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Publication Date
2007-06-04

Peer reviewed

STRICHARTZ ESTIMATES AND LOCAL SMOOTHING ESTIMATES FOR ASYMPTOTICALLY FLAT SCHRÖDINGER EQUATIONS

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Abstract. In this article we study global-in-time Strichartz estimates for the Schrödinger evolution corresponding to long-range perturbations of the Euclidean Laplacian. This is a natural continuation of a recent article [28] of the third author, where it is proved that local smoothing estimates imply Strichartz estimates.

By [28] the local smoothing estimates are known to hold for small perturbations of the Laplacian. Here we consider the case of large perturbations in three increasingly favorable scenarios: (i) without non-trapping assumptions we prove estimates outside a compact set modulo a lower order spatially localized error term, (ii) with non-trapping assumptions we prove global estimates modulo a lower order spatially localized error term, and (iii) for time independent operators with no resonance or eigenvalue at the bottom of the spectrum we prove global estimates for the projection onto the continuous spectrum.

1. Introduction

This article is a natural continuation of the third author's work in [28], which studies the connection between long-time Strichartz estimates and local smoothing estimates for Schrödinger equations with C^2, asymptotically flat coefficients.

Given a time dependent second order elliptic operator in $\mathbb{R}^n$

$$A(t,x,D) = D_i a^{ij}(t,x) D_j + b^i(t,x) D_i + D_i b^i(t,x) + c(t,x)$$

we consider the dispersive properties of solutions to the Schrödinger evolution

$$Pu := (D_t + A(t,x,D))u = f, \quad u(0) = u_0. \tag{1.1}$$

Two of the most stable ways of measuring dispersion are the local smoothing estimates and the Strichartz estimates. The local smoothing estimates give L^2 time integrability for the spatially localized energy, with a half-derivative gain. To state them we use a local smoothing space X which will be defined shortly, and its dual X',

$$\|u\|_{X \cap L^\infty_t L^2_x} \lesssim \|u_0\|_{L^2} + \|f\|_{X' + L^1_t L^2_x} \tag{1.2}$$

where in a first approximation one may set

$$\|u\|_X \sim \|\langle x\rangle^{-\frac{1}{2}-} |D|^{\frac{1}{2}} u\|_{L^2_{t,x}}.$$

The second author was supported by the NSF through a MSPRF, and the other two authors were partially supported by the NSF through grants DMS0354539 and DMS 0301122.

The Strichartz estimates on the other hand measure the space-time integrability of solutions and have the form

$$\|u\|_{L_t^{p_1} L_x^{q_1}} \lesssim \|u_0\|_{L^2} + \|f\|_{L_t^{p_2'} L_x^{q_2'}} \tag{1.3}$$

where the indices (p_1, q_1) and (p_2, q_2) satisfy the relation

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \qquad 2 \le p, q \le \infty$$

and $(p, q) \neq (2, \infty)$ if $n = 2$. Any pair (p, q) satisfying these requirements will be called a Strichartz pair.[1]

The local smoothing estimates have been long known to hold in the flat case $A = -\Delta$ and for certain small perturbations. For operators with variable coefficients, local in time smoothing estimates were first established in [7] and [9]. Global in time estimates on the other hand are considerably more difficult to obtain and are known only in some very special cases. See, e.g., [20] for time independent, non-trapping, smooth, compactly supported, though not necessarily small, perturbations of the Laplacian.

There are also some known results which show global-in-time smoothing estimates in the presence of certain trapped rays. Here, the estimates involve a different spatial weight and a loss of regularity due to the trapping. See [6], [21] and the references therein.

The Strichartz estimates hold globally in the flat case $A = -\Delta$. Local-in-time Strichartz estimates for variable coefficient operators have also been established in [23], [12], and [19] provided, amongst other things, that the coefficients are non-trapping. We also refer the interested reader to the simplified approaches of [15] and [27]. Again, global in time estimates are more difficult and have been obtained only recently in [20] (time independent, non-trapping, smooth, compactly supported perturbations of the Laplacian) respectively [28] (small, C^2 long range perturbations of the Laplacian).

The above references would be incomplete without mentioning the vast body of work on dispersive and Strichartz estimates for lower order perturbations of the Laplacian. For this we refer the reader to some of the more recent papers [10, 11] and the references therein.

The third author's article [28] is one of the starting points of this work. The main result in [28] is to construct a global in time outgoing parametrix for the equation (1.1) for C^2 long range perturbations of the Laplacian. This construction uses the FBI transform, an approach that is reminiscent of the earlier works [24, 25, 26] for the wave equation. See, also, [27] for a survey of these techniques and the closely related work [22] which is based instead on a wave packet decomposition.

The errors associated to the parametrix are handled using the local smoothing estimates. Consequently one is led to the second result of [28], which roughly asserts that

$$\text{Local Smoothing Estimates} \implies \text{Strichartz Estimates.}$$

[1] For simplicity of exposition, we shall not directly address the $q = \infty$ endpoint estimate. This permits us in the sequel to use Littlewood-Paley theory. See [14] for the corresponding endpoint argument in the flat case.

Local smoothing estimates are also proved in [28], but only for small long range perturbations of the Laplacian. The aim of the present work is to consider large long range perturbations of the Laplacian.

