Title
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Journal
Algebraic & Geometric Topology, 17(1)

ISSN
1472-2739

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Publication Date
2017-01-26

DOI
10.2140/agt.2017.17.439

Peer reviewed

Comments, corrections, and related references welcomed, as always!

TEXed February 1, 2016

SIMPLICIAL COMPLEXES WITH LATTICE STRUCTURES

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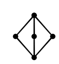
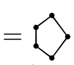
Abstract. If L is a finite lattice, we show that there is a natural topological lattice structure on the geometric realization of its order complex $\Delta(L)$ (definition recalled below). Lattice-theoretically, the resulting object is a subdirect product of copies of L. We note properties of this construction and of some variants thereof, and pose several questions. For M_3 the 5-element nondistributive modular lattice, $\Delta(M_3)$ is modular, but its underlying topological space does not admit a structure of distributive lattice, answering a question of Walter Taylor.

We also describe a construction of "stitching together" a family of lattices along a common chain, and note how $\Delta(M_3)$ can be obtained as a case of this construction.

1. A lattice structure on $\Delta(L)$

I came upon the construction studied here from a direction unrelated to the concept of order complex; so I will first motivate it in roughly the way I discovered it, then recall the order complex construction, which turns out to describe the topological structure of these lattices.

1.1. **The construction.** The motivation for this work comes from Walter Taylor's paper [17], which examines questions of which topological spaces – in particular, which finite-dimensional simplicial complexes – admit various sorts of algebraic structure, including structures of lattices. He asks in Question 9.4.7 whether there exist spaces which admit structures of lattice, but not of distributive lattice. (More precisely, he asks whether there are spaces admitting structures of modular lattice but not of distributive lattice, and whether there are spaces admitting structures of lattice but not of modular lattice. But the above simplified question is enough to motivate our construction.)

The most familiar examples of nondistributive lattices are finite lattices such as $M_3 =$  and $N_5 =$ , whose underlying sets, looked at as discrete spaces, certainly also admit structures of distributive lattice (e.g., they can be rearranged into chains). But we may ask whether, starting with such a finite lattice L, there is some way of building from it a more geometric sort of lattice, whose underlying topological space has a distinctive geometry which perhaps precludes a distributive lattice structure.

As a first attempt, one might identify each $x \in L$ with the $[0,1]$-valued function on L having value 1 at x and 0 elsewhere, and try to extend the lattice structure on L to convex linear combinations of these functions. However, there is no evident way of defining meets and joins of such linear combinations so that they extend the given operations on L, while continuing to satisfy the lattice identities.

But suppose, instead, that we identify each element x of our finite lattice L with the $[0,1]$-valued function having value 1 on the principal ideal $\downarrow x = \{y \in L \mid y \leq x\}$ generated by x, and 0 elsewhere. I claim that these functions belong to a family which does have a natural lattice structure: the space $F(L)$ of all $[0,1]$-valued functions f on L such that for each $t \in [0,1]$, the set

(1) $\qquad f_t \ = \ \{x \in L \mid f(x) \geq t\}$

is a principal ideal of L; and that the characteristic functions we started with form a sublattice isomorphic to L.

2010 *Mathematics Subject Classification.* Primary: 05E45, 06B30. Secondary: 06A07, 57Q99.

Key words and phrases. Order complex of a poset or lattice; topological lattice; distributive lattice; modular lattice; breadth of a lattice.

This preprint is readable online at `http://math.berkeley.edu/~gbergman/papers/` and `http://arxiv.org/abs/1602.00034`. The former version is likely to be updated more frequently than the latter.

Indeed, let us order functions f with the above property by pointwise comparison, writing $f \leq g$ if $f(x) \leq g(x)$ for all $x \in L$. A greatest lower bound of any $f, g \in F(L)$ is given by their pointwise infimum $f \wedge g$, since for each $t \in [0,1]$, the set $(f \wedge g)_t$ is $f_t \cap g_t$, an intersection of principal ideals, and hence again a principal ideal of L.

That f and g have a least upper bound can be seen in various ways. One the one hand, one can define $f \vee g$ to be the pointwise infimum of all upper bounds of f and g in $F(L)$, and note that for all $t \in [0,1]$ this makes $(f \vee g)_t$ the intersection of a set of principal ideals, which, since L is finite, is again a principal ideal. Hence $f \vee g \in F(L)$, and by construction it will be the least member of $F(L)$ majorizing both f and g. Alternatively, one can guess that the desired function $f \vee g$ on L will have the property that for each $t \in [0,1]$, the set $(f \vee g)_t$ is the join of f_t and g_t as ideals of L, and verify that this condition indeed determines an element $f \vee g \in F(L)$, and gives the desired least upper bound. Finally one can, for each $x \in L$, take $(f \vee g)(x)$ to be the largest $t \in [0,1]$ such that x is majorized by the join of an element y such that $f(y) \geq t$ and an element z such that $g(z) \geq t$, and again verify that the function $f \vee g$ so defined has the desired properties. Since least upper bounds are unique when they exist, all these constructions give the same operation on $F(L)$.

So $F(L)$, regarded as a partially ordered set under coordinatewise inequality, forms a lattice, and we see that the elements of $F(L)$ that are $\{0,1\}$-valued form a sublattice isomorphic to L. We shall see in §3.1 that when applied to M_3, this construction gives a modular topological lattice whose underlying topological space does not admit a distributive lattice structure, and so indeed answers one part of the question mentioned.

1.2. **The order-complex construction.** For $F(L)$ defined as above, any $f \in F(L)$ can be written as a *convex linear combination* of the characteristic functions of a *chain* of principal ideals, namely, the sets f_t ($t \in [0,1]$). The characteristic functions in such a chain are linearly independent, and the principal ideals in terms of which f is so expressed are unique if we require that each characteristic function appear with nonzero coefficient. For each such chain of n characteristic functions, the n-tuples of coefficients that can be applied to them to get a convex linear combination are the n-tuples of real numbers in $[0,1]$ which sum to 1, and these form, geometrically, an $n-1$-simplex. The observation about unique representations with nonzero coefficients shows that though a given point of $F(L)$ may lie in the simplices corresponding to more than one chain, it will lie in the interior of only one. Thus, $F(L)$ has the form of a simplicial complex, with simplices corresponding to the chains of principal ideals of L; equivalently, to the chains of elements of L.

The geometric construction just described is an instance of a standard concept, the *order complex* of a poset P. This is the simplicial complex $\Delta(P)$ having an n-simplex for each chain of $n+1$ elements in P, and such that the faces of the simplex determined by a chain C are the simplices determined by subchains of C.

The term "simplicial complex" is commonly used both for (i) an abstract object given by a set V ("vertices") and a family D of subsets of V ("simplices"; an $n+1$-element set being called an n-simplex), such that D is closed under passing to subsets (faces of simplices), and also for (ii) the *geometric realization* of such an abstract simplicial complex: a topological space in which the abstract simplices are replaced by subspaces – a point for each 0-simplex, a line segment for each 1-simplex, a triangle for each 2-simplex, and so forth – with the appropriate face-relations among them. The whole simplicial complex, in this sense, is their union. So what we have shown is that the *geometric realization* of the order complex of a lattice has a natural lattice structure.

In the literature, the geometric realization of an abstract simplicial complex D may either be denoted by a symbol such as $|D|$ or $||D||$, or by the same symbol D, allowing context to determine which is meant (cf. [19], sentence beginning at bottom of p. 2). Here we shall always understand simplicial complexes in the geometric sense, unless they are specified as "abstract". We will occasionally say "geometric realization" for emphasis.

From now on, in place of the notation $F(L)$ used above, I will use the standard notation, $\Delta(L)$.

The construction of the order complex of a partially ordered set P appears to have been introduced in 1937 by P. Alexandroff [1, §2.1], who called it the "barycentric subdivision" of P, because the barycentric subdivision, in the classical sense, of a simplicial complex D is given by the order complex of the partially ordered set of simplices of D, ordered by inclusion. (In [19, p. 7] the operation of barycentric subdivision on simplicial complexes is described, more precisely, as the composite of the construction taking a simplicial complex to the partially ordered set of its simplices, and the construction taking a partially ordered set to its order complex.) The concept seems to have been revived, or perhaps rediscovered, by Jon Folkman [10, next-to-last sentence before Theorem 3.1], and is now a standard tool [19], [20]. What is perhaps new in the approach of this note is the representation of the points of these complexes as convex linear combinations of characteristic functions of principal ideals. The set of functions that can be so described has a natural partial ordering by pointwise comparison, and when P is a lattice, the resulting poset has the lattice structure discussed above, which would otherwise be hard to see.

For L a finite lattice, let us define for each $t \in [0,1]$ the function $h_t : \Delta(L) \to L$, associating to each $f \in \Delta(L)$ the generator $h_t(f)$ of the ideal $f_t = \{y \in L \mid f(y) \geq t\}$. It follows from our description of the lattice operations on $\Delta(L)$ that each h_t is a lattice homomorphism; and clearly the family of these functions as t ranges over $[0,1]$ separates points of $\Delta(L)$.

In the next theorem we summarize the above observations, putting them initially in the context of a general finite poset P. We will not distinguish notationally between a poset or lattice and its underlying set; hence $[0,1]^P$ will mean the set of all $[0,1]$-valued functions on the set P. We will write "principal ideal" for "principal downset" in speaking about posets, so as to use the same language for posets and lattices. The reason why the condition that f_t be a principal ideal of P is imposed below only for $t \in (0,1]$, rather than for all $t \in [0,1]$ as above, will be noted in §1.3.

Theorem 1. *If P is a finite poset, then the geometric realization of its order complex, $\Delta(P)$, can be identified with the set of all functions $f \in [0,1]^P$ having the property that for each $t \in (0,1]$, the subset $f_t = \{x \in L \mid f(x) \geq t\}$ is a principal ideal of P. For such f and t, we shall write $h_t(f)$ for the element of P such that $f_t = \downarrow h_t(f)$. Regarding $\Delta(P)$, so constructed, as partially ordered by pointwise comparison, each map h_t is an isotone map $\Delta(P) \to P$, and this family of maps separates points of $\Delta(P)$.*

If the finite poset P has pairwise meets (greatest lower bounds) of elements, then so does $\Delta(P)$. These are given by pointwise infima of functions $P \to [0,1]$; equivalently, they may be described by the formula $(f \wedge g)_t = f_t \wedge g_t = f_t \cap g_t$ $(f, g \in \Delta(P),\ t \in (0,1])$. This operation $\wedge$ is continuous on $\Delta(P)$, and is respected by the maps h_t.

Similarly, if P has pairwise joins (least upper bounds) of elements, then so does $\Delta(P)$. These may be described by the formula $(f \vee g)_t = f_t \vee g_t$, where $f_t \vee g_t$ denotes the join as principal ideals, and the resulting operation $\vee$ on $\Delta(P)$ is, likewise, continuous and respected by the maps h_t. If in addition to pairwise joins, P has a join of the empty family, i.e., a least element 0, then the elements of $\Delta(P)$ can be described as those maps $f : P \to [0,1]$ which satisfy the identities $f(0) = 1$ and $f(x \vee y) = f(x) \wedge f(y)$ $(x, y \in P)$.

Hence if L is a finite lattice, then $\Delta(L)$ is a topological lattice, which as an abstract lattice is a subdirect product of copies of L. Again, its elements can be described as all maps $f : P \to [0,1]$ satisfying $f(0) = 1$ and $f(x \vee y) = f(x) \wedge f(y)$. □

Sketch of proof. Everything has been covered in the preceding remarks except the last sentence of the third paragraph (and its echo in the final paragraph). To see that $f(x \vee y) = f(x) \wedge f(y)$ holds when $f \in \Delta(P)$, let $s = f(x \vee y)$ and $t = f(x) \wedge f(y)$. Then the facts that f_s and f_t are principal downsets give the relations $\leq$ and $\geq$ respectively. The identity $f(0) = 1$, and the fact that these two identities imply membership in $\Delta(P)$, are straightforward. □

(One may ask: if P is a finite partially ordered set merely having meets, why can't we get joins $f \vee g$ in $\Delta(P)$ by one of the constructions described earlier, namely, as the pointwise infimum of all upper bounds of f and g in $\Delta(P)$? The answer is that if we don't assume P also has joins, the set of upper bounds of f and g in $\Delta(P)$ may be empty.)

Note that the maps h_t in the above theorem, since they send the connected space $\Delta(P)$ to the discrete space P, are necessarily discontinuous; hence the resulting map $\Delta(P) \to P^{(0,1]}$ defining our subdirect product structure is also discontinuous with respect to the natural topology on $P^{(0,1]}$. More on this in §5.2.

1.3. **Notes on 0 and 1.** In the construction of $\Delta(P)$ in Theorem 1, we required f_t to be a principal ideal for all $t \in (0,1]$, but not for $t = 0$. This is because every $[0,1]$-valued function f on P satisfies $f_0 = P$; hence if P has no greatest element, the set of functions $f : P \to [0,1]$ such that f_0 is a principal ideal is empty; so $\Delta(P)$ would be empty if we required its elements to have that property. On the other hand, for posets P *with* greatest element, such as lattices, the condition that $f_0 = P$ be a principal ideal is vacuous, so its omission makes no difference. If we were only considering lattices, it would be natural to word our condition as saying f_t is a principal ideal for all $t \in [0,1]$; but since we will be proving many of our results for general posets, we require this only for $t > 0$.

At the other end, notice that for $f \in \Delta(P)$, since f_1 is a principal ideal, it is nonempty. Thus, as P is finite, f must have the value 1 at some minimal element of P. If P has a least element 0_P – in particular, if it is lattice – this tells us that $f(0_P)$ is automatically 1. So in discussing the values of a function $f \in \Delta(L)$ for L a lattice, we may take for granted the condition $f(0_L) = 1$, and, for instance, describe f in terms of its values on $L - \{0_L\}$.

1.4. **Some conventions.** Let us make precise some terms we have already used.

Convention 2. *In this note, all lattices, semilattices and partially ordered sets will be assumed nonempty.*

This is mainly so that finite lattices are guaranteed to have a least element 0. (Alternatively, we could have supplemented the lattice operations with a zeroary join operation, giving 0 as the join of the empty family, alongside pairwise joins. Still another possibility would have been to omit Convention 2, and allow the empty lattice along with the others, noting, for instance, that $\Delta(\emptyset) = \emptyset$, since the unique $[0,1]$-valued function f on the empty set does *not* have the property that the sets f_s are principal ideals – because the empty lattice has no principal ideals. But I chose to avoid the distractions that that special case would have entailed.)

