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Stratification of Derived Categories of Tate Motives

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STRATIFICATION OF DERIVED CATEGORIES OF TATE MOTIVES

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Contents

Abstract **v**

Dedication **vi**

Acknowledgments **vii**

1 Introduction **1**

2 Preliminaries **10**
2.1 Definitions and Basic Properties 10
2.2 Verdier and Bousfield Localization 17
2.3 Compact Objects and Compact Generation 21
2.4 Finite Localization 23

3 Tensor-Triangulated Categories **25**
3.1 Rigidly-Compactly Generated tt-Categories 26
3.2 Smashing Localization 28
3.3 The Balmer Spectrum and Classification 31
3.3.1 Balmer Spectrum 31
3.3.2 Classification Theorem 33
3.3.3 Examples 35

4 Balmer–Favi Support and Stratification **37**
4.1 Idempotent Triangles 37
4.2 Balmer–Favi Support 39
4.3 Stratification 42

5 Motives **47**
5.1 Derived Categories of Motives 47
5.2 Derived Categories of Étale Motives 48
5.3 Balmer Spectrum of Tate Motives 50

6 Minimality at the Height Zero Prime **53**
6.1 t-structures 53
6.2 t-structure for Rational Motives 58

7 Minimality at the Height One Prime **64**

8 Brown–Adams Representability and Stratification **67**
8.1 Brown–Adams Representability 69
8.2 Brown–Adams Representability and Stratification 70

9 Minimality at the Height Two Prime **78**

10 Stratification of Tate Motives **84**

Bibliography **85**

Abstract

Stratification of Derived Categories of Tate Motives

by

David Rubinstein

We classify the localizing tensor ideals of the derived categories of mixed Tate motives over certain algebraically closed fields. More precisely, we prove that these categories are stratified in the sense of Barthel, Heard and Sanders. A key ingredient in the proof is the development of a new technique for transporting stratification between categories by means of Brown–Adams representability, which may be of independent interest.

לזכּרון די באָבע־זיידע מייַנע ע"ה. זאָלן זיי האָבן אַ ליכטיקן גן־עדן

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איך שעפּ גאָר אַ סך נחת פֿון דיר. איך האָב דיך שטאַרק ליב, מײַן באַשערטע.

Chapter 1

Introduction

Categories are pervasive in mathematics, and whenever there is a category there is a notion of isomorphism, or sameness. A goal we might strive towards therefore is to classify all the objects in a category up to this notion of sameness. However, it was understood early on that we are often confronted with 'wild' classification problems: there is usually no hope of classifying up to isomorphism all finite dimensional representations of a group G in positive characteristic; no hope of classifying topological spaces up to homotopy equivalence; no hope of classifying all vector bundles on an algebraic variety up to isomorphism; and so on and so forth.

In light of this impossibility, there has been an important paradigm shift in the last few decades: one changes perspective and works 'stably.' In other words, one changes the category and asks for a weaker notion of classification in an associated 'stable' category. For example, rather than working in the category of representations of a group, $rep(kG)$, one works in the category of 'stable representations,' $stab(kG)$. Similarly, rather than considering the category of topological spaces up to homotopy, $\mathrm{Ho}(Top)$, one considers the 'stable homotopy category,' SH. A unifying theme is that these stable categories have a tensor-triangulated structure —

they are 'tensor-triangulated categories.' In these tensor-triangulated categories, one asks not for a classification of objects 'on the nose,' but rather a classification of objects up to a weaker notion of equivalence in which two objects are deemed equivalent if they can build each other using the tensor-triangulated structure. What this technically amounts to is a classification of the so called 'thick tensor ideals' of the stable category. This new perspective led to a myriad of landmark theorems in the late 1990s in algebraic topology [HS98], modular representation theory [BCR97] and algebraic geometry [Nee92], [Tho97].

These important theorems were greatly clarified and unified via the work of Paul Balmer [Bal05], who gave an abstract classification theorem for tensor-triangulated categories. In short, Balmer constructed a topological space $Spc(\mathcal{K})$, called the Balmer spectrum, out of a tensor triangulated category $\mathcal{K}$. This topological space comes equipped with the universal notion of support for the objects in $\mathcal{K}$, and this universal notion of support always classifies the objects in $\mathcal{K}$ up to the above weak notion of equivalence; namely, two objects are equivalent if and only if their supports coincide. From this perspective, the classification of objects in $\mathcal{K}$ is equivalent to a computation of the Balmer spectrum $Spc(\mathcal{K})$. Indeed, the aforementioned landmark theorems can be phrased as statements about the Balmer spectrum.

While the theory of the Balmer spectrum is conceptually very satisfying and powerful, there is an issue with the scope of its applicability: many of the stable categories of interest arise as a 'finite' piece of a larger stable category (e.g. finite dimensional vs infinite dimensional stable representations $stab(kG) \subset Stab(kG)$; or perfect complexes rather than arbitrary complexes $D^{\text{perf}}(R) \subset D(R)$). Technically, these categories arise as the subcategories of "compact" objects of a larger stable category, and Balmer's classification theorem only applies to these categories of compact objects. However, these larger objects are some of the most interesting: for example, one of the main insights of stable homotopy theory is that all cohomology theories become representable when working stably, but the objects that represent them are not compact.

Barthel, Heard, and Sanders, motivated by earlier work of Neeman and Benson, Iyengar and Krause, have made significant progress in deducing a classification theorem for such 'large' tensor-triangulated categories [BHS23b]. In particular, they develop a theory of stratification for a rigidly-compactly generated tensor-triangulated category $\mathcal{T}$ which provides a framework for classifying the localizing ideals of $\mathcal{T}$. At its heart is a support theory *Supp* for the large category $\mathcal{T}$ that takes value in the Balmer spectrum of compact objects $Spc(\mathcal{T}^c)$. This support theory is given by tensoring with certain tensor idempotent objects $g(\mathcal{P}) \in \mathcal{T}$ associated to each $\mathcal{P} \in Spc(\mathcal{T}^c)$. Explicitly, we have

$$Supp(t) = \{\mathcal{P} \in Spc(\mathcal{T}^c) : t \otimes g(\mathcal{P}) \neq 0\}$$

for each $t \in \mathcal{T}$. The category $\mathcal{T}$ is ***stratified*** if this support theory provides an order-preserving bijection between the collection of localizing tensor ideals of $\mathcal{T}$ and the collection of all subsets of $Spc(\mathcal{T}^c)$.

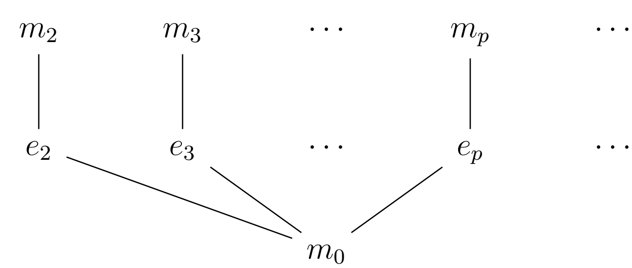
In this thesis, we study a particular example of a rigidly-compactly generated tensor-triangulated category, arising from the theory of Motives. Motives originated in the work of Grothendieck and his students in the 1960s in their study of algebraic cycles and cohomology theories in algebraic geometry (see [Gro69, Man68]). The unifying idea was that there should be a "universal cohomology theory" in algebraic geometry, which they called motivic cohomology. Unfortunately, they never succeeded in constructing the conjectured category of motives, from which motivic cohomology arises. However, much later on, Voevodsky [Voe00] introduced a "derived category" of motives, denoted $DM(\mathbb{F}, R)$, which is morally the 'stable version' of the conjectured category of motives. His groundbreaking work (see [Voe98, MV99, VSF00]) greatly increased our understanding of motivic cohomology, and using this theory, Voevodsky was able to solve some longstanding fundamental conjectures in algebraic geometry: namely, the Milnor conjecture and the Bloch–Kato conjecture [SV00, Voe11, Voe10].

This category $DM(\mathbb{F}, R)$ contains an extraordinary amount of information: for example, every algebraic variety over the field $\mathbb{F}$ gives rise to an object in this category. To avoid some of this complexity, a common approach is to restrict our attention to a smaller tensor-triangulated subcategory. One of these is the subcategory of *Tate Motives*, denoted $DTM(\mathbb{F}, R) \subset DM(\mathbb{F}, R)$.

Recent progress has been made in understanding the structure of the compact part of this category over certain algebraically closed fields, through the beautiful computation of its Balmer spectrum by Martin Gallauer [Gal19].

Theorem 1.0.1 (Gallauer)**.** *The Balmer spectrum of $DTM(\overline{\mathbb{Q}}, \mathbb{Z})^c$ is the following topological space:*

$$Spc(DTM(\overline{\mathbb{Q}}, \mathbb{Z})^c) =$$



These primes correspond to vanishing of various étale and motivic cohomology groups. We will explain in more detail this computation in Chapter 5. Our main theorem is the following (see Theorem 10.0.1):

Theorem 1.0.2 (Stratification)**.** *The category $DTM(\overline{\mathbb{Q}}, \mathbb{Z})$ is stratified.*

To reiterate, this provides classification of Tate motives in $DTM(\overline{\mathbb{Q}}, \mathbb{Z})$: two Tate motives t_1, t_2 are equivalent precisely when their supports agree, that is $Supp(t_1) = Supp(t_2)$. There are many consequences of a category $\mathcal{T}$ being stratified (see, e.g. [BHS23b, Part III]). For example, if $\mathcal{T}$ is stratified and $Spc(\mathcal{T}^c)$ satisfies a mild topological assumption, then we can answer a version of the *telescope conjecture* (see [BHS23b, Theorem 9.11]). This famous conjecture [DHS88, Rav84] was phrased originally for SH (and has recently been shown to be false in SH! see [BHLS23]) but it can be translated to any tensor-triangulated category, see Remark 3.2.5. The mild topological condition holds for $Spc(DTM(\overline{\mathbb{Q}}, \mathbb{Z})^c)$ for example, so as a corollary to our main theorem, we get (see Theorem 10.0.2):

Theorem 1.0.3 (Telescope Conjecture)**.** *The Telescope Conjecture holds for $DTM(\overline{\mathbb{Q}}, \mathbb{Z})$.*

Proving that a category is stratified amounts to establishing two properties: (a) the 'local-to-global' principle, which morally says every object can be reconstructed from its local pieces; and (b), the 'minimality' property, which says the localizing ideals generated by the tensor idempotents $g(\mathcal{P})$ are minimal. Importantly, Barthel, Heard and Sanders, building off the earlier work of Stevenson, prove that if the spectrum is noetherian, as the above spectrum of Tate motives is for example, the local-to-global principle always holds, and so the problem reduces to proving the minimality property. Moreover, minimality is a local condition, so the bulk of the thesis is devoted to proving that minimality holds at the unique closed points in the corresponding local categories for each prime in $Spc(DTM(\overline{\mathbb{Q}}, \mathbb{Z})^c)$. While there are infinitely many primes in $Spc(DTM(\overline{\mathbb{Q}}, \mathbb{Z})^c)$, there are qualitatively only 3 distinct primes to consider: $m_0, e_p,$ and m_p.

The local category associated to the generic point m_0 is equivalent to the category of rational motives $DTM(\overline{\mathbb{Q}}, \mathbb{Q})$. We prove this category is stratified in Chapter 6 using an extension of work on t-structures for rational motives by Peter and Levine [Pet13, Lev93].

For the heights one and two primes, e_p and m_p, we use a result similar in spirit to the quasi-finite descent of [BCH$^+$23] to reduce to the following two local categories:

1. $DTM^{\text{ét}}(\overline{\mathbb{Q}}, \mathbb{Z}/p)$, which corresponds to e_p (see Chapter 7), and

2. $DTM(\overline{\mathbb{Q}}, \mathbb{Z}/p)$, which corresponds to m_p (see Chapter 9).

Stratification of $DTM^{\text{ét}}(\overline{\mathbb{Q}}, \mathbb{Z}/p)$ is an immediate consequence of the 'Rigidity Theorem' for étale motives with finite coefficients (see Theorem 7.0.2). On the other hand, in proving $DTM(\overline{\mathbb{Q}}, \mathbb{Z}/p)$ is stratified in Chapter 9, we are led naturally to the following question, which may be of independent interest.

Question 1.0.4. Suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ are two rigidly-compactly generated tensor-triangulated categories with a tt-equivalence between their compact parts $\mathcal{T}_1^c \simeq \mathcal{T}_2^c$. If $\mathcal{T}_1$ is stratified, does it follow that $\mathcal{T}_2$ is stratified as well?

In Chapter 8 we investigate this question through the lens of Brown–Adams representability. More precisely, given a rigidly-compactly generated tt-category $\mathcal{T}$, we study the module category $\mathcal{A} := Add((\mathcal{T}^c)^{op}, \mathcal{A}b)$ along with the restricted Yoneda functor

$$\begin{aligned} h : \mathcal{T} &\to \mathcal{A} \\ t &\to \hat{t} := Hom(-, t)|_{\mathcal{T}^c}. \end{aligned}$$

The essential image of restricted Yoneda lies in the subcategory of homological functors, and in certain examples a much stronger relationship holds. For example, Adams showed in [Ada71] that for the stable homotopy category $\mathcal{T} = \text{SH}$, every homological functor $\mathcal{H} : (\mathcal{T}^c)^{op} \to \mathcal{A}b$ is the restriction to $\mathcal{T}^c$ of a representable functor on $\mathcal{T}$, and every map between homological functors is induced by a (non-unique) map between the representing spectra. Due to this historical example, a tt-category is said to satisfy Brown–Adams representability if it satisfies those above two properties. Determining whether or not a category satisfies Brown–Adams representability is a challenging problem in general. Importantly for us however, Neeman, generalizing a theorem of Brown, showed in [Nee97] that a sufficient condition is that the subcategory of compact objects $\mathcal{T}^c$ is equivalent to a countable category.

The category of modules $\mathcal{A}$, and its relation to $\mathcal{T}$, has also been studied by Balmer, Krause and Stevenson in a series of recent papers [BKS19, BKS20, Bal20a, Bal20b]. In particular, they use $\mathcal{A}$ to define a new spectrum, the so-called homological spectrum $Spc^h(\mathcal{T}^c)$. Associated to each 'homological prime' $\beta \in Spc^h(\mathcal{T}^c)$ is a certain pure-injective object $E_\beta \in \mathcal{T}$, and these can be used to define a new support theory for $\mathcal{T}$. These E_β objects are often much better behaved than the analogous $g(\mathcal{P})$ objects are. For example, they are often field objects (see Remark 8.2.3). Our main theorem for this section is the following (see Theorem 8.2.16):

Theorem 1.0.5. *Suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ are two rigidly-compactly generated tensor-triangulated categories with a tt-equivalence between their compact parts $\mathcal{T}_1^c \simeq \mathcal{T}_2^c$. Suppose further that:*

(1) $\mathcal{T}_1$ is stratified;

(2) $\mathcal{T}_1^c$ is equivalent to a countable category;

(3) The E_β objects in $\mathcal{T}_1$ are field objects; and

(4) *For every nonzero homological functor $\hat{t} \in \mathcal{A}_1$ there exists a nonzero map $\hat{E}_\beta \otimes \hat{x} \to \hat{t}$ for some $\beta \in Spc^h(\mathcal{T}_1^c)$ and compact $x \in \mathcal{T}_1^c$.*

Then $\mathcal{T}_2$ is also stratified.

As mentioned above, we use this theorem to prove that $DTM(\overline{\mathbb{Q}}, \mathbb{Z}/p)$ is stratified. The compact part of this category is known to be equivalent to the (bounded) derived category of filtered $\mathbb{Z}/p$ vector spaces (see Chapter 9), and this category is well understood and amenable to the techniques of stratification. In particular, it satisfies the hypothesis of the above theorem. This allows us to conclude that $DTM(\overline{\mathbb{Q}}, \mathbb{Z}/p)$ is stratified (see Theorem 9.0.11), and hence deduce that minimality holds at the height two prime m_p.

A final comment to make is that, while we are only considering motives over $\overline{\mathbb{Q}}$ in this thesis, our results hold for other algebraically closed fields satisfying a certain vanishing condition, as in [Gal19, Hypothesis 6.6]. In particular, this vanishing condition holds for $\overline{\mathbb{F}}_p$, see [Gal19, Remark 6.8]. Moreover, our results are conjectured to hold for $\mathbb{C}$. More specifically, the conjecture is as follows: for any real closed field $\mathbb{F}$, there is a canonical surjective comparison map (see [Bal10, §5])

$$\rho : Spc(DTM(\mathbb{F}, \mathbb{Z})^c) \to Spec(\mathbb{Z})$$

and so the Balmer spectrum can be found by identifying the fibers of this map. For fields satisfying the vanishing hypothesis, the fiber of the generic point (0) is known to be a singleton (for example, the fiber of (0) is m_0 for the case $DTM(\overline{\mathbb{Q}}, \mathbb{Z})^c$), whereas the fiber is only conjecturally a singleton for $\mathbb{F} = \mathbb{C}$ (see [BG22, Remark 11.8]). If the fiber is a singleton as conjectured, then our proofs hold in their entirely for $\mathbb{C}$ as well.

The thesis is structured as follows. In Chapter 2 we establish some preliminary material about triangulated categories. In particular we introduce the various localization theories they admit. In Chapter 3 we turn to tensor-triangulated categories, and provide a bare-bones account of the Balmer spectrum for an essentially small tensor-triangulated category $\mathcal{K}$, and the corresponding classification theorem. In Chapter 4 we introduce the theory of stratification, and prove the needed form of descent we will use in later chapters. Chapter 5 is dedicated to describing the computation of the Balmer spectrum of $DTM(\overline{\mathbb{Q}}, \mathbb{Z})^c$ by Gallauer. In Chapter 6 we prove minimality holds at the height 0 prime, m_0. In Chapter 7 we show how our descent proposition, combined with the Rigidity Theorem discussed earlier immediately gives minimality at the height 1 prime, e_p. This establishes stratification for the derived category of étale Tate motives, $DTM^{\text{ét}}(\overline{\mathbb{Q}}, \mathbb{Z})$. The final two chapters are dedicated towards proving minimality at the height 2 prime, m_p. Chapter 8 provides an account of Brown–Adams representability for a rigidly-compactly generated tensor-triangulated category, culminating in our Theorem 1.0.5. We then apply this theorem in Chapter 9 to prove minimality at the final prime, which establishes that $DTM(\overline{\mathbb{Q}}, \mathbb{Z})$ is stratified.

Chapter 2

Preliminaries

In this chapter we will review the basic theory of triangulated categories. We will not prove much in what follows, and will instead refer the reader to standard references, the main one being [Nee01].

2.1 Definitions and Basic Properties

Definition 2.1.1. Let $\mathcal{T}$ be an additive category and let $\Sigma : \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ be an auto-equivalence. Then we say that:

1. A *triangle* in $\mathcal{T}$ is a diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

2. A morphism of triangles is a commutative diagram of the form

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \Sigma\alpha_1} \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & \Sigma X' \end{array}$$

Definition 2.1.2. An additive category $\mathcal{T}$ is said to be a *triangulated category* if we have an auto-equivalence $\Sigma : \mathcal{T} \xrightarrow{\sim} \mathcal{T}$, which we refer to as the suspension, along with a distinguished class of triangles satisfying the following axioms:

(TR0) Triangles are closed under isomorphisms. Moreover the triangle

$$X \xrightarrow{id} X \to 0 \to \Sigma X$$

is a distinguished triangle for all $X \in \mathcal{T}$.

(TR1) Triangles are closed under rotations. That is, a triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

is a distinguished triangle if and only if

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is a distinguished triangle.