A difficulty one encounters is the possible presence of trapped rays, i.e. geodesics which are confined to a compact spatial region. This brings us to our second starting point, namely Bouclet and Tzvetkov's work [2]. For smooth, time independent, long range perturbations of the Laplacian, they prove that local in time Strichartz estimates hold in the exterior of a sufficiently large ball, in other words that the loss due to trapping is also confined to a bounded region. Another aim of the present work is to provide an analogous result which is global in time and holds for C^2 time-dependent coefficients.

1.1. **Estimates outside a ball.** We begin with our assumptions on the coefficients. Consider a dyadic spatial decomposition of $\mathbb{R}^n$ into the sets

$$D_0 = \{|x| \leq 2\}, \quad D_j = \{2^j \leq |x| \leq 2^{j+1}\}, \quad j = 1, 2, \dots$$

and for $j \geq 0$ set

$$A_j = \mathbb{R} \times D_j, \quad j \geq 0, \qquad A_{<j} = \mathbb{R} \times \{|x| \leq 2^j\} = \bigcup_{l<j} A_l.$$

Our weak asymptotic flatness condition has the form

$$\sum_{j\in\mathbb{N}} \sup_{A_j} \Big[\langle x\rangle^2 \Big(|\partial_x^2 a(t,x)| + |\partial_t a(t,x)| \Big) + \langle x\rangle |\partial_x a(t,x)| + |a(t,x) - I_n| \Big] \leq \kappa < \infty \tag{1.4}$$

and for the lower order terms we have a related condition,

$$\sum_{j\in\mathbb{N}} \sup_{A_j} \langle x\rangle |b(t,x)| \leq \kappa \tag{1.5}$$

$$\begin{cases} \sup \langle x\rangle^2 (|c(t,x)| + |\mathrm{div}\ b(t,x)|) \leq \kappa \\ \limsup\limits_{|x|\to\infty} \langle x\rangle^2 (|c(t,x)| + |\mathrm{div}\ b(t,x)|) < \varepsilon \ll 1 \end{cases} \qquad n \neq 2$$
$$\begin{cases} \sup \langle x\rangle^2 (\ln(2+|x|^2))^2 (|c(t,x)| + |\mathrm{div}\ b(t,x)|) \leq \kappa, \\ \limsup\limits_{|x|\to\infty} \langle x\rangle^2 (\ln\langle x\rangle)^2 (|c(t,x)| + |\mathrm{div}\ b(t,x)|) < \varepsilon \ll 1 \end{cases} \qquad n = 2. \tag{1.6}$$

Here ε is a fixed sufficiently small parameter. For any κ, (1.4) restricts the trapped rays to finitely many of the regions A_j. If κ is sufficiently small, which we do not assume, then it is known that trapped rays do not exist. Notice that we may choose $M = M(\varepsilon)$ sufficiently large so that

$$\sum_{j\geq M} \sup_{A_j} \Big[\langle x\rangle^2 \Big(|\partial_x^2 a(t,x)| + |\partial_t a(t,x)| \Big) + \langle x\rangle |\partial_x a(t,x)| + |a(t,x) - I_n| \Big] \leq \varepsilon \tag{1.7}$$

and

$$\sum_{j\geq M} \sup_{A_j} \langle x\rangle |b(t,x)| \leq \varepsilon \tag{1.8}$$

$$\begin{aligned}(1.9)\qquad &\sup_{A_{\geq M}} \langle x\rangle^2(|c(t,x)|+|\text{div }b(t,x)|)\leq \varepsilon, \qquad n\neq 2\\ &\sup_{A_{\geq M}} \langle x\rangle^2(\ln\langle x\rangle)^2(|c(t,x)|+|\text{div }b(t,x)|)\leq \varepsilon, \qquad n=2.\end{aligned}$$

To describe the local smoothing space X, we use a dyadic partition of unity of frequency

$$1=\sum_{k=-\infty}^{\infty} S_k(D).$$

The functions at frequency 2^k are measured using the norms

$$\|u\|_{X_k}=\|u\|_{L^2_{t,x}(A_{<0})}+\sup_{j\geq 0}\|\langle x\rangle^{-1/2}u\|_{L^2_{t,x}(A_j)}, \quad k>0$$

$$\|u\|_{X_k}=2^{\frac{k}{2}}\|u\|_{L^2_{t,x}(A_{<-k})}+\sup_{j\geq -k}\|(|x|+2^{-k})^{-1/2}u\|_{L^2_{t,x}(A_j)}, \quad k\leq 0.$$

The local smoothing space X is the completion of the Schwartz space with respect to the norm

$$\|u\|_X^2=\sum_{k=-\infty}^{\infty}2^k\|S_k u\|^2_{X_k}.$$

Its dual X' has norm

$$\|f\|_{X'}^2=\sum_{k=-\infty}^{\infty}2^{-k}\|S_k f\|^2_{X'_k}.$$

In dimension $n\geq 3$ the space X is a space of distributions, and we have the Hardy type inequality

$$\|\langle x\rangle^{-1}u\|_{L^2_{t,x}}\lesssim \|u\|_X. \tag{1.10}$$

On the other hand in dimensions $n=1,2$, the space X is a space of distributions modulo constants, and we have the BMO type inequality

$$\sum_{j\geq 0}\|\langle x\rangle^{-1}(u-u_{D_j})\|^2_{L^2_{t,x}(A_j)}\lesssim \|u\|_X^2 \tag{1.11}$$

where u_{D_j} represents the (time dependent) average of u in D_j. At the same time X' contains only functions with integral zero. We refer the reader to [28] for more details.