Convention 3. *In this note, a* topological lattice *means a Hausdorff topological space with a lattice structure under which the lattice operations are continuous.*

We remark that some natural-seeming topologies on lattices fail to satisfy these conditions. For instance, the lattice of vector subspaces of $\mathbb{R}^2$ may be regarded as composed of a circle, comprising the 1-dimensional subspaces, and two isolated points, the trivial subspace and the whole space. But the meet and join operations are discontinuous on the circle, since their outputs jump from the isolated points back to the circle whenever their two arguments fall together. We also remark that in the literature, a partially ordered set is sometimes given the non-Hausdorff "Alexandrov (or Alexandroff) topology" introduced in [1], in which the open subsets are the downsets. But we will not consider those topological spaces here.

We shall find it convenient to make a notational identification:

Convention 4. *If x is an element of a partially ordered set P, then the symbol $\downarrow x$, denoting the principal ideal of P generated by x, will also be used, in discussing $\Delta(P)$, for the characteristic function of that set.*

1.5. **Dependence among sections of this note.** All sections of this paper assume the material of §1. The remaining major sections, §§2-6, are largely independent of one another, with the following exceptions: §4 assumes §3; the construction of §2.2 is used in §§4.2-4.3 and in §5.1, and §5.1 also refers to the construction of §3.3.

Within each of the main sections, later subsections often depend on earlier ones.

2. Generalizations and variants of the construction $\Delta(-)$

Before studying the construction $\Delta(-)$, let us digress and record some ways in which it can be modified.

2.1. **Infinite L or P.** We have been assuming that our given lattice L is finite. If we allow infinite L, we have to decide, first, what topology to put on $[0,1]^L$. The product topology will not, in general, make joins continuous. For instance, suppose L consists of 0, 1, and infinitely many pairwise incomparable elements x_i $(i \in \mathbb{N})$ lying between them, so that for $i \neq j$ we have $x_i \vee x_j = 1$ and $x_i \wedge x_j = 0$. Then the sequence $\downarrow x_1$, $\downarrow x_2$, $\dots$ in $\Delta(L)$, regarded as a sequence of $[0,1]$-valued functions, approaches $\downarrow 0$ in the product topology on $[0,1]^L$; but $\lim_{i\to\infty}(\downarrow x_0 \vee \downarrow x_i) = \lim_{i\to\infty} \downarrow 1 = \downarrow 1$, which is not equal to $(\lim_{i\to\infty} \downarrow x_0) \vee (\lim_{i\to\infty} \downarrow x_i) = \downarrow x_0 \vee \downarrow 0 = \downarrow x_0$.

If we use instead the topology on $[0,1]^L$ given by the metric $d(f,g) = \sup_{x\in L} |f(x) - g(x)|$, which for finite L also gives the standard topology, the above problem goes away. Indeed, note that if $d(f,f') < \varepsilon$, then for all t we have $f_{t+\varepsilon} \subseteq f'_t \subseteq f_{t-\varepsilon}$ (where we replace $t+\varepsilon$ by 1 if $t+\varepsilon > 1$, and $t-\varepsilon$ by 0 if $t-\varepsilon < 0$). It is easy to deduce from this and the description of joins in Theorem 1 that whenever $d(f,f') < \varepsilon$ and $d(g,g') < \varepsilon$, we get $d(f \vee g, f' \vee g') < \varepsilon$, and similarly for meets.

Using this topology, if we assume that all chains in L have finite length, then the situation is quite similar to what we saw in §1. The set of functions f such that all the sets f_t are principal ideals will be a possibly infinite simplicial complex, but still composed of finite-dimensional *facets* (maximal simplices). Each simplex is still compact, though the whole space in general is not. (For an example of noncompactness, let L again be the lattice whose elements are 0, 1 and x_i $(i \in \mathbb{N})$. Then the elements $\downarrow x_i \in \Delta(L)$ have pairwise distance 1, hence can have no convergent subsequence in our metric. For another noncompact topology, see [20, p. 98, line 10].)

If L has infinite chains, there are more choices to be made. We can again look at the set of $f \in [0,1]^L$ which are finite convex linear combinations of characteristic functions of chains of principal ideals; or at the larger set of those f such that every f_t $(t \in (0,1])$ is a principal ideal; these can be described as the *possibly infinite* convex linear combinations of chains of such characteristic functions. The former space is still made up of finite-dimensional simplices, though such simplices will not in general be contained in maximal simplices; the latter space is less like a simplicial complex. In another direction, we can generalize principal ideals to nonempty ideals ($=$ directed unions of principal ideals). If we do so, we must again choose between using convex linear combinations of finite chains of such ideals, and of arbitrary chains.

Most of the above considerations apply not only to lattices, but also to $\vee$-semilattices, $\wedge$-semilattices, and general posets P. However, in the cases of $\wedge$-semilattices and posets, there is no evident reason to reject the topology induced by the product topology on $[0,1]^P$ in favor of the one based on the metric $\sup_{x\in P} |f(x) - g(x)|$. The meet operation is continuous under either topology, and the order-relation $\leq$, as a subset of $\Delta(P) \times \Delta(P)$, is closed in either topology.

2.2. **Constructions based on closure operators (etc.)** Returning for simplicity to finite lattices L, what if we abstract the underlying set of L, and its family of principal ideals, to a general finite set X, and its closed sets under an arbitrary closure operator cl on X?

If we write $\Delta(X, \mathrm{cl})$ for the set of functions $f : X \to [0,1]$ such that for each $t \in (0,1]$, the set $f_t = \{x \in X \mid f(x) \geq t\}$ is closed under cl, we see that $\Delta(X, \mathrm{cl})$ will again be a union of simplices, indexed now by the chains of cl-closed subsets of X; and that we can give it a lattice structure in the same way we did for $\Delta(L)$. In fact, if we write L_{cl} for the finite lattice of cl-closed subsets of X, the above simplicial-complex-with-lattice-structure will be isomorphic to $\Delta(L_{\mathrm{cl}})$. The isomorphism $\Delta(X, \mathrm{cl}) \to \Delta(L_{\mathrm{cl}})$ can be described as sending each $f \in \Delta(X, \mathrm{cl})$ to the function on L which takes each $A \in L_{\mathrm{cl}}$ to the minimum value of f on $A \subseteq X$, or to 1 if $A = \emptyset$; its inverse sends each $f \in \Delta(L_{\mathrm{cl}})$ to the function taking each $x \in X$ to the maximum value of f on those members of L_{cl} which (as subsets of X) contain x; equivalently, to $f(\mathrm{cl}(\{x\}))$. So the construction $\Delta(X, \mathrm{cl})$ does not give new topological lattices; but at times we will find it more convenient than the construction $\Delta(L)$.

The above construction extends to the situation where we are merely given a finite set X and a family P of subsets of X. The functions $f : X \to [0,1]$ for which all of the sets f_t belong to P forms a simplicial complex isomorphic to $\Delta(P)$, where P is regarded as ordered by inclusion. The isomorphism can be described as in the preceding paragraph (with $\Delta(P)$ for $\Delta(L_{\mathrm{cl}})$), except that, of the two equivalent descriptions of one direction of the isomorphism, the one using the sets $\mathrm{cl}(\{x\})$ must be dropped.

2.3. **Generalizing** $[0,1]$**.** In a different direction, suppose we replace the interval $[0,1]$ in our construction of $\Delta(L)$ with any *complete lattice* M. Thus, given a finite lattice L, let $\Delta_M(L)$ denote the set of all set-maps $f : L \to M$ such that for each $t \in M$, the set $f_t = \{x \in L \mid f(x) \geq t\}$ is a principal ideal of L. (Our use of $t \in (0,1]$ elsewhere in this note suggests that we say, "for each $t \in M - \{0_M\}$". But as noted, this makes no difference since L has a greatest element.) Partially ordering these functions f by pointwise comparison, we again see that meets of arbitrary families of elements can be constructed as pointwise meets. (This includes the meet of the empty family, which is the constant function with value everywhere $1_M \in M$.) It follows that every subset $S \subseteq \Delta_M(L)$ also has a join, namely the meet of all members of $\Delta_M(L)$ that majorize all members of S. However, when M is not a chain, we can no longer describe $f \vee g$ by the condition that $(f \vee g)_t$ be the join of f_t and g_t as ideals of L. Indeed, let L be a 2-element lattice $\{0_L, 1_L\}$, and f, g set-maps $L \to M$ whose values at 0_L are both 1_M, and whose values at 1_L are incomparable elements $p, q \in M$. We see that $f, g \in \Delta_M(L)$; and writing $t = p \vee q \in M$, we see that the value of $f \vee g$ at 1_L will be t, and deduce that $(f \vee g)_t = \downarrow 1_L$, though $f_t \vee g_t = \downarrow 0_L \vee \downarrow 0_L = \downarrow 0_L$.

Note that if we associate to every $f \in \Delta_M(L)$ the set $\overline{f} = \{(x, y) \in L \times M \mid y \leq f(x)\}$, which clearly determines it, then the sets $\overline{f} \subseteq L \times M$ that we get are characterizable by a pair of conditions symmetric in L and M; namely, they are those subsets such that for each $x \in L$, the set $\{y \in M \mid (x, y) \in \overline{f}\}$ is a principal ideal of M, and such that for each $y \in M$, the set $\{x \in L \mid (x, y) \in \overline{f}\}$ is a principal ideal of L. This is, in fact, a case of a known construction, the *tensor product* $L \otimes M$ of lattices [15]. So for L a finite lattice, the lattice $\Delta(L)$ that is the main subject of this note can be described as $L \otimes [0,1]$. If L and M are not both complete, their tensor product may fail to be a lattice (e.g., this happens if one is M_3, and the other is a free lattice on 3 generators), but in that situation, variants of the construction have been described which do give lattices [15].

The construction we have called $\Delta_M(L)$ can, of course, be generalized to allow M and/or L be semilattices or posets rather than lattices. In doing this, one must make choices on what to do when one or both of these does not have a greatest element (cf. §1.3).

2.4. **A more up-down symmetric construction.** Returning to the construction $\Delta(L)$, or more generally, $\Delta(P)$, note that our description of this complex in terms of $[0,1]$-valued functions is very asymmetric with respect to the order relation: $\Delta(P)$ consists of convex linear combinations of chains of characteristic functions of principal *ideals*, rather than principal *filters* (dual ideals). A consequence is that for L a lattice, though meets in $\Delta(L)$ are pointwise infima, joins have a less trivial description. In contrast, the standard description of the abstract order complex $\Delta(P)$ in terms of chains of elements of P is completely symmetric.

We could, of course, *reverse* the asymmetric feature of our construction of the spaces $\Delta(P)$ and $\Delta(L)$, and use convex linear combinations of characteristic functions of principal filters $\uparrow x$ instead of principal ideals $\downarrow x$. A minor difficulty is that larger elements generate smaller filters; so the resulting lattice would reverse the ordering of L. However, we could fix this by interchanging 0 and 1 in our characteristic functions; equivalently, by using characteristic functions of complements of principal filters. (The set of such complements has properties dual to those of a closure system: every subset of L *contains* a largest member of this set.)

In fact, our original construction, and the above dual approach, and the generalization in §2.2 where the lattice of principal ideals of L was replaced by the lattice of closed sets of any closure operator on a finite set, can be subsumed under one construction. The version of this construction for a general poset P was sketched in the last paragraph of §2.2, so below, we simply describe the lattice case.

Lemma 5. *Let X be a finite set, and L a family of subsets of X which, under the partial ordering by inclusion, forms a lattice; i.e., such that for every $a, b \in L$ there is a least member of L containing $a \cup b$, denoted $a \vee b$, and a greatest member of L contained in $a \cap b$, denoted $a \wedge b$.*

Let $\Delta(X, L)$ be the set of functions $X \to [0,1]$ such that for every $t \in (0,1]$, the set $f_t = \{y \in X \mid f(y) \geq t\}$ belongs to L.

Then $\Delta(X, L)$ is a simplicial complex with a lattice structure, isomorphic as such to the object $\Delta(L)$ constructed as in Theorem 1 from the abstract lattice L.

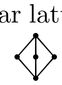
Sketch of proof. It is easy to see that every element of $\Delta(L)$ has a unique representation as a convex linear combination with nonzero coefficients of the characteristic functions of a chain of members of L, and that this gives an isomorphism as ordered simplicial complexes with $\Delta(L)$.

Let us describe explicitly the lattice operations of $\Delta(X, L)$. Given $f, g \in \Delta(L)$ the function $f \wedge g \in \Delta(L)$ can be determined by specifying the sets $(f \wedge g)_t$ for all t. If we let $(f \wedge g)_t \subseteq X$ be the meet $f_t \wedge g_t$ in L of the sets f_t, g_t, it is straightforward to verify that this gives an element $f \wedge g \in \Delta(X, L)$, which will be a greatest lower bound of f and g.

Since our hypotheses are up-down symmetric, the construction of $f \vee g$ is analogous (and is, in fact, the construction we used in Theorem 1). □

3. Two questions of Walter Taylor, an answer to one of them, and some approaches to the other

3.1. **Modular but not distributive.** As mentioned earlier, this paper was motivated by the study by Walter Taylor [17] of topological spaces admitting algebraic structures of various sorts. One question he asked [17, second part of Question 9.4.7] was whether there exist topological spaces – in particular, "nice" ones such as simplicial complexes – which admit structures of modular lattice, but not of distributive lattice.

Consider the topological lattice $\Delta(M_3)$, where M_3 is the lattice .

Since M_3 is modular but not distributive, the same is true of $\Delta(M_3)$. In earlier drafts of this note, I suggested that the underlying topological space of this lattice might not admit a distributive lattice structure, and so would answer the above question. Walter Taylor eventually discovered that the nonexistence of such a distributive lattice structure indeed follows on combining two results in the literature. With his permission, I will give this argument. We recall,

Definition 6. *A finite nonempty subset of a lattice L is said to be* meet-irredundant *if the meet of that set is not equal to the meet of any proper subset.* (*We shall regard the greatest element of L, it this exists, as the meet of the empty set. Thus a singleton $\{x\} \subseteq L$ is meet-irredundant if and only if x is not a greatest element of L.*)

The breadth $\mathrm{Br}(L)$ *of L is the supremum of the cardinalities of all meet-irredundant subsets, a natural number or $+\infty$.* (*By the above observation on singleton families, the 1-element lattice has breadth 0, while the lattices of breadth 1 are the chains with more than one element.*)

The above definition, as formulated, is very meet-join asymmetric; but it easily shown to be equivalent to its dual. (In [14, §1.7], a symmetric definition of the breadth of an arbitrary partially ordered set is given, which for lattices is equivalent to the above definition.)

The next result refers to the "inductive dimension" of a topological space. This is a topological invariant about which we only need to know that on simplicial complexes, it is equal to the usual dimension.

Proposition 7 (Walter Taylor, personal communication)**.** *Let X be a compact connected metrizable topological space of inductive dimension n, such that the set of points at which X has dimension n has nonempty interior. Then if X admits a structure of distributive topological lattice, X is embeddable, as a topological space, in $\mathbb{R}^n$.*

Hence the underlying topological space of $\Delta(M_3)$ does not admit a structure of distributive lattice.