(TR2) Any morphism $X \xrightarrow{f} Y$ in $\mathcal{T}$ can be completed to a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{c_f} C_f \xrightarrow{\partial} \Sigma X$$

This object C_f is called a cone of the morphism f.

(TR3) A commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & & & \downarrow{\scriptstyle \Sigma\alpha_1} \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & \Sigma X' \end{array}$$

where the rows are distinguished triangles can be filled in to get a morphism of triangles

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\
{\scriptstyle \alpha_1}\downarrow & & {\scriptstyle \alpha_2}\downarrow & & {\scriptstyle \alpha_3}\,\vdots & & \downarrow{\scriptstyle \Sigma\alpha_1} \\
X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & \Sigma X'
\end{array}$$

(TR4) The "octahedral axiom:" This axiom tells us how cones of composible morphisms interact. Given two composible maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

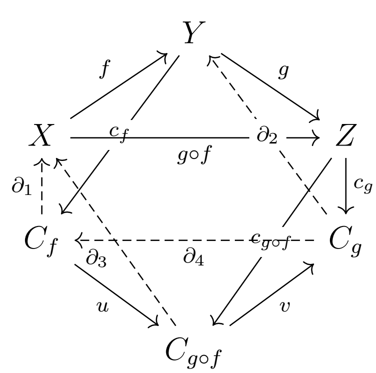
and three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{c_f} C_f \xrightarrow{\partial_1} \Sigma X$$
$$Y \xrightarrow{g} Z \xrightarrow{c_g} C_g \xrightarrow{\partial_2} \Sigma Y$$
$$X \xrightarrow{g \circ f} Z \xrightarrow{c_{g \circ f}} C_{g \circ f} \xrightarrow{\partial_3} \Sigma X$$

the octahedral axiom states that there exists a distinguished triangle

$$C_f \xrightarrow{u} C_{g \circ f} \xrightarrow{v} C_g \xrightarrow{\partial_4} \Sigma C_f$$

such that, if we denote maps into a suspension as dotted maps, the following diagram commutes:



Moreover, we require the two maps from the top to the bottom to coincide, and we require the two maps from the bottom to the top to coincide.

Remark 2.1.3. It follows from Lemma 2.1.8 below that the third object appearing in the triangle C_f as in $(TR2)$ is unique up to isomorphism. However, a fundamental issue with triangulated categories is that C_f is only unique up to a non-unique isomorphism. Nonetheless, we will often refer to it as 'the cone' instead of the more accurate 'a cone.'

Remark 2.1.4. While triangulated categories are rarely abelian categories, one often views the collection of distinguished triangles as being modeled on the collection of exact sequences in an abelian category. Indeed, consider the following lemma:

Lemma 2.1.5. *Let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

be a distinguished triangle. Then $g \circ f = h \circ g = 0$.

Proof. By (TR0) the triangle $X \xrightarrow{id} X \to 0 \to \Sigma X$ is a distinguished triangle. Then we have maps

$$\begin{array}{ccccccc} X & \xrightarrow{id} & X & \longrightarrow & 0 & \longrightarrow & \Sigma X \\ {\scriptstyle id}\downarrow & & {\scriptstyle f}\downarrow & & & & \downarrow \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z & \xrightarrow[h]{} & \Sigma X \end{array}$$

which by (TR3) can be completed to a morphism of distinguished triangles

$$\begin{array}{ccccccc} X & \xrightarrow{id} & X & \longrightarrow & 0 & \longrightarrow & \Sigma X \\ {\scriptstyle id}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle u}\downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z & \xrightarrow[h]{} & \Sigma X \end{array}$$

which shows that $g \circ f = 0$. A similar proof shows that $h \circ g = 0$. □

Remark 2.1.6. A similar proof as in the above lemma can also be given to demonstrate that g is a 'weak cokernel' of f, in the sense that any other map out of f that composes to 0 must factor, *non-uniquely*, through g. In particular, for any morphism $X \xrightarrow{f} Y$ we have that C_f is a weak cokernel of f. Let us for a moment then drastically abuse notation and write the cone as $C_f = Y/X$. Doing so provides some motivation for the octahedral axiom (TR4), which is by far the most mysterious, and the one that initially brings the most dread to any budding stable homotopy theorist. Indeed, with this abusive notation, (TR4) is the 'triangulated' version of the third isomorphism theorem, ie that $Z/X \simeq (Z/Y)/(Y/X)$.

Remark 2.1.7. For any $a \in \mathcal{T}$ the functor $Hom(a, -) : \mathcal{T} \to \mathcal{A}b$ sends a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

to an exact sequence of abelian groups

$$Hom(a, X) \xrightarrow{f_*} Hom(a, Y) \xrightarrow{g_*} Hom(a, Z)$$

(see [Nee01, Lemma 1.1.10]). Moreover, if we define $H_i := Hom(a, \Sigma^{-i}(-))$ then the rotation axiom implies we get a long exact sequence

$$\cdots \to H_1(Z) \to H_0(X) \to H_0(Y) \to H_0(Z) \to H_{-1}(Z) \to \cdots$$

In this sense, the morphism $Z \xrightarrow{h} \Sigma X$ in a distinguished triangle can be thought of as the 'connecting morphism' one typically has in long exact sequences of homology.

Lemma 2.1.8 (5-lemma)**.** *Consider a morphism of distinguished triangles:*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \Sigma\alpha_1} \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & \Sigma X' \end{array}$$

If two of the α_i's are isomorphisms then so is the third.

Proof. By the rotation axiom we can assume that α_1 and α_2 are isomorphisms and show that α_3 is as well. Fix $a \in \mathcal{T}$ and let us apply $H_i := Hom(a, \Sigma^{-i}(-))$ to obtain

$$\begin{array}{ccccccccc} H_0(X) & \xrightarrow{H_0(f)} & H_0(Y) & \xrightarrow{H_0(g)} & H_0(Z) & \xrightarrow{H_0(h)} & H_{-1}(X) & \xrightarrow{H_{-1}(f)} & H_{-1}(Y) \\ \downarrow{\scriptstyle H_0(\alpha_1)} & & \downarrow{\scriptstyle H_0(\alpha_2)} & & \downarrow{\scriptstyle H_0(\alpha_3)} & & \downarrow{\scriptstyle H_{-1}(\alpha_1)} & & \downarrow{\scriptstyle H_{-1}(\alpha_2)} \\ H_0(X') & \xrightarrow[H_0(f')]{} & H_0(Y') & \xrightarrow[H_0(g')]{} & H_0(Z') & \xrightarrow[H_0(h')]{} & H_{-1}(X') & \xrightarrow[H_{-1}(f')]{} & H_{-1}(Y') \end{array}$$

a commutative diagram of abelian groups with exact rows. Hence by the 'normal' 5-lemma in abelian categories, we get that $H_0(\alpha_3)$ is an isomorphism since the outer 4 vertical maps are by assumption. Since a was arbitrary, Yoneda tells us α_3 is also an isomorphism. □

Remark 2.1.9. Continuing to naively think of C_f as a 'quotient' Y/X can be instructive. Indeed, for $f : X \to Y$ the cone C_f provides a measurement of how far off f is from being an isomorphism:

Proposition 2.1.10. *Let $f : X \to Y$ be a morphism. Then $C_f \simeq 0$ if and only if f is an isomorphism.*

Proof. Suppose that $f : X \to Y$ is an isomorphism. Then we consider the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{id} & X & \longrightarrow & 0 & \longrightarrow & \Sigma X \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle f} & & & & \downarrow{\scriptstyle id} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[c_f]{} & C_f & \longrightarrow & \Sigma X \end{array}$$

where the rows are distinguished triangles. We can fill this in to a morphism of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{id} & X & \longrightarrow & 0 & \longrightarrow & \Sigma X \\ {\scriptstyle id}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle id}\downarrow \\ X & \xrightarrow[f]{} & Y & \xrightarrow[c_f]{} & C_f & \longrightarrow & \Sigma X \end{array}$$

By lemma 2.1.8 this shows h is an isomorphism. Now assume that $C_f \simeq 0$. Then we can obtain a morphism of triangles

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{id} & X & \longrightarrow & \Sigma X \\ {\scriptstyle id}\downarrow & & {\scriptstyle id}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle id}\downarrow \\ 0 & \longrightarrow & X & \xrightarrow[f]{} & Y & \longrightarrow & \Sigma X \end{array}$$

and so again by lemma 2.1.8 we get that f is an isomorphism. □

Remark 2.1.11. At this point, the author feels the reader may be getting antsy waiting for an example of a triangulated category. While the author does not necessarily expect the reader to know about the following examples of triangulated categories, he nonetheless hopes the reader takes solace in seeing this theory is not a vacuous one:

1. Let R be a commutative ring. The homotopy category $K(R)$ and the derived category $D(R)$ are triangulated. More generally, for $\mathcal{X}$ a nice enough scheme, the derived category of complexes of $\mathcal{O}_{\mathcal{X}}$-modules with quasi-coherent homology, $D(\mathcal{X})$ is triangulated.

2. Let G be a finite group and k a field of characteristic dividing the order of G. The stable module category $Stab(kG)$ is triangulated.

3. The stable homotopy category SH is triangulated.

4. The derived category of (étale) motives over a field $\mathbb{F}$, $DM^{(\text{ét})}(\mathbb{F})$ is triangulated.

5. The stable $\mathbb{A}^1$-homotopy category over a field $\mathbb{F}$, $\mathrm{SH}^{\mathbb{A}^1}(\mathbb{F})$ is triangulated.

Definition 2.1.12. Let $\mathcal{T}$ be a triangulated category and let $\mathcal{S} \subseteq \mathcal{T}$ be a full, replete, additive subcategory. Then we say that:

(a) $\mathcal{S}$ is *triangulated* if $\mathcal{S}$ is closed under (de)suspensions and cones of morphisms between objects in $\mathcal{S}$.

(b) $\mathcal{S}$ is *thick* if it is a triangulated subcategory closed under direct summands.

(c) $\mathcal{S}$ is *localizing* if it is a thick subcategory closed under coproducts.

Definition 2.1.13. For any subset $\mathcal{S} \subset \mathcal{T}$ we will write $Thick(\mathcal{S})$ and $Loc(\mathcal{S})$ to be the smallest thick and localizing subcategory containing $\mathcal{S}$.

Remark 2.1.14. In a triangulated category which admits all coproducts, any triangulated subcategory which is closed under coproducts is automatically also thick. Indeed, any triangulated category which admits all coproducts is idempotent complete (see [Nee01, Proposition 1.6.8]), and so any category closed under coproducts contains all direct summands by an Eilenberg swindle argument.

2.2 Verdier and Bousfield Localization

Definition 2.2.1. Let $\mathcal{T}_1, \mathcal{T}_2$ be triangulated categories. Then a *triangulated functor* $F : \mathcal{T}_1 \to \mathcal{T}_2$ is an additive functor equipped with a natural isomorphism of functors $F \circ \Sigma_{\mathcal{T}_1} \simeq \Sigma_{\mathcal{T}_2} \circ F$ such that for every distinguished triangle in $\mathcal{T}_1$

$$X \to Y \to Z \to \Sigma X$$

the triangle

$$F(X) \to F(Y) \to F(Z) \to \Sigma F(X)$$

is distinguished in $\mathcal{T}_2$.

Definition 2.2.2. Let $F : \mathcal{T}_1 \to \mathcal{T}_2$ be a triangulated functor. The *kernel* of F is

$$Ker(F) := \{X \in \mathcal{T}_1 : F(X) \simeq 0\}.$$

Lemma 2.2.3. *Let $F : \mathcal{T}_1 \to \mathcal{T}_2$ be a triangulated functor. The kernel of F is a thick subcategory. If moreover, $\mathcal{T}_1$ and $\mathcal{T}_2$ admit all coproducts, and F is coproduct preserving, then $Ker(F)$ is a localizing subcategory.*

Proof. That $Ker(F)$ is a thick subcategory is [Nee01, Lemma 2.1.5]. If F preserves coproducts, then it is clear that $Ker(F)$ is closed under coproducts and hence localizing. □

Remark 2.2.4. When the author was first learning this material, he found it useful to think of triangulated categories as abelian groups, and tensor-triangulated categories (to be discussed shortly) as rings. In this analogy, thick subcategories are morally normal subgroups, and triangulated functors are group homomorphisms. Keeping in mind the classic result in group theory that normal subgroups are nothing more than kernels of group homomorphisms, the following result became more clear to the author.

Proposition 2.2.5 (Verdier Localization)**.** *Let $\mathcal{S} \subseteq \mathcal{T}$ be a thick category. Then there exists a triangulated category denoted $\mathcal{T}/\mathcal{S}$ along with a triangulated functor*

$$F : \mathcal{T} \to \mathcal{T}/\mathcal{S}$$

which is the universal functor with $\mathcal{S} \subseteq Ker(F)$. Moreover, $\mathcal{T}/\mathcal{S}$ can be described as follows: let $Mor_{\mathcal{S}}$ denote the collection of all morphisms $f : X \to Y$ in $\mathcal{T}$ such that $C_f \in \mathcal{S}$. Then $\mathcal{T}/\mathcal{S} \simeq \mathcal{T}[Mor_{\mathcal{S}}^{-1}]$. That is, $\mathcal{T}/\mathcal{S}$ is the category obtained by *inverting all morphisms whose cone lies in $\mathcal{S}$. Furthermore, $\mathcal{T}[Mor_{\mathcal{S}}^{-1}]$ admits a calculus of left-fractions in the sense of [GZ67].*

Proof. See [Nee01, Theorem 2.1.8]. □

Example 2.2.6 (Derived Category). *Let $\mathcal{A}$ be a Grothendieck abelian category, and let $K(\mathcal{A})$ be the homotopy category of chain complexes. Let*

$$\mathcal{S} = \{X^\bullet \in K(\mathcal{A}) : H^i(X^\bullet) = 0 \text{ for all } i \in \mathbb{Z}\}$$

be the subcategory of acyclic complexes. Then the derived category of $\mathcal{A}$ is the Verdier localization

$$D(\mathcal{A}) = K(\mathcal{A})/S.$$

Moreover, in this example, $f \in Mor_{\mathcal{S}}$ if and only if f is a quasi-isomorphism. Hence in the derived category, all acyclic complexes are (isomorphic to) 0 and all quasi-isomorphisms are honest isomorphisms.

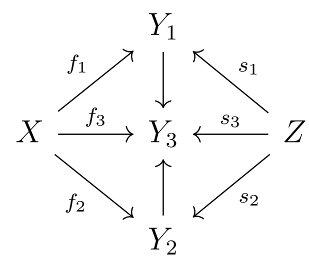
Remark 2.2.7. Since $\mathcal{T}/\mathcal{S}$ has a calculus of left-fractions, we can provide a simple description of the category as:

(a) $Obj(\mathcal{T}/\mathcal{S}) = Obj(\mathcal{T})$, and

(b) $Mor(\mathcal{T}/\mathcal{S})$ =equivalence classes of morphisms

$$X \xrightarrow{f} Y \xleftarrow{s} Z$$

with $s \in Mor_{\mathcal{S}}$, where two fractions $X \xrightarrow{f_1} Y_1 \xleftarrow{s_1} Z$ and $X \xrightarrow{f_2} Y_2 \xleftarrow{s_2} Z$ are equivalent if there exists a commutative diagram



with $s_3 \in Mor_{\mathcal{S}}$.

Nonetheless, we still may have set-theoretic size issues in general. A way to insure there are no size issues is the following.

Definition 2.2.8. Let $\mathcal{T}$ be a triangulated category and let $\mathcal{S} \subseteq \mathcal{T}$ be a subset. Then we denote:

$$\mathcal{S}^{\perp} = \{X \in \mathcal{T} : Hom(Y, X) = 0 \text{ for all } Y \in \mathcal{S}\}$$
$${}^{\perp}\mathcal{S} = \{X \in \mathcal{T} : Hom(X, Y) = 0 \text{ for all } Y \in \mathcal{S}\}.$$

Definition 2.2.9. A *Bousfield localization* on a triangulated category $\mathcal{T}$ is a triangulated functor $L : \mathcal{T} \to \mathcal{T}$ along with a natural transformation $\eta : id \to L$ such that

1. $L\eta = \eta L$, and

2. $L\eta$ is an isomorphism.

Given a Bousfield localization L, we say that

1. A morphism $f : X \to Y$ is an *L-equivalence* if Lf is an isomorphism.

2. An object X is *L-local* if X is in the essential image of L.

3. An object X is *L-acyclic* if $X \in Ker(L)$.

Proposition 2.2.10. *Let $\mathcal{T}$ be a triangulated category and let $\mathcal{S} \subseteq \mathcal{T}$ be a thick subcategory. Then the following are equivalent:*

1. *There exists a Bousfield localization $L : \mathcal{T} \to \mathcal{T}$ with $Ker(L) = \mathcal{S}$.*

2. *The inclusion functor $\mathcal{S} \hookrightarrow \mathcal{T}$ admits a right adjoint.*

3. *For each $X \in \mathcal{T}$ there is a distinguished triangle*

$$X' \to X \to X'' \to \Sigma X''$$

   with $X' \in \mathcal{S}$ and $X'' \in \mathcal{S}^{\perp}$.

4. *The quotient functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ admits a fully faithful right adjoint.*

5. *The composite*

$$\mathcal{S}^{\perp} \hookrightarrow \mathcal{T} \to \mathcal{T}/\mathcal{S}$$

   is an equivalence.

6. *The inclusion $\mathcal{S}^{\perp} \hookrightarrow \mathcal{T}$ admits a left adjoint and ${}^{\perp}(\mathcal{S}^{\perp}) = \mathcal{S}$.*

Moreover, in this case we have $\mathcal{S} = Ker(L) = {}^{\perp}Im(L)$ and $\mathcal{S}^{\perp} = Im(L)$

Proof. See [Kra10, Proposition 4.9.1] and [Kra10, Proposition 4.10.1]. □

2.3 Compact Objects and Compact Generation

In this section, we make precise the notion of 'finite objects' for a triangulated category, and discuss how many of the important examples are generated from these objects.

Definition 2.3.1. Let $\mathcal{T}$ be a triangulated category. An object $x \in \mathcal{T}$ is said to be *compact* if the functor

$$Hom_{\mathcal{T}}(x, -) : \mathcal{T} \to \mathcal{A}b$$

preserves coproducts. We will let $\mathcal{T}^c \subseteq \mathcal{T}$ denote the subcategory of compact objects. It is an easy exercise to check that $\mathcal{T}^c$ is a thick subcategory of $\mathcal{T}$.

Remark 2.3.2. An equivalent characterization, which helps explain the name 'compact,' is that any map from x to an infinite coproduct of objects factors through a finite coproduct.

Definition 2.3.3. We say that $\mathcal{T}$ is *compactly generated* if $\mathcal{T}$ has small coproducts and there exists a set of compact objects $\mathcal{G}$ such that $Loc(\mathcal{G}) = \mathcal{T}$.

Remark 2.3.4. Compactly generated triangulated categories satisfy a very important representability condition, originally discussed by Brown in the papers [Bro62, Bro63], and then generalized by Neeman (see below).

Theorem 2.3.5 (Brown Representability)**.** *Let $\mathcal{T}$ be a compactly generated triangulated category. Let*

$$\mathcal{H} : \mathcal{T}^{op} \to \mathcal{A}b$$

be a functor that sends distinguished triangles to exact sequences, and sends coproducts in $\mathcal{T}$ to products in $\mathcal{A}b$. Then $\mathcal{H}$ is representable.