In [28] the case of a small perturbation of the Laplacian is considered, and it is proved that

Theorem 1.1. [28]. *Assume that either*

(i) $n\geq 3$ *and* (1.4), (1.5),(1.6) *hold with a sufficiently small* κ *or*

(ii) $n=1,2$, $b^i=0$, $c=0$ *and* (1.4) *holds with a sufficiently small* κ.

Then the local smoothing estimate

$$\|u\|_{X\cap L^\infty_t L^2_x}\lesssim \|u_0\|_{L^2}+\|f\|_{X'+L^1_t L^2_x} \tag{1.12}$$

holds for all solutions u *to* (1.1).

As one can see, the assumptions are more restrictive in low dimensions. This is related to the spectral structure of the operator A, precisely to the presence of a resonance at zero. This is the case if $A = -\Delta$ or, more generally, if $b^i = 0$ and $c = 0$. However the zero resonance is unstable with respect to lower order perturbations. To account for non-resonant situations, it is convenient to introduce a stronger norm which removes the quotient structure,

$$\|u\|_{\tilde{X}}^2 = \|\langle x\rangle^{-1} u\|_{L^2_{t,x}}^2 + \sum_{k=-\infty}^{\infty} 2^k \|S_k u\|_{X_k}^2, \qquad n \neq 2$$

$$\|u\|_{\tilde{X}}^2 = \|\langle x\rangle^{-1} (\ln(2+|x|))^{-1} u\|_{L^2_{t,x}}^2 + \sum_{k=-\infty}^{\infty} 2^k \|S_k u\|_{X_k}^2, \qquad n = 2.$$

Its dual is

$$\tilde{X}' = X' + \langle x\rangle L^2_{t,x}, \quad n \neq 2, \qquad \tilde{X}' = X' + \langle x\rangle (\ln(2+|x|)) L^2_{t,x}, \quad n = 2.$$

Due to the Hardy inequality above, if $n \geq 3$ we have $\tilde{X} = X$. On the other hand in low dimension the $\tilde{X}$ norm adds some local square integrability to the X norm. Precisely, we have

Lemma 1.2. *Let $n = 1, 2$. Then*

$$\|u\|_{\tilde{X}} \lesssim \|u\|_X + \|u\|_{L^2_{t,x}(\{|x|\leq 1\})}. \tag{1.13}$$

The first goal of this article is to show, without any trapping assumption, that loss-less (with respect to regularity), global-in-time local smoothing and Strichartz estimates hold exterior to a sufficiently large ball, modulo a localized error term. It is hoped that this error term can be separately estimated for applications of interest. Moreover, in the case of finite times, this error term can be trivially estimated by the energy inequality and immediately yields a C^2, long range, time dependent analog of the result of [2].

For M fixed and sufficiently large so that (1.7), (1.8) and (1.9) hold, we consider a smooth, radial, nondecreasing cutoff function ρ which is supported in $\{|x| \geq 2^M\}$ with $\rho(|x|) \equiv 1$ for $|x| \geq 2^{M+1}$. Then we define the exterior local smoothing space $\tilde{X}_e$ with norm

$$\|u\|_{\tilde{X}_e} = \|\rho u\|_{\tilde{X}} + \|(1-\rho)u\|_{L^2_{t,x}}$$

and the dual space $\tilde{X}'_e$ with norm

$$\|f\|_{\tilde{X}'_e} = \inf_{f=\rho f_1 + (1-\rho) f_2} \|f_1\|_{\tilde{X}'} + \|f_2\|_{L^2_{t,x}}.$$

Now we can state our exterior local smoothing estimates.

Theorem 1.3. *Let $n \geq 1$. Assume that the coefficients a^{ij}, b^i and c are real and satisfy* (1.4)*,* (1.5)*,* (1.6)*. Then the solution u to* (1.1) *satisfies*

$$\|u\|_{\tilde{X}_e \cap L^\infty_t L^2_x} \lesssim \|u_0\|_{L^2} + \|f\|_{\tilde{X}'_e + L^1_t L^2_x} + \|u\|_{L^2_{t,x}(\{|x|\leq 2^{M+1}\})}. \tag{1.14}$$

In the low dimensional resonant case the situation is a bit more delicate. First of all, the above theorem does not give a meaningful estimate in the $n = 1, 2$ resonant case as the last term in the right of (1.14) blows up for constant functions, which correspond to the zero resonance. Since we do not control the local L^2 norm for X functions, truncation by the cutoff function ρ does not preserve the X space. To remedy this we define a time dependent local average for u, namely

$$u_\rho = \left(\int_{\mathbb{R}^n} (1-\rho)\, dx\right)^{-1} \int_{\mathbb{R}^n} (1-\rho)u\, dx,$$

and define a modified truncation by the self-adjoint operator

$$T_\rho u = \rho u + (1-\rho)u_\rho.$$

We note that T_ρ leaves constant functions unchanged, as well as the integral of u (if finite).

Then we set

$$\|u\|_{X_e} = \|T_\rho u\|_X + \|u - T_\rho u\|_{L^2_{t,x}}$$

and have the dual space X_e' with norm

$$\|f\|_{X_e'} = \inf_{f = T_\rho f_1 + (1-T_\rho) f_2} \|f_1\|_{X'} + \|f_2\|_{L^2_{t,x}}.$$

We now have the following alternative to Theorem 1.3 which is consistent with operators with a constant zero resonance:

Theorem 1.4. *Let $n = 1, 2$. Assume that*

(i) the coefficients a^{ij} are real and satisfy (1.4)*;*

(ii) the coefficients b^i are real, satisfy (1.5)*, and $\partial_i b^i = 0$;*

(iii) there are no zero order terms, $c = 0$.