Proof. For a distributive topological lattice L whose underlying space has the properties stated, Choe [6] shows that $\mathrm{Br}(L) = n$, while Baker and Stralka [3, Corollary 3.5] show that if $\mathrm{Br}(L) = n$, then L embeds homeomorphically in an n-cell, hence in $\mathbb{R}^n$. This gives the first assertion. (The results cited from [6] and [3] each have a somewhat more general hypothesis; the hypothesis stated above is roughly their intersection.)

Since the underlying space of $\Delta(M_3)$ consists of three triangles meeting at a common edge, it is 2-dimensional but not embeddable $\mathbb{R}^2$. Hence it does not admit a structure of topological distributive lattice. □

The object $\Delta(M_3)$ has made earlier appearances in the literature; see the picture on p. 4 of [13], and the paragraph preceding that picture.

Before Taylor supplied Proposition 7, my thought on how one might prove nonexistence of a distributive lattice structure on the underlying space of $\Delta(M_3)$ centered on the following question, to which I still do not know the answer.

Question 8. *If X is a connected finite simplicial complex having a subcomplex Y such that the subspace $X - Y$ has at least 3 connected components, each having all of Y in its closure, can X admit a structure of distributive topological lattice?*

We end this section with a result of the same nature as the one quoted from [6] in the proof of Proposition 7, but for not-necessarily-distributive lattices. We could not have used it in that proof, since it is restricted to lattices of the form $\Delta(L)$, and we were interested in arbitrary possible topological lattice structures on $\Delta(M_3)$. But the parallelism with Choe's result is interesting.

Lemma 9. *If L is a finite lattice, then the breadth of the lattice $\Delta(L)$ is equal to the supremum of the lengths of all chains in L* (*where by the length of a finite chain we understand one less than the number of elements in the chain*)*. In other words* (*in view of the structure of $\Delta(L)$*), $\mathrm{Br}(\Delta(L)) = \dim(\Delta(L))$.

Proof. For the easy direction, suppose that $0 = x_0 < \dots < x_n$ is a chain in L. Choose any real numbers $1 > r_1 > \dots > r_n \geq 0$, and for $i = 1, \dots, n$, let $f_i \in \Delta(L)$ be defined to have the value 1 on $\downarrow x_{i-1}$, and r_i elsewhere. Then we see that $\bigwedge f_i$ has value r_i at x_i; but if we omit any f_i from this meet, the result has the larger value r_{i-1} at x_i; so the meet is irredundant, so $\Delta(L)$ has breadth at least n.

For the converse, suppose $f_1, \dots, f_n \in \Delta(L)$ are distinct elements forming a meet-irredundant set, and let us show that L has a chain of length n.

For each i, we have $f_i \not\geq \bigwedge_{j \neq i} f_j$, so for each i there must exist $x_i \in L$ such that $f_i(x_i) < f_j(x_i)$ for all $j \neq i$. Let us choose such an x_i for each i, and reindex our elements, if necessary, so that

$$(2) \qquad f_1(x_1) \;\leq\; \dots \;\leq\; f_n(x_n).$$

Then I claim that in L,

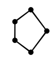
(3) $\quad 0 \ < \ x_1 \ < \ x_1 \vee x_2 \ < \ \dots \ < \ x_1 \vee \dots \vee x_n.$

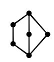
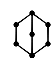
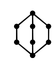
Since the relation $\leq$ holds between successive terms of (3), it suffices to show that for each i we have $\bigvee_{j<i} x_j \neq \bigvee_{j \leq i} x_j$. We shall do this by showing that f_i has distinct values on these two elements. By the third paragraph of Theorem 1, f_i turns joins in L into infima in $[0,1]$, so what we must show is that $\bigwedge_{j<i} f_i(x_j) \neq \bigwedge_{j \leq i} f_i(x_j)$; in other words, that each term $f_i(x_j)$ $(j<i)$ is larger than $f_i(x_i)$. And indeed, $f_i(x_j) > f_j(x_j) \geq f_i(x_i)$, the first inequality by choice of x_j, the second by (2). Thus, L has a chain (3) of length n, as desired. □

This suggests

Question 10. *Can the equality* $\mathrm{Br}(M) = \dim(M)$ *be proved for larger classes of topological lattices* M *than those of the form* $\Delta(L)$ *? (E.g., all topological lattices whose underlying spaces are connected simplicial complexes? Are connected compact Hausdorff spaces?)*

Can one at least obtain one or the other inequality between these invariants?

3.2. **What about an example with no modular lattice structure?** Recall that M_3 is one of two 5-element non-distributive lattices, the other being the nonmodular lattice $N_5 =$ . Thus, we might hope that $\Delta(N_5)$ would give an example answering Taylor's other question, of whether a topological space can admit a structure of lattice, but not of modular lattice [17, first part of Question 9.4.7]. As a simplicial complex, $\Delta(N_5)$ consists of a tetrahedron with a triangle attached along one edge. Now it is certainly true that a simplicial complex $\Delta(L)$ for L a modular lattice cannot have this form; for in a finite modular lattice L, all maximal chains have the same length [14, Theorem 374], hence in its order complex, all maximal simplices have the same dimension. Nevertheless, it is easy to construct a topological distributive (and hence modular) lattice, *not* of the form $\Delta(L)$, whose underlying space is homeomorphic to $\Delta(N_5)$. Simply take the 2-dimensional lattice $[0,1]^2$, "glue" a copy of $[0,1]$ to its top, identifying the top element of $[0,1]^2$ with the bottom element of $[0,1]$, and take the direct product of the resulting lattice with the lattice $[0,1]$. Geometrically, the result is a cube with a square attached along one edge; and since a cube is homeomorphic to a tetrahedron, by a homeomorphism that can be made to carry one edge of the cube to an edge of the tetrahedron, and a square is similarly homeomorphic to a triangle, the above example is indeed homeomorphic to $\Delta(N_5)$.

On the other hand, if one started with a lattice L such as , or, if need be,  or , we can hope that the underlying topological space of $\Delta(L)$ will not admit any modular lattice structure. The intuition is that although when one glues one topological lattice on top of another, the common connecting sublattice can have much lower dimension than the two parts we are connecting, there is no evident construction that preserves distributivity *or* modularity and unites *more than two* parts along a common connection Y having codimension greater than 1. So we ask,

Question 11. *Suppose* X *and* Y *are as in Question 8, and moreover suppose that at least one (at least two? at least three?) of the connected components of* $X - Y$ *having* Y *in their closures have dimension exceeding that of* Y *by at least two. Can* X *admit a structure of* modular *topological lattice?*

The order complexes of the three nonmodular lattices shown above have the properties described in the above question, so a positive answer to any of the three versions of that question would answer this part of Taylor's question. We will look at another example that might do so in §3.4.

3.3. **A construction suggested by the glued example.** The distributive lattice constructed in the preceding section by gluing $[0,1]$ onto $[0,1]^2$ exhibits a property that we noted could not occur in the order complex $\Delta(L)$ of a modular lattice L. Thus, not every simplicial complex with lattice structure arises as a $\Delta(L)$. Can we modify the construction $\Delta(L)$ to embrace this example? In a somewhat ad hoc way, we can.

Proposition 12. *If* P *is a finite partially ordered set, and* $S \subseteq P \times P$ *a family of pairs* (x,y) *with* $x \leq y$, *let* $\Delta(P,S)$ *be the subspace of* $\Delta(P)$ *consisting of functions* $f \in \Delta(P)$ *with the property that for all* $(x,y) \in S$, *if* $f(x) < 1$ *then* $f(y) = 0$ *(equivalently, such that* $f(x)$ *and* $f(y)$ *do not both lie strictly between* 0 *and* 1*).*

Then

(i) $\Delta(P,S)$ *is a subcomplex of* $\Delta(P)$ *with the same set of vertices.*

(ii) *If* P *has meets of all pairs of elements, then the meet operation of* $\Delta(P)$ *carries* $\Delta(P,S)$ *into itself.*

(iii) *If* P *has joins of all pairs of elements, and if the second coordinates* y *of the pairs* $(x,y)\in S$ *are all join-prime (definition recalled below), then the join operation of* $\Delta(P)$ *carries* $\Delta(P,S)$ *into itself.*

Thus, if P *is a lattice* L*, and the second coordinates of all members of* S *are join-prime in* L*, then* $\Delta(L,S)$ *is a sublattice of* $\Delta(L)$*, containing the sublattice of vertices.*

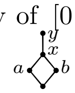
Proof. Given $f\in\Delta(P)$, if we write f as a convex linear combination with nonzero coefficients of the characteristic functions of the principal ideals generated by elements $z_0<\dots<z_n$, then we find that f will belong to $\Delta(P,S)$ if and only if for each $(x,y)\in S$, either $z_0\geq x$ (which is equivalent to $f(x)=1$) or $z_n\not\geq y$ (equivalent to $f(y)=0$). The class of nonempty chains $z_0<\dots<z_n$ satisfying this condition for a given $(x,y)\in S$ is clearly closed under taking nonempty subchains, hence the points f associated with chains having this property for all $(x,y)\in S$ form a subcomplex of $\Delta(P)$.

At every vertex of $\Delta(P)$, all coordinates are 0 or 1, so the condition that for all $(x,y)\in S$, $f(x)$ and $f(y)$ do not both lie strictly between 0 and 1 is certainly satisfied, completing the proof of (i).

If P has meets, as assumed in (ii), so that $\Delta(P)$ is closed under pointwise infima, I claim that $\Delta(P,S)$ is also closed under these infima. Indeed, for $f,g\in\Delta(P,S)$ and $(x,y)\in S$, either $f(x)=g(x)=1$, in which case the infimum $f\wedge g$ certainly satisfies the required condition, or one of them, say $f(x)$, is <1, in which case $f(y)=0$, and the infimum again satisfies our condition.

Finally, suppose that P has pairwise joins, and that for each $(x,y)\in S$, the element y is join-prime, meaning that $w\vee z\geq y$ only if $w\geq y$ or $z\geq y$. Again let $f,g\in\Delta(P,S)$ and $(x,y)\in S$. If $(f\vee g)(y)=t>0$, then by the construction of joins in the semilattice $\Delta(P)$, we must have $f(y_1)\geq t$ and $g(y_2)\geq t$ for some y_1, y_2 with join $\geq y$. Since y is join-prime, one of y_1, y_2 must be $\geq y$; say $y_1\geq y$. Then $f(y)\geq f(y_1)\geq t>0$, so $f(x)=1$, so $(f\vee g)(x)=1$, as required to show $f\vee g\in\Delta(P,S)$.

The final assertion of the proposition is now clear. □

To see that the lattice obtained by gluing a copy of $[0,1]$ to the top of a copy of $[0,1]^2$ is an example of this construction, let L be the 5-element lattice 

, and S the singleton $\{(x,y)\}$. We see that the elements of $\Delta(L)$ with $f(y)=0$ form a sublattice isomorphic to $[0,1]^2$, each element being determined by its values at a and b, while the elements with $f(x)=1$ form a sublattice isomorphic to $[0,1]$, each being determined by its value at y. These sublattices together comprise $\Delta(L,S)$, and the greatest element of the former is the least element of the latter, giving the asserted structure.

One can, in turn, get from this the cube-with-a-square-attached, as $\Delta(L\times\{0,1\},S')$ where $S'=\{((x,0),(y,0))\}$.

Incidentally, the need for the condition in Question 8 that each of the connected components of $X-Y$ considered have Y in its closure is illustrated by the line-glued-on-top-of-a-square lattice $\Delta(L,S)$ of the next-to-last paragraph above. Though $\Delta(L,S)$ is distributive, if we denote by 0 the least element of L, then removing the 1-simplex $Y=\Delta(\{0,x\})$, i.e., the vertical diagonal of the square, breaks $\Delta(L,S)$ into three pieces, whose closures are respectively the triangles $\Delta(\{0,a,x\})$ and $\Delta(\{0,b,x\})$ and the line-segment $\Delta(\{x,y\})$; but the last of these closures does not contain all of Y.

We remark that in the context of the first sentence of Proposition 12, for any four elements $x\leq x'\leq y'\leq y$ of P, the condition on points of $\Delta(P)$ determined by the pair (x',y') implies the condition determined by (x,y); so if (x',y') lies in S, it makes no difference whether (x,y) does. Hence we can restrict S in that construction to be an antichain in the poset of pairs of elements, ordered by inclusion of intervals.

3.4. **Lattices with whiskers.** In §3.2 we noted some variants of the topological lattice $\Delta(N_5)$ which might have the property that their underlying spaces would not admit a modular lattice structure.

There is another very simple example that might have this property: the topological lattice gotten by gluing a copy of the unit interval to the top of $\Delta(N_5)$. (I.e., the union of $\Delta(N_5)$ with $[0,1]$, where the greatest element of the former lattice is identified with the least element of the latter. In the notation of [14, end of §I.1.6], this is $\Delta(N_5)\dot{+}[0,1]$; in that of §3.3 above, it is $\Delta(L,S)$, where L is the result of adjoining to N_5 a new element 1^+ above its existing top element 1, and $S=\{(1,1^+)\}$.)

Note that if we take the *distributive* topological lattice which we saw in §3.2 was *homeomorphic* to $\Delta(N_5)$, and similarly glue a copy of $[0,1]$ to its top, the resulting distributive lattice is *not* homeomorphic to the nonmodular lattice just described: the "top" of the distributive lattice homeomorphic to $\Delta(N_5)$ lies on the outer boundary of its 2-dimensional part, while the "top" of $\Delta(N_5)$ is one end of the edge along which its 2-dimensional part and 3-dimensional part meet; so gluing $[0,1]$ to the "top" has different effects.

We shall see below that any topological lattice structure on the underlying space of our above extension of the genuine topological lattice $\Delta(N_5)$ has the property that the homeomorphic copy of $\Delta(N_5)$ is again a sublattice, and that the point to which we have attached $[0,1]$ remains the greatest element, or becomes the least element, of that sublattice. This might prove useful in showing that no such lattice structure can be modular. The key tool will be

Proposition 13. *Suppose a topological lattice L is connected, and has a point p such that $L-\{p\}$ is disconnected.*

Then $L-\{p\}$ has precisely two connected components, and these are $(\downarrow p)-\{p\}$ and $(\uparrow p)-\{p\}$. (So in the notation of [14, §I.1.6], *$L=\downarrow p \dot{+} \uparrow p$.)*

Proof. Let us first prove that any two elements of L neither of which lies in $\downarrow p$ must lie in the same connected component of $L-\{p\}$. Indeed, if x_0 and x_1 are such elements, then the map $f: L\to L$ defined by $f(y)=x_0\vee y$ has range in $L-\{p\}$, and has connected image since it has connected domain; so $f(x_0)=x_0$ and $f(x_0)=x_0\vee x_1$ lie in the same connected component of $L-\{p\}$. Similarly, x_1 and $x_0\vee x_1$ lie in the same connected component of that space; so x_0 and x_1 lie in the same connected component, as claimed.