Proof. See [Nee01, Theorem 8.3.3]. □

Corollary 2.3.6. *Let $\mathcal{T}$ be a compactly generated triangulated categories and let $\mathcal{S}$ be any triangulated category. If $F : \mathcal{T} \to \mathcal{S}$ is a coproduct preserving triangulated functor then F has a right adjoint.*

Proof. This is [Nee01, Theorem 8.4.4]. The point is that, for any $s \in \mathcal{S}$ the functor

$$\mathcal{H} := Hom_{\mathcal{S}}(F(-), s)$$

satisfies the hypothesis in the theorem above. Hence, there is an object, denoted $G(s) \in \mathcal{T}$ such that

$$Hom_{\mathcal{S}}(F(-), s) \simeq Hom_{\mathcal{T}}(-, G(s))$$

This can then be extended to give a functor $G : \mathcal{S} \to \mathcal{T}$, which is the sought after adjoint. $\square$

2.4 Finite Localization

In this section, we show how Brown-representability can be used to produce localization functors.

Definition 2.4.1. Let $L : \mathcal{T} \to \mathcal{T}$ be a Bousfield localization functor on a triangulated category $\mathcal{T}$. We say L is a *finite localization* if the subcategory of L-acylic objects, $Ker(L)$ is generated as a localizing subcategory by a set of compact objects.

Theorem 2.4.2. *Let $\mathcal{T}$ be a compactly generated triangulated category and let $S \subset \mathcal{T}$ be a localizing subcategory. If there exist a set of compact objects $\mathcal{C}$ such that $\mathcal{S} = Loc(\mathcal{C})$ then there exists a finite localization $L : \mathcal{T} \to \mathcal{T}$ with $Ker(L) = \mathcal{S}$.*

Proof. The assumptions on $\mathcal{S}$ mean it is itself compactly generated. Moreover, since it is a localizing subcategory of $\mathcal{T}$, the inclusion $\iota : \mathcal{S} \to \mathcal{T}$ is a coproduct preserving triangulated functor. Hence by Corollary 2.3.6 it admits a right adjoint, which, by Proposition 2.2.10, shows there is a Bousfield localization functor

$$L : \mathcal{T} \to \mathcal{T}$$

with $Ker(L) = \mathcal{S}$. □

Chapter 3

Tensor-Triangulated Categories

In this chapter, we cover the basic theory of tensor-triangulated categories. In particular, we go over the appropriate analogue of compact generation in the tensor-world. We end with an account of Balmer's abstract classification theorem for an essentially small tensor-triangulated category, and give some example classifications. Our goal for this chapter is to set the stage for the theory of stratification in the following chapter.

Definition 3.0.1. A *tensor-triangulated category* is a triangulated category $\mathcal{T}$ equipped with a compatible closed symmetric monoidal structure, as in [HPS97, Appendix A]. In particular, the functors

$$- \otimes - : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$$
$$[-,-] : \mathcal{T}^{\mathrm{op}} \times \mathcal{T} \to \mathcal{T}$$

are triangulated functors, where $[-,-]$ denotes the internal-hom. We will write $\mathbb{1}_{\mathcal{T}}$ for the unit object, and will often drop the subscript and just write $\mathbb{1}$ if it is clear which category we are talking about.

Definition 3.0.2. Let $\mathcal{S} \subseteq \mathcal{T}$ be a full, replete subcategory of a tensor-triangulated category. We say $\mathcal{S}$ is a *tensor-ideal* if $a \otimes s \in \mathcal{S}$ for all $a \in \mathcal{T}, s \in \mathcal{S}$. We write $Thickid(\mathcal{S})$ and $Locid(\mathcal{S})$ for the smallest thick tensor-ideal and smallest localizing tensor-ideal containing $\mathcal{S}$.

Remark 3.0.3. To further the author's analogy of a triangulated category as abelian group from Remark 2.2.4, a tensor-triangulated can be thought of as a kind of suped-up ring. This naive intuition can actually get us quite far, as we shall see in §3.3.

Definition 3.0.4. Let $\mathcal{T}$ and $\mathcal{S}$ be tensor-triangulated categories. Then a *tensor-triangulated functor* $F : \mathcal{T} \to \mathcal{S}$ is a triangulated functor that is also a strong-monoidal functor. In particular, we have natural isomorphisms $F(\mathbb{1}_{\mathcal{T}}) \simeq \mathbb{1}_{\mathcal{S}}$ and $F(a \otimes_{\mathcal{T}} b) \simeq F(a) \otimes_{\mathcal{S}} F(b)$ for all $a, b \in \mathcal{T}$.

3.1 Rigidly-Compactly Generated tt-Categories

Definition 3.1.1. An object x in a tensor-triangulated category $\mathcal{K}$ is said to be *dualizable* if there exists an object Dx and morphisms $\mathbb{1} \xrightarrow{\eta} Dx \otimes x$ and $x \otimes Dx \xrightarrow{\epsilon} \mathbb{1}$ such that the two maps

$$x \simeq x \otimes \mathbb{1} \xrightarrow{1 \otimes \eta} x \otimes Dx \otimes x \xrightarrow{\epsilon \otimes 1} \mathbb{1} \otimes x \simeq x$$

$$Dx \simeq \mathbb{1} \otimes Dx \xrightarrow{\eta \otimes 1} Dx \otimes x \otimes Dx \xrightarrow{1 \otimes \epsilon} Dx \otimes \mathbb{1} \simeq Dx$$

are the identity.

Definition 3.1.2. A tensor-triangulated category $\mathcal{K}$ is called *rigid* if every object $x \in \mathcal{K}$ is dualizable.

Remark 3.1.3. Let $\mathcal{T}$ be a tensor-triangulated category that has all small coproducts. Then we have two candidate subcategories of 'finite' objects:

(a) The subcategory of compact objects $\mathcal{T}^c$, and

(b) The subcategory of dualizable objects.

An analogue for a compactly generated triangulated category in the tensor world are those where these two candidates for finite objects coincide.

Definition 3.1.4. Let $\mathcal{T}$ be a tensor-triangulated category that has all small coproducts. We say that $\mathcal{T}$ is a *rigidly-compactly generated tensor-triangulated category* if:

(a) $\mathcal{T}$ is compactly generated, and

(b) The compact objects coincide with the subcategory of dualizable objects. In particular, the unit object $\mathbb{1}$ is compact.

Remark 3.1.5. Many of the examples given in Remark 2.1.11 are rigidly-compactly generated tensor-triangulated categories (see for example, [HPS97, Example 1.2.3]):

(a) Let R be a commutative ring. Then $D(R)$ is rigidly-compactly generated. The tensor structure is the derived tensor product, $- \otimes^{\mathbb{L}} -$, and the unit is given by R. The subcategory of perfect complexes, $D^{\text{perf}}(R) := D^b(proj(R))$, i.e the subcategory of bounded complexes of finitely-generated projective modules, is the subcategory of compact-dualizable objects.

(b) Let G be a finite group and k a field of characteristic dividing the order of G. Then $Stab(kG)$ is a rigidly-compactly generated tensor-triangulated category. The tensor structure is given by tensoring over k with the diagonal action of the group, and the unit is the trivial module k. The subcategory of finite-dimensional kG modules, $stab(kG)$ forms the subcategory of compact-dualizable objects.

(c) The stable homotopy category SH is rigidly-compactly generated. The tensor product is the smash product $- \wedge -$, and the unit is the sphere-spectrum $\mathbb{S}$. The subcategory of compact-dualizable objects, $\mathrm{SH}^{\mathrm{fin}}$, is the Spanier–Whitehead category of finite CW-complexes.

Remark 3.1.6. Moreover, the various forms of localizations discussed in the previous sections all apply to tensor-triangulated categories. We state them without proof below:

Proposition 3.1.7. *Let $\mathcal{S} \subset \mathcal{T}$ be a thick tensor-ideal of a tensor-triangulated category $\mathcal{T}$. The localization $\mathcal{T}/\mathcal{S}$ is a tensor-triangulated category and the canonical localization functor*

$$F : \mathcal{T} \to \mathcal{T}/\mathcal{S}$$

is a tensor-triangulated functor.

Remark 3.1.8. The above proposition applies to Bousfield localizations as well. Moreover, if the kernel of a Bousfield localization $Ker(L)$ is a tensor-ideal, then $Im(L)$ is itself a tensor-triangulated category through the equivalence $Im(L) \simeq \mathcal{T}/Ker(L)$ from Proposition 2.2.10. One should be warned however that $Im(L)$ is not a tensor-triangulated subcategory of $\mathcal{T}$! For example, its unit is $L(\mathbb{1}_{\mathcal{T}})$.

3.2 Smashing Localization

In this section we introduce the final kind of localization we will be interested in. We put off this form of localization until after having introduced tensor-triangulated categories, because we feel its most natural setting (and how to explain the name) lies in the world of tensor-triangulated categories.

Definition 3.2.1. Let $\mathcal{T}$ be a triangulated category (not necessarily tensor for now). A Bousfield localization $L : \mathcal{T} \to \mathcal{T}$ is a *smashing localization* if it preserves coproducts. The reason for the name is the following proposition:

Proposition 3.2.2. *Let $\mathcal{T}$ be a rigidly-compactly generated tensor-triangulated category and let $L : \mathcal{T} \to \mathcal{T}$ be a Bousfield localization such that $Ker(L)$ is a tensor ideal. Then L is smashing if and only if the natural map $\alpha_t : L(\mathbb{1}) \otimes t \to L(t)$ is an isomorphism for all $t \in \mathcal{T}$.*

Proof. This is a standard result, see for example [HPS97, Definition 3.3.2]. Nonetheless, we give the proof below so that the reader can see some general techniques one uses in proofs in the world of rigidly-compactly generated tt-categories. The map α_t is defined as follows. We have the natural map $\eta : \mathbb{1} \to L(\mathbb{1})$ since L is a Bousfield localization. Then this map fits into a distinguished triangle

$$X \to \mathbb{1} \xrightarrow{\eta_{\mathbb{1}}} L(\mathbb{1}) \to \Sigma X$$

with $X \in Ker(L)$ by Proposition 2.2.10. Now fix $t \in \mathcal{T}$ and apply $L(- \otimes t)$ to the triangle. $L(X \otimes t) = 0$ by assumption, and so we have

$$L(\eta_{\mathbb{1}} \otimes X) : L(t) \to L(L(\mathbb{1}) \otimes t)$$

is an isomorphism. The map α_t is thus defined as the composite

$$L(\mathbb{1}) \otimes t \xrightarrow{\eta_{L(\mathbb{1}) \otimes t}} L(L(\mathbb{1}) \otimes t) \xrightarrow{L(\eta_{\mathbb{1}} \otimes X)^{-1}} L(t).$$

Since the tensor-product preserves coproducts we get that if the natural map above is an isomorphism then L preserves coproducts. Now a general argument shows that the set of all objects for which α_t is an isomorphism is thick, and if L preserves coproducts it is also localizing. Since $\mathcal{T}$ is rigidly-compactly generated, it thus suffices to show that α_t is an isomorphism for t dualizable. Moreover, again by Proposition 2.2.10, it suffices to show that $L(\mathbb{1}) \otimes t \in Ker(L)^{\perp}$. So take $s \in Ker(L)$ and $t \in \mathcal{T}^c$ and we compute:

$$\begin{aligned} Hom(s, L(\mathbb{1}) \otimes t) &\simeq Hom(s, L(\mathbb{1}) \otimes D^2(t)) \\ &\simeq Hom(s, L(\mathbb{1}) \otimes [D(t), \mathbb{1}]) \\ &\simeq Hom(s, [D(t), L(\mathbb{1})]) \\ &\simeq Hom(s \otimes D(t), L(\mathbb{1})) \\ &= 0 \end{aligned}$$

where the last equality holds since $s \otimes D(t) \in Ker(L)$ since we are assuming $Ker(L)$ is a tensor-ideal. □

Remark 3.2.3. This explains the origin of the name. The above proposition tell us the smashing localization is given by tensoring by $L(\mathbb{1})$, and recall from Remark 3.1.5 the tensor product in SH is called the smash product.

Corollary 3.2.4. *Every finite localization is a smashing localization.*

Proof. See, for example [Kra10, Theorem 5.6.1]. □

Remark 3.2.5. The content of the so-called telescope conjecture is the reverse implication. We say a rigidly-compactly generated tt-category $\mathcal{T}$ satisfies the telescope conjecture if every smashing localization is a finite localization. An equivalent characterization of this conjecture was originally phrased for SH in [Rav84] (and was recently shown to be false! [BHLS23]), while its current form can be asked in any rigidly-compactly generated tt-category.

Remark 3.2.6. There is another instructive way to think about smashing and finite localizations, via so-called idempotent triangles. We put off this perspective until the beginning of the next chapter so that we can discuss it in the context of stratification.

3.3 The Balmer Spectrum and Classification

For the rest of this chapter, we will let $\mathcal{K}$ denote an essentially small tensor-triangulated category (for example, $\mathcal{K} = \mathcal{T}^c$ can be the compact-dualizable objects of some rigidly-compactly generated category $\mathcal{T}$). In this section we provide an account of the Balmer spectrum of $\mathcal{K}$, which allows for a geometric approach to the study of $\mathcal{K}$, much as $Spec(R)$ allows for a geometric approach to the study of a commutative ring R. We shall see that computing the Balmer spectrum of $\mathcal{K}$ is essentially equivalent to computing the thick-tensor ideals of $\mathcal{K}$. The original reference for what follows is [Bal05].

3.3.1 Balmer Spectrum

Definition 3.3.1. We say a proper thick tensor-ideal $\mathcal{P} \subset \mathcal{K}$ is a *prime ideal*, if, whenever $a \otimes b \in \mathcal{P}$ then $a \in \mathcal{P}$ or $b \in \mathcal{P}$. The *Balmer spectrum* of $\mathcal{K}$, $Spc(\mathcal{K})$ is the set:

$$Spc(\mathcal{K}) = \{\mathcal{P} \subset \mathcal{K} : \mathcal{P} \text{ is prime}\}.$$

For a collection of objects $\mathcal{G} \subseteq \mathcal{K}$ let

$$Z(\mathcal{G}) = \{\mathcal{P} \in Spc(\mathcal{K}) : \mathcal{G} \cap \mathcal{P} = \varnothing\}.$$

The collection $\{Z(\mathcal{G}) : \mathcal{G} \subseteq \mathcal{K}\}$ are the closed subsets for a topology on $Spc(\mathcal{K})$. We call this corresponding topology the Balmer topology.

Definition 3.3.2. The *support* of an object $a \in \mathcal{K}$ is the set

$$supp(a) = \{\mathcal{P} \in Spc(\mathcal{K}) : a \notin \mathcal{P}\}.$$

Note that these are precisely the primes for which a is not killed in the localization $\mathcal{K}/\mathcal{P}$. Moreover, the support satisfies the following properties:

(i) $supp(a)$ is closed for all $a \in \mathcal{K}$.

(ii) $supp(0) = \varnothing$ and $supp(\mathbb{1}) = Spc(\mathcal{K})$.

(iii) $supp(\Sigma a) = supp(a)$.

(iv) $supp(a \oplus b) = supp(a) \cup supp(b)$.

(v) $supp(c) \subseteq supp(a) \cup supp(b)$ for any distinguished triangle

$$a \to b \to c \to \Sigma a.$$

(vi) $supp(a \otimes b) = supp(a) \cap supp(b)$.

Remark 3.3.3. The collection $\{supp(a) : a \in \mathcal{K}\}$ forms a basis of closed subsets for $Spc(\mathcal{K})$, see [Bal05, Remark 2.7].

Theorem 3.3.4. *The pair $(Spc(\mathcal{K}), supp)$ is the universal notion of support for $\mathcal{K}$ that satisfies properties (a)-(f) above.*

Proof. See [Bal05, Theorem 3.2] for the precise statement and proof. □

Proposition 3.3.5. *Let $\mathcal{P} \in Spc(\mathcal{K})$. Then its closure is given by*

$$\overline{\{\mathcal{P}\}} = \{\mathcal{Q} \in Spc(\mathcal{K}) : \mathcal{Q} \subseteq \mathcal{P}\}.$$

In particular, the minimal primes are the closed points.

Proof. See [Bal05, Proposition 2.9]. □

Proposition 3.3.6. *Let $F : \mathcal{K}_1 \to \mathcal{K}_2$ be a tensor-triangulated functor. Then we obtain an induced continuous map on spectra*

$$Spc(F) : Spc(\mathcal{K}_2) \to Spc(\mathcal{K}_1)$$

given by $Spc(F)(\mathcal{Q}) = F^{-1}(\mathcal{Q})$ for all $\mathcal{Q} \in Spc(\mathcal{K}_2)$.

Proof. See [Bal05, Proposition 2.6]. □

3.3.2 Classification Theorem

In this section we will show how the Balmer spectrum gives us a classification of thick tensor-ideals of $\mathcal{K}$. First, some preliminary definitions.

Remark 3.3.7. The Balmer spectrum $Spc(\mathcal{K})$ is a spectral space, in the sense of [Hoc69], see [Bal05].

Definition 3.3.8. We say a subset $Z \subseteq Spc(\mathcal{K})$ is a *Thomason subset* if it is a union of closed subsets, where each has quasi-compact complement. If $Spc(\mathcal{K})$ is moreover a noetherian space, then Z being Thomason is equivalent to Z being specialization closed.

Definition 3.3.9. Let $\mathcal{J} \subseteq \mathcal{K}$ be a thick tensor-ideal of $\mathcal{K}$. Then the *radical* of $\mathcal{J}$ is

$$\sqrt{\mathcal{J}} := \{a \in \mathcal{K} : a^{\otimes n} \in \mathcal{J} \text{ for some } n \geq 1\}.$$

Moreover, we say $\mathcal{J}$ is radical if $\mathcal{J} = \sqrt{\mathcal{J}}$.

Remark 3.3.10. When $\mathcal{K}$ is rigid, for example when $\mathcal{K} = \mathcal{T}^c$ is the category of compact-dualizable objects of a rigidly-compactly generated category $\mathcal{T}$, then every thick tensor-ideal is radical, see [Bal05, Proposition 4.4].

Definition 3.3.11. We say $\mathcal{K}$ is a *local tensor-triangulated category* if $Spc(\mathcal{K})$ has a unique closed point. This is equivalent to any of the following conditions (see [Bal10, Proposition 4.2]):

(a) $Spc(\mathcal{K})$ is a local topological space; ie, every open cover $Spc(\mathcal{K}) = \bigcup_{i \in I} \mathcal{U}_i$ is trivial, in the sense $Spc(\mathcal{K}) = \mathcal{U}_j$ for some j.

(b) $\mathcal{K}$ has a unique minimal prime ideal.

(c) The ideal $\sqrt{0} \subseteq \mathcal{K}$ is the unique minimal prime.

(d) Whenever we have $a \otimes b = 0$ we have that a is tensor-nilpotent or b is tensor-nilpotent.

If $\mathcal{K}$ is rigid, then by Remark 3.3.10 these conditions are further equivalent to saying that (0) is the unique minimal prime.