Then the solution u to (1.1) *satisfies*

$$\|u\|_{X_e \cap L^\infty_t L^2_x} \lesssim \|u_0\|_{L^2} + \|f\|_{X_e' + L^1_t L^2_x} + \|u - u_\rho\|_{L^2_{t,x}(\{|x| \le 2^{M+1}\})}. \tag{1.15}$$

Once we have the local smoothing estimates, the parametrix construction in [28] allows us to obtain corresponding Strichartz estimates. If (p, q) is a Strichartz pair we define the exterior space $\tilde{X}_e(p, q)$ with norm

$$\|u\|_{\tilde{X}_e(p,q)} = \|u\|_{\tilde{X}_e} + \|\rho u\|_{L^p_t L^q_x}$$

and the dual space $\tilde{X}'(p, q)$ with norm

$$\|f\|_{\tilde{X}_e'(p,q)} = \inf_{f = f_1 + \rho f_2} \|f_1\|_{\tilde{X}_e'} + \|f_2\|_{L^{p'}_t L^{q'}_x}.$$

Theorem 1.5. *Let $n \ge 1$. Assume that the coefficients a^{ij}, b^i and c are real and satisfy* (1.4)*,* (1.5)*,* (1.6)*. Then for any two Strichartz pairs (p_1, q_1) and (p_2, q_2), the solution u to* (1.1) *satisfies*

$$\|u\|_{\tilde{X}_e(p_1,q_1) \cap L^\infty_t L^2_x} \lesssim \|u_0\|_{L^2} + \|f\|_{\tilde{X}_e'(p_2,q_2) + L^1_t L^2_x} + \|u\|_{L^2_{t,x}(\{|x| \le 2^{M+1}\})}. \tag{1.16}$$

Correspondingly, in the resonant case we define

$$\|u\|_{X_e(p,q)} = \|u\|_{X_e} + \|\rho u\|_{L^p_t L^q_x}$$

and the dual space $X_e'(p, q)$ with norm

$$\|f\|_{X_e'(p,q)} = \inf_{f = f_1 + \rho f_2} \|f_1\|_{X_e'} + \|f_2\|_{L^{p'}_t L^{q'}_x}.$$

Then we have

Theorem 1.6. *Let $n = 1, 2$. Assume that the coefficients of P are as in Theorem 1.4. Then for any two Strichartz pairs (p_1, q_1) and (p_2, q_2), the solution u to (1.1) satisfies*

$$\|u\|_{X_e(p_1,q_1)\cap L_t^\infty L_x^2} \lesssim \|u_0\|_{L^2} + \|f\|_{X_e'(p_2,q_2)+L_t^1 L_x^2} + \|u - u_\rho\|_{L^2_{t,x}(\{|x|\leq 2^{M+1}\})}. \tag{1.17}$$

In both cases the space-time norms are over $[0, T] \times \mathbb{R}^n$ for any time $T > 0$ with constants independent of T. If the time T is finite, then we may use energy estimates to trivially bound the error term. Doing so results in the following, which is a C^2-analog of the exterior Strichartz estimates of [2].

Corollary 1.7. *(a.) Assume that the coefficients a^{ij}, b^i, and c are as in Theorem 1.3. Then for any two Strichartz pairs (p_1, q_1) and (p_2, q_2), the solution u to (1.1) satisfies*

$$\|u\|_{\tilde{X}(p_1,q_1)\cap L_t^\infty L_x^2} \lesssim_T \|u_0\|_{L^2} + \|f\|_{\tilde{X}'(p_2,q_2)+L_t^1 L_x^2}. \tag{1.18}$$

(b.) Assume that the coefficients a^{ij} and b^i are as in Theorem 1.4. Then for any two Strichartz pairs (p_1, q_1) and (p_2, q_2), the solution u to (1.1) satisfies

$$\|u\|_{X(p_1,q_1)\cap L_t^\infty L_x^2} \lesssim_T \|u_0\|_{L^2} + \|f\|_{X'(p_2,q_2)+L_t^1 L_x^2}. \tag{1.19}$$

In both cases, the space-time norms are over $[0, T] \times \mathbb{R}^n$ and $T > 0$ is finite.

We conclude this subsection with a few remarks concerning several alternative set-ups for these results.

1.1.1. *Boundary value problems.* Our proof of Theorems 1.3,1.4, 1.5 1.6 treats the interior of the ball $B = \{|x| < 2^M\}$ as a black box with the sole property that the energy is conserved by the evolution. Hence the results remain valid for exterior boundary problems. Precisely, take a bounded domain $\Omega \subset B$ and consider either the Dirichlet problem

$$\begin{cases} Pu = f & \text{in } \Omega^c \\ u(0) = u_0 \\ u = 0 & \text{in } \partial\Omega \end{cases} \tag{1.20}$$

or the Neumann problem

$$\begin{cases} Pu = f & \text{in } \Omega^c \\ u(0) = u_0 \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{in } \partial\Omega \end{cases} \tag{1.21}$$

where

$$\frac{\partial}{\partial \nu} = \nu_i(a^{ij} D_j + b^i)$$

and ν is the unit normal to $\partial\Omega$.