The dual argument shows that any two elements not in $\uparrow p$ lie in the same connected component.

Since every element of $L-\{p\}$ is either $\notin\downarrow p$ or $\notin\uparrow p$, our space has at most two connected components. Moreover, if any element were both $\notin\downarrow p$ and $\notin\uparrow p$, those connected components would be the same, contradicting our assumption of disconnectedness; so every element lies in one or the other, and we see that the two connected components are just $\uparrow p$ and $\downarrow p$. □

(The fact that in the context of the above result, $L-\{p\}$ cannot have more than two connected components is proved by Taylor [17, third through fifth paragraphs of §7.2.1], generalizing a result asserted by A. D. Wallace [21].)

We shall use this result in the case where one of $S_0\cup\{p\}$, $S_1\cup\{p\}$ is a space we are interested in, while the other is the simplest nontrivial connected topological space, $[0,1]$.

Corollary 14. *Let X be a nontrivial connected topological space, p a point of X, and X' the space obtained by attaching a "whisker" W to X at p; that is, by taking the disjoint union of X with a copy W of the unit interval, and identifying $p\in X$ with one endpoint of W (and giving the resulting space the natural topology).*

Then structures of topological lattice on X' correspond bijectively to structures of topological lattice on X under which p is either a greatest or a least element. Namely, if p is the greatest element of X under a topological lattice structure, then in the corresponding structure on X', the "whisker" W is a lattice-theoretic copy of the unit interval, glued at its bottom to the top of that lattice, while if p is the least element of our lattice structure on X, then W is a copy of $[0,1]$ glued at its top to the bottom of that lattice.

Likewise, suppose p, q are elements of X, and X'' is the space obtaining by attaching to X one whisker, W, at p, and another whisker, W', at q. Then X'' cannot admit a structure of topological lattice unless $p\neq q$, and in that case, such structures correspond to structures of topological lattice on X under which one of p, q is the greatest element and the other is the least element, by making W, W' copies of $[0,1]$ glued to X as above at p and q.

Sketch of proof. If we are given a point $p\in X$, it is clear that for any structure of topological lattice on X under which p is the greatest element, the result of "gluing" a copy of $[0,1]$ to the top of X will give a topological lattice homeomorphic to X'; and we have the obvious corresponding construction if p is the least element of our lattice structure. Conversely, if we are given a lattice structure on X', then, noting that the space $X'-\{p\}$ is disconnected, namely, that it is the union of its open subsets $S_0=X-\{p\}$ and $S_1=W-\{p\}$, we can apply Proposition 13 to conclude that one of X, W is precisely $\downarrow p$, and the other is $\uparrow p$. By the symmetry of the result we want to prove, it will suffice to consider the case where $X=\downarrow p$, $W=\uparrow p$.

Clearly, this makes X and W sublattices of X', such that X' is the lattice formed by gluing W on top of X. It remains only to say why the lattice structure on W must agree with the usual lattice structure on $[0,1]$. It is probably known that, up to isomorphism, that is the only topological lattice structure on that space; in any case, that description of the lattice structure of W is not hard to deduce here by noting that for any interior point p' of W, deletion of p' also disconnects X', so Proposition 13 can be applied again to show that all the elements of W on the "X" side of p' are $< p'$, and those on the other side are $> p'$.

To get the assertions of the final paragraph, first note that if $p = q$, then X'' falls into at least three pieces on deleting p; hence by Proposition 13 it cannot admit a structure of topological lattice. Assuming that $p \neq q$ and that we are given a topological lattice structure on X'', the desired result can be obtained by applying twice the one-whisker result proved above: first to X'', viewed as obtained by adjoining the whisker W' to the space X', and then to X', viewed as obtained by adjoining the whisker W to X. (In the latter case, we apply our result to the lattice structure on X' that the first step shows us it inherits from X'', as the sublattice $\uparrow q$ or $\downarrow q$.) □

It follows from the first statement of the above corollary that if the underlying topological space of $\Delta(N_5)$ admits no modular topological lattice structure having the *same greatest element* as that of $\Delta(N_5)$ itself, then the result of gluing a whisker to the top of $\Delta(N_5)$ will be a nonmodular topological lattice whose underlying space cannot be given *any* structure of modular topological lattice, and so will give our desired example. (Here I am using the fact that if the underlying space of $\Delta(N_5)$ admits no modular lattice structure having the same greatest element as its standard structure, then by symmetry, it also admits no modular lattice structure having for least element the greatest element of that structure.)

We can hedge our bets by noting that the second assertion of the corollary similarly tells us that unless the underlying space of $\Delta(N_5)$ admits a modular lattice structure having *both* the same greatest element *and* the same least element as in $\Delta(N_5)$, then the result of attaching one whisker to the top of $\Delta(N_5)$ and one to the bottom will be an example of the desired sort.

Incidentally, a different application of the construction of Proposition 13 shows that the necessary condition for a simplicial complex that admits a lattice structure to admit a distributive lattice structure given by Proposition 7 is not sufficient. If we glue to the top of $\Delta(M_3)$ the topological lattice $[0,1]^3$, the result will be a topological lattice whose underlying space has dimension 3 and is clearly embeddable in $\mathbb{R}^3$; but if it admitted a distributive topological lattice structure, Proposition 13 shows us that its $\Delta(M_3)$ part would also, which we have seen is not so.

We remark that the method of proof Proposition 13 also yields information, though not as strong, about topological *semilattices.* If P is a connected $\vee$-semilattice which becomes disconnected on deleting a point p, one sees as in that proof that any two elements $\notin \downarrow p$ lie in the same connected component of $P - \{p\}$; in other words, all but at most one of its connected components lie in $\downarrow p$.

Examples of such semilattices with many connected components in $\downarrow p$ can be constructed by starting with an arbitrary family of connected topological $\vee$-semilattices P_i $(i \in I)$, each having a greatest element p_i, and possibly one additional connected topological $\vee$-semilattice P_0 $(0 \notin I)$ with an arbitrary point p_0 chosen, identifying all the points p_i (for $i \in I$ or $I \cup \{0\}$ depending on whether we have a P_0), and giving the resulting set P the topology and order-structure determined in the obvious ways by those of the given structures. (In particular, the join of elements $x_i \in P_i$, $x_j \in P_j$ where $i \neq j$ and $i, j \in I$ is p; while if there is a P_0, the join of elements $x_i \in P_i$, $x_0 \in P_0$ is $p \vee x_0 \in P_0$.)

However, not all examples have this simple form. For instance, consider the $\vee$-subsemilattice P of $[0,1]^2$ consisting of the points (x, x) and $(x, 1)$ for all $x \in [0,1]$; and take $p = (1,1)$. Then $P - \{p\}$ has two connected components, but its join operation does not have the form described; e.g., $(0,0) \vee (0,1) = (0,1)$, rather than being $p = (1,1)$.

Returning to the lattice-theoretic version of Proposition 13, this can be generalized by replacing the singleton $\{p\}$ with any sublattice $P \subseteq L$ such that for any two elements $x < y \in P$ one has $[x, y] \subseteq P$ (any "convex sublattice" of L), and such that $L - P$ is disconnected. Again, $L - P$ will have just two connected components. The unions of these components with P will be the sublattices $\{x \in L \mid (\exists\, p \in P)\ x \leq p\}$ and $\{x \in L \mid (\exists\, p \in P)\ x \geq p\}$. In the language of [14, §IV.2.1], L will be the lattice obtained by gluing these two sublattices together over P.

4. More questions and examples

4.1. **A question on local distributivity.** A notable property of lattices of the form $\Delta(L)$, and hence also of their sublattices $\Delta(L, S)$, is that each of their simplices is a distributive sublattice. (Each simplex has the form $\Delta(C)$ for C a chain in L, and every chain C in a lattice is a distributive sublattice; hence so is the induced lattice $\Delta(C)$.) Thus, every interior point of every maximal simplex of such a topological lattice has a neighborhood which is a distributive sublattice. This suggests

Question 15. *Let L be a topological lattice whose underlying set is a finite simplicial complex* (*or perhaps belongs to some wider class, such as finite CW-complexes*)*. Suppose we call an element $p \in L$ a "point of local distributivity" if p has a neighborhood which is a distributive sublattice of L. Must the set of points of local distributivity* (*an open subset of L*) *be dense in L?*

4.2. **A short-lived variant of the above question, and another construction.** The observations that led to Question 15 also suggest a simpler question: whether every topological lattice which is homeomorphic to an n-cell is distributive.

None of the examples we have seen contradicts this, but a counterexample has long been known [8]. We will discuss that example in §4.4; here I will describe a simpler one, obtained by slightly "thickening" the lattice $\Delta(M_3)$. In developing this example it will be convenient to represent $\Delta(M_3)$ as $\Delta(X, \mathrm{cl})$ (notation as in §2.2), where $X = \{0,1,2\}$ and cl is the closure operator under which the singletons are closed, while the closure of each 2-element set is all of X. This has the effect that we will be "thickening" our copy of $\Delta(M_3)$ within the 3-cube $[0,1]^3$, rather than having to work in the 5-cube $[0,1]^L$.

To see what we will mean by "thickening", note that for this X and cl, $\Delta(X, \mathrm{cl})$ consists of all functions $f : \{0,1,2\} \to [0,1]$ such that if the value of f at two of these elements is $\geq t$, then so is its value at the third element. This forces the two lowest values of f to be equal, so that $\Delta(X, \mathrm{cl})$ is 2-dimensional. Now let us fix a positive constant $c < 1$, and let $\Delta_c(X, \mathrm{cl})$ denote the set of those functions $f : \{0,1,2\} \to [0,1]$ such that if the value of f at two points is $\geq t$, then its value at the third point is $\geq t - c$. It is not hard to see visually that this space is homeomorphic to the 3-ball; one can give a formal proof starting with the observation that every ray through the point $(1/2,\ 1/2,\ 1/2)$ meets the space in a nontrivial closed interval. To see that it is a lattice under ordering by componentwise inequality, note that a least upper bound of any two elements (e.g., $(1,0,0)$ and $(0,0,1)$) can be gotten by taking their pointwise supremum and, if the value of one of the coordinates thereof is less than the minimum of the other two by more than c, increasing it to precisely that minimum minus c. (So in the above example, the least upper bound is $(1, 1{-}c, 1)$.) Greatest lower bounds are, as usual, calculated coordinatewise.

This lattice structure is still nondistributive, since we find that

$$(0,1,0) \wedge ((1,0,0) \vee (0,0,1)) \;=\; (0, 1{-}c, 0),$$

(4) while

$$((0,1,0) \wedge (1,0,0)) \vee ((0,1,0) \wedge (0,0,1)) \;=\; (0,0,0).$$

Thus, it answers the suggested question in the negative.

Curiously, this lattice, though obtained by modifying a lattice isomorphic to $\Delta(M_3)$, contains no copies of M_3. Indeed, suppose $p_0,\ p_1,\ p_2 \in \Delta_c(X, \mathrm{cl})$ satisfy

(5) $$p_0 \vee p_1 = p_0 \vee p_2 = p_1 \vee p_2, \qquad \text{and} \qquad p_0 \wedge p_1 = p_0 \wedge p_2 = p_1 \wedge p_2.$$

Let $a \in [0,1]$ be the greatest of the $3 \times 3 = 9$ coordinates of p_0, p_1 and p_2, and suppose without loss of generality that this value a is attained at the 0-coordinate of p_0. Then the 0-coordinate of the element $p_0 \vee p_1 = p_0 \vee p_2 = p_1 \vee p_2$ must be a, as can be seen from either of the first two expressions. Hence a is also the 0-coordinate of the last expression, $p_1 \vee p_2$, and, being the largest coordinate in any of the p_i, it cannot arise in that join by the operation of increasing the smallest coordinate of a 3-tuple to the value of the second-largest coordinate minus c (because $c > 0$); so it must be the 0-coordinate of p_1 or p_2. Assume without loss of generality that it is the 0-coordinate of p_1. Then a will also be the 0-coordinate of $p_0 \wedge p_1 = p_0 \wedge p_2 = p_1 \wedge p_2$ (because it is the 0-coordinate of the first of these); moreover, since a is the largest of the coordinates of the p_i, the coordinate a in these meets cannot arise as the lesser of two distinct values; so it must be the 0-coordinate of *all* of p_0, p_1, p_2. So these three elements have the same 0-coordinate. Next, let b be largest of the 1- and 2-coordinates of p_0, p_1 and p_2, and assume without loss of generality that it occurs as a 1-coordinate. Then the same argument shows that b is the 1-coordinate of all three elements (since the second-largest coordinate also cannot arise by the "increase the smallest coordinate" operation). So p_0, p_1, p_2 agree in their first *two* coordinates. Hence they form a chain in $\Delta_c(X, \mathrm{cl})$; so the relations (5) force $p_0 = p_1 = p_2$, completing the proof that $\Delta_c(X, \mathrm{cl})$ contains no copy of M_3.

Being nondistributive by 4, and containing no M_3, the lattice $\Delta_c(X,\mathrm{cl})$ must contain an N_5. And, indeed, if $c\in(0,1/2]$ it is easy to verify that the two elements $(1,0,0)<(1,c,0)$ belong to $\Delta(X,\mathrm{cl})$, and have equal joins with $(0,0,1)$, and also equal meets with that element; while if $c\in[1/2,1)$ the same is true with the middle coordinate c of the second element replaced by $1-c$. In each case, this gives an explicit 5-element sublattice of the indicated form. So though $\Delta(X,\mathrm{cl})\cong\Delta(M_3)$ is, like M_3, modular, the thickened lattice $\Delta_c(X,\mathrm{cl})$ is not.

4.3. **The general thickening construction (and variants).** Abstracting the construction of the preceding section, let us, for any closure operator cl on a finite set X, and any $c\in[0,1]$, define

(6) $$\Delta_c(X,\mathrm{cl}) \ = \ \{f:X\to[0,1] \mid (\forall\, t\in(0,1])\ f_{t-c}\supseteq \mathrm{cl}(f_t)\,\}.$$

(Here we define f_t for all real t by the same formula (1) that we have been using when $t\in[0,1]$. Thus, in (6), for $t\le c$ the set f_{t-c} is all of X. The same set (6) could therefore be defined with t ranging only over $(c,1]$, which would avoid the need to extend the definition of f_t, at the price of looking a little less straightforward.)

It is not hard to see that $\Delta_c(X,\mathrm{cl})$ will again form a lattice. (However, though the structure of $\Delta(X,\mathrm{cl})$ is determined by that of the lattice L_{cl}, the structure of $\Delta_c(X,\mathrm{cl})$ is not determined by c and L_{cl}. Indeed, for $c>0$ its dimension is $\mathrm{card}(X)$, which is not determined by those data.)

There are various further generalizations of this construction.

If we are merely given a family P of subsets of a set X, then a definition which reduces to the above when P is the lattice of closed subsets of a closure operator is to make $\Delta_c(X,P)$ the poset of those functions $f:X\to[0,1]$ such that for all $t>c$, the set f_{t-c} contains some member of P majorizing f_t.