Definition 3.3.12. Let $\mathcal{G} \subseteq \mathcal{K}$ be a collection of objects. Define

$$supp(\mathcal{G}) := \bigcup_{a \in \mathcal{G}} supp(a).$$

For $Y \subseteq Spc(\mathcal{K})$ define

$$\mathcal{K}_Y := \{a \in \mathcal{K} : supp(a) \subseteq Y\}.$$

Theorem 3.3.13 (Classification Theorem)**.** *The two assignments $supp(\mathcal{G})$ and $\mathcal{K}_Y$ above induce order-preserving bijections between*

$$\{\textit{radical thick tensor-ideals of } \mathcal{K}\} \leftrightarrow \{\textit{Thomason subsets of } Spc(\mathcal{K})\}$$

Proof. See [Bal05, Theorem 4.10]. □

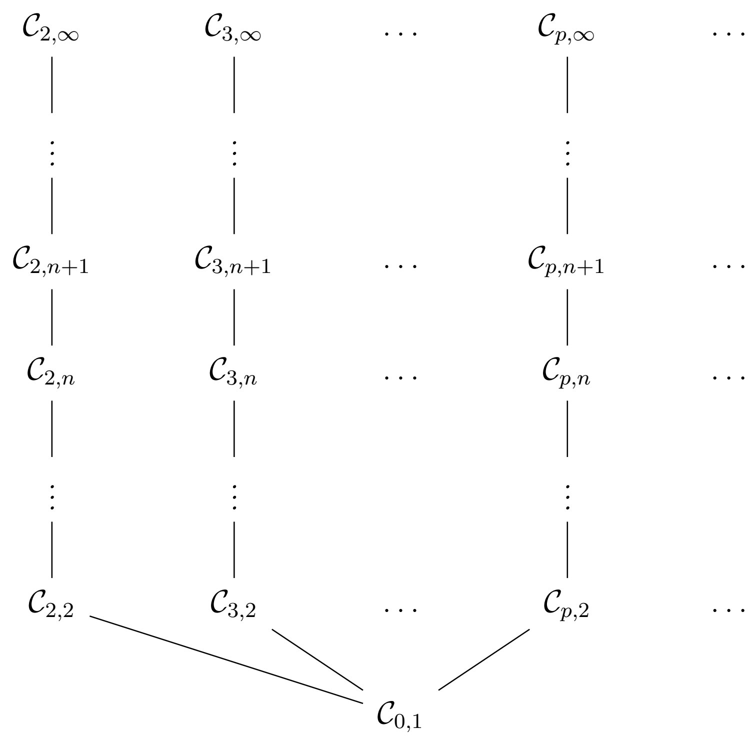
Corollary 3.3.14. *Let $a_1, a_2 \in \mathcal{K}$. Then $Thickid(a_1) = Thickid(a_2)$ if and only if $supp(a_1) = supp(a_2)$.*

Remark 3.3.15. Keeping in mind Remark 3.3.10, we can often strengthen the theorem to provide a classification for all thick tensor-ideals.

3.3.3 Examples

As mentioned in the introduction, various landmark classification theorems from the 1990*s* can be expressed as the computation of their Balmer spectrum.

Theorem 3.3.16 ([HS98, Bal10])**.** *The Balmer spectrum of SH^c is the following topological space:*



In this picture, each point $\mathcal{C}_{p,n}$ is the kernel of the $(n-1)^{th}$ p–local Morava K-theory, and the specialization relations are depicted by the lines going upwards.

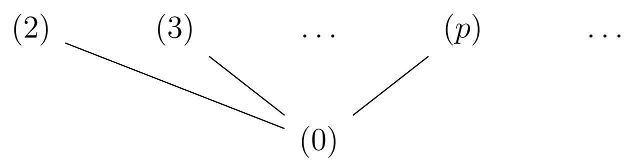
Theorem 3.3.17 ([BCR97, Bal10])**.** *Let G be a finite group and k a field of characteristic p dividing the order of G. The Balmer spectrum of the stable* *module category is*

$$Spc(stab(kG)) \simeq Proj(H^\bullet(G,k)).$$

Here $Proj(H^\bullet(G,k))$ is the projective variety associated to the group cohomology ring.

Theorem 3.3.18 ([Nee92, Tho97, Bal10])**.** *Let R be a commutative ring. The Balmer spectrum for the derived category of perfect-complexes $D^{perf}(R)$ is given by $Spc((D^{perf}(R)) \simeq Spec(R)$. For example, the Balmer spectrum for the derived category of the integers is homeomorphic to $Spec(\mathbb{Z})$:*

$Spc(D^{perf}(\mathbb{Z})) \simeq$



Remark 3.3.19. Unfortunately we cannot simply extend Theorem 3.3.13 to a rigidly-compactly generated tensor-triangulated category $\mathcal{T}$. The problem is that the axioms for Balmer's universal notion of support are not appropriate for big objects. For example, the support of a compact object should be closed but this should not be expected for the support of an arbitrary object. Nonetheless, in the next chapter, we introduce how another notion of support, the so-called Balmer–Favi support, can be used to classify the localizing tensor-ideals of $\mathcal{T}$ in certain nice contexts.

Chapter 4

Balmer–Favi Support and Stratification

Throughout this chapter $\mathcal{T}$ will denote a rigidly-compactly generated tensor-triangulated category and we will let $\mathcal{T}^c$ denote the subcategory of compact-dualizable objects. The goal for this chapter is to provide a framework for classifying the localizing tensor ideals of $\mathcal{T}$.

4.1 Idempotent Triangles

Let us first explain a useful equivalent characterization of smashing localizations from Balmer and Favi [BF11].

Proposition 4.1.1. *Let $e \xrightarrow{\gamma} \mathbb{1} \xrightarrow{\lambda} f \to \Sigma e$ be a distinguished triangle. Then the following are equivalent:*

(1) $\gamma \otimes id_e : e \otimes e \xrightarrow{\sim} e$ is an isomorphism.

(2) $e \otimes f = 0$.

(3) $\lambda \otimes id_f : f \xrightarrow{\sim} f \otimes f$ is an isomorphism.

Moreover, in this case $Ker(- \otimes f) = Im(e \otimes -)$ and $Ker(e \otimes -) = Im(f \otimes -)$.

Definition 4.1.2. We call a distinguished triangle

$$e \xrightarrow{\gamma} \mathbb{1} \xrightarrow{\lambda} f \to \Sigma e$$

an *idempotent triangle* if it satisfies any of the above equivalent conditions.

Proof. This is [BF11, Proposition 3.1]. □

Theorem 4.1.3. *All smashing localizations for $\mathcal{T}$ arise as idempotent triangles. More precisely we have:*

(1) Let $L : \mathcal{T} \to \mathcal{T}$ be a smashing localization. Then the distinguished triangle we get for the unit, as in Proposition 2.2.10 (Part 3)

$$e \to \mathbb{1} \to f \to \Sigma e$$

is an idempotent triangle.

(2) Let

$$e \xrightarrow{\gamma} \mathbb{1} \xrightarrow{\lambda} f \to \Sigma e$$

be an idempotent triangle. Then the functor $L_f := - \otimes f : \mathcal{T} \to \mathcal{T}$ is a smashing localization.

Proof. This is [BF11, Theorem 3.5]. □

Remark 4.1.4. Let $Y \subseteq Spc(\mathcal{T}^c)$ be a Thomason subset, and denote by

$$\mathcal{T}^c_Y = \{x \in \mathcal{T}^c : supp(x) \subseteq Y\}$$

the corresponding thick tensor ideal, as in Theorem 3.3.13. Then we get an associated idempotent triangle

$$e_Y \to \mathbb{1} \to f_Y \to \Sigma e_Y$$

where

$$Ker(- \otimes f_Y) = Im(e_Y \otimes -) = Locid(e_Y) = Locid(\mathcal{T}_Y^c)$$

Now let $V := Spc(\mathcal{T}^c) \setminus Y$ denote the complement of Y. We denote

$$\mathcal{T}(V) := \mathcal{T}/Locid(\mathcal{T}_Y^c) \simeq Im(- \otimes f_Y) = Locid(f_Y)$$

to be the associated localization. Since the localization functor $\mathcal{T} \to \mathcal{T}(V)$ is smashing, it preserves coproducts and hence compact objects, so we get an associated map on spectrum $Spc(\mathcal{T}(V)^c) \to Spc(\mathcal{T}^c)$. Moreover, (see for example, [BHS23b, Remark 1.23]) this map induces a homeomorphism

$$Spc(\mathcal{T}(V)^c) \simeq V \hookrightarrow Spc(\mathcal{T}^c).$$

Definition 4.1.5. As a specific case of the above discussion, let $\mathcal{P} \in Spc(\mathcal{T}^c)$ and consider the associated Thomason subset $Y_{\mathcal{P}} := \bigcup_{x \in \mathcal{P}} supp(x)$ from the Classification Theorem 3.3.13. We let $\mathcal{T}_{\mathcal{P}}$ denote the finite localization $\mathcal{T}/Locid(\mathcal{P})$ as in the above remark. This is a local category as in Definition 3.3.11, and we will refer to it as the local category at the prime $\mathcal{P}$. Its Balmer spectrum is the set of generalizations of $\mathcal{P}$,

$$Spc(\mathcal{T}_{\mathcal{P}}^c) \simeq gen(\mathcal{P}) = \{\mathcal{Q} \in Spc(\mathcal{T}^c) : \mathcal{P} \in \overline{\{\mathcal{Q}\}}\}.$$

4.2 Balmer–Favi Support

We now introduce the main support theory we will use to study localizing tensor ideals. This notion of support was first introduced by Balmer and Favi in [BF11], and further properties of it were studied in [BHS23b].

Definition 4.2.1. Let $\mathcal{P} \in Spc(\mathcal{T}^c)$ be a prime. We say $\mathcal{P}$ is *weakly visible* if

$\{\mathcal{P}\}$ is the intersection of a Thomason subset and the complement of a Thomason subset. We say $\mathcal{P}$ is *visible* if its closure is Thomason. Note that a visible prime is weakly visible. If every prime $\mathcal{P} \in Spc(\mathcal{T}^c)$ is weakly visible (respectively visible) we say $Spc(\mathcal{T}^c)$ is *weakly noetherian* (respectively noetherian).

Definition 4.2.2. Associated to each weakly visible prime $\mathcal{P} \in Spc(\mathcal{T}^c)$ is a nonzero tensor idempotent object $g(\mathcal{P})$ defined by

$$g(\mathcal{P}) = e_{Y_1} \otimes f_{Y_2}$$

for any choice of Thomason subsets Y_1, Y_2 of $Spc(\mathcal{T}^c)$ such that $\{\mathcal{P}\} = Y_1 \cap Y_2^c$. This does not depend on choice of Thomason subsets, see [BF11, Corollary 7.5].

Remark 4.2.3. Let us make a quick remark about these $g(\mathcal{P})$ objects in a special case. Let $\mathcal{T}$ be a rigidly-compactly generated tt-category, and suppose $Spc(\mathcal{T}^c)$ consists of two, connected points

$$Spc(\mathcal{T}^c)$$

$$\begin{array}{c} \mathcal{M} \\ | \\ \mathcal{P}. \end{array}$$

That is, a closed point $\mathcal{M}$ and a generic point $\mathcal{P}$. Consider the idempotent triangle

$$e_{\mathcal{M}} \to \mathbb{1} \to f_{\mathcal{M}} \to \Sigma e_{\mathcal{M}}$$

associated to the finite localization for $Y = \{\mathcal{M}\}$. We have that

(1) $g(\mathcal{M}) = e_{\mathcal{M}}$, and

(2) $g(\mathcal{P}) = f_{\mathcal{M}}$.

Definition 4.2.4 (Balmer–Favi)**.** Suppose that $Spc(\mathcal{T}^c)$ is weakly noetherian and let $t \in \mathcal{T}$. The *Balmer-Favi support* of t is

$$Supp(t) := \{\mathcal{P} \in Spc(\mathcal{T}^c) : t \otimes g(\mathcal{P}) \neq 0\}.$$

Remark 4.2.5. This theory satisfies the expected properties of a support theory, that is:

1. $Supp(0) = \varnothing$ and $Supp(\mathbb{1}) = Spc(\mathcal{T}^c)$.

2. $Supp(\Sigma t) = Supp(t)$ for all $t \in \mathcal{T}$.

3. $Supp(c) \subseteq Supp(a) \cup Supp(b)$ whenever we have a triangle $a \to b \to c \to \Sigma a$.

4. $Supp(\coprod_{i \in I} t_i) = \bigcup_{i \in I} Supp(t_i)$ for any collection of objects $t_i \in \mathcal{T}$.

Remark 4.2.6. The Balmer-Favi support extends the universal Balmer support for compact objects. That is, for $x \in \mathcal{T}^c$ we have $Supp(x) = supp(x)$; see [BHS23b, Lemma 2.18].

Definition 4.2.7 (Local-to-Global Principle)**.** Suppose that $Spc(\mathcal{T}^c)$ is weakly noetherian. We say $\mathcal{T}$ satisfies the *local-to-global principle* if

$$Locid(t) = Locid\langle t \otimes g(\mathcal{P}) : \mathcal{P} \in Spc(\mathcal{T}^c)\rangle$$

for any $t \in \mathcal{T}$.

Remark 4.2.8. Note that if $\mathcal{T}$ satisfies the local-to-global principle, then it also satisfies the so-called detection property:

$$Supp(t) = \varnothing \iff t = 0.$$

Remark 4.2.9. While there is no general characterization for when the local-to-global principle holds, a sufficient condition is that $Spc(\mathcal{T}^c)$ is noetherian. This is established in [BHS23b, Theorem 3.22], building off the work of Stevenson [Ste13, Ste17].

4.3 Stratification

Now we come to the main characterization of stratification established in [BHS23b, Theorem 4.1]:

Theorem 4.3.1 (Stratification)**.** *Suppose $Spc(\mathcal{T}^c)$ is weakly noetherian. Then the following are equivalent:*

(a) The local-to-global principle holds for $\mathcal{T}$ and for each prime $\mathcal{P} \in Spc(\mathcal{T}^c)$ we have that $Locid(g(\mathcal{P}))$ is a minimal localizing tensor ideal of $\mathcal{T}$.

(b) For all $t \in \mathcal{T}$, $Locid(t) = Locid\langle g(\mathcal{P}) : \mathcal{P} \in Supp(t)\rangle$.

(c) The map

$$\{Localizing\ ideals\ of\ \mathcal{T}\} \xrightarrow{Supp} \{Subsets\ of\ Spc(\mathcal{T}^c)\}$$
$$\mathcal{L} \to Supp(\mathcal{L}) := \bigcup_{t \in \mathcal{L}} Supp(t)$$

is a bijection.

Definition 4.3.2. We say that $\mathcal{T}$ is *stratified* if any of the equivalent conditions (a)-(c) from above hold. We will say that $\mathcal{T}$ has "*minimality at $\mathcal{P}$*", or *satisfies the minimality property at $\mathcal{P}$*, if $Locid(g(\mathcal{P}))$ is a minimal localizing ideal. Importantly, the minimality property is a condition that can be checked locally:

Theorem 4.3.3. *[BHS23b, Proposition 5.2] Suppose $\mathcal{T}$ satisfies the local-to-global principle and has weakly noetherian spectrum. Then $\mathcal{T}$ has minimality at the prime*

$\mathcal{P} \in Spc(\mathcal{T}^c)$ *if and only if the local category* $\mathcal{T}_{\mathcal{P}}$ *has minimality at its unique closed point.*

Remark 4.3.4. Thus, assuming the local-to-global principle holds, to establish $\mathcal{T}$ is stratified it suffices to check if each local category $\mathcal{T}/Locid(\mathcal{P})$ has minimality at its unique closed point.

Remark 4.3.5. The theory of stratification is amenable to various descent techniques (see, for example [BCH$^+$23, BCHS23]). Let us close out this section with a slight modification of 'quasi-finite descent' [BCHS23, Theorem 17.16] that we will use throughout the paper.

Definition 4.3.6. Let $\mathcal{T}$ and $\mathcal{S}$ be rigidly-compactly generated tt-categories. A coproduct preserving tt-functor $f^* : \mathcal{T} \to \mathcal{S}$ is called a *geometric functor*. We denote the right adjoint by f_* (which exists by Corollary 2.3.6). Moreover, there is the so-called projection formula

$$f_*(f^*(t)) \simeq f_*(\mathbb{1}) \otimes t$$

for all $t \in \mathcal{T}$, see for example [BDS16].

Proposition 4.3.7. *Let* $\mathcal{T}$ *and* $\mathcal{S}$ *be rigidly-compactly generated tensor-triangulated categories. Let* $f^* : \mathcal{T} \to \mathcal{S}$ *be a geometric functor and let*

$$\varphi := Spc(f^*) : Spc(\mathcal{S}^c) \to Spc(\mathcal{T}^c)$$

denote the induced map on spectra. Suppose that:

(a) $\mathcal{T}$ *is local and satisfies the detection property;*

(b) There exists a unique $\mathcal{P} \in Spc(\mathcal{S}^c)$ *such that* $\varphi(\mathcal{P}) = \mathcal{M}$*, the unique minimal prime in* $Spc(\mathcal{T}^c)$*; and*

(c) $g(\mathcal{M}) \in Locid(f_*(\mathbb{1}))$.

Then minimality at $\mathcal{P}$ *implies minimality at* $\mathcal{M}$.

Proof. First we will show that f^* is conservative on all objects $0 \neq t$ such that $Supp(t) \subseteq \{\mathcal{M}\}$. Let $0 \neq t$ be such an object supported on $\mathcal{M}$ and suppose that $f^*(t) = 0$. Then we have

$$0 = f_*(f^*(t)) = f_*(\mathbb{1}) \otimes t$$

showing that $f_*(\mathbb{1}) \in Ker(- \otimes t)$. This gives a containment of localizing tensor ideals $Locid(f_*(\mathbb{1})) \subseteq Ker(- \otimes t)$. By assumption however $g(\mathcal{M}) \in Locid(f_*(\mathbb{1}))$ so we get that $g(\mathcal{M}) \in Ker(- \otimes t)$. This implies however that $Supp(t) = \varnothing$ which means that $t = 0$ since $\mathcal{T}$ satisfies the detection property by assumption. Hence f^* is conservative on objects supported on $\mathcal{M}$. Now let $0 \neq t \in Locid(g(\mathcal{M}))$ and we note that:

$$0 \neq f^*(t) \in Locid(f^*(g(\mathcal{M}))) = Locid(g(\mathcal{P}))$$

where $f^*(g(\mathcal{M})) = g(\varphi^{-1}(\{\mathcal{M}\})) = g(\mathcal{P})$ by [BS17, Proposition 5.11] and by the hypothesis on the pre-image of $\{\mathcal{M}\}$. Since we are assuming minimality at $\mathcal{P}$ we have an equality

$$Locid(f^*(t)) = Locid(f^*(g(\mathcal{M}))).$$

Hence we get $f_*f^*(g(\mathcal{M})) \in f_*(Locid(f^*(t))) \subseteq Locid(t)$ by [BCHS23, 13.4]. Then observe that

$$g(\mathcal{M}) \in Locid(f_*(\mathbb{1})) \implies g(\mathcal{M}) \simeq g(\mathcal{M}) \otimes g(\mathcal{M}) \in Locid(g(\mathcal{M}) \otimes f_*(\mathbb{1})).$$

Using the projection formula we have $g(\mathcal{M}) \otimes f_*(\mathbb{1}) \simeq f_*f^*(g(\mathcal{M}))$ which gives us $g(\mathcal{M}) \in Locid(f_*f^*(g(\mathcal{M}))) \subseteq Locid(t)$ as desired. Hence minimality holds at the closed point $\mathcal{M}$. $\square$

Remark 4.3.8. The inspiration for the previous proposition comes from the following example. Let R be a commutative, local noetherian ring with maximal ideal $\mathfrak{m}$. Then we have the geometric functor

$$f^* := \kappa(\mathfrak{m}) \otimes - : D(R) \to D(\kappa(\mathfrak{m}))$$

where $f_*(\mathbb{1}) = \kappa(\mathfrak{m})$. Then one has, see for example [BHS23b, Theorem 5.8], that $g(\mathcal{M}) \in \mathit{Locid}(K_{\mathfrak{m}})$ is in the localizing ideal of the Koszul complex. Since $K_{\mathfrak{m}}$ has finite length-homology, $K_{\mathfrak{m}} \in \mathit{Locid}(\kappa(\mathfrak{m}))$. Hence we are in the setting of Proposition 4.3.7 and we can conclude that minimality holds at $\mathcal{M}$. This hints to the author that the previous proposition can be used in situations where we have a more concrete description about $g(\mathcal{M})$. We we will not use it however in what follows. The following result however, will be used throughout. Moreover, one should compare how similar the proof of the following proposition is to the proof of the above.