Then we have

Corollary 1.8. *a) The results in Theorems 1.3 and 1.5 remain valid for both the Dirichlet problem* (1.20) *and the Neumann problem* (1.21).

b) The results in Theorems 1.4 and 1.6 remain valid for the Neumann problem (1.21) *with the additional condition $b^i \nu_i = 0$ on $\partial\Omega$.*

The more restrictive hypothesis in part (b) is caused by the requirement that constant functions solve the homogeneous problem.

1.1.2. *Complex coefficients.* The only role played in our proofs by the assumption that the coefficients b^i and c are real is to insure the energy conservation in the interior region. Hence we can allow complex coefficients in the region $\{|x| > 2^{M+1}\}$ where the coefficients satisfy the smallness condition.

In addition, allowing c to be complex in the interior region does not affect energy conservation either, since we are assuming an a priori control of the local L^2 space-time norm of the solution. Hence we have

Remark 1.9. *a) The results in Theorems 1.3 and 1.5 remain valid for complex coefficients b^i, c with the restriction that b^i are real in the region $\{|x| < 2^{M+1}\}$.*

b) The results in Theorems 1.4 and 1.6 remain valid for coefficients b^i which are real in the region $\{|x| < 2^{M+1}\}$.

1.2. **Non-trapping metrics.** The second goal of the article is to consider the previous setup but with an additional non-trapping assumption. To state it we consider the Hamilton flow H_a for the principal symbol of the operator A, namely

$$a(t, x, \xi) = a^{ij}(t, x)\xi_i\xi_j.$$

The spatial projections of the trajectories of the Hamilton flow H_a are the geodesics for the metric $a_{ij}dx^i dx^j$ where $(a_{ij}) = (a^{ij})^{-1}$.

Definition 1.10. *We say that the metric (a_{ij}) is non-trapping if for each $R > 0$ there exists $L > 0$ independent of t so that any portion of a geodesic contained in $\{|x| < R\}$ has length at most L.*

The non-trapping condition allows us to use standard propagation of singularities techniques to bound high frequencies inside a ball in terms of the high frequencies outside. Then the cutoff function ρ which was used before is no longer needed, and we obtain

Theorem 1.11. *Let $R > 0$ be sufficiently large. Assume that the coefficients a^{ij}, b^i and c are real and satisfy* (1.4), (1.5), (1.6). *Assume also that the metric a_{ij} is non-trapping. Then the solution u to* (1.1) *satisfies*

$$\|u\|_{\tilde{X}} \lesssim \|u_0\|_{L^2} + \|f\|_{\tilde{X}'} + \|u\|_{L^2_{t,x}(\{|x|\le 2R\})}, \tag{1.22}$$

respectively

Theorem 1.12. *Let $R > 0$ be sufficiently large, and let $n = 1, 2$. Assume that the coefficients of P are as in Theorem 1.4. Assume also that the metric a_{ij} is non-trapping. Then the solution u to* (1.1) *satisfies*

$$\|u\|_X \lesssim \|u_0\|_{L^2} + \|f\|_{X'} + \|u - u_\rho\|_{L^2_{t,x}(\{|x|\le 2R\})}. \tag{1.23}$$

We note that the high frequencies in the error term on the right are controlled by the X norm on the left. Also the low frequencies ($\ll 1$) are controlled by the X norm using the uncertainty principle. Hence the only nontrivial part of the error term corresponds to intermediate (i.e. ≈ 1) frequencies.

The proof combines the arguments used for the exterior estimates with a standard multiplier construction from the theory of propagation of singularities. Adding to the above results the parametrix obtained in [28] we obtain

Theorem 1.13. *Let $R > 0$ be sufficiently large. Assume that the coefficients a^{ij}, b^i and c are real and satisfy* (1.4), (1.5), (1.6). *Assume also that the metric a_{ij} is non-trapping. Then for any two Strichartz pairs (p_1, q_1) and (p_2, q_2), the solution u to* (1.1) *satisfies*

$$\|u\|_{\tilde{X}\cap L_t^{p_1}L_x^{q_1}} \lesssim \|u_0\|_{L^2} + \|f\|_{\tilde{X}'+L_t^{p_2'}L_x^{q_2'}} + \|u\|_{L^2_{t,x}(\{|x|\leq 2R\})}, \tag{1.24}$$

respectively

Theorem 1.14. *Let $n = 1, 2$, and let $R > 0$ be sufficiently large. Assume that the coefficients of P are as in Theorem 1.4. Assume also that the metric a_{ij} is non-trapping. Then for any two Strichartz pairs (p_1, q_1) and (p_2, q_2), the solution u to* (1.1) *satisfies*

$$\|u\|_{X\cap L_t^{p_1}L_x^{q_1}} \lesssim \|u_0\|_{L^2} + \|f\|_{X'+L_t^{p_2'}L_x^{q_2'}} + \|u - u_\rho\|_{L^2_{t,x}(\{|x|\leq 2R\})}. \tag{1.25}$$

1.2.1. *An improved result for trapped metrics.* A variation on the above theme is obtained in the case when there are trapped rays, but not too many. If they exist, they must be confined to the interior region $\{|x| \leq 2^M\}$. Then we can define the conic set

$$\begin{aligned}\Omega^L_{trapped} = \{\ &(t, x, \xi) \in \mathbb{R} \times T^*B(0, 2^M);\ \text{the } H_a \text{ bicharacteristic through } (t, x, \xi)\\ &\text{has length at least } L \text{ within } |x| \leq 2^M\}.\end{aligned}$$

Given a smooth zero homogeneous symbol $q(x, \xi)$ which equals 1 for $|x| > 2^M$, we define modified exterior spaces by

$$\|u\|_{\tilde{X}_q} = \|q(x, D)u\|_{\tilde{X}} + \|u\|_{L^2(\{|x|\leq 2^{M+1}\})}$$

with similar modifications for $\tilde{X}'_q$, X_q and X'_q.