Returning, for simplicity, to the case where P is the set of closed subsets of X under a closure operator cl, suppose we think of the condition on f in (6) as saying that for each $Y\subseteq X$, and $z\in\mathrm{cl}(Y)$, we have $f(z)\ge(\min_{y\in Y} f(y))-c$. Then one may generalize the construction of (6) to one in which the c in that condition is allowed to vary with the pair (Y,z).

Finally, we can replace the map $t\mapsto t-c$ by other functions; for instance, $t\mapsto(1-c)\,t$.

Might some instance of the construction $\Delta_c(X,\mathrm{cl})$ provide a counterexample to Question 15? No; for let $\mathrm{card}(X)=n$, and consider the $n!$ bijections $e:\{0,\dots,n{-}1\}\to X$. Each such bijection determines a simplex

(7) $$D_e \ = \ \{f\in[0,1]^X \mid f(e(0))\ge\cdots\ge f(e(n{-}1))\}\ \subseteq\ [0,1]^X,$$

and the union of these simplices is the whole n-cube. (These are, in fact, the n-simplices comprising $\Delta(X,\mathrm{cl}')=[0,1]^X$, where cl' is the trivial closure operation on X.) Let us show that

(8) Each D_e intersects $\Delta_c(X,\mathrm{cl})$ in a distributive sublattice of the latter.

This will make $\Delta_c(X,\mathrm{cl})$ a finite union of closed distributive sublattices, from which it is easy to deduce a positive answer to Question 15 for lattices $\Delta_c(X,\mathrm{cl})$.

To get (8), it will suffice to show that for each e, and all $f,g\in D_e\cap\Delta_c(X,\mathrm{cl})$, we have

(9) The coordinatewise supremum and infimum of f and g again lie in $\Delta_c(X,\mathrm{cl})$.

This will imply that these coordinatewise operations give the meet and join of such a pair f, g in $\Delta_c(X,\mathrm{cl})$, as they clearly do in D_e, hence they will carry $D_e\cap\Delta_c(X,\mathrm{cl})$ into itself, making it a distributive sublattice of $\Delta_c(X,\mathrm{cl})$.

The assertion of (9) is straightforward for the infimum. To prove the statement about the supremum, consider any $t\in[0,1]$, any subset $Y\subseteq X$, and any $z\in\mathrm{cl}(Y)$. We must show that for $f,g\in D_e\cap\Delta_c(X,\mathrm{cl})$,

(10) If the coordinatewise supremum of f and g is $\ge t$ at all points of Y, then it is $\ge t-c$ at z.

Let i be the largest member of $\{0,\dots,n{-}1\}$ such that $e(i)\in Y$. If f and g satisfy the hypothesis of (10), their coordinatewise supremum has value at least t at $e(i)$; assume without loss of generality that $f(e(i))\ge t$. Then by choice of i, and the condition $f\in D_e$ (see (7)), we have $f(y)\ge t$ for all $y\in Y$; so $f(z)\ge t-c$. Hence the coordinatewise supremum of f and g also has value $\ge t-c$ at z, establishing (10), hence (9), hence (8), hence a positive answer to Question 15 for lattices $\Delta_c(X,\mathrm{cl})$.

The above argument goes over to the generalization mentioned above in which c is allowed to vary with Y and z.

We remark that in view of the observation in the preceding section that the lattice constructed there contained no copy of M_3, the 'short-lived question' with which that section began might be revived in the weakened form: Must a *modular* topological lattice that is homeomorphic to an n-cell be distributive? This is known to be true for $n \leq 3$ [2] [13].

4.4. **A construction of D. Edmondson.** In an earlier draft of this note, I had a simpler version of Question 15, asking whether every topological lattice whose underlying space is a simplicial complex must be a finite union of closed distributive sublattices. But it turned out that the example in [8] of a nonmodular lattice structure on the 3-cell has properties that make the existence of such a decomposition very unlikely. Below, we recall and generalize that construction, and note why it probably does not admit such a decomposition.

The example of [8], after the change of coordinates $(x, y, z) \mapsto (x, y, z + xy)$, which turns the partial ordering described there into the ordering by componentwise inequality, is

$$(11) \qquad E \;=\; \{(x, y, z) \in [0, 1]^3 \mid xy \leq z \leq x\}.$$

If we write $L = [0, 1]^2$, $M = [0, 1]$, each regarded as a lattice under componentwise operations, and define two set-maps $L \to M$ by $a(x, y) = xy$, $b(x, y) = x$, then the construction of (11) can be abstracted as follows. (Note that the pair of coordinates (x, y) in (11) and in our definitions of a and b become the x of (12), and the coordinate z becomes the y of (12).)

Proposition 16 (after D. Edmondson [8]). *Let L and M be topological lattices, and $a, b : L \to M$ continuous isotone maps* (*not assumed to be lattice homomorphisms*) *such that $a(x) \leq b(x)$ for all $x \in L$. Then the set*

$$(12) \qquad E(L, M; a, b) \;=\; \{(x, y) \in L \times M \mid a(x) \leq y \leq b(x)\},$$

partially ordered by componentwise inequality, forms a topological lattice, with operations

$$(13) \qquad \begin{aligned} (x, y) \vee (x', y') \;&=\; (x \vee x',\ y \vee y' \vee a(x{\vee}x')),\\ (x, y) \wedge (x', y') \;&=\; (x \wedge x',\ y \wedge y' \wedge b(x{\wedge}\, x')).\end{aligned}$$

Proof. That the expressions in (13) give a least upper bound and a greatest lower bound for (x, y) and (x', y') is clear assuming that the elements described indeed lie in (12). In each case, one of the two inequalities in the condition for membership in (12) is handled by the last joinand or meetand in the formula; we must in each case verify the other inequality; namely, for the join we must show that $y \vee y' \vee a(x{\vee}x') \leq b(x{\vee}x')$, and for the meet, that $y \wedge y' \wedge b(x{\wedge}\, x') \geq a(x{\wedge}\, x')$.

To get the former inequality, it suffices to verify that each of the joinands on the left is less than or equal to the term on the right. This is true of the first joinand, y, because $y \leq b(x) \leq b(x{\vee}x')$. The second joinand, y', is handled in the same way, while the inequality for $a(x{\vee}x')$ follows from our hypothesis $a \leq b$. The corresponding result for meets holds by the dual argument. Finally, the operations (13) are continuous because the operations of L and M, and the maps a and b used in (13), are assumed continuous. □

The condition that the map a of the above construction be isotone shows that $a(x) \vee a(x') \leq a(x \vee x')$. The first statement of the next result shows that when that inequality is strict, so that a fails to be a $\vee$-semilattice homomorphism, we get a copy of N_5 in our lattice, unless the map b constrains things too tightly.

Lemma 17. *In the situation of Proposition 16, if $x, x' \in L$ are such that*

$$(14) \qquad a(x) \vee a(x') \;<\; a(x{\vee}x') \wedge b(x),$$

then the three elements

$$(15) \qquad (x,\, a(x) \vee a(x')) < (x,\, a(x{\vee}x') \wedge b(x)) \;\; \text{and} \;\; (x',\, a(x')) \;\; \text{of} \;\; E(L, M; a, b)$$

generate a sublattice isomorphic to N_5.

Likewise, if

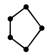
$$(16) \qquad b(x) \wedge b(x') \;>\; b(x{\wedge}\, x') \vee a(x),$$

then the three elements

$$(17) \qquad (x,\, b(x) \wedge b(x')) > (x,\, b(x{\wedge}\, x') \vee a(x)) \;\; \text{and} \;\; (x',\, b(x')) \;\; \text{of} \;\; E(L, M; a, b)$$

generate such a sublattice.

Proof. We shall prove the first assertion. We begin by noting that the first element of (15) indeed satisfies the upper bound on its second component required by (12): this can be seen from (14). The remaining conditions for our three elements to lie in $E(L, M; a, b)$ are immediate.

The inequalities $a(x') \leq a(x) \vee a(x') < a(x \vee x') \wedge b(x)$ (the last by (14)) show that on taking the componentwise meet of either of the first two elements of (15) with the third element, we get $(x \wedge x',\, a(x'))$, hence the corresponding meets in $E(L, M; a, b)$ are both $(x \wedge x',\, a(x') \wedge b(x \wedge x'))$. When we take the corresponding joins, both likewise give $(x \vee x',\, a(x \vee x'))$ (cf. (13)). So the lattice they generate indeed has the form .

The second assertion of the lemma holds by the dual argument. □

Turning back to (11) and the sentence following it, which concern the lattice E based on the particular maps $a,\, b : [0,1]^2 \to [0,1]$ given by $a(x,y) = xy$, $b(x,y) = x$, observe that for any real numbers $0 < x < x' < 1$ and $0 < y' < y < 1$ (note the opposite orders of primed and unprimed terms in the two inequalities) we have $a(x,y) \vee a(x',y') = xy \vee x'y'$, while $a((x,y) \vee (x',y')) = a(x',y) = x'y$, which is strictly larger than the maximum of xy and $x'y'$. Moreover, if (x,y) is close enough to (x',y'), then this strict inequality continues to hold on taking the meet with $b(x)$ as in (14). Thus, by Lemma 17 there are little copies of N_5 resting all over the lower surface of the lattice E. This is very different from the behavior of lattices $\Delta(L)$ and the variants of these that we have constructed, where any nondistributive sublattice that occurs must have points from more than one simplex. It appears to me unlikely that the present lattice can, like those, be written as a finite union of closed distributive sublattices, as the earlier version of Question 15 suggested; though I don't see a proof that this is impossible.

On the other hand, it is easy to see that every cube

$$(18) \qquad [x,\, x+\varepsilon] \times [y,\, y+\varepsilon] \times [z,\, z+\varepsilon]$$

lying wholly within our lattice E forms a distributive sublattice; so points of local distributivity form the whole interior of E; so E does not give a counterexample to the present form of Question 15. (Incidentally, the need for our comment in the preceding paragraph, that for (x,y) close enough to (x',y') the operation $- \wedge b(x)$ of (14) did not matter, can be eliminated if we take $b(x,y) = 1$ in place of $b(x,y) = x$ in (11). I used the latter only for conformity with [8].)

4.5. **Some different sorts of questions.** The above observation on "tiny" sublattices (18) suggests the question: To what extent is it true, in a topological lattice, that "What starts in a small neighborhood stays in a small neighborhood"? Since a free lattice on three or more generators is infinite, it seems possible for the sublattice generated by three elements p, q, r lying in an arbitrarily small neighborhood of a point x to "spread" far from x. An obstruction to this is that this sublattice will lie in the interval between $p \wedge q \wedge r$ and $p \vee q \vee r$, each of which will be close to $x \wedge x \wedge x = x \vee x \vee x = x$. But might the interval between two points that are close to x be very "wide", and contain points not close to x?

Yes and no. If L is a compact topological lattice, then an immediate consequence of [7, Lemma 3] is that every neighborhood U of a point $x \in L$ has a subneighborhood V such that for all $y < z$ in V, all points of the interval $[y,z] = \{w \in L \mid y \leq w \leq z\}$ lie in U. (This is the case $y < z$ of that lemma, since when $y < z$ we have $(L \wedge z) \vee y = [y,z]$.) On the other hand, we will see in §4.6 that this can fail in noncompact lattices.

If we look at the sublattice generated by *infinitely many* points in a neighborhood of x, we lose the trick based on the interval $[p \wedge q \wedge r,\ p \vee q \vee r]$. Here is what little I know about that case.

Lemma 18. *Let L be a compact Hausdorff topological lattice, and x a point of L. Then the following conditions are equivalent.*

(i) *Every neighborhood U of x contains a neighborhood V of x which is a sublattice of L.*

(ii) *Every neighborhood U of x contains a neighborhood V of x which is an interval $[y,z]$ in L.*

Proof. (ii) $\Longrightarrow$ (i) is immediate, since every interval is a sublattice.

Conversely, assume (i) holds. As noted above, [7, Lemma 3] gives a subneighborhood $U' \subseteq U$ of x such that for all $y < z$ in U', the interval $[y,z]$ lies in U. Because L is compact Hausdorff, we can find a *closed* neighborhood $U'' \subseteq U'$ of x; and (i) gives us a neighborhood $U''' \subseteq U''$ of x which is a sublattice.

The closure of U''' will be a compact sublattice contained in the closed neighborhood U'', and because it is compact, it will have a least and a greatest element, $y < z$. By choice of U', the interval $[y,z]$ is contained in U, and since it contains U''', it is a neighborhood of x, establishing (ii). □

One can give further variant statements equivalent to (i) and (ii) above; for instance, that every neighborhood U of x contain elements y and z such that $[y,z]$ is a neighborhood of x. This in turn is easily seen to be equivalent to the conjunction of the two conditions that U contain a y such that $\uparrow y$ is a neighborhood of x, and that it contain a z such that $\downarrow z$ is a neighborhood of x.

By [7, Theorem 3], every compact *distributive* topological lattice satisfies condition (ii) above, hence both conditions; but for general L, let us ask

Question 19. *If L is a compact Hausdorff topological lattice (perhaps subject to further conditions, such as being a simplicial complex), must every $x \in L$ satisfy the equivalent conditions of Lemma 18?*

In another direction, to get a sense of how far things can get from the sort of examples we have seen, we ask

Question 20. *If L is a connected topological lattice (perhaps assumed compact), must L be pathwise connected?*

4.6. **Noncompact lattices.** We give below a family of examples showing that much of what the preceding section tells us is true, or suggests may be true, for compact topological lattices fails in the noncompact case.

The idea will be the following. It is not hard to find lattices L with metrics $d: L \times L \to \mathbb{R}^{\geq 0}$ in which (say) the interval between a fixed pair of elements $p < q$ contains elements arbitrarily far from p and q. E.g., if L is a lattice consisting of a least and a greatest element, and an infinite set of pairwise incomparable elements lying between them, and $c: L \to \mathbb{N}^{>0}$ is any unbounded function assigning a positive integer to each member of L, then letting $d(x,y) = c(x) + c(y)$ when $x \neq y$, and 0 otherwise, gives an unbounded metric. (Think of $c(x)$ as the distance from x to an outside point $*$, and imagine the only way to get between distinct points x and y is via $*$.) The topology on L defined by this metric is of no interest to us, since it is discrete; but the idea will be to start with such a metric on L, and form a lattice of functions $[0,1] \to L$, metrized so that appropriate families of functions that differ only on small subintervals of $[0,1]$ give "miniaturized" copies of L. The construction is a bit contrived, but will do what we need. The next lemma describes it; the one that follows applies it to get the desired examples.