Proposition 4.3.9. *Let $f^* : \mathcal{T} \to \mathcal{S}$ be a geometric functor and let*

$$\varphi := \mathit{Spc}(f^*) : \mathit{Spc}(\mathcal{S}^c) \to \mathit{Spc}(\mathcal{T}^c)$$

denote the induced map on spectra. Assume that:

(1) $\mathcal{T}$ is local and satisfies the detection property;

(2) $f_*(\mathbb{1}_{\mathcal{S}}) \in \mathcal{T}$ is compact; and

(3) There exists a unique $\mathcal{P} \in \mathit{Spc}(\mathcal{S}^c)$ such that $\varphi(\mathcal{P}) = \mathcal{M}$ is the unique closed point in $\mathit{Spc}(\mathcal{T}^c)$.

Then minimality at $\mathcal{P}$ implies minimality at $\mathcal{M}$.

Proof. Our first step will again be to show that f^* is conservative on all objects supported on $\mathcal{M}$. Let $Supp(t) \subseteq \{\mathcal{M}\}$ and suppose $f^*(t) = 0$. Then $0 = f_* f^*(t) \simeq f_*(\mathbb{1}_{\mathcal{S}}) \otimes t$. Hence

$$\varnothing = Supp(f_*(\mathbb{1}) \otimes t) = supp(f_*(\mathbb{1}_{\mathcal{S}})) \cap Supp(t)$$

where the second equality holds from the half-tensor product formula (see [BHS23b, Lemma 2.18]). Since $\mathcal{S} \neq 0$ we have that $\mathbb{1}_{\mathcal{S}} = f^*(\mathbb{1}_{\mathcal{T}}) \neq 0$ and so $f_*(\mathbb{1}_{\mathcal{S}}) \neq 0$ since f_* is conservative on the image of f^* (see [BCHS23, Remark 13.9]). Now $supp(f_*(\mathbb{1}_{\mathcal{S}}))$ is closed since we are assuming $f_*(\mathbb{1}_{\mathcal{S}})$ is compact, and so it contains the unique closed point $\mathcal{M}$, which forces $Supp(t) = \varnothing$. Hence we conclude that $t = 0$ since $\mathcal{T}$ satisfies the detection property. This establishes that f^* is conservative on on all objects supported on $\mathcal{M}$. Now suppose $0 \neq t \in Locid(g(\mathcal{M}))$ and note that

$$0 \neq f^*(t) \in Locid(f^*(g(\mathcal{M}))) = Locid(g(\mathcal{P}))$$

where $f^*(g(\mathcal{M})) = g(\varphi^{-1}(\{\mathcal{M}\})) = g(\mathcal{P})$ by the same argument from the above proposition. Moreover, since we are assuming minimality at $\mathcal{P}$ we again obtain an equality

$$Locid(f^*(t)) = Locid(f^*(g(\mathcal{M}))).$$

Just as before, this implies $f_* f^*(g(\mathcal{M})) \in f_*(Locid(f^*(t))) \subseteq Locid(t)$. Finally we compute:

$$\begin{aligned} g(\mathcal{M}) &\in Locid(g(\mathcal{M}) \otimes f_*(\mathbb{1})) \text{ (by [BHS23b, Lemma 3.7])} \\ &= Locid(f_* f^*(g(\mathcal{M}))) \\ &\subseteq Locid(t) \end{aligned}$$

which establishes minimality at the prime $\mathcal{M}$ as desired. □

Chapter 5

Motives

In this chapter we recall the computation of the Balmer spectra of certain motivic categories by Martin Gallauer. We will follow [CD19, CD16] for the general theory of motives, and will also refer heavily to [Gal19] and the references therein.

5.1 Derived Categories of Motives

Definition 5.1.1. [CD19, 11.1.1, 11.1.2] We fix a commutative ring R, and the field $\overline{\mathbb{Q}}$. The *derived category of motives* over $\overline{\mathbb{Q}}$ is the tt-category, denoted $DM(\overline{\mathbb{Q}}, R)$, that is constructed out of the derived category of Nisnevich sheaves with transfers of R–modules on the category of smooth, finite type $\overline{\mathbb{Q}}$–schemes; see [CD19, 11.1.1, 11.1.2] for further details.

Remark 5.1.2. This category comes equipped with a functor

$$R(-) : Sm/\overline{\mathbb{Q}} \to DM(\overline{\mathbb{Q}}, R)$$

that sends a smooth $\overline{\mathbb{Q}}$-scheme X to its underlying 'motive', which is denoted $R(X)$. Moreover, the construction of $DM(\overline{\mathbb{Q}}, R)$ inverts the motive of the projective line, which we denote by $R(1)$. For any integer n, we then denote the 'Tatetwist' of weight n by $R(n) := R(1)^{\otimes n}$, and one gets that $R(i) \otimes R(j) = R(i+j)$ for all $i, j \in \mathbb{Z}$.

Definition 5.1.3. The *derived category of Tate motives*, denoted $DTM(\overline{\mathbb{Q}}, R)$, is defined as $DTM(\overline{\mathbb{Q}}, R) = Loc(R(n) : n \in \mathbb{Z})$. It is a rigidly-compactly generated tensor-triangulated category; see for example [San22, Example 5.17] and the references therein.

Remark 5.1.4. Given a ring homomorphism $R \to R'$ there is an induced geometric functor

$$\gamma^* : DM(\overline{\mathbb{Q}}, R) \to DM(\overline{\mathbb{Q}}, R')$$

which restricts to a geometric functor on Tate motives

$$\gamma^* : DTM(\overline{\mathbb{Q}}, R) \to DTM(\overline{\mathbb{Q}}, R').$$

The right adjoint γ_* of this geometric functor is always conservative (see [CD16, 5.4.2]). Moreover, if $R' \in D(R)^c$, then γ_* preserves compact objects [Gal19, §3].

5.2 Derived Categories of Étale Motives

There is a similar, parallel story to tell about the so-called étale motives:

Definition 5.2.1. [CD16, Gal19] We again fix a commutative ring R, and the field $\overline{\mathbb{Q}}$. The *derived category of étale motives* over $\overline{\mathbb{Q}}$ is the tensor-triangulated category denoted $DM^{\text{ét}}(\overline{\mathbb{Q}}, R)$, that is constructed out of the derived category of étale sheaves with transfers of R–modules on the category of smooth, finite type $\overline{\mathbb{Q}}$-schemes; see [CD16] and [Gal19, §4] for further details.

This category again comes equipped with an 'étale' motive functor:

$$R^{\text{ét}}(-) : Sm/\overline{\mathbb{Q}} \to DM^{\text{ét}}(\overline{\mathbb{Q}}, R),$$

and we again have invertible étale Tate objects $R^{\text{ét}}(n)$ arising from the étale motive of the projective line.

Definition 5.2.2. The *derived category of étale Tate motives* is the tt-category $DTM^{\text{ét}}(\overline{\mathbb{Q}}, R) = Loc(R^{\text{ét}}(n) : n \in \mathbb{Z})$. When R is any localization or quotient of the integers $\mathbb{Z}$ then $DTM^{\text{ét}}(\overline{\mathbb{Q}}, R)$ is also rigidly-compactly generated (see [Gal19, §4]).

Remark 5.2.3. Given a ring map $R \to R'$, we again have a geometric functor $\gamma^* : D(T)M^{\text{ét}}(\overline{\mathbb{Q}}, R) \to D(T)M^{\text{ét}}(\overline{\mathbb{Q}}, R')$ with conservative right adjoint γ_* which preserves compacts whenever $R' \in D(R)^c$.

Remark 5.2.4. Essentially since every Nisnevich sheaf is also an étale sheaf, there is a canonical 'étalification' tt-functor

$$\alpha_{\text{ét}} : DTM(\overline{\mathbb{Q}}, R) \to DTM^{\text{ét}}(\overline{\mathbb{Q}}, R)$$

that commutes with change of coefficients. Moreover, whenever $\mathbb{Q} \subseteq R$ this functor is an equivalence of categories [CD19, 16.1.2].

An important result by Gallauer provides another deep connection between étale Tate motives and Tate motives:

Theorem 5.2.5. *[Gal19, Theorem C.4] Let p be a prime number and let β_p : $\mathbb{Z}/p(0) \to \mathbb{Z}/p(1)$ denote the Bott map. Consider the geometric functor*

$$\gamma^* : DTM(\overline{\mathbb{Q}}, \mathbb{Z}_{(p)}) \to DTM(\overline{\mathbb{Q}}, \mathbb{Z}/p)$$

induced by the ring map $\mathbb{Z}_{(p)} \to \mathbb{Z}/p$ *and let*

$$\gamma_* : DTM(\overline{\mathbb{Q}}, \mathbb{Z}/p) \to DTM(\overline{\mathbb{Q}}, \mathbb{Z}_{(p)})$$

denote its right adjoint. Then the étalification functor

$$\alpha_{ét} : DTM(\overline{\mathbb{Q}}, \mathbb{Z}_{(p)}) \to DTM^{ét}(\overline{\mathbb{Q}}, \mathbb{Z}_{(p)})$$

is a finite localization with kernel $Locid(\gamma_*(cone(\beta_p)))$*. In particular, we have an equivalence*

$$DTM(\overline{\mathbb{Q}}, \mathbb{Z}_{(p)})/Locid(\gamma_*(cone(\beta_p))) \simeq DTM^{ét}(\overline{\mathbb{Q}}, \mathbb{Z}_{(p)}).$$

5.3 Balmer Spectrum of Tate Motives

Armed with this, we can now describe Gallauer's computation of the Balmer spectrum of Tate motives. Let us first state the spectrum for finite coefficients:

Proposition 5.3.1. *[Gal19, Proposition 8.2] Let* $\beta_p : \mathbb{Z}/p(0) \to \mathbb{Z}/p(1)$ *denote the Bott map. Then* $Spc(DTM(\overline{\mathbb{Q}}, \mathbb{Z}/p)^c)$ *consists of two connected points:*

$$Spc(DTM(\overline{\mathbb{Q}}, \mathbb{Z}/p)^c) = \begin{array}{c} (0) \\ | \\ \langle cone(\beta_p) \rangle \end{array}$$

The last result needed before the main computation by Gallauer is the following computation of the spectrum with rational coefficients in [Pet13, Theorem 4.15]:

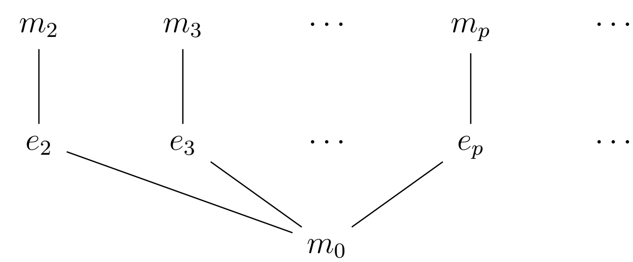
Theorem 5.3.2 (Peter)**.** *The Balmer spectrum of rational motives is*

$$Spc(DTM(\overline{\mathbb{Q}}, \mathbb{Q})^c) \simeq Spec(\mathbb{Q}) \simeq (0)$$

We can now bring these results together for integral coefficients [Gal19, Theorem 8.6]:

Theorem 5.3.3 (Gallauer)**.** *The Balmer spectrum of $DTM(\overline{\mathbb{Q}}, \mathbb{Z})^c$ is the following noetherian topological space*

$$Spc(DTM(\overline{Q}, \mathbb{Z})^c) =$$



where the p's are prime numbers and the specialization relations are depicted by the lines going upwards. The primes are given as follows:

(height 2) : *m_p is the kernel of the tt-functor*

$$\gamma_p^* : DTM(\overline{\mathbb{Q}}, \mathbb{Z})^c \to DTM(\overline{\mathbb{Q}}, \mathbb{Z}/p)^c$$

induced from the ring map $\mathbb{Z} \to \mathbb{Z}/p$. Moreover, this prime coincides with the motives whose mod-p motivic cohomology vanishes.

(height 1) : *e_p is the kernel of the following composite*

$$DTM(\overline{\mathbb{Q}}, \mathbb{Z})^c \xrightarrow{\gamma_p^*} DTM(\overline{\mathbb{Q}}, \mathbb{Z}/p)^c \xrightarrow{\alpha_{ét}} DTM^{ét}(\overline{\mathbb{Q}}, \mathbb{Z}/p)^c.$$

This corresponds to the motives whose mod-p étale cohomology vanishes.

(height 0) : *The generic point m_0 is the kernel of the finite localization*

$$\gamma_{\mathbb{Q}}^* : DTM(\overline{\mathbb{Q}}, \mathbb{Z})^c \to DTM(\overline{\mathbb{Q}}, \mathbb{Q})^c$$

induced from the ring localization $\mathbb{Z} \to \mathbb{Q}$. This prime corresponds to those motives whose rational motivic cohomology vanishes.

Moreover, the étalification tt-functor $\alpha_{ét} : DTM(\overline{\mathbb{Q}}, \mathbb{Z}) \to DTM^{ét}(\overline{\mathbb{Q}}, \mathbb{Z})$ *induces an inclusion on spectra*

$$Spec(\mathbb{Z}) \simeq Spc(DTM^{ét}(\overline{\mathbb{Q}}, \mathbb{Z})^c) \xhookrightarrow{Spc(\alpha_{ét})} Spc(DTM(\overline{\mathbb{Q}}, \mathbb{Z})^c)$$

which is a homeomorphism onto the subspace $\{m_0, e_p : \ p \text{ prime}\}$.

Remark 5.3.4. Since $Spc(DTM(\overline{\mathbb{Q}}, \mathbb{Z})^c)$ is noetherian, $DTM(\overline{\mathbb{Q}}, \mathbb{Z})$ automatically satisfies the local-to-global principle (recall Remark 4.2.9). The rest of this thesis will therefore be focused on proving minimality holds at each prime.

Remark 5.3.5. Looking at the computation of $Spc(DTM(\overline{\mathbb{Q}}, \mathbb{Z})^c)$ above, we see that there are 3 'types' of primes. Indeed, each vertical slice in the spectrum is obtained from a finite localization $DTM(\overline{\mathbb{Q}}, \mathbb{Z}) \to DTM(\overline{\mathbb{Q}}, \mathbb{Z}_{(p)})$ ([Gal19, Example 6.12, Proposition 10.2]) induced from the ring map $\mathbb{Z} \to \mathbb{Z}_{(p)}$. These are local categories, with m_p being the (0) ideal now. Pictorially, for example, we have

$$Spc(DTM(\overline{\mathbb{Q}}, \mathbb{Z}_{(p)})^c) = \begin{array}{c} m_p = (0) \\ | \\ e_p \\ | \\ m_0 \end{array}$$

Moreover, by [BHS23b, Proposition 1.32] in order to prove minimality holds at the primes m_p and e_p it suffices to prove it in the local category $DTM(\overline{\mathbb{Q}}, \mathbb{Z}_{(p)})$.

Chapter 6

Minimality at the Height Zero Prime

In this chapter, we will prove minimality holds at the the height zero prime m_0.

Remark 6.0.1. Recall that the height zero prime m_0 is obtained from the algebraic localization going from integral to rational coefficients. That is

$$DTM(\overline{\mathbb{Q}}, \mathbb{Z})/Locid(m_0) \simeq DTM(\overline{\mathbb{Q}}, \mathbb{Q})$$

By Theorem 4.3.3, to prove that minimality holds at the prime m_0 it suffices to prove minimality holds at the unique prime in $DTM(\overline{\mathbb{Q}}, \mathbb{Q})$. Let us now recall the computation of $Spc(DTM(\overline{\mathbb{Q}}, \mathbb{Q})^c)$ by Peter [Pet13].

6.1 t-structures

Definition 6.1.1. A *t-structure* on $\mathcal{T}$ is a pair of subcategories $\mathbf{t} = (\mathcal{U}, \mathcal{V})$ such that:

1. $\mathcal{U}, \mathcal{V}$ are both closed under summands;

2. $\Sigma\mathcal{U} \subseteq \mathcal{U}$ and $\Sigma^{-1}\mathcal{V} \subseteq \mathcal{V}$;

3. $Hom_{\mathcal{T}}(\mathcal{U}, \mathcal{V}) = 0$; and

4. Every object $X \in \mathcal{T}$ fits into a triangle

$$U \to X \to V[-1] \to U[1]$$

with $U \in \mathcal{U}, V \in \mathcal{V}$. This is equivalent to saying the inclusion $\mathcal{U} \hookrightarrow \mathcal{T}$ (respectively $\mathcal{V} \hookrightarrow \mathcal{T}$) admits a right adjoint $\tau^{\leq 0} : \mathcal{T} \to \mathcal{U}$ (respectively a left adjoint $\tau^{\geq 1} : \mathcal{T} \to \mathcal{V}$) called the 'truncation functors.'

Given a t-structure on $\mathcal{T}$ we say that:

1. $\mathcal{U}$ is the *aisle* of the t-structure, and $\mathcal{V}$ is the **coaisle** of the t-structure.

2. The category $\mathcal{T}^{\heartsuit} := \mathcal{U} \cap \mathcal{V}$ is called the *heart* of the t-structure. A major reason t-structures are of interest is that the heart of any t-structure $\mathcal{T}^{\heartsuit}$ is an abelian category (see [BBD82]). Moreover, we get an 'internal' cohomological functor $\mathcal{H}^0 : \mathcal{T} \to \mathcal{T}^{\heartsuit}$ given by

$$\mathcal{H}^0 = \tau^{\geq 0} \circ \tau^{\leq 0} \simeq \tau^{\leq 0} \circ \tau^{\geq 0}$$

and we define the higher cohomologies to be $H^n(X) := H^0(\Sigma^n X)$ for all $n \in \mathbb{Z}$.

The prototypical example of a t-structure is the following:

Example 6.1.2. *Let R be a commutative ring and consider the unbounded derived category $D(R)$. Then the following is a t-structure on $D(R)$, called the 'canonical' or 'standard' t-structure:*

$$\mathcal{U} = \{X \in D(R) : H^i(X) = 0 \textit{ for all } i > 0\}$$
$$\mathcal{V} = \{X \in D(R) : H^i(X) = 0 \textit{ for all } i < 0\}$$

The heart of this t-structure will be canonically isomorphic to $Mod(R)$.

Remark 6.1.3. To orient the reader for why we are interested in the following definitions: our goal will be to eventually produce a triangulated equivalence between $DTM(\overline{\mathbb{Q}}, \mathbb{Q})$ and the derived category of the heart of a t-structure on $DTM(\overline{\mathbb{Q}}, \mathbb{Q})$. Such an equivalence can be obtained, provided the t-structure satisfies extra properties, which we now explain.

Definition 6.1.4. Let $\mathbf{t} = (\mathcal{U}, \mathcal{V})$ denote a t-structure on a rigidly-compactly generated tt-category $\mathcal{T}$ and let us denote the heart by $\mathcal{T}^{\heartsuit}$. The t-structure $\mathbf{t}$ is said to be:

(a) *Non Degenerate* if

$$\bigcap_{k\in\mathbb{Z}} \Sigma^k\mathcal{U} = 0 = \bigcap_{k\in\mathbb{Z}} \Sigma^k\mathcal{V}.$$

(b) *Smashing* if the coaisle $\mathcal{V}$ is closed under coproducts.

