 \\
=&rz(u_r^2 + r_z^2) + V_2^2 rz^{-1} u_\theta^2 + zr^{-1}(f_2 u_\theta - u_\tau)^2.
\end{aligned}
\tag{7.3.25}
$$

Or, if u is invariant in θ, then $u_\theta = 0$, we get

$$
\begin{aligned}
&rz(u_r^2 + r_z^2) + V_n^2 rz^{-1} u_\theta^2 + zr^{-1}(f_n u_\theta - u_\tau)^2 \\
=&rz(u_r^2 + r_z^2) + zr^{-1} u_\tau^2 \\
=&rz(u_r^2 + r_z^2) + V_2^2 rz^{-1} u_\theta^2 + zr^{-1}(f_2 u_\theta - u_\tau)^2.
\end{aligned}
\tag{7.3.26}
$$

In either case, it would imply the numerator of the Yamabe functional satisfies

$$
\int_{\mathcal{O}(-n)} |\nabla u|^2 dVol_{g_{LEB(n)}} \geq \int_{\mathcal{O}(-2)} |\nabla u|^2 dVol_{g_{LEB(2)}}. \tag{7.3.27}
$$

On the other hand, by (7.3.22), we have

$$
\begin{aligned}
\Big(\int_{\mathcal{O}(-n)} u^4 \ dVol_{g_{LEB(n)}}\Big)^{1/2} =&\Big(\int u^4 rzV_n \ dVol_0\Big)^{1/2} \\
<&\sqrt{\frac{n}{2}} \cdot \Big(\int u^4 rzV_2 \ dVol_0\Big)^{1/2} \\
=&\sqrt{\frac{n}{2}} \cdot \Big(\int_{\mathcal{O}(-2)} u^4 \ dVol_{g_{LEB(2)}}\Big)^{1/2}.
\end{aligned}
\tag{7.3.28}
$$

Overall, we get

$$
Q_n(u) > \sqrt{\frac{2}{n}} \cdot Q_2(u). \tag{7.3.29}
$$

From [50], we know that there is no Yamabe solution for $n = 2$. By either [1, Theorem 3.1] or our Theorem 6.1.1, it implies

$$\inf Q_2(u) \geq \frac{Q(S^4)}{\sqrt{2}} \tag{7.3.30}$$

where the inf is taken over all functions that compactify $\mathcal{O}(-2)$ to a compact orbifold.

Eventually, it shows

$$Q_n(u) > \frac{Q(S^4)}{\sqrt{n}}. \tag{7.3.31}$$

Moreover, by Theorem 6.1.1, we know that the minimal energy $\inf Q_n(u) \leq Q(S^4)/\sqrt{n}$, so there is no such u satisfying assumptions of the theorem, that solves the Yamabe equation and also minimizes the Yamabe energy. $\square$

Remark 7.3.2. Especially, the theorem holds if u depends only on $\mathcal{H}^3$. This gives us a hope to prove (7.3.23) without assuming condition 2) in the theorem, by using some tricks like the moving plane method on $\mathcal{H}^3$.

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