Then the same argument as in the proof of the above Theorems gives

Corollary 1.15. *Assume that q is supported outside $\Omega^L_{trapped}$ for some $L > 0$. Then the results in Theorems 1.3, 1.5, 1.4 and 1.6 remain valid with $\tilde{X}_e$, $\tilde{X}'_e$, X_e and X'_e replaced by $\tilde{X}_q$, $\tilde{X}'_q$, X_q and X'_q.*

We also note that if A has time independent coefficients then $\Omega^L_{trapped}$ is translation invariant. Hence a compactness argument allows us to replace $\Omega^L_{trapped}$ by $\Omega^\infty_{trapped}$, which contains all the trapped geodesics.

1.2.2. *Boundary value problems.* Consider solutions u for either the Dirichlet problem (1.20) or the Neumann problem (1.21). Then singularities will propagate along generalized broken bicharacteristics (see [17, 18],[13],[4]). Hence the non-trapping condition needs to be modified accordingly.

Definition 1.16. *We say that the metric (a_{ij}) is non-trapping if for each $R > 0$ there exists $L > 0$ independent of t so that any portion of a generalized broken bicharacteristic is contained in $\{|x| < R\}$ has length at most L.*

With this modification the results of Theorems 1.11, 1.12, remain valid. However, some care must be taken with the results on propagation of singularities near the boundary, as not all of them are known to be valid for operators with only C^2 coefficients.

On the other hand we do not know whether the bounds in Theorems 1.13, 1.14 are true or not. These hinge on the validity of local Strichartz estimates near the boundary. This is currently an unsolved problem.

1.2.3. *Complex coefficients.* Again, one may ask to what extent are our results in this section are valid if complex coefficients are allowed. We have

Remark 1.17. *The results in Theorems 1.11, 1.12, 1.13, 1.14 remain valid if the coefficients b^i and c are allowed to be complex.*

This result is obtained without making any changes to our proofs provided that the constant κ in (1.5) is sufficiently small. Otherwise, the multiplier q used in the proof has to change too much along bicharacteristics from entry to exit from $B(0, 2^M)$; this in turn forces a modified multiplier for the exterior region. See, e.g., [8, 9] and [23].

1.3. **Time independent metrics.** It is natural to ask when can one eliminate the error term altogether. This is a very delicate question, which hinges on the local in space evolution of low frequency solutions. For general operators A with time dependent coefficients this question seems out of reach for now.

This leads us to the third part of the paper where, in addition to the flatness assumption above and the non-trapping hypothesis on a_{ij}, we take our coefficients a^{ij}, b^i, c to be time-independent. Then the natural obstruction to the dispersive estimates comes from possible eigenvalues and zero resonances of the operator A.

Since the operator A is self-adjoint, it follows that its spectrum is real. More precisely, A has a continuous spectrum $\sigma_c = [0, \infty)$ and a point spectrum σ_p consisting of discrete finite multiplicity eigenvalues in $\mathbb{R}^-$, whose only possible accumulation point is 0.

From the point of view of dispersion there is nothing we can do about eigenvalues. Consequently we introduce the spectral projector P_c onto the continuous spectrum, and obtain dispersive estimates only for $P_c u$ for solutions u to (1.1).

The resolvent

$$R_\lambda = (\lambda - A)^{-1}$$

is well defined in $\mathbb{C}\setminus(\sigma_c \cup \sigma_p)$. One may ask whether there is any meromorphic continuation of the resolvent R_λ across the positive real axis, starting on either side. This is indeed possible. The poles of this meromorphic continuation are called resonances. This is of interest to us because the resonances which are close to the real axis play an important role in the long time behavior of solutions to the Schrödinger equation.

In the case which we consider here (asymptotically flat), there are no resonances nor eigenvalues inside the continuous spectrum i.e. in $(0, \infty)$. However, the bottom of the continuous spectrum, namely 0, may be either an eigenfunction (if $n \geq 5$) or a resonance (if $n \leq 4$). For zero resonances we use a fairly restrictive definition:

Definition 1.18. *We say that* 0 *is a resonance for A if there is a function $u \in \tilde{X}^0$ so that $Au = 0$. The function u is called a zero resonant state of A.*

Here $\tilde{X}^0$ denotes the spatial part of the $\tilde{X}$ norm. I.e. $\tilde{X} = L_t^2 \tilde{X}^0$.

The main case we consider here is when 0 is neither an eigenfunction (if $n \geq 5$) nor a resonance (if $n \leq 4$). This implies that there are no eigenvalues close to 0. Then A has at most finitely many negative eigenvalues, and the corresponding eigenfunctions decay exponentially at infinity.