(c) *Compatible* if $\mathcal{T}^{\heartsuit} \otimes \mathcal{T}^{\heartsuit} \subseteq \mathcal{T}^{\heartsuit}$ and $\mathbb{1} \in \mathcal{T}^{\heartsuit}$. In this case $\mathcal{T}^{\heartsuit}$ is itself a tensor-abelian category, whose tensor structure is exact in both variables. Moreover the unit of $\mathcal{T}^{\heartsuit}$ is just the same unit as in $\mathcal{T}$ (see [Pet13, Remark 3.7]).

(d) *Strongly hereditary* if $Hom_{\mathcal{T}}(X, \Sigma^i Y) = 0$ for $X, Y \in \mathcal{T}^{\heartsuit}$ and $i \geq 2$. Note that if $\mathbf{t}$ is strongly hereditary then $\mathcal{T}^{\heartsuit}$ is a hereditary abelian category, that is $Ext^i(M, N) = 0$ for all $i \geq 2$ and $M, N \in \mathcal{T}^{\heartsuit}$, see [Lev93, Proposition 1.6].

Example 6.1.5.

(a) The standard t-structure on $D(R)$ for a commutative ring R is both non-degenerate and smashing. If the tensor structure is exact on $Mod(R)$ then it is compatible.

(b) If $\mathcal{A}$ is a hereditary algebra then the standard t-structure on $D(\mathcal{A})$ is strongly hereditary.

Remark 6.1.6. A t-structure **t** is smashing if and only if the cohomology functor $\mathcal{H}^0 : \mathcal{T} \to \mathcal{T}^\heartsuit$ preserves coproducts, see for example [AHMV17, Lemma 3.3].

Lemma 6.1.7. *If a t-structure* **t** *is strongly hereditary, then, for all $M, N \in \mathcal{T}^\heartsuit$ and for all $i \in \mathbb{Z}$ we have $Ext^i(M,N) \simeq Hom_{\mathcal{T}}(M, \Sigma^i N)$.*

Proof. Indeed, for $i = 0$ we have $Ext^0(M,N) = Hom_{\mathcal{T}}(M,N) = Hom_{\mathcal{T}^\heartsuit}(M,N)$ since $\mathcal{T}^\heartsuit$ is a full subcategory. For $i = 1$ we have $Ext^1(M,N) = Hom_{\mathcal{T}}(M, \Sigma N)$ for any heart $\mathcal{T}^\heartsuit$ (see [BBD82]). For $i \geq 2$ we have $Ext^i(M,N) = 0 = Hom_{\mathcal{T}}(M, \Sigma^i N)$ by assumption and since $\mathcal{T}^\heartsuit$ is a hereditary category. Finally for $i < 0$ we always have $Ext^i(M,N) = 0$ and $Hom_{\mathcal{T}}(M, N[i]) = 0$ since every t-structure **t** has this vanishing condition. □

Remark 6.1.8. One has the following two inclusion maps

$$\begin{array}{ccc} \mathcal{T}^\heartsuit & \overset{i_{\mathcal{T}}}{\hookrightarrow} & \mathcal{T} \\ {\scriptstyle i_{D(\mathcal{T}^\heartsuit)}} \downarrow & & \\ D(\mathcal{T}^\heartsuit). & & \end{array}$$

A very natural question to ask is whether we can extend the inclusion of $\mathcal{T}^\heartsuit$ into $\mathcal{T}$ into a triangulated functor on $D(\mathcal{T}^\heartsuit)$ (called the realization functor)

$$\begin{array}{ccc} \mathcal{T}^\heartsuit & \overset{i_{\mathcal{T}}}{\hookrightarrow} & \mathcal{T} \\ {\scriptstyle i_{D(\mathcal{T}^\heartsuit)}} \downarrow & \nearrow {\scriptstyle \exists real} & \\ D(\mathcal{T}^\heartsuit) & & \end{array}$$

in such a way that $real(i_{D(\mathcal{T}^\heartsuit)}) \simeq i_{\mathcal{T}}$? For the bounded derived category, an affirmative answer is reached provided there is a so-called 'f-category' living above the category $\mathcal{T}$ (see [Bei06, Appendix]). Such a condition is not too strict in practice, as Modoi proved in [Mod19] that every triangulated category which is the underlying category of a stable derivator admits such an f-category. In particular, as every stable model category gives rise to a stable derivator (see for example [Vir19, Example 1.11]), any triangulated category that arises as the homotopy category of a stable model category admits an f-category.

Remark 6.1.9. In [Vir19], Virili is able to construct an unbounded realization functor for a t-structure with some assumptions about some higher structures living above $\mathcal{T}$. Let us briefly set the stage for the following theorem. Suppose that $\mathcal{T}$ arises as the base of a strong, stable derivator $\mathbb{D}$, that is $\mathcal{T} = \mathbb{D}(e)$, and let $\mathbf{t}$ be a t-structure on $\mathcal{T}$. Then Virili is able to construct a morphism of derivators

$$real_{\mathbf{t}} : Ch_{\mathcal{T}^\heartsuit} \to \mathbb{D}$$

where $Ch_{\mathcal{T}^\heartsuit}$ is the derivator defined by $Ch_{\mathcal{T}^\heartsuit}(I) = Ch((\mathcal{T}^\heartsuit)^I)$ for any diagram category I. Moreover, this morphism of derivators takes quasi-isomorphisms to isomorphisms ([Vir19, Theorem 6.7]), so, assuming the derived category $D(\mathcal{T}^\heartsuit)$ exists, the morphism factors through the derivator underlying the derived category to give a morphism of derivators

$$real_{\mathbf{t}} : D_{\mathcal{T}^\heartsuit} \to \mathbb{D}$$

Moreover, assuming the t-structure satisfies certain properties we can say more about this morphism. We summarize all this below, rephrased in the language of triangulated categories.

Theorem 6.1.10. *[Vir19, §6.3] Suppose $\mathcal{T}$ arises as the base of a strong and stable derivator $\mathbb{D}$ (for example, $\mathcal{T}$ can be the homotopy category of a stable model category), and let $\boldsymbol{t}$ be a t-structure on $\mathcal{T}$. Denote by*

$$real^b : D^b(\mathcal{T}^\heartsuit) \to \mathcal{T}$$

the bounded realization functor (which exists by [Mod19]). Then we can lift the bounded realization functor to an unbounded one

$$real : D(\mathcal{T}^{\heartsuit}) \to \mathcal{T}.$$

Moreover, suppose that

(a) $\boldsymbol{t}$ *is non-degenerate,*

(b) $real^b$ *is fully faithful, and*

(c) $\boldsymbol{t}$ *is smashing.*

Then the unbounded realization functor remains fully faithful and is coproduct preserving.

6.2 t-structure for Rational Motives

We will now introduce the t-structure that exists for $DTM(\overline{\mathbb{Q}}, \mathbb{Q})$. This was first studied by Levine in [Lev93], and then expanded upon by Wildeshaus in [Wil16]. They state their results at the level of compact objects, however, as we will see their constructions generalize to the unbounded categories. For ease of notation let us write $\mathcal{T} := DTM(\overline{\mathbb{Q}}, \mathbb{Q})$ for what follows.

Definition 6.2.1. Define $\mathcal{T}_{[a,b]} := Loc(\mathbb{Q}(n) : a \leq -2n \leq b)$. We allow for both $a, b \in \{-\infty, \infty\}$, in which case we just have $\mathcal{T} = \mathcal{T}_{(-\infty,\infty)}$. We write $\mathcal{T}_a := \mathcal{T}_{[a,a]}$.

Remark 6.2.2. It follows that $\mathcal{T}_a \simeq D(\mathbb{Q})$ (see [Lev93, §1]).

We can define a t-structure on $\mathcal{T}_{[a,b]}$ as follows:

Lemma 6.2.3. *Let* $a \leq b \leq c$*. Then* $\boldsymbol{t} := (\mathcal{T}_{[a,b-1]}, \mathcal{T}_{[b,c]})$ *is a t-structure on* $\mathcal{T}_{[a,c]}$*.*

Proof. This is [Lev93, Lemma 1.2]. □

Remark 6.2.4. Let $a \in \mathbb{Z}$. The cohomology functor associated to the t-structure $(\mathcal{T}_{[\infty,a-1]}, \mathcal{T}_{[a,\infty]})$ is denoted $gr_a := H^0_a$. Note that this t-structure is smashing by construction, so gr_a is coproduct preserving by Remark 6.1.6.

Definition 6.2.5. [Lev93, Definition 1.4] Let a be even. Define $\mathcal{T}_a^{\geq 0}$ to be the full, additive category generated by $\mathbb{Q}(\frac{a}{2})[n]$ for $n \leq 0$. Similarly define $\mathcal{T}_a^{\leq 0}$ to be the full, additive category generated by $\mathbb{Q}(\frac{a}{2})[n]$ for $n \geq 0$. Let $\mathcal{T}^{\geq 0}$ be the full subcategory of $\mathcal{T}$ of objects $X \in \mathcal{T}$ such that $gr_c(X) \in \mathcal{T}_c^{\geq 0}$ for all $c \in \mathbb{Z}$. Similarly, define $\mathcal{T}^{\leq 0}$ be the full subcategory of $\mathcal{T}$ of objects $X \in \mathcal{T}$ such that $gr_c(X) \in \mathcal{T}_c^{\leq 0}$ for all $c \in \mathbb{Z}$.

Remark 6.2.6. Under the equivalence $D(\mathbb{Q}) \simeq \mathcal{T}_a$ from Remark 6.2.2, we have that the pair $(\mathcal{T}_a^{\leq 0}, \mathcal{T}_a^{\geq 0})$ on $\mathcal{T}_a$ corresponds to the standard t-structure on $D(\mathbb{Q})$.

Remark 6.2.7. [Lev93, §4] To spell it out a little more, we have that:

$$X \in \mathcal{T}^{\leq 0} \iff gr_a(X) \simeq \coprod_{n \geq 0} \Sigma^n \mathbb{Q}(\frac{a}{2})^{m_n} \text{ for all } a \in \mathbb{Z}.$$

$$X \in \mathcal{T}^{\geq 0} \iff gr_a(X) \simeq \coprod_{n \leq 0} \Sigma^n \mathbb{Q}(\frac{a}{2})^{m_n} \text{ for all } a \in \mathbb{Z}.$$

Here we come to the t-structure we are interested in.

Theorem 6.2.8. *The pair $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is a t-structure on $\mathcal{T}$. Moreover,*

(a) The t-structure is non-degenerate, compatible and strongly hereditary.

(b) The heart $\mathcal{AT}$ is generated as a full abelian category closed under extensions and coproducts by $\mathbb{Q}(n), n \in \mathbb{Z}$.

(c) Every object $X \in \mathcal{AT}$ admits a functorial filtration by sub-objects

$$0 \subset \cdots \subset Z_n(X) \subset Z_{n+1}(X) \subset \cdots \subset X$$

whose sub-quotients $Z_n(X)/Z_{n-1}(X)$ are coproducts of shifts of $\mathbb{Q}(n)$.

(d) The t-structure is smashing.

Proof. Parts (a)-(c) are [Lev93, Theorem 4.2] and [Pet13, Lemma 3.3]. The only change is in Peter's lemma, where our t-structure will no longer be bounded, but that is unimportant for us. Moreover, part (d) follows from our definition of $\mathcal{T}_a$, the description of the aisle and co-aisle in the preceding remark, and that gr_a commutes with coproducts (recall Remark 6.2.4). □

Corollary 6.2.9. *There is an equivalence of triangulated categories*

$$real : D(\mathcal{AT}) \simeq DTM(\overline{\mathbb{Q}}, \mathbb{Q}).$$

Proof. We can apply Theorem 6.1.10 to get an exact and coproduct preserving functor $real : D(\mathcal{AT}) \to DTM(\overline{\mathbb{Q}}, \mathbb{Q})$. To prove that $real^b$ is fully faithful, since $\mathcal{AT}$ generates $D^b(\mathcal{AT})$ as a triangulated category, it suffices to prove that

$$\begin{aligned} Ext^i(M,N) &\xrightarrow{real^b} hom_{DTM(\overline{\mathbb{Q}},\mathbb{Q})}(real^b(M), real^b(N)[i]) \\ &= hom_{DTM(\overline{\mathbb{Q}},\mathbb{Q})}(M, N[i]) \end{aligned}$$

is a bijection for objects $M, N \in \mathcal{AT}$. Since the t-structure is strongly hereditary, this follows from Lemma 6.1.7. Hence $real^b$ is fully faithful, which implies $real$ is fully faithful by Theorem 6.1.10. Thus the essential image is a localizing subcategory of $DTM(\overline{\mathbb{Q}}, \mathbb{Q})$. This category contains all the Tate twists by Theorem 6.2.8(b). Since the Tate twists generate $DTM(\overline{\mathbb{Q}}, \mathbb{Q})$ we get that the essential image is everything. □

This equivalence immediately gives us the following:

Corollary 6.2.10. *Every object* $t \in DTM(\overline{\mathbb{Q}}, \mathbb{Q})$ *is isomorphic to a coproduct of shifts of its cohomologies,* $t \simeq \coprod_i \Sigma^{-i} H^i(t)$*.*

Proof. This isomorphism holds in $D(\mathcal{AT})$ because $\mathcal{AT}$ is hereditary, see for example [Kra07]. The result thus follows for $DTM(\overline{\mathbb{Q}}, \mathbb{Q})$ since *real* is a coproduct preserving equivalence, and is just the identity on the heart. □

We now turn to classifying the localizing ideals of $DTM(\overline{\mathbb{Q}}, \mathbb{Q})$. We will closely follow Peter's approach of computing the thick tensor ideals of $DTM(\overline{\mathbb{Q}}, \mathbb{Q})^c$.

Remark 6.2.11. Let us fix some notation for the rest of this section. We have the additive functor

$$\phi : DTM(\overline{\mathbb{Q}}, \mathbb{Q}) \to \mathcal{AT}$$
$$X \longrightarrow \coprod_i H^i(X)$$

and the inclusion functor

$$\iota : \mathcal{AT} \hookrightarrow DTM(\overline{\mathbb{Q}}, \mathbb{Q}).$$

Note that the functor ϕ is also a tensor-functor, because the t-structure is compatible; see [Big07]. Following the terminology of Peter we say $\mathcal{M} \subseteq \mathcal{AT}$ is a coherent tensor ideal of $\mathcal{AT}$ if it is closed under extensions, kernels, cokernels, and tensoring by arbitrary elements of $\mathcal{AT}$. Let us denote by $Coh^{\amalg}(\mathcal{AT})$ to be the set of coherent tensor ideals of $\mathcal{AT}$ closed under coproducts.

Proposition 6.2.12. *The maps*

$$\{\textit{Localizing ideals of } DTM(\overline{\mathbb{Q}}, \mathbb{Q})\} \underset{\phi^*}{\overset{\iota^*}{\rightleftarrows}} Coh^{\amalg}(\mathcal{AT})$$

defined by

$$\iota^*(\mathcal{L}) = \mathcal{L} \cap \mathcal{AT} = \iota^{-1}(\mathcal{L})$$
$$\phi^*(\mathcal{M}) = \phi^{-1}(\mathcal{M})$$

are bijective functions inverse to each other.

Proof. This is the unbounded analogue of [Pet13, Theorems 4.4, 4.11]. We include a sketch of the argument here for the reader's convenience. The first step is to prove ϕ^* is well defined, so let $\mathcal{M} \subseteq \mathcal{AT}$ be a coherent tensor ideal, closed under coproducts. The same proof as in [Pet13, Theorem 4.4] shows that $\phi^*(\mathcal{M})$ is closed under extensions and tensors. It is closed under coproducts because the cohomology functors preserve coproducts since the t-structure is smashing, and because $\mathcal{M}$ is closed under coproducts by assumption. Now we observe that since $\phi \circ \iota \simeq id_{\mathcal{AT}}$ we have $\iota^*(\phi^*(\mathcal{M})) = (\phi \circ \iota)^{-1}(M) = M$, so we get that ι^* is well defined on the image of ϕ^*, and is left inverse to it. Thus, it remains to show ϕ^* is surjective. To do so, we can apply [Pet13, Theorem 4.11] after noting that we still have an isomorphism

$$t \simeq \coprod_{i \in \mathbb{Z}} \Sigma^{-i} H^i(t)$$

for any $t \in DTM(\overline{\mathbb{Q}}, \mathbb{Q})$. □

Remark 6.2.13. We can now complete our last reduction step: we continue to denote $\mathcal{T} = DTM(\overline{\mathbb{Q}}, \mathbb{Q})$. For each $a \in \mathbb{Z}$ recall we have the coproduct preserving functor

$$gr_a : \mathcal{T} \to \mathcal{T}_a \simeq D(\mathbb{Q}) \simeq Gr_{\mathbb{Z}}(Vect_{\mathbb{Q}})$$

Moreover, Biglari showed ([Big09, Proposition 3.8]) the following:

Proposition 6.2.14. *The functor*

$$gr := \coprod_{a \in \mathbb{Z}} \Sigma^{-a} gr_a : \mathcal{T} \to D(\mathbb{Q})$$
$$X \mapsto \coprod_{a \in \mathbb{Z}} \Sigma^{-a} gr_a(X)$$

is a tensor-triangulated functor.

If we restrict the domain of this functor to just the heart, we obtain the following [Lev93, Corollary 4.3]:

Proposition 6.2.15. *The functor* $gr|_{\mathcal{AT}} : \mathcal{AT} \to D(\mathbb{Q})$ *gives an equivalence of* $\mathcal{AT}$ *with a tensor subcategory of* $Gr_{\mathbb{Z}}(Vect_{\mathbb{Q}})$*.*

With these two propositions in hand, we can run the exact argument as in [Pet13, Theorem 4.15] to conclude:

Corollary 6.2.16. *The only coherent tensor ideals closed under coproducts of* $\mathcal{AT}$ *are the* (0) *ideal and* $\mathcal{AT}$*.*

Theorem 6.2.17. *The category* $DTM(\overline{\mathbb{Q}}, \mathbb{Q})$ *is stratified.*

Proof. Since $Spc(DTM(\overline{\mathbb{Q}}, \mathbb{Q})^c) = \{*\}$ is just a point, we have that $g(*) = \mathbb{1} \neq 0$, and the local-to-global principle holds. Moreover, by combining Proposition 6.2.12 and Proposition 6.2.16, we get that there are only two localizing ideals of $DTM(\overline{\mathbb{Q}}, \mathbb{Q})$. Hence the localizing ideal $g(*)$ generates has to be minimal. □

Again keeping in mind Remark 6.0.1, this gives us:

Corollary 6.2.18. *Minimality holds at the height zero prime* m_0*.*

Chapter 7

Minimality at the Height One Prime

In this chapter we establish minimality for the height one prime e_p. Using this and results from the previous chapter, we prove that $DTM^{\text{ét}}$ is stratified.