Lemma 21. *Let L be a lattice, and $\Gamma(L)$ the quotient of the set of all L-valued functions on $[0,1]$ that assume only finitely many distinct values, with each value assumed on a finite union of intervals, by the equivalence relation that identifies functions which differ at only finitely many points. (Because of that identification, we don't have to say whether the abovementioned intervals are open, closed, or half-open.) Then the pointwise lattice operations determine a lattice structure on $\Gamma(L)$ under which it is a subdirect product of copies of L.*

If, moreover, we are given a real-valued function $c: L \to [1,\infty)$, such that for all $x, y \in L$,

$$(19) \qquad c(x \vee y) \leq c(x) + c(y) \quad \text{and} \quad c(x \wedge y) \leq c(x) + c(y),$$

and we let $\Gamma(L,c)$ denote the lattice $\Gamma(L)$ given with the metric

$$(20) \qquad d(f,f') = \int_{\{t \in [0,1] \mid f(t) \neq f'(t)\}} (c(f(t)) + c(f'(t)))\,dt,$$

then in the topology induced by that metric, the lattice operations are continuous, and $\Gamma(L,c)$ is contractible.

Proof. To express $\Gamma(L)$ as a subdirect power of L, let us, for each $s \in (0,1)$ define $h_s: \Gamma(L) \to L$ to take each $f \in \Gamma(L)$ to its eventual constant value as t approaches s from above. Then we see that each h_s is a homomorphism, and these homomorphisms separate points.

The verification that for c satisfying (19), the formula (20) defines a metric is straightforward. (It is the integral of the $c(x)+c(y)$-metric of our motivating remarks.)

In proving continuity of the meet and join operations, it suffices by symmetry to consider joins. Given $f, g \in \Gamma(L,c)$, to prove continuity of $\vee$ at (f,g), we must show that

(21) For every $\varepsilon > 0$ there exists $\delta > 0$ such that for all f', g' satisfying $d(f,f') < \delta$ and $d(g,g') < \delta$, we have $d(f \vee g, f' \vee g') < \varepsilon$.

Now (abbreviating $c(f(t))$ to $c(f)$, etc., for simplicity), we see from (20) and (19) that

$$(22)\qquad d(f\vee g,\ f'\vee g')\ \leq\ \int_{\{t\,|\,f\neq f' \text{ or } g\neq g'\}} (c(f)+c(g)+c(f')+c(g'))\,dt.$$

So to prove (21), it suffices to show that by taking the δ of that statement small enough, we can make the integrals of $c(f)+c(f')$ and of $c(g)+c(g')$ over the range shown in (22) each less than $\varepsilon/2$. By symmetry, it suffices to prove this for $c(f)+c(f')$. In doing so, let us regard the range of integration as the union of the set where $f\neq f'$, and the set where $f=f'$ and $g\neq g'$, and show that we can make the integral of $c(f)+c(f')$ over each of these sets less than $\varepsilon/4$. The integral over the former set is precisely $d(f,f')$, so we can make it $<\varepsilon/4$ by taking $\delta\leq\varepsilon/4$. On the other hand, on the latter set, since $f=f'$, our integrand is $2\,c(f)$. Since f is fixed, the function $c(f)$ has an upper bound m. If we take $\delta\leq\varepsilon/(4m)$, then the condition $d(g,g')<\varepsilon/(4m)$, implies, in view of the definition (20) and the assumption that c is everywhere ≥ 1, that the total length of the set where $g\neq g'$ must be $<\varepsilon/(8m)$. Hence the integral of $2\,c(f)\leq 2m$ over that set will be $\leq\varepsilon/4$, completing the proof of continuity.

Finally, to see contractibility, let us, for $f,g\in\Gamma(L,c)$ and $s\in[0,1]$ define $H_s(f,g)\in\Gamma(L,c)$ to agree with g on $[0,s)$ and with f on $(s,1]$. (Since members of $\Gamma(L,c)$ are defined only up to the relation of agreeing at all but finitely many points, we don't have to specify the value at s itself.) I claim that $H_s(f,g)$ is jointly continuous in s, f, and g. Given $f,g,f',g'\in\Gamma(L,c)$ and $s,s'\in[0,1]$, the triangle inequality in $\Gamma(L,c)$ gives $d(H_s(f,g),\ H_{s'}(f',g'))\leq d(H_s(f,g),\ H_{s'}(f,g))+d(H_{s'}(f,g),H_{s'}(f',g'))$. If we now fix f and g, it is not hard to see that we can make the distance $d(H_s(f,g),\ H_{s'}(f,g))$ small by making $s-s'$ small; on the other hand, we see that $d(H_{s'}(f,g),H_{s'}(f',g'))$ is bounded by $d(f,f')+d(g,g')$, giving the asserted joint continuity. Hence holding g, fixed, $H_s(f,g)$ is jointly continuous in s and f; so letting s vary from 0 to 1, we get a homotopy from $H_0(-,g)$, the identify function of $\Gamma(L,c)$, to $H_1(-,g)$, the constant function with value g, establishing contractibility. □

We can now give the promised counterexamples. In the next lemma, note that the condition introducing (i) and (ii) holds, in particular, if L is an infinite chain with a least and a greatest element, regarded as a lattice, and c any unbounded function $L\to[1,\infty)$. The condition introducing (iii) holds both for free lattices on $n\geq 3$ generators and for free modular lattices on $n\geq 4$ generators (see [5, Exercise 6.3:9] and [14, Exercise I.5.11] respectively).

Lemma 22. *Suppose that L and c are as in Lemma 22, that L has a least element 0 and a greatest element 1, and that c is unbounded. Then*

(i) *Every nonempty open set $V\subseteq\Gamma(L,c)$ contains elements $p<q$ such that the interval $[p,q]\subseteq\Gamma(L,c)$ is unbounded (i.e., has infinite radius under the metric (20)).*

(ii) *Every nonempty open set $V\subseteq\Gamma(L,c)$ generates a sublattice which is unbounded. In fact, both the $\vee$-subsemilattice and the $\wedge$-subsemilattice generated by V are unbounded.*

If, rather, L is any infinite lattice generated by $n<\infty$ elements, and we define c to take each $x\in L$ to the least length of an expression for x in terms of those generators (where we understand the length *of an expression to mean the total number of occurrences in that expression of the symbols for the generators), then*

(iii) *Every nonempty open subset of $V\subseteq\Gamma(L,c)$ contains n elements which generate an unbounded sublattice of $\Gamma(L,c)$.*

Proof. To prove (i), let us show that for all $r\in\Gamma(L,c)$ and $\varepsilon>0$, we can find the desired p and q within distance ε of r. Suppose we take a small subinterval $I\subseteq[0,1]$, and let p agree with r *except* on I, where it has the value 0, and let q likewise agree with r except on I, where it has the value 1. Then if I is small enough (for instance, of length $<\varepsilon/(2m)$, where m is the maximum of $c(0)$, $c(1)$, and the images under c of the finitely many values of the function r), then p and q will each be within distance ε of r. Let us now consider any $s\in[p,q]$ which likewise agrees with r except on I, but has for value there an arbitrary element $x\in L$. If we take x such that $c(x)$ is sufficiently large, then s will be arbitrarily far from r; so $[p,q]$ is indeed unbounded.

To prove the $\vee$-semilattice case of (ii), we shall show that for any $r\in\Gamma(L,c)$, V contains a finite family of points p_i such that $\bigvee p_i$ is "far" from r. The construction is similar to the preceding, so I shall abbreviate the details. We again start by replacing the values of r on a small subinterval $I\subseteq[0,1]$ with $0\in L$, getting an element $p\in\Gamma(L,c)$. We then take $x\in L$ such that $c(x)$ has large value, and let N be the integer $\lceil c(x)\rceil$. We subdivide I into N^2 equal small subintervals I_i $(0\leq i<N^2)$, and let each p_i be formed

from p by changing the value on I_i from 0 to x. Since the length of each I_i is $1/N^2$ times that of I, the distance $d(p, p_n)$ is $\leq (N + c(0))/N^2$ times the length of I, hence is small if $c(x)$, and hence N, is large enough; so the p_i can all be made close to p, which, if I has been taken small, is close to r. But if we form the join of these N^2 elements in $\Gamma(L, c)$, this has value x on *all* of I; so for fixed I and large enough $c(x)$, this join is arbitrarily far from p, and hence from r. For the $\wedge$-semilattice case of (ii) we use the dual construction.

Turning to (iii), it is easy to see that the function c defined in the sentence preceding that statement satisfies (19) and is unbounded. To get an unbounded sublattice of $\Gamma(L, c)$ generated by n elements near an element $r \in \Gamma(L, c)$, again choose a small interval $I \subseteq [0, 1]$, and this time construct n elements, by replacing the values of r on I by one or another of our n generators of L. It is easy to see that for I small enough, those n elements indeed lie in V, but that no matter how small I is, the unboundedness of c implies that the sublattice they generate has elements at unbounded distances from r. □

By choosing L as indicated immediately before the above lemma, we can get distributive $\Gamma(L, c)$ satisfying conditions (i) and (ii), and modular $\Gamma(L, c)$ satisfying (iii).

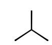
The construction $\Gamma(L, c)$ really uses nothing specific to lattices, and might have other uses in universal algebra, if it is not already known.

5. Further notes on the order-complex construction

5.1. **Why topologists haven't looked at $\Delta(L)$.** In the literature I am aware of, e.g., [19], the order complex construction $\Delta(\ )$ is regularly applied, not to finite lattices L, but to their subposets $L - \{0, 1\}$. This is because $\Delta(L - \{0, 1\})$ can have nontrivial homotopy and homology, while $\Delta(L)$ cannot, nor can $\Delta(L - \{0\})$ or $\Delta(L - \{1\})$. Indeed,

Lemma 23. *If a finite partially ordered set P has a least element, a greatest element, or more generally, an element z that is comparable to all elements of P, then $\Delta(P)$ is contractible.*

Proof. Given z comparable to all elements of P, let us define for each $s \in [0, 1]$ a map $H_s : \Delta(P) \to \Delta(P)$; namely, for $f \in \Delta(P)$, let $H_s(f) = (1 - s) \cdot f + s \cdot (\downarrow z)$. To see that $H_s(f)$ lies in $\Delta(P)$, note that f is a convex linear combination of characteristic functions of principal ideals corresponding to a chain of elements of P, and since z is comparable with all elements of L, throwing it in still leaves a chain. Since H_0 is the identity map of $\Delta(P)$ and H_1 is a constant map (i.e., $H_1(f)$ is independent of f), $\Delta(P)$ is contractible. □

Examining the above proof, one sees that the simplicial complex $\Delta(P)$ is in fact a cone on $\Delta(P - \{z\})$. In particular, if L is a lattice with more than one element, then $\Delta(L)$ is a *cone on a cone* on $\Delta(L - \{0, 1\})$. For instance, if $L = M_3$, then $L - \{0, 1\}$ is a 3-element antichain, hence $\Delta(L - \{0, 1\})$ is a 3-point discrete space. The simplicial complex $\Delta(L - \{0\})$ or $\Delta(L - \{1\})$ is a cone on that set, and so looks like  (with the center point as greatest or least element, respectively); while $\Delta(L)$, a cone on the latter space, has, as we have seen, the form of three triangles connected along a common edge.

Another triviality result, specific to lattices, is

Lemma 24. *If L is a finite lattice, then $\Delta(L)$ is a retract of $[0, 1]^L$ by a piecewise linear order-preserving map. More generally, if* cl *a closure operator on a finite set X, then $\Delta(X, \mathrm{cl})$ is a retract of $[0, 1]^X$ by such a map.*

Sketch of proof. To get a retraction r in the more general situation, map each $f \in [0, 1]^X$ to the function $r(f)$ such that for $t \in (0, 1]$, $r(f)_t = \mathrm{cl}(f_t)$; equivalently, such that for $x \in L$, $r(f)(x)$ is the largest t such that $x \in \mathrm{cl}(f_t)$. □

For another such result, see [18, Theorem 6.2].

Neither of the above lemmas generalizes to lattices of the form $\Delta(L, S)$ described in Proposition 12. For example, if $L = \{0, 1\}$ and $S = \{(1, 1)\}$, then $\Delta(L, S)$ is a 2-point disconnected lattice. However, I don't know whether those lemmas hold for lattices $\Delta(L, S)$ in which S contains no pairs of the form (y, y). Such lattices will be connected; indeed, the proof of Proposition 12(i) adapts easily to show that for such S, $\Delta(L, S)$ contains the 1-skeleton of $\Delta(L)$. (This makes $\pi_0(\Delta(L))$ trivial, and by [18, Theorem 6.2] the higher homotopy groups of any topological semilattice are trivial; so contractibility indeed seems likely.)

5.2. $\Delta(L)$ **as a subspace of** $L^{[0,1]}$**.** We saw in §2.3 that the construction $\Delta(L)$ is the case $M=[0,1]$ of a general construction that is symmetric in two complete lattices L and M. Hence, though we have mainly been regarding $\Delta(L)$ as a subspace of $[0,1]^L$, the symmetric description shows that it can also be viewed as a subspace of $L^{[0,1]}$. Namely, to each $f\in\Delta(L)\subseteq[0,1]^L$ we can associate the member of $L^{[0,1]}$ taking each $t\in[0,1]$ to the element $h_t(f)\in L$ such that $f_t=\downarrow h_t(f)$. (Cf. last sentence of Theorem 1.)

Now for L a finite lattice, $L^{[0,1]}$, being an infinite direct product of finite sets, has a natural totally disconnected compact Hausdorff topology. This induces a topology on $\Delta(L)$, in general stronger (having more open sets) than its topology as a simplicial complex. To see that it is at least as strong as the latter, note that given any $\varepsilon>0$, if we choose $0<t_1<\cdots<t_n<1$ in $[0,1]$ with successive differences $<\varepsilon$, then the coordinates indexed by $t_1,\dots,t_n$ of the image in $L^{[0,1]}$ of an element $f\in\Delta(L)$ determine to within ε all the coordinates of f as a member of $[0,1]^L$. Thus, in $\Delta(L)$, every neighborhood of f with respect to the $[0,1]^L$-topology contains a neighborhood of f with respect to the $L^{[0,1]}$-topology. These topologies on $\Delta(L)$ are distinct if L has more than one element, since one is totally disconnected and the other is contractible. The $L^{[0,1]}$-topology is not, of course, compact, since distinct compact Hausdorff topologies on a space cannot be comparable. It follows that $\Delta(L)$, though closed in $[0,1]^L$, is not, in general, closed in $L^{[0,1]}$.

To see concretely the difference between the two topologies, note that if one moves $f\in\Delta(L)$ continuously in the $[0,1]^L$-topology, so that its value at some $x\in L$ comes up from below to a value $t\in(0,1]$, then when it reaches that value, the image of f in the t-indexed coordinate of $L^{[0,1]}$ jumps discontinuously from an element of L that does not majorize x to one that does. (However, we do have "continuity from above": if we move f *downward* continuously in $\Delta(L)$ with respect to the $[0,1]^L$-topology, then its image in $L^{[0,1]}$ at any moment *is* the limit of its earlier values.)