Theorem 1.19. *Suppose that a^{ij}, b^i, c are real, time-independent, and satisfy the conditions* (1.4),(1.5)*, and* (1.6)*. We also assume that the Hamiltonian vector field H_a permits no trapped geodesics and that* 0 *is not an eigenvalue or a resonance of A. Then for all solutions u to* (1.1) *we have*

$$\|P_c u\|_{\tilde{X}} \lesssim \|u_0\|_2 + \|f\|_{\tilde{X}'}. \tag{1.26}$$

From this, using the parametrix of [28], we immediately obtain the corresponding global-in-time Strichartz estimates:

Theorem 1.20. *Suppose that a^{ij}, b^i, c are real, time-independent, and satisfy the conditions* (1.4),(1.5)*, and* (1.6)*. Moreover, assume that the Hamiltonian vector field H_a permits no trapped geodesics. Assume, also, that* 0 *is not an eigenvalue or a resonance of A. Then for all solutions u to* (1.1)*, we have*

$$\|P_c u\|_{L_t^{p_1} L_x^{q_1} \cap \tilde{X}} \lesssim \|u_0\|_2 + \|f\|_{L_t^{p_2'} L_x^{q_2'} + \tilde{X}'}, \tag{1.27}$$

for any Strichartz pairs (p_1, q_1) and (p_2, q_2).

One can compare this with the result of [20], where the authors consider a smooth compactly supported perturbation of the metric in $3+1$ dimensions where no eigenvalues are present. Estimates in the spirit of (1.27) have also recently be shown by [3], though only for smooth coefficients and with a more restrictive spectral projection. We also note the related work [10] on Schrödinger equations with magnetic potentials. In their work, the second order operator is taken to be $-\Delta$. Theorem 1.20 is a more general version of the main theorem in [10] in the sense that it allows a more general leading order operator and that it assumes less flatness on the coefficients.

In dimension $n \geq 3$ zero is not an eigenvalue or a resonance for $-\Delta$, nor for small perturbations of it. However, in dimension $n = 1, 2$, zero is a resonance and the corresponding resonant states are the constant functions. This spectral picture is not stable with respect to lower order perturbations, but it does remain stable with respect to perturbations of the metric a^{ij}. Hence there is some motivation to also investigate this case in more detail. We prove the following result.

Theorem 1.21. *Assume that the coefficients of P are time-independent, but otherwise as in Theorem 1.4. Assume also that the Hamiltonian vector field H_a permits no trapped geodesics, and that there are no nonconstant zero resonant states of A. Then for all solutions u to* (1.1)*, we have*

$$\|u\|_X \lesssim \|u_0\|_2 + \|f\|_{X'}. \tag{1.28}$$

In terms of Strichartz estimates, this has the following consequence:

Theorem 1.22. *Assume that the coefficients of P are time-independent, but otherwise as in Theorem 1.4. Assume also that the Hamiltonian vector field H_a permits no trapped* *geodesics, and that there are no nonconstant zero resonant states of A. Then for all solutions u to* (1.1)*, we have*

$$\|u\|_{L_t^{p_1} L_x^{q_1} \cap X} \lesssim \|u_0\|_2 + \|f\|_{L_t^{p_2'} L_x^{q_2'} + X'} \tag{1.29}$$

for any Strichartz pairs (p_1, q_1) and (p_2, q_2).

Implicit in the above theorems is the fact that there are, under their hypothesis, no eigenvalues for A. There is another simplification if we make the additional assumption that $b = 0$.

Remark 1.23. *If in addition $b = 0$, then there are no nonconstant generalized zero eigenvalues of A.*

In order to prove Theorems 1.19 and 1.21, we restate the bounds (1.26) and (1.28) in terms of estimates on the resolvent using the Fourier transform in t. We then argue via contradiction. Using the positive commutator method, we show an outgoing radiation condition (see Steps 8-10 of the proof), which allows us to pass to subsequences and claim that if (1.26) were false, then there is a resonance or an eigenvalue v within the continuous spectrum. By hypothesis this cannot occur at 0. We use another multiplier and the radiation condition to then show that $v \in L^2$ and thus cannot be a resonance. As results of [16] show that there are no eigenvalues embedded in the continuous spectrum, we reach a contradiction. If instead (1.28) were false, then the same argument produces a nonconstant zero resonance, again reaching a contradiction.

The paper is organized as follows. In the next section, we fix some further notations and our paradifferential setup. It is here that we show that we may permit the lower order terms in the local smoothing estimates in a perturbative manner. In the third section, we prove the local smoothing estimates using the positive commutator method, first in the exterior local smoothing spaces and then in the non-trapping case. The fourth section is devoted to non-trapping, time-independent operators. In the final section, we review the parametrix of [28] and use it to show how the Strichartz estimates follow from the local smoothing estimates.

Acknowledgements: The authors thank W. Schlag and M. Zworski for helpful discussions regarding some of the spectral theory, and in particular the behavior of resonances, contained herein.

2. Notations and the paradifferential setup

2.1. **Notations.** We shall be using dyadic decompositions of both space and frequency. For the spatial decomposition, we let χ_k denote smooth functions satisfying

$$1 = \sum_{j=0}^{\infty} \chi_j(x), \quad \text{supp } \chi_0 \subset \{|x| \le 2\}, \quad \text{supp } \chi_j \subset \{2^{j-1} < |x| < 2^{j+1}\} \text{ for } j \ge 1.$$

We also set

$$\chi_{<k} = \sum_{0 \le j < k} \chi_j$$

with the obvious modification for $\chi_{>k}$. In frequency, we use a smooth Littlewood-Paley decomposition

$$1 = \sum_{j=-\infty}^{\infty} S_j(D), \quad \text{supp } s_j \subset \{2^{j-1} < |\xi| < 2^{j+1}\}$$

and similar notations for $S_{<k}, S_{>k}$ are applied.