Remark 7.0.1. From Proposition 4.3.3 establishing minimality at the prime e_p is equivalent to establishing minimality at the closed point in

$$DTM(\overline{\mathbb{Q}}, \mathbb{Z}_{(p)})/Locid(e_p).$$

Combining Theorems 5.2.5, 5.3.1 and 5.3.3, we get that

$$DTM(\overline{\mathbb{Q}}, \mathbb{Z}_{(p)})/Locid(e_p) \simeq DTM^{\text{ét}}(\overline{\mathbb{Q}}, \mathbb{Z}_{(p)})$$

and so we are reduced to establishing minimality at the closed point in the category $DTM^{\text{ét}}(\overline{\mathbb{Q}}, \mathbb{Z}_{(p)})$. To do so, we will consider the geometric functor

$$\gamma^* : DTM^{\text{ét}}(\overline{\mathbb{Q}}, \mathbb{Z}_{(p)}) \to DTM^{\text{ét}}(\overline{\mathbb{Q}}, \mathbb{Z}/p) \tag{7.1}$$

induced by the ring map $\mathbb{Z}_{(p)} \to \mathbb{Z}/p$. Note that $\mathbb{Z}_{(p)}$ is a regular ring and so we have $\mathbb{Z}/p \in D(\mathbb{Z}_{(p)})^c$. Hence $\gamma_*(\mathbb{1}_{DTM^{\text{ét}}(\overline{\mathbb{Q}}, \mathbb{Z}/p)}) \in DTM^{\text{ét}}(\overline{\mathbb{Q}}, \mathbb{Z}_{(p)})^c$ by Remark 5.1.4.

This category $DTM^{\text{ét}}(\overline{\mathbb{Q}}, \mathbb{Z}/p)$ is quite simple in fact.

Theorem 7.0.2 (Rigidity Theorem)**.** *There is a tt-equivalence*

$$DM^{\acute{e}t}(\overline{\mathbb{Q}}, \mathbb{Z}/p) \simeq D(\mathbb{Z}/p)$$

Proof. This is [CD16, Theorem 4.5.2]. □

Corollary 7.0.3. *The category $DTM^{\acute{e}t}(\overline{\mathbb{Q}}, \mathbb{Z}_{(p)})$ is stratified.*

Proof. As the derived category of a field is, of course, stratified, we obtain the same for $DM^{\text{ét}}(\overline{\mathbb{Q}}, \mathbb{Z}/p)$. We can then apply [BKS19, Proposition 5.22] to conclude that $DTM^{\text{ét}}(\overline{\mathbb{Q}}, \mathbb{Z}/p)$ is itself a tt-field, and thus is also stratified (see [BCHS23, Theorem 18.4]). □

Remark 7.0.4. On spectra the geometric functor (7.1) gives

$$Spc(DTM^{\text{ét}}(\overline{\mathbb{Q}}, \mathbb{Z}/p)^c) \xrightarrow{Spc(\gamma_*)} Spc(DTM^{\text{ét}}(\overline{\mathbb{Q}}, \mathbb{Z}_{(p)})^c)$$

$$\begin{array}{ccc} (0) & \overset{Spc(\gamma_*)}{\hookrightarrow} & e_p \\ & & | \\ & & m_0 \end{array}$$

We can therefore apply Theorem 4.3.9 and we immediately get:

Corollary 7.0.5. *Minimality holds at the height 1 primes e_p.*

Proof. By Remark 7.0.1 we need to prove the category $DTM^{\text{ét}}(\overline{\mathbb{Q}}, \mathbb{Z}_{(p)})$ satisfies minimality at the unique closed point. Moreover we can apply Proposition 4.3.9, since we have that $\gamma_*(\mathbb{1}_{DTM^{\text{ét}}(\overline{\mathbb{Q}}, \mathbb{Z}/p)}) \in DTM^{\text{ét}}(\overline{\mathbb{Q}}, \mathbb{Z}_{(p)})^c$, to focus on establishing minimality at the unique prime in $DTM^{\text{ét}}(\overline{\mathbb{Q}}, \mathbb{Z}/p)$. This then follows from Corollary 7.0.3. □

Theorem 7.0.6. *The derived category of étale motives, $DTM^{\acute{e}t}(\overline{\mathbb{Q}}, \mathbb{Z})$ is stratified.*

Proof. As $Spc(DTM^{\text{ét}}(\overline{\mathbb{Q}}, \mathbb{Z})^c)$ is noetherian it automatically satisfies the local-to-global principle. Hence we just need to show minimality holds at each prime. By Theorem 5.3.3, these primes are m_0 and e_p for p prime, and we showed minimality holds at these primes in Corollaries 7.0.5 and 6.2.18. □

Chapter 8

Brown–Adams Representability and Stratification

In this chapter, we return to Question 1.0.4 from the introduction in order to prove minimality holds at the height two prime m_p in Chapter 9. To remind the reader, we suppose we are given two rigidly-compactly generated tt-categories $\mathcal{T}_1$ and $\mathcal{T}_2$, where $\mathcal{T}_1$ is stratified and that we have a tt-equivalence between their compact parts $\mathcal{T}_1^c \simeq \mathcal{T}_2^c$. Our question is whether this then implies that $\mathcal{T}_2$ is stratified as well. While we are not able to answer this question in full generality, we are able to achieve positive results assuming our category satisfies Brown–Adams Representability, which we now explain.

Let $\mathcal{T}$ be a rigidly-compactly generated tt-category. The *category of modules* on $\mathcal{T}$ is the Grothendieck abelian category

$$\mathcal{A} := Mod(\mathcal{T}^c) := Add((\mathcal{T}^c)^{op}, Ab)$$

of contravarient additive functors from $\mathcal{T}^c$ to abelian groups. The subcategory of finitely presented modules $\mathcal{A}^{fp} := mod(\mathcal{T}^c)$ coincides with the usual Freyd-envelope of $\mathcal{T}^c$ [Nee01, Chapter 5]. For relevant information about this category of modules, we will be primarily following the recent series of papers by Balmer, Krause and Stevenson [Bal20a, BKS20, Bal20b, BKS19].

Remark 8.0.1. We have the restricted Yoneda functor

$$h : \mathcal{T} \to \mathcal{A}$$
$$t \to \hat{t} := Hom(-, t)|_{\mathcal{T}^c}$$

which fits into the commutative square

$$\begin{array}{ccc} \mathcal{T}^c & \overset{h}{\hookrightarrow} & \mathcal{A}^{fp} \\ \downarrow & & \downarrow \\ \mathcal{T} & \xrightarrow{h} & \mathcal{A}. \end{array}$$

Note that restricted Yoneda is no longer an embedding in general, due to the potential existence of so-called phantom maps. Nevertheless, $h : \mathcal{T} \to \mathcal{A}$ is conservative, that is $\hat{t} = 0 \implies t = 0$, because $\mathcal{T}$ is compactly generated.

Definition 8.0.2. For $S \subseteq \mathcal{A}$ we write

1. $Loc_{\mathcal{A}}(S)$ to be the smallest Serre subcategory containing S closed under coproducts and suspension.

2. $Locid_{\mathcal{A}}(S)$ the smallest Serre subcategory containing S closed under coproducts and tensor products (and hence is automatically closed under suspension, see [BKS19, Remark 2.2]).

We summarize the facts we will need about $\mathcal{A}$ below. This can be found in [BKS19, §§2-3].

(1) $\mathcal{A}$ inherits a suspension $\Sigma_{\mathcal{A}}$ such that $h \circ \Sigma_{\mathcal{T}} = \Sigma_{\mathcal{A}} \circ h$.

(2) $\mathcal{A}$ is closed symmetric monoidal under Day convolution. Under this tensor product $h : \mathcal{T} \to \mathcal{A}$ is symmetric monoidal, and moreover $\hat{t} \otimes -$ is exact and colimit preserving for any $t \in \mathcal{T}$.

8.1 Brown–Adams Representability

Definition 8.1.1. Sitting inside of $\mathcal{A}$ is the full subcategory of *Homological Functors*, which we will denote $Hol(\mathcal{T}^c)$, consisting of those contravariant functors that send exact triangles to long exact sequences. Clearly, $\hat{t} \in Hol(\mathcal{T}^c)$ for any $t \in \mathcal{T}$. Moreover, in certain important historic examples there has been a much stronger relationship between the essential image of restricted Yoneda and the subcategory of homological functors:

Theorem 8.1.2. *[Ada71] Let $\mathcal{T} = SH$ denote the stable homotopy category, and $\mathcal{T}^c$ the category of finite spectra. Then any homological functor*

$$\mathcal{H} : (\mathcal{T}^c)^{op} \to \mathcal{A}b$$

is isomorphic to $h(t)$ for some $t \in \mathcal{T}$. Moreover, any natural transformation

$$h(t) \to h(s)$$

is induced by some (non-unique) map

$$t \to s.$$

Remark 8.1.3. Neeman, Keller and Christensen have investigated the extent to which this result generalizes to other rigidly-compactly generated tt-categories in [Nee97, CKN01]. Following their terminology, we say that $\mathcal{T}$ satisfies:

(BRO) If every $\mathcal{H} \in Hol(\mathcal{T}^c)$ is isomorphic to $h(t)$ for some $t \in \mathcal{T}$.

(BRM) If every natural transformation $h(t) \to h(s)$ is induced by some (potentially non-unique) morphism $t \to s$.

Remark 8.1.4. It follows from results of Beligiannis (see [Bel00, Theorem 11.8]) that (BRM) implies (BRO).

Definition 8.1.5. We say that $\mathcal{T}$ satisfies *Brown–Adams representability* if condition (BRM) holds.

Remark 8.1.6. Neeman shows [Nee97, Proposition 4.11, Theorem. 5.1] that not every triangulated category satisfies Brown–Adams representability. However, he established a sufficient condition, namely that $\mathcal{T}$ satisfies Brown–Adams representability if $\mathcal{T}^c$ is equivalent to a countable category (that is, a category with only countably many objects and morphisms between them).

Remark 8.1.7. Suppose $\mathcal{T}$ satisfies Brown–Adams representability. Then it follows (see [Nee97, Remark 3.2]) that any isomorphism $\hat{s} \xrightarrow{\sim} \hat{t}$ in $\mathcal{A}$ is induced by an isomorphism $s \xrightarrow{\sim} t$ in $\mathcal{T}$.

8.2 Brown–Adams Representability and Stratification

Let us now connect this back to the theory of stratification:

Definition 8.2.1. [Bal20b, Remark 3.4] The *homological spectrum* of $\mathcal{T}^c$, denoted $Spc^h(\mathcal{T}^c)$, is the set of all maximal Serre tensor ideals in $\mathcal{A}^{fp}$. We will refer to $\beta \in Spc^h(\mathcal{T}^c)$ as a *homological prime.*

Remark 8.2.2. [Bal20a, Proposition 2.4] Let $\beta \in Spc^h(\mathcal{T}^c)$ be a homological prime, and consider the quotient

$$\overline{h} : \mathcal{T} \to \mathcal{A} \to \mathcal{A}/Locid_{\mathcal{A}}(\beta).$$

There is a unique pure-injective object $E_\beta \in \mathcal{T}$ such that

$$Locid_{\mathcal{A}}(\beta) = Ker(\hat{E}_\beta \otimes -).$$

Remark 8.2.3. These pure-injective objects E_β, while abstractly defined, are often nice, recognizable objects. For example, in the derived category of a ring, we have that $Spc^h(D(R)^c) \simeq Spc((D(R)^c)$, and recalling that $Spc(D(R)^c) \simeq Spec(R)$, the E_β object corresponding to $p \in Spec(R)$ is isomorphic to the residue field $\kappa(p)$. Similarly in SH, we have $Spc^h(\mathrm{SH}^c) \simeq Spc(\mathrm{SH}^c)$ and under this correspondence the E_β objects are isomorphic to the Morava K-theories $K(p,n)$ (see [BC21, Corollaries 3.3, 3.6]).

Definition 8.2.4. We say an object $F \in \mathcal{T}$ is a *field object* if, for any $t \in \mathcal{T}$, we have that $t \otimes F$ is a coproduct of suspensions of F.

Remark 8.2.5. In both examples in Remark 8.2.3, the E_β objects are field objects.

Remark 8.2.6. Let us assume that $\mathcal{T}$ is stratified. Then it follows that $\mathcal{T} = Locid(E_\beta : \beta \in Spc^h(\mathcal{T}^c))$. Indeed, the local-to-global principle for $\mathcal{T}$ tells us we have $\mathcal{T} = Locid(g(\mathcal{P}) : \mathcal{P} \in Spc(\mathcal{T}^c))$. Moreover, since $\mathcal{T}$ is stratified, the canonical comparison map $\phi : Spc^h(\mathcal{T}^c) \to Spc(\mathcal{T}^c)$ is a homeomorphism [BHS23a, Thrm. 4.7], so let us denote $\beta_{\mathcal{P}}$ to be the unique homological prime corresponding to $\mathcal{P} \in Spc(\mathcal{T}^c)$. It follows [BHS23a, Lemma. 3.7] that $E_{\beta_{\mathcal{P}}} \in Locid(g(\mathcal{P}))$, and since $\mathcal{T}$ is stratified, we get an equality $Locid(E_{\beta_{\mathcal{P}}}) = Locid(g(\mathcal{P}))$.

Remark 8.2.7. Let us now show how this can help us pass stratification from one category to another. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be rigidly-compactly generated tt-categories and let $F : \mathcal{T}_1^c \xrightarrow{\sim} \mathcal{T}_2^c$ be a tt-equivalence. Then this equivalence induces:

(1) an exact, tensor equivalence $\hat{F} : \mathcal{A}_1 \xrightarrow{\sim} \mathcal{A}_2$; and

(2) a homeomorphism $Spc^h(F) : Spc^h(\mathcal{T}_2^c) \xrightarrow{\sim} Spc^h(\mathcal{T}_1^c)$.

Let $\beta_1 \in Spc^h(\mathcal{T}_1^c)$ and let β_2 be the unique homological prime in $Spc^h(\mathcal{T}_2^c)$ mapping to β_1. Consider the following diagram:

$$\begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{\widehat{F}} & \mathcal{A}_2 \\ \widehat{L}_1 \downarrow\uparrow \widehat{R}_1 & & \widehat{L}_2 \downarrow\uparrow \widehat{R}_2 \\ \overline{\mathcal{A}_1} := \mathcal{A}_1/Locid_{\mathcal{A}}(\beta_1) & \xrightarrow[\overline{F}]{} & \mathcal{A}_2/Locid_{\mathcal{A}}(\beta_2) := \overline{\mathcal{A}_2}. \end{array}$$

Proposition 8.2.8. *Keeping the notation and set up as in the above remark 8.2.7, we have*

(a) $\overline{F} \circ \widehat{L}_1 \simeq \widehat{L}_2 \circ \widehat{F}$.

(b) $\overline{F} : \overline{\mathcal{A}_1} \to \overline{\mathcal{A}_2}$ *is an equivalence.*

(c) $\widehat{F} \circ \widehat{R}_1 \simeq \widehat{R}_2 \circ \overline{F}$.

Proof. Parts (a) and (b) follow by definition of the $\beta_i's$ and the fact that giving a Serre subcategory of $\mathcal{A}^{fp}$ is equivalent to giving the localizing category it generates (see for example, [BKS20, Appendix A. Remark 8]). Let us now prove (c). First note that (a) and (b) imply that we also have $\overline{F}^{-1} \circ \hat{L}_2 \simeq \hat{L}_1 \circ \hat{F}^{-1}$. Then for an arbitrary object $c \in \overline{\mathcal{A}}_1$ we compute

$$\begin{aligned}\mathcal{A}_2(-, \hat{R}_2\overline{F}c) &\simeq \overline{\mathcal{A}}_2(\hat{L}_2(-), \overline{F}(c)) \\ &\simeq \overline{\mathcal{A}}_1(\overline{F}^{-1}\hat{L}_2(-), c) \\ &\simeq \overline{\mathcal{A}}_1(\hat{L}_1\hat{F}^{-1}(-), c) \\ &\simeq \mathcal{A}_1(\hat{F}^{-1}(-), \hat{R}_1(c)) \\ &\simeq \mathcal{A}_2(-, \hat{F}\hat{R}_1(c))\end{aligned}$$

and then we summon Yoneda. □

Remark 8.2.9. The objects E_{β_i} are uniquely determined by the injective hull of the unit in $\overline{\mathcal{A}_i}$. That is, letting $\mathbb{1} \to \overline{E}_{\beta_i}$ be this injective hull, we have that E_β is the unique object in $\mathcal{T}$ such that $\widehat{E}_{\beta_i} = \hat{R}_i(\overline{E}_{\beta_i}) \in \mathcal{A}_i$ (see, for example [Bal20a, 2.11]).

Proposition 8.2.10. *Let $\mathcal{T}_i, \mathcal{A}_i, \beta_i$ be as in Remark 8.2.7. Then $\hat{F}(\hat{E}_{\beta_1}) = \hat{E}_{\beta_2}$.*

Proof. We have that $\overline{F}$ will send the injective hull of the unit in $\overline{\mathcal{A}_1}$ to the injective hull of the unit in $\overline{\mathcal{A}_2}$. Hence we know that $\overline{F}(\overline{E}_{\beta_1}) \simeq \overline{E}_{\beta_2}$. Then we compute that

$$\begin{aligned}\widehat{E}_{\beta_2} &= \widehat{R}_2(\overline{E}_{\beta_2}) \\ &= \widehat{R}_2(\overline{F}(\overline{E}_{\beta_1})) \\ &= \widehat{F}(\widehat{R}_1(\overline{E}_{\beta_1})) \\ &= \widehat{F}(\widehat{E}_{\beta_1})\end{aligned}$$

which is precisely what we wanted. □

Hypothesis 8.2.11. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be rigidly-compactly generated tt-categories. Suppose

(1) $\mathcal{T}_1$ is stratified;

(2) $\mathcal{T}_1^c$ is equivalent to a countable category;

(3) $F : \mathcal{T}_1^c \to \mathcal{T}_2^c$ is a tt-equivalence; and

(4) For every nonzero homological functor $\hat{t} \in \mathcal{A}_1$ there exists a nonzero map $\hat{E}_\beta \otimes \hat{x} \to \hat{t}$ for some $\beta \in Spc^h(\mathcal{T}_1^c)$ and compact $x \in \mathcal{T}_1^c$.

We shall also fix $\beta_1 \in Spc^h(\mathcal{T}_1^c)$ and let β_2 denote the unique homological prime in $Spc^h(\mathcal{T}_2^c)$ mapping to β_1 as in Remark 8.2.7.

Remark 8.2.12. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be rigidly-compactly generated tt-categories and suppose they satisfy conditions $(1) - (3)$ of Hypothesis 8.2.11. Then it follows from Proposition 8.2.10 that $\mathcal{A}_1$ satisfies condition (4) if and only if $\mathcal{A}_2$ satisfies it.