The failure of $\Delta(L)$ to be closed in $L^{[0,1]}$, deduced above, can be seen directly. Let L be the two-element lattice $\{0,1\}$. Then $\Delta(L)$, as a subset of $L^{[0,1]}=\{0,1\}^{[0,1]}$, corresponds to the set of characteristic functions of the principal ideals of $[0,1]$, i.e., the characteristic functions of the subintervals $[0,t]$ $(0\leq t\leq 1)$. It is easy to see that the closure of this set in the $L^{[0,1]}$-topology consists of these functions, together with the characteristic functions of the nonempty *non-principal* ideals, i.e., the sets $[0,t)$ $(0<t\leq 1)$.

Under both topologies, the meet and join operations of $\Delta(L)$ are continuous.

In subsequent sections we will, as before, consider our lattices $\Delta(L)$ to have the $[0,1]^L$-topology (and similarly for the objects $\Delta(P)$ where P is a finite partially ordered set or a $\vee$- or $\wedge$-semilattice, to which most of the above discussion applies).

5.3. **Functoriality.** Like the familiar power-set construction on sets, our construction $\Delta(\ )$ on posets, semilattices and lattices can be made into both covariant and contravariant functors; though we shall find that the contravariant constructions are more limited and less straightforward than the covariant ones. We begin with the covariant cases.

Proposition 25. *The construction $P\mapsto\Delta(P)$ of §1 can be made a covariant functor from the category of finite posets and isotone maps to the category of partially ordered simplicial complexes and isotone simplicial maps. Namely, given an isotone map $h:P\to Q$, we define the induced map $\Delta(h):\Delta(P)\to\Delta(Q)$ to carry an element f which is a convex linear combination of characteristic functions $\downarrow x_1,\dots,\downarrow x_n$ $(x_i\in P)$ to the convex linear combination of $\downarrow h(x_1),\dots,\downarrow h(x_n)$ with the same coefficients. Equivalently, for each $y\in Q$, $(\Delta(h)(f))(y)$ is the maximum of $f(x)$ over all $x\in P$ with $h(x)\geq y$.*

When applied to posets P in which every pair of elements has a join (respectively, a meet), and morphisms h that respect joins (meets), the functor gives morphisms $\Delta(h)$ that likewise respect joins (meets). Thus, it also gives covariant functors from the categories of finite $\vee$-semilattices, $\wedge$-semilattices, and lattices to those of topological $\vee$-semilattices, $\wedge$-semilattices and lattices.

Sketch of proof. It is straightforward to see that the construction described in the first paragraph gives a functor, and has the equivalent description noted in the last sentence of that paragraph.

In view of the descriptions of meets and joins in $\Delta(P)$ given in Theorem 1, the proof of the remaining assertions comes down to showing that if P has joins, respectively meets, and Q does likewise, and these are respected by h, then for all $f,g\in\Delta(P)$ and $t\in(0,1]$, we have $(\Delta(h)(f\vee g))_t=(\Delta(h)(f))_t\vee(\Delta(h)(g))_t$, respectively $(\Delta(h)(f\wedge g))_t=(\Delta(h)(f))_t\wedge(\Delta(h)(g))_t$, as principal ideals of Q. Now in both P and Q, joins (respectively, meets) of principal ideals correspond to the joins (respectively, meets) of their generating elements. Hence if h respects joins and/or meets of elements of P and Q, the map $\Delta(h)$, as we have defined it, will respect these operations on elements of $\Delta(P)$ and $\Delta(Q)$. □

We now give the more limited result holding for $\Delta(\ )$ as a contravariant functor. In connection with the statement below, note that in a *finite* $\vee$-semilattice P with 0, every subset has not only a least upper bound, but also a greatest lower bound; and dually for finite $\wedge$-semilattices with 1. Hence the difference between such semilattices and finite lattices merely concerns which maps among them are considered morphisms. In all these cases, the existence of both least upper bounds and greatest lower bounds necessarily carries over to the ordered simplicial complex $\Delta(P)$, since the construction of that object is the same whether we regard P as a poset, semilattice, or lattice.

Proposition 26. *On the category whose objects are finite partially ordered sets, and whose morphisms are set-maps h satisfying*

(23) *The inverse image under h of every principal ideal is a principal ideal,*

(*a subset of the isotone maps*)*, the construction $P \mapsto \Delta(P)$ of §1 can be made a* contravariant *functor Δ' to the category of partially ordered simplicial complexes and isotone simplicial maps. Namely, for $h : P \to Q$ satisfying* (23)*, we let $\Delta'(h) : \Delta(Q) \to \Delta(P)$ carry $f \in \Delta(Q)$ to $f \circ h \in \Delta(P)$.*

In particular, Δ' yields a contravariant functor from the category of finite $\vee$-semilattices with least element, 0*, and morphisms h preserving joins and least elements, to that of partially ordered simplicial complexes. However, the resulting morphisms $\Delta'(h)$ need* not *respect the join operations inherited by the complexes $\Delta(P)$* (*as described in Theorem 1 or Proposition 25*)*, even when applied to* 0*- and* 1*-respecting lattice homomorphisms between finite distributive lattices: but* will *respect the meet operations and greatest elements* 1 *of these complexes, even if the given morphism of $\vee$-semilattices does not.*

Proof. To see that a map $h : P \to Q$ satisfying (23) must be isotone, let $x \leq y$ in P. Then $h^{-1}(\downarrow h(y))$ will be a principal ideal of P containing y, hence containing $x \leq y$, which says that $h(x) \in \downarrow h(y)$, i.e., $h(x) \leq h(y)$, as required.

Given such an $h : P \to Q$, condition (23) shows that if $f : Q \to [0,1]$ belongs to $\Delta(Q)$, then $\Delta'(h)(f) = f \circ h$ belongs to $\Delta(P)$. It is straightforward to verify that $\Delta'(h)$ is an isotone simplicial map, and that the construction Δ', so defined, respects composition of morphisms, and identity morphisms.

To see that when P and Q are $\vee$-semilattices with 0, every homomorphism $h : P \to Q$ satisfies (23), note that the inverse image under h of any principal ideal of Q is nonempty (because it contains 0_P), is a downset, and is closed under $\vee$; hence, since P is finite, it will be a principal ideal.

For an example showing that even for h a homomorphism of finite distributive lattices respecting greatest and least elements, the map $\Delta'(h)$ need not respect joins, let Q be the 8-element lattice $\{0,1\}^3$, P its sublattice consisting of the four elements $(i,j,0)$ and the one element $(1,1,1)$, and h the inclusion of P in Q. Then the characteristic functions of $\downarrow(1,0,1)$ and $\downarrow(0,1,1)$ in $\Delta(Q)$ have for inverse images in $\Delta(P)$ the characteristic functions of $\downarrow(1,0,0)$ and $\downarrow(0,1,0)$, whose join is $\downarrow(1,1,0)$. However, the join of the given characteristic functions in $\Delta(Q)$ is $\downarrow(1,1,1)$, whose inverse image in $\Delta(P)$ is $\downarrow(1,1,1)$.

One sees, however, that the maps $\Delta'(h)$ respect meets, including the empty meet, by noting that $(f \circ h)_t = h^{-1}(f_t)$, and combining this with the facts that taking inverse images under a set map respects intersections of subsets (including the empty intersection, i.e., the greatest element), and that intersection as subsets gives the meet operation on principal ideals in lattices. □

5.4. **Representability.** The covariant version of our order complex construction, and to a more limited extent, the contravariant variant version, are *representable.*

The superficially more obvious case is the contravariant one: For P a finite partially ordered set, $\Delta(P)$ is a collection of antitone (i.e., order-reversing) maps from P to the interval $[0,1]$, so its points should be the morphisms $P \to [0,1]^{\mathrm{op}}$ in an appropriate category. If we denote by $\mathbf{Poset}_{\text{prin-id}}$ the category whose objects are all partially ordered sets, and where the morphisms are the maps under which the inverse image of every principal ideal is a principal ideal, then this works – except where the anomalous treatment of $0 \in [0,1]^{\mathrm{op}}$, discussed in §1.3, interferes. We find that at the underlying-set level we have

$$(24) \qquad \mathbf{Poset}_{\text{prin-id}}(P,\,[0,1]^{\mathrm{op}}) \;=\; \begin{cases} \Delta'(P) & \text{if } P \text{ has a greatest element} \\ \emptyset & \text{otherwise.} \end{cases}$$

We shall not look further at this case.

For the covariant functor Δ, things work out more nicely. We saw in §1 that each $f \in \Delta(P)$ is determined by the chain of principal ideals f_t as t ranges over $(0,1]$. Each such principal ideal f_t has the form $\downarrow h_t(f)$ for some $h_t(f) \in P$ (§1.2), and f is determined by this chain of elements, which we can regard as a morphism $h(f) : (0,1]^{\mathrm{op}} \to P$.

In what category? The above category $\mathbf{Poset}_{\text{prin-id}}$ would do; but in fact, morphisms $(0,1]^{\mathrm{op}} \to P$ in that category are the same as the morphisms $(0,1]^{\mathrm{op}} \to P$ in the much larger category $\mathbf{Poset}_{\text{chain-sup}}$ of isotone maps that respect least upper bounds of chains (when these exist). So for finite partially ordered sets P, we can make the identification

(25) $$\mathbf{Poset}_{\text{chain-sup}}((0,1]^{\mathrm{op}}, P) \ = \ \Delta(P).$$

Note that isotone maps among *finite* partially ordered sets trivially respect least upper bounds of chains, explaining why in Proposition 25 we were able to define the covariant version of Δ on all such maps. It is straightforward to verify that the behavior of $\mathbf{Poset}_{\text{chain-sup}}((0,1]^{\mathrm{op}}, -)$ on morphisms agrees with the behavior described there; in particular, it gives isotone maps that are continuous, and in fact, are simplicial maps.

Covariant representable set-valued functors are known to preserve *limits* in the category-theoretic sense (products of objects, equalizers of pairs of morphisms, and constructions that can be obtained from these) [16, Theorem V.4.1], [5, Theorem 8.8.7]. Let us show that Δ respects limits, not only as a set-valued functor, but as a functor to partially ordered topological spaces. Since we are only considering Δ to be defined on *finite* partially ordered sets, our result will concern limits over finite diagrams.

Theorem 27. *As a covariant functor from finite partially ordered sets to partially ordered compact Hausdorff spaces, Δ respects limits over finite diagrams.*

Hence the same is true of Δ as a covariant functor from finite $\vee$-semilattices, $\wedge$-semilattices, or lattices to compact Hausdorff spaces with structures of topological $\vee$-semilattice, $\wedge$-semilattice, or lattice.

Sketch of proof. Let **FinPoset** be the category of finite partially ordered sets, which we have noted is a *full* subcategory of $\mathbf{Poset}_{\text{chain-sup}}$. Let $\mathbf{D}$ be a finite category (one with finitely many objects and finitely many morphisms), and $F : \mathbf{D} \to \mathbf{FinPoset}$ a functor. Since both **FinPoset** and the category of partially ordered compact Hausdorff spaces admit finite limits, we get a comparison morphism

(26) $$C : \ \Delta(\varprojlim_{\mathbf{D}} F) \ \to \ \varprojlim_{\mathbf{D}} (\Delta \circ F).$$

[5, Definition 8.8.13 et seq.].

Because at the set level Δ is representable within the category $\mathbf{Poset}_{\text{chain-sup}}$, and therefore respects limits, the continuous map (26) will be a bijection. But a continuous bijection between compact Hausdorff spaces is a homeomorphism.

To show that C is an order isomorphism, let us again call on the fact that within the larger category $\mathbf{Poset}_{\text{chain-sup}}$, the object $(0,1]^{\mathrm{op}}$ represents the underlying set of Δ. It follows that if we form the object $(0,1]^{\mathrm{op}} \times \{0,1\}$, with the product ordering, then a morphism from this object to a finite partially ordered set P will correspond to a pair (f,g) of members of $\Delta(P)$ with $f \leq g$; hence $(0,1]^{\mathrm{op}} \times \{0,1\}$ represents the functor giving the graph of the order relation on $\Delta(P)$. Since this functor is representable, it respects limits; so the orderings on $\Delta(\varprojlim F)$ and $\varprojlim(\Delta \circ F)$ agree, as claimed.

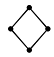
The final sentence holds because limits of the indicated algebraic structures have as their underlying partially ordered sets, respectively topological spaces, the limits of those sets or spaces. ☐

The case of the above result where $\mathbf{D}$ is a 2-object category with only the two identity morphisms says that for any two posets P and Q we have

(27) $$\Delta(P \times Q) \ \cong \ \Delta(P) \times \Delta(Q).$$

This isomorphism is classical [9, Lemma II.8.9], [20, Theorem 3.2], but the proofs in those works are quite complicated. In §5.6 we will give a "hands-on" proof of the same result, which still seems a bit simpler than the proofs cited.

Can we say anything about the behavior of Δ on *colimits*? In general, covariant representable functors do not respect colimits; but in some particular classes of cases they do [5, §8.9] [4]. I claim that our functor Δ respects finite coproducts of partially ordered sets. Note that a coproduct $P \amalg Q$ of posets is their disjoint union, with each poset having its original order structure, and no order-relations holding between elements of P and elements of Q. Hence, an isotone map from $(0,1]^{\mathrm{op}}$ (or, indeed, any chain) to $P \amalg Q$ must have image entirely in P or entirely in Q. So $\Delta(P \amalg Q)$ is a disjoint union of $\Delta(P)$ and $\Delta(Q)$; and it is easy to see that it also has the right structure of ordered topological space to be their coproduct.

But this does not yield corresponding statements for coproducts of semilattices or lattices. Colimits of algebraic structures, unlike limits, do not typically arise from colimits of their underlying sets (or, in this case, their underlying posets), so the above argument does not imply that Δ respects coproducts of lattices or semilattices; and in fact, it does not. For instance, if P and Q are each 1-element lattices, then their coproduct as lattices is the 4-element lattice , and applying Δ to that, we get a lattice consisting of two triangles (2-simplices) fitted together to form a diamond. In contrast, the topological lattices $\Delta(P)$ and $\Delta(Q)$ are, like P and Q, 1-point lattices, whose coproduct is, as just noted, a 4-element lattice.

5.5. **A discrete analog of $\Delta(P) \cong \Delta'(P)$.** The fact that the geometric construction $\Delta(P)$ can be regarded as the values of either a covariant or a contravariant functor has an analog purely in the realm of discrete posets P, in which moreover, there is no reason to restrict attention to finite P. I sketch it below.

The idea is to replace the interval $[0,1]$ in the construction Δ with the 2-element object $\{0,1\}$. Thus, we associate to every poset P the set of $\{0,1\}$-valued functions $f: P \to \{0,1\}$ such that the set $f^{-1}(1)$ is a principal ideal. This, of course, is just the set of characteristic functions of principal ideals of P, and can be identified with P; and indeed, if we make this construction a covariant functor as in Proposition 25, it is just the identity functor of **Poset**.