Proposition 8.2.13. *Keeping everything as in Hypothesis 8.2.11, we have*

$$\mathcal{T}_2 = Locid(E_\beta : \beta \in Spc^h(\mathcal{T}_2^c))$$

Proof. First note that $\mathcal{T}_2$ also satisfies Brown–Adams representability since $\mathcal{T}_2^c$ is also countable due to the equivalence $F : \mathcal{T}_1^c \to \mathcal{T}_2^c$. Hence restricted Yoneda is full so we have that for any nonzero $t \in \mathcal{T}_2$ there is a nonzero map $E_\beta \otimes x \to t$ for some compact $x \in \mathcal{T}_2^c$, and $\beta \in Spc^h(\mathcal{T}_2^c)$. Denoting $[-,-]$ to be the internal hom in $\mathcal{T}_2$, we have

$$Hom_{\mathcal{T}_2}(E_\beta \otimes x, t) \simeq Hom_{\mathcal{T}_2}(x, [E_\beta, t])$$

which implies that $[E_\beta, t]$ must be nonzero. Hence we have the following string of implications (compare the following proof to [BCHS23, Theorem 6.4]):

$$\begin{aligned} t = 0 &\iff [E_\beta, t] = 0 \text{ for all } \beta \in Spc^h(\mathcal{T}_2^c) \\ &\iff t \in \{E_\beta : \beta \in Spc^h(\mathcal{T}_2^c)\}^\perp \\ &\iff t \in (Locid(E_\beta : \beta \in Spc^h(\mathcal{T}_2^c)))^\perp. \end{aligned}$$

In other words, letting $\mathcal{L} := Locid(E_\beta : \beta \in Spc^h(\mathcal{T}_2^c))$, we have that $\mathcal{L}^\perp = 0$. Now, because $\mathcal{L}$ is set-generated, it is a strictly localizing tensor ideal, see for example [BHS23b, Proposition 3.5]. As a consequence we have that $\mathcal{L} = {}^\perp(\mathcal{L}^\perp)$; see [BCHS23, Remark 2.11]. Thus we have $\mathcal{L} = {}^\perp(\mathcal{L}^\perp) = {}^\perp 0 = \mathcal{T}_2$. □

Proposition 8.2.14. *Keep the assumptions as in Hypothesis 8.2.11 and suppose further that the E_β objects in $\mathcal{T}_1$ are field objects, as in Definition 8.2.4. Then the same is true for the E_β objects in $\mathcal{T}_2$.*

Proof. Note that, because of Brown–Adams representability, it suffices to prove that

$$\hat{E}_{\beta_2} \otimes \hat{t} \simeq \coprod_{j \in I} \Sigma^{m_j} \hat{E}_{\beta_2}$$

in $\mathcal{A}_2$ (recall Remark 8.1.7). Moreover, since $\hat{F}$ is an equivalence, we can check this after applying $\hat{F}^{-1}$. Take $0 \neq t \in \mathcal{T}_2$. Then we have that

$$\begin{aligned} \widehat{F}^{-1}(\hat{t} \otimes \widehat{E}_{\beta_2}) &\simeq \hat{F}^{-1}(\hat{t}) \otimes \hat{F}^{-1}(\widehat{E}_{\beta_2}) \\ &\simeq \hat{F}^{-1}(\hat{t}) \otimes \widehat{E}_{\beta_1} \\ &\simeq \coprod_{j \in I} \Sigma^{m_j} \widehat{E}_{\beta_1} \\ &\simeq \coprod_{j \in I} \Sigma^{m_j} \hat{F}^{-1}(\widehat{E}_{\beta_2}) \\ &\simeq \hat{F}^{-1}\big(\coprod_{j \in I} \Sigma^{m_j}(\widehat{E}_{\beta_2})\big) \end{aligned}$$

Hence we have that $\hat{t} \otimes \hat{E}_{\beta_2} \simeq \coprod_j \Sigma^{m_j} \hat{E}_{\beta_2}$. Since Brown–Adams representability holds for $\mathcal{T}_2$ this isomorphism in $\mathcal{A}_2$ is witnessed by an isomorphism in $\mathcal{T}_2$. □

Remark 8.2.15. The above assumption that the E_β objects are field objects is not as strong as it may sound. Indeed, recall from Remark 8.2.3 that they are field objects in many examples of interest.

Theorem 8.2.16. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be as in Hypothesis 8.2.11. If the E_β objects are field objects (in either $\mathcal{T}_1$ or $\mathcal{T}_2$), then $\mathcal{T}_2$ is stratified.*

Proof. To prove that $\mathcal{T}_2$ is stratified, we need to prove that the local-to-global principle holds and that $Locid(g(\mathcal{P}))$ is a minimal localizing ideal for all primes $\mathcal{P} \in Spc(\mathcal{T}_2^c)$. Now, since $\mathcal{T}_1$ is stratified, we have a bijection $Spc^h(\mathcal{T}_1^c) \simeq Spc(\mathcal{T}_1^c)$ (see [BHS23a, Theorem 4.7]). Moreover, under the equivalence $F : \mathcal{T}_1^c \to \mathcal{T}_2^c$ we also obtain $Spc^h(\mathcal{T}_2^c) \simeq Spc(\mathcal{T}_2^c)$. Fix a $\mathcal{P} \in Spc(\mathcal{T}_2^c)$ and let $\beta_\mathcal{P} \in Spc^h(\mathcal{T}_2^c)$ be the corresponding homological prime. It follows from [BHS23a, Lemma 3.7] that $g(\mathcal{P}) \otimes E_{\beta_\mathcal{P}} \neq 0$, and $g(\mathcal{P}) \otimes E_\beta = 0$ for all other β. Then we have

$$\begin{aligned}
Locid(g(\mathcal{P})) &= Locid(g(\mathcal{P})) \otimes \mathcal{T}_2 \\
&= Locid(g(\mathcal{P})) \otimes Locid(E_\beta : \beta \in Spc^h(\mathcal{T}_2^c)) \quad \text{(by Proposition 8.2.13)} \\
&= Locid(g(\mathcal{P}) \otimes E_\beta : \beta \in Spc^h(\mathcal{T}_2^c)) \\
&= Locid(g(\mathcal{P}) \otimes E_{\beta_\mathcal{P}}) \\
&\subseteq Locid(E_{\beta_\mathcal{P}}).
\end{aligned}$$

Since $E_{\beta_\mathcal{P}}$ is a field object it generates minimal localizing ideals. Indeed we have

$$0 \neq \coprod_{j \in I} \Sigma^{m_j} E_{\beta_\mathcal{P}} \simeq g(\mathcal{P}) \otimes E_{\beta_\mathcal{P}} \in Locid(g(\mathcal{P}))$$

Since localizing ideals are thick, $Locid(g(\mathcal{P}))$ contains E_β which gives the reverse containment, $Locid(E_{\beta_{\mathcal{P}}}) \subseteq Locid(g(\mathcal{P}))$. Note that this also proves that the localizing ideals $Locid(g(\mathcal{P}))$ are minimal. Finally we note that, by Proposition 8.2.13

$$\mathbb{1} \in Locid(E_\beta : \beta \in Spc^h(\mathcal{T}^c)) = Locid(g(\mathcal{P}) : \mathcal{P} \in Spc(\mathcal{T}^c)),$$

which shows the local-to-global principle also holds. Hence $\mathcal{T}_2$ is stratified. □

Chapter 9

Minimality at the Height Two Prime

In this chapter we prove minimality at the last remaining prime m_p. This allows us to conclude that $DTM(\overline{\mathbb{Q}}, \mathbb{Z})$ is stratified.

Remark 9.0.1. We again consider the geometric functor

$$\gamma_p^* : DTM(\overline{\mathbb{Q}}, \mathbb{Z}_{(p)}) \to DTM(\overline{\mathbb{Q}}, \mathbb{Z}/p)$$

induced by the ring map $\mathbb{Z}_{(p)} \to \mathbb{Z}/p$. On spectra, we get the following picture

$$Spc(DTM(\overline{\mathbb{Q}}, \mathbb{Z}/p)^c) \hookrightarrow Spc(DTM(\overline{\mathbb{Q}}, \mathbb{Z}_{(p)})^c)$$

$$\begin{array}{ccc} \mathcal{M} := (0) & & m_p \\ | & & | \\ \mathcal{P} := \langle cone(\beta_p) \rangle & \hookrightarrow & e_p \\ & & | \\ & & m_0 \end{array}$$

We again have that $\gamma_*(\mathbb{1}_{DTM(\overline{\mathbb{Q}}, \mathbb{Z}/p)}) \in DTM(\overline{\mathbb{Q}}, \mathbb{Z}_{(p)})^c$, so by Proposition 4.3.9 to prove minimality holds at the height 2 prime m_p, we are reduced to proving minimality holds at the unique closed point $\mathcal{M} \in Spc(DTM(\overline{\mathbb{Q}}, \mathbb{Z}/p)^c)$.

Remark 9.0.2. The category of Tate motives with finite coefficients has another useful characterization as the derived category of filtered vector spaces, which we now describe. Let us denote $\mathbb{Z}^{op}Mod(\mathbb{Z}/p)$ to be the category of presheaves on the poset category $\mathbb{Z}$ with coefficients in the category $Mod(\mathbb{Z}/p)$ of $\mathbb{Z}/p$-modules. A presheaf $M \in \mathbb{Z}^{op}Mod(\mathbb{Z}/p)$ is called a *filtered-module* if $M_{n,n+1}$ is a monomorphism for all n. This is a quasi-abelian category, in the sense of [Sch99], and so it can be derived, which we will denote by $D_{fil}(\mathbb{Z}/p)$. Details of its construction can be found in [Gal18, §3]. Importantly for us, it is rigidly-compactly generated; see [Gal18, Corollary 3.4]. Moreover, we have the following two equivalences:

Theorem 9.0.3. *[SS16, 3.16] There is an equivalence of tt-categories between $D_{fil}(\mathbb{Z}/p)$ and $D(\mathbb{Z}^{op}Mod(\mathbb{Z}/p))$.*

Proposition 9.0.4. *[Gal19, Propositions 7.7, 7.9] There is an equivalence of triangulated categories*

$$pos : D_{fil}(\mathbb{Z}/p)^c \xrightarrow{\sim} DTM(\overline{\mathbb{Q}}, \mathbb{Z}/p)^c$$

that induces a bijection of thick tensor ideals.

Remark 9.0.5. The proof of this last proposition uses the existence of a strongly hereditary t-structure on $D_{fil}(\mathbb{Z}/p)^c$ [Gal18, Lemma 7.6]. Moreover, to obtain the bijection of thick tensor ideals between the two categories, Gallauer only needed to show the functor is tensor on a certain subcategory of $D_{fil}(\mathbb{Z}/p)^c$. In particular, the functor is tensor on the heart of the t-structure. Since the t-structure is strongly hereditary, and generates $D_{fil}(\mathbb{Z}/p)^c$, every object is a finite sum of shifts of objects in the heart, see for example [Hub16]. Using this, one can show that the functor is really tensor everywhere.

Corollary 9.0.6. *The functor $pos : D_{fil}(\mathbb{Z}/p)^c \to DTM(\overline{\mathbb{Q}}, \mathbb{Z}/p)^c$ is a tensor triangulated equivalence.*

Remark 9.0.7. Let $M \in \mathbb{Z}^{op}Mod(\mathbb{Z}/p)$. Associated to M is the graded $\mathbb{Z}/p[\beta]$-module $\bigoplus_{n\in\mathbb{Z}} M_n$ where β has degree -1 and acts by $\beta : M \to M(1)$. Conversely, given a graded $\mathbb{Z}/p[\beta]$-module $\bigoplus_{n\in\mathbb{Z}} M_n$ we get a presheaf sending n to M_n with transition maps given by $\beta : M_n \to M_{n-1}$. This provides a tensor-equivalence between the Grothendieck categories $\mathbb{Z}^{op}Mod(\mathbb{Z}/p)$ and $Mod_{gr}(\mathbb{Z}/p[\beta])$, see for example [DS13, Lemma. 2.2] and the references therein.

Remark 9.0.8. Hence, by combining Theorem 9.0.3 and Remark 9.0.7, we obtain a tt-equivalence between $D_{fil}(\mathbb{Z}/p)$ and $D(Mod_{gr}(\mathbb{Z}/p[\beta]))$. The tt-geometry of graded $\mathbb{Z}/p[\beta]$-modules has been studied by Stevenson and Dell'Ambrogio, and more recently by Barthel, Heard and Sanders. Let us summarize what is needed for us below:

Theorem 9.0.9. *[DS13, BHS23a] The category $\mathcal{T} := D(Mod_{gr}(\mathbb{Z}/p[\beta]))$ is stratified. As a consequence, $D_{fil}(\mathbb{Z}/p)$ is stratified as well. Moreover, we have that:*

(1) $\mathcal{T}^c$ is countable, and hence satisfies Brown–Adams representability.

(2) The natural map $\varphi : Spc^h(\mathcal{T}^c) \xrightarrow{\sim} Spc(\mathcal{T}^c)$ is a homeomorphism. Moreover, the 'naive' homological support

$$Supp^h_{naive}(t) := \{\beta \in Spc^h(\mathcal{T}^c) : E_\beta \otimes t \neq 0\}$$

corresponds with the actual homological support.

(3) Letting p denote the unique prime corresponding to $\beta \in Spc^h(\mathcal{T}^c)$, the homological primes $E_\beta \simeq k(p)$ are field objects in $\mathcal{T}$.

(4) For all nonzero $t \in \mathcal{T}$ there exists a non-zero map from $E_\beta \otimes x \to t$ for some $\beta \in Spc^h(\mathcal{T}^c)$ and $x \in \mathcal{T}^c$.

Proof. That $\mathcal{T}$ is stratified is [DS13, Theorem 5.7]. Part (2) is a general consequence of stratification, see [BHS23a, Theorem 4.7] and the remark after it. Part (3) is [BHS23a, Example 5.3] and [DS13, Lemma 4.2]. Finally, part (4) is [DS13, Proposition 4.7]. □

Proposition 9.0.10. *Continue to let $\mathcal{T} := D(Mod_{gr}(\mathbb{Z}/p[\beta]))$ and denote by $\mathcal{A} = Add((\mathcal{T}^c)^{op}, \mathcal{A}b)$ the category of modules on $\mathcal{T}$ as in Chapter 8. Then for all non-zero homological functors $0 \neq \hat{t} \in \mathcal{A}$ there exists a non-zero map $\hat{E}_\beta \otimes \hat{x} \to \hat{t}$ for some $\beta \in Spc^h(\mathcal{T}^c)$ and $x \in \mathcal{T}^c$.*

Proof. As the corresponding statement holds for $\mathcal{T}$ by Theorem 9.0.9, we really have to just show there are no phantom maps out of $E_\beta \otimes x$ for any $\beta \in Spc^h(\mathcal{T}^c)$ and $x \in \mathcal{T}^c$. To do so, first note that the graded ring $R := \mathbb{Z}/p[\beta]$ is a graded-local, regular, noetherian ring (see [BH93, §1.5] for example). Letting $\mathfrak{m}$ denote the unique maximal ideal in R, we have $E_{\beta_\mathcal{M}} \simeq k(\mathfrak{m}) \in \mathcal{T}^c$ because R is graded-regular. Now let $0 \neq t \in \mathcal{T}$ be arbitrary. If there is a nonzero map $f : E_{\beta_\mathcal{M}} \otimes x \to t$ for some $x \in \mathcal{T}^c$, then the map $\hat{f} : \hat{E}_{\beta_\mathcal{M}} \otimes \hat{x} \to \hat{t}$ remains nonzero since there can be no phantom maps out of compact objects. In this case the proof is complete.

Otherwise, assume there are no-nonzero maps $E_{\beta_\mathcal{M}} \otimes x \to t$ for any compact object $x \in \mathcal{T}^c$. This implies that $[E_{\beta_\mathcal{M}}, t] = 0$, which, since $E_{\beta_\mathcal{M}} \in \mathcal{T}^c$, tells us that $[E_{\beta_\mathcal{M}}, \mathbb{1}] \otimes t = 0$. Since $E_{\beta_\mathcal{M}}$ is a direct summand of $E_{\beta_\mathcal{M}} \otimes [E_{\beta_\mathcal{M}}, \mathbb{1}] \otimes E_{\beta_\mathcal{M}}$ this forces $E_{\beta_\mathcal{M}} \otimes t = 0$ and so $E_{\beta_\mathcal{M}} \notin Supp^h(t)$. Again using that $\mathcal{T}$ is stratified, we conclude that $\mathcal{M} \notin Supp(t)$. Now recall from Remark 4.2.3 that $g(\mathcal{M}) = e_\mathcal{M}$ and $g(\mathcal{P}) = f_\mathcal{M}$, and so we get $e_\mathcal{M} \otimes t = 0$, and $t \simeq t \otimes f_\mathcal{M}$. Let $f^* : \mathcal{T} \to \mathcal{T}_\mathcal{P}$ be the finite localization associated to $\mathcal{P}$. Recall that f_* is fully faithful, and that $f_*(\mathbb{1}) = f_\mathcal{M}$. Then using the projection formula we get $f_* f^*(t) = t \otimes f_\mathcal{M} \simeq t$. Moreover, Gallauer showed that $\mathcal{T}_\mathcal{P} \simeq D(\mathbb{Z}/p)$ as tensor-triangulated categories (see [Gal18, Lemma 3.7, Lemma 5.3]). Indeed, when thinking of $\mathcal{T}$ as the category of filtered $\mathbb{Z}/p$-vector spaces, the functor $f^* : \mathcal{T} \to \mathcal{T}_{\mathcal{P}}$ corresponds to forgetting the filtration. Now, by assumption we know that there is a nonzero map $\alpha : E_{\beta_{\mathcal{P}}} \otimes x \to t$ for some compact x. Our goal is to show that $\hat{\alpha}$ remains nonzero in $\mathcal{A}$. To do so, consider the following commutative diagram:

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{h} & \mathcal{A} \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle \hat{f}^*} \\ \mathcal{T}_{\mathcal{P}} & \underset{h}{\hookrightarrow} & \mathcal{A}_{\mathcal{P}} \end{array}$$

Since $\mathcal{T}_{\mathcal{P}}$ is the derived category of a field, it is phantomless, and so to show that $\hat{\alpha}$ is nonzero in $\mathcal{A}$, by the commutativity of the above diagram, it suffices to show that $f^*(\alpha) \neq 0$. To do so, let η denote the counit of the adjunction $f^* \dashv f_*$. Recall that since f_* is fully faithful η is an isomorphism. Finally, consider the following commutative diagram:

$$\begin{array}{ccc} E_{\beta_{\mathcal{P}}} \otimes x & \xrightarrow{\alpha} & t \\ {\scriptstyle \eta}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow{\scriptstyle \eta} \\ f_*f^*(E_{\beta_{\mathcal{P}}} \otimes x) & \xrightarrow[f_*f^*(\alpha)]{} & f_*f^*(t) \end{array}$$

This implies that $f_*f^*(\alpha) \neq 0$ and so $f^*(\alpha) \neq 0$ as desired. $\square$

This allows us to use the results from Chapter 8 to immediately conclude:

Theorem 9.0.11. *The category $DTM(\overline{\mathbb{Q}}, \mathbb{Z}/p)$ is stratified.*

Proof. We have a tt-equivalence $D_{fil}(\mathbb{Z}/p)^c \simeq DTM(\overline{\mathbb{Q}}, \mathbb{Z}/p)^c$ where $D_{fil}(\mathbb{Z}/p)$ is stratified, the subcategory of compact objects is countable, and whose homological primes E_β are field objects by Theorem 9.0.9. The last proposition gives us the final assumption needed to apply Theorem 8.2.16 to conclude that $DTM(\overline{\mathbb{Q}}, \mathbb{Z}/p)$ is stratified as well. $\square$

As a result, keeping in mind Remark 9.0.1, we conclude that:

Corollary 9.0.12. *Minimality holds at the height 2 prime* m_p*.*

Chapter 10

Stratification of Tate Motives

Let us bring the results from Chapters 9, 7, and 6 together into our main theorem.

Theorem 10.0.1. *The category* $DTM(\overline{\mathbb{Q}}, \mathbb{Z})$ *is stratified.*

Proof. Since $DTM(\overline{\mathbb{Q}}, \mathbb{Z})$ satisfies the local-to-global principle automatically because $Spc(DTM(\overline{\mathbb{Q}}, \mathbb{Z})^c)$ is noetherian, we just have to show minimality holds at each prime. This is precisely what we showed in Corollaries 9.0.12, 7.0.5, and 6.2.18. □

As remarked in the introduction, an important consequence of a category being stratified is an affirmative answer to the abstract *telescope conjecture* [BHS23b, Theorem 9.11]. Hence as a corollary to the above theorem we immediately get:

Theorem 10.0.2. *Every smashing ideal of* $DTM(\overline{\mathbb{Q}}, \mathbb{Z})$ *is compactly generated. That is, every smashing localization is a finite localization.*

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