The contravariant functor analogous to Δ' should be a functor on $\mathbf{Poset}_{\text{prin-id}}$, carrying each morphism $h: P \to Q$ in that category to the map that takes every principal ideal $\downarrow q$ to the principal downset $h^{-1}(\downarrow q)$ of P; or, looked at as a map on elements, that takes q to the generator of $h^{-1}(\downarrow q)$. Let us call this functor G.

As in the case of Δ', the maps $G(h): Q \to P$ will be isotone. However, they will not in general be morphisms of $\mathbf{Poset}_{\text{prin-id}}$. Rather, they turn out to have the dual property of carrying each principal *filter* $\uparrow p$ to a principal filter of Q, namely $\uparrow h(p)$. In fact, one finds that G gives a contravariant equivalence between $\mathbf{Poset}_{\text{prin-id}}$ and $\mathbf{Poset}_{\text{prin-filt}}$.

Given any two partially ordered sets P and Q, the structure given by a pair of morphisms related under this equivalence,

$$(28) \qquad P \xrightarrow[\mathbf{Poset}_{\text{prin-id}}]{h} Q \qquad \text{and} \qquad Q \xrightarrow[\mathbf{Poset}_{\text{prin-filt}}]{G(h)} P$$

turns out to be what is called a *Galois connection* between P and Q, with h the "lower adjoint" and $G(h)$ the "upper adjoint" [22].

The functor Δ' of §5.3 can be described in terms of the above construction, as the composite $\Delta \circ G$.

If P and Q are complete lattices, it is not hard to show that the morphisms $P \to Q$ in $\mathbf{Poset}_{\text{prin-id}}$ are precisely the maps $P \to Q$ that respect all joins (empty, finite, and infinite), while morphisms in $\mathbf{Poset}_{\text{prin-filt}}$ are those that similarly respect meets. Hence applied to such subjects, the functor G yields a contravariant equivalence between the category of complete lattices and maps that respect all joins – in other words, the category of complete $\vee$-semilattices with least element, understood to have for morphisms the maps that respect all the operations of such semilattices (zeroary, finitary, finite and infinitary) – and the category with the same objects, but having for morphisms the maps that similarly respect all meets – the category of complete $\wedge$-semilattices with greatest element. Restricting attention to finite lattices, and composing with Δ, this explains the properties of Δ' noted in the last paragraph of Proposition 26.

5.6. **A hands-on construction of $\Delta(P \times Q)$.** Returning to the geometric construction $\Delta(-)$, let us end this section by giving, as promised, a proof of the case of Theorem 27 where the limit in question is a pairwise direct product which does not rely on general properties of representable functors, but only on §5.3. (I wrote this up before coming on the approach of §5.4. If the referee recommends dropping this section, I shall do so.)

Proposition 28 ([9, Lemma II.8.9], [20, Theorem 3.2])**.** *For P and Q finite posets, $\Delta(P) \times \Delta(Q) \cong \Delta(P \times Q)$ as partially ordered topological spaces, via the map taking $(f_P, f_Q) \in \Delta(P) \times \Delta(Q)$ to the element $f \in \Delta(P \times Q)$ defined by $f(x,y) = f_P(x) \wedge f_Q(y)$ $(x \in P,\ y \in Q,$ infimum "$\wedge$" taken in $[0,1])$.*

Proof. That the function $f : P \times Q \to [0,1]$ given by $f(x,y) = f_P(x) \wedge f_Q(y)$ belongs to $\Delta(P\times Q)$ is easily verified: for each $t \in (0,1]$, f_t will be $(f_P)_t \times (f_Q)_t$, so writing $(f_P)_t = \downarrow h_t(f)$, $(g_P)_t = \downarrow h_t(g)$, we see that $f_t = \downarrow h_t(f) \times \downarrow h_t(g) = \downarrow (h_t(f), h_t(g))$, a principal ideal of $P \times Q$.

The resulting map $\Delta(P) \times \Delta(Q) \to \Delta(P \times Q)$ is clearly isotone; i.e., if $(f_P, f_Q) \leq (g_P, g_Q)$ in $\Delta(P) \times \Delta(Q)$, then the elements of $\Delta(P \times Q)$ constructed from these satisfy $f \leq g$. Conversely, if $(f_P, f_Q) \not\leq (g_P, g_Q)$, then assuming without loss of generality that $f_P \not\leq g_P$, we can choose $x \in P$ such that $f_P(x) > g_P(x)$. Taking a $y \in Q$ such that $f_Q(y) = 1$ (i.e., any element of the principal ideal $(f_Q)_1$), we get $f(x,y) = f_P(x) \wedge f_Q(y) = f_P(x) > g_P(x) \geq g_P(x) \wedge g_Q(y) = g(x,y)$; so $f \not\leq g$. Thus, the map described is an embedding of posets.

It remains to show that it is surjective. Given $f \in \Delta(P \times Q)$, let us choose $(p,q) \in P \times Q$ such that $f(p,q) = 1$, and define $f_P(x) = f(x,q)$ and $f_Q(y) = f(p,y)$ for all $x \in P$, $y \in Q$. We want to show that f_P and f_Q belong to $\Delta(P)$ and $\Delta(Q)$ respectively, and that the element $f' \in \Delta(P \times Q)$ defined by

(29) $$f'(x,y) = f_P(x) \wedge f_Q(y)$$

is equal to f. Thus, we need to show for every $t \in (0,1]$ that the sets $(f_P)_t$ and $(f_Q)_t$ are principal ideals, and that the sets f'_t and f_t coincide.

Given $t \in (0,1]$, let $f_t = \downarrow (x_0, y_0)$ for $x_0 \in P$, $y_0 \in Q$. Since $f(p,q) = 1$, we have $(p,q) \in f_t = \downarrow(x_0,y_0)$, whence

(30) $$p \in \downarrow x_0 \text{ and } q \in \downarrow y_0.$$

We now compute $(f_P)_t = \{z \in P \mid f_P(z) \geq t\} = \{z \in P \mid f(z,q) \geq t\} = \{z \in P \mid (z,q) \in f_t\} = \{z \in P \mid (z,q) \in \downarrow(x_0,y_0)\} = \downarrow x_0$ (the last step by the second relation of (30)). Similarly $(f_Q)_t = \downarrow y_0$. So as required, $(f_P)_t$ and $(f_Q)_t$ are principal, and by (29), $f'_t = \downarrow(x_0,y_0) = f_t$. □

6. Stitching lattices together

6.1. **The construction.** We noted in §3.1 that $\Delta(M_3)$ consists of three triangles, each a distributive sublattice corresponding to a maximal chain in M_3, joined along a common edge. Is this an instance of a general way that lattices can be attached together?

In [14, §IV.2, esp. IV.2.3], several ways of attaching lattices to one another are discussed, under the names *gluing, pasting, S-gluing* and *multipasting.* The next lemma shows that the way $\Delta(M_3)$ is put together from sublattices can also be generalized. We shall see in §6.3 that it does not fall under any of those previously studied patterns.

Lemma 29. *Suppose $(P_i)_{i\in I}$ is a nonempty family of posets, whose order-relations we shall write $\leq_i$, such that the P_i are pairwise disjoint except for a common chain C which has the same order structure in all the P_i. Suppose moreover that for each $x \in P_i$ $(i \in I)$ there is a least element $u_i(x) \in C$ that is $\geq_i x$, and a greatest element $d_i(x) \in C$ that is $\leq_i x$.* (*Note that if C is* finite, *then this condition holds if and only if each P_i has a least and a greatest element, and these lie in C. The letters u and d are mnemonic for "up" and "down"*).

Then the set $P = \bigcup_I P_i$ may be partially ordered by taking elements $x \in P_i$, $y \in P_j$ to satisfy $x \leq y$ if and only if either $i = j$ and $x \leq_i y$, or $i \neq j$ and the following equivalent conditions hold: (i) $u_i(x) \leq d_j(y)$, (ii) $u_i(x) \leq_j y$, (iii) $x \leq_i d_j(y)$, (iv) $x \leq_i z \leq_j y$ *for some* $z \in C$.

If the P_i are $\vee$-semilattices, then the resulting poset P is also a $\vee$-semilattice; namely, for $x \in P_i$, $y \in P_j$, we have $x \vee y = x \vee_i y$ if $i = j$, while if $i \neq j$, and we assume without loss of generality that $u_i(x) \geq u_j(y)$, then $x \vee y = x \vee_i u_j(y) \in P_i \subseteq P$.

If the P_i are $\wedge$-semilattices, then so is P, by the dual construction.

Hence if the P_i are lattices, P is a lattice.

Sketch of proof. The parenthetical observation on the case where C is finite in the first paragraph is immediate, as is the verification of the equivalence of conditions (i)-(iv) of the second paragraph. The verification that the relation $\leq$ defined in that paragraph is a partial ordering is routine; this includes the fact that it is well-defined; namely, that if x and/or y lies in C, the condition for $x \leq y$ to hold does not depend on which posets P_i and/or P_j those elements are regarded as lying in.

Turning to the description of $x \vee y$, it clearly gives in each case an upper bound to x and y, so we must verify that this is majorized by any upper bound z of those elements, say lying in P_k.

In the case $i = j$, the desired result is clear if $k = i$. If $k \neq i$, then if z majorizes x and y, it must majorize $u_i(x)$ and $u_i(y)$, hence it majorizes the larger of these, which is $\geq x \vee_i y$, as required.

In the case where $i \neq j$ and $u_i(x) \geq u_j(y)$, we note that k must be distinct from at least one of i and j, hence by definition of the ordering of P, z majorizes at least one of $u_i(x)$, $u_j(y)$; hence it majorizes the smaller of them, $u_j(y)$. So in P it majorizes both x and $u_j(y)$, which both lie in P_i; so by the preceding case it majorizes $x \vee_i u_j(y)$, the asserted join of x and y, as desired.

The dual assertion follows by symmetry. Hence when the P_i are lattices we get the final assertion. □

We remark that the assumption that C is a chain was not needed for our construction of the partial order on P, but only for the verification that if the P_i have meets and/or joins, so does P; and for the parenthetical note on the case where C is finite.

Following the theme of the terms "gluing" and "pasting", let us call the P of Lemma 29 the poset, semilattice or lattice obtained by *stitching* the P_i together along C.

6.2. **When $\Delta(P)$ can be obtained by stitching.** What instances of the stitching construction can we hope will be respected by the construction Δ? For this to hold, not only C, but also $\Delta(C)$ must be a chain; hence C must have ≤ 2 elements. Ignoring the trivial cases where it has zero or one element, the next lemma shows that for C having two elements the desired result holds. In particular, it gives the description of $\Delta(M_3)$ which motivated these considerations.

Lemma 30. *Let $(P_i)_{i \in I}$ be a finite nonempty family of finite posets, disjoint except for a common least element 0 and a common greatest element 1, and let P be the poset obtained by stitching the P_i together along $C = \{0, 1\}$.*

Then the poset $\Delta(P)$ can be obtained by stitching together the $\Delta(P_i)$ along their common 1-simplex $\Delta(\{0, 1\})$.

Proof. The case where the index-set I has just one element is trivial, so assume the contrary. It is easy to see that for P as described, elements of distinct sets $P_i - C$, $P_j - C$ will be incomparable, hence every chain in P must be contained in one of the P_i, hence will lie in more than one of them if and only if it lies in $\{0, 1\}$. It follows from these observations and the description of the operator Δ in terms of chains of elements that $\Delta(P)$ can be identified with the union of the $\Delta(P_i)$, and that the $\Delta(P_i)$ are pairwise disjoint except for the common 1-simplex $\Delta(\{0, 1\})$.

Note that for $x \in P_i$, the principal ideal generated by x in P is the same as the principal ideal generated by x within P_i, except when $x = 1$, in which case it is all of P. From this we can see that the embedding of $\Delta(P_i)$ in $\Delta(P)$ carries each $f \in \Delta(P_i)$ to the function that agrees with f on P_i, and takes the value $f(1)$ everywhere else in P. Likewise, the embeddings of $\Delta(\{0, 1\})$ in the spaces $\Delta(P_i)$ and $\Delta(P)$ take each $f \in \Delta(\{0, 1\})$ to the function which has the value $f(1)$ at all elements other than 0.

From this it follows that in $\Delta(P_i)$, each element f is majorized by a smallest element $u_i(f) \in \Delta(\{0, 1\})$, namely, the function whose value at 1 is the greatest of the values of f at points of P_i other than 0. Likewise, f majorizes a largest element $d_i(f) \in \Delta(\{0, 1\})$, namely, the function on $\{0, 1\}$ whose value at 1 is $f(1)$.

It remains to verify that the order relation $f \leq g$ of $\Delta(P)$ is described as in the second paragraph of Lemma 29. We first note for f and g in the same set $\Delta(P_i)$, our description of how elements of that space extend to functions on P shows that they satisfy $f \leq g$ in $\Delta(P)$ if and only if they satisfy the same inequality in $\Delta(P_i)$. From this case, it easily follows that for f and g lying in distinct spaces $\Delta(P_i)$, $\Delta(P_j)$, the equivalent conditions (i)-(iv) of Lemma 29 also imply that when f and g are extended to elements of $\Delta(P)$, they satisfy $f \leq g$ there.

Conversely, suppose $f \leq g$ in $\Delta(P)$, where $f \in \Delta(P_i)$ and $g \in \Delta(P_j)$ with $i \neq j$. In particular, the values of f at all points of $P_i - \{0\}$ must be $\leq$ the values of g at these points; but the latter are $g(1)$; so f is majorized by the function on P which is $g(1)$ everywhere on $P - \{0\}$. Since that function is $d_j(g)$, we get condition (iii) of Lemma 29. □

6.3. **Stitching $\neq$ gluing, etc.** To verify that stitching of lattices does not fall under the list of lattice constructions given in [14, §IV.2], we recall that for all of those constructions, the lattice constructed has the universal mapping property of the colimit of the diagram formed from the given lattices and sublattices. (In the case of multipasting, this is made part of the definition, [12, Definition 6].) But the same is not true of stitching. For instance, if we stitch together two three-element chains $\{0 < a < 1\}$, $\{0 < b < 1\}$ along the common chain $\{0 < 1\}$, the result is a 4-element lattice, isomorphic to $\{0,1\}^2$; but this does not have the universal property referred to, since the inclusions of the two given lattices in, say, the chain $\{0 < a < b < 1\}$ do not factor through $\{0,1\}^2$.

Note also that gluing and pasting preserve the class of modular lattices [11, Theorem 16], [14, Theorem 303]; but stitching together the chains $\{0 < a < b < 1\}$ and $\{0 < c < 1\}$ along $\{0,1\}$ gives the nonmodular lattice N_5.

7. Acknowledgements

I am indebted to George Grätzer, Patricia Hersh, Walter Taylor, Fred Wehrung, and Volkmar Welker for invaluable pointers to what is known on some of these topics.

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