We say that a function is frequency localized at frequency 2^k if its Fourier transform is supported in the annulus $\{2^{k-1} < |\xi| < 2^{k+1}\}$. An operator K is said to be frequency localized if Kf is supported in $\{2^{k-10} < |\xi| < 2^{k+10}\}$ for any function f which is frequency localized at 2^k.

For κ as in (1.4), we may choose a positive, slowly varying sequence $\kappa_j \in \ell^1$ satisfying

$$\sup_{A_j} \langle x\rangle^2 |\partial_x^2 a(t,x)| + \langle x\rangle |\partial_x a(t,x)| + |a(t,x) - I_n| \le \kappa_j, \tag{2.1}$$

$$\sum \kappa_j \lesssim \kappa,$$

and

$$|\ln \kappa_j - \ln \kappa_{j-1}| \le 2^{-10}.$$

When the lower order terms are present, we may choose κ_j so that each dyadic piece of (1.5) is also controlled similarly. We may also assume that M in (1.7) is chosen sufficiently large that

$$\sum_{j \ge M} \kappa_j \lesssim \varepsilon.$$

Associated to this slowly varying sequence, we may choose functions $\kappa_k(s)$ with

$$\kappa_0 < \kappa_k(s) < 2\kappa_0, \quad 0 \le s < 2,$$
$$\kappa_j < \kappa_k(s) < 2\kappa_j, \quad 2^j < s < 2^{j+1}, \quad j \ge 1,$$

for $k \ge 0$,

$$\kappa_k < \kappa_k(s) < 2\kappa_k, \quad 0 \le s < 2^{-k},$$
$$\kappa_j < \kappa_k(s) < 2\kappa_j, \quad 2^j < s < 2^{j+1}, \quad j \ge -k$$

for $k < 0$, and

$$|\kappa_k'(s)| \le 2^{-5} s^{-1} \kappa_k(s).$$

2.2. **Embeddings for the X spaces.** Here we prove Lemma 1.2. For the purpose of this section we can entirely neglect the time variable. Let ψ be a smooth, spherically symmetric Schwartz function with $\psi(0) = 1$ which is frequency localized in the unit annulus. Set

$$\psi_k(x) = \psi(2^k x).$$

Given $u \in \tilde{X}$, we split it into

$$u = u^{in} + u^{out}$$

where

$$u^{in} = \sum_{k<0} T_k S_k u$$

and T_k is the operator

$$T_k v = v(t,0)\psi_k(x).$$

For frequencies $k > 0$, we have the dyadic bound

$$\|\langle x\rangle^{-1} S_k u\|_{L^2} \lesssim \|S_k u\|_{X_k}$$

which we can easily sum over k to obtain

$$\|\langle x\rangle^{-1} S_{>0} u\|_{L^2} \lesssim \|u\|_X.$$

For frequencies $k < 0$ it is easy to see that

$$\|(1 - T_k) S_k u\|_{X_k} \lesssim \|S_k u\|_{X_k} \tag{2.2}$$

follows from the bound

$$\|\chi_{<-k} S_k u\|_{L^2_t L^\infty_x} \lesssim 2^{\frac{n-1}{2}k} \|S_k u\|_{X_k}, \quad k \leq 0. \tag{2.3}$$

which is a consequence of Bernstein's inequality.

The gain is that $(1 - T_k) S_k u(t, 0) = 0$. This leads to the improved pointwise bound

$$|x|^{-1} |(1 - T_k) S_k u| \lesssim 2^{\frac{n+1}{2}k} \|S_k u\|_{X_k}, \quad |x| < 2^{-k}$$

and further to the improved L^2 bound

$$\sup_j \|(2^k |x| + 2^{-k} |x|^{-1})^{\frac{1}{2}} |x|^{-1} (1 - T_k) S_k u\|_{L^2(A_j)} \lesssim 2^{\frac{k}{2}} \|S_k u\|_{X_k}. \tag{2.4}$$

Then, by orthogonality with respect to spatial dyadic regions, we can sum up

$$\|\langle x\rangle^{-1} \sum_{k<0} (1 - T_k) S_k u\|^2_{L^2} \lesssim \|u\|^2_X$$

which combined with the previous high frequency bound yields

$$\|\langle x\rangle^{-1} u^{out}\|_{L^2} \lesssim \|u\|_X. \tag{2.5}$$

For the terms in u^{in}, differentiation yields a 2^k factor, and therefore we can estimate

$$\|u^{in}\|_{\dot H^1} \lesssim \|u\|_X. \tag{2.6}$$

It remains to prove the bounds

$$\|\langle x\rangle^{-1} v\|_{L^2} \lesssim \|v\|_{L^2(B(0,1))} + \|v\|_{\dot H^1}, \qquad n = 1 \tag{2.7}$$

respectively

$$\|\langle x\rangle^{-1} (\ln(1 + \langle x\rangle))^{-1} v\|_{L^2} \lesssim \|v\|_{L^2(B(0,1))} + \|v\|_{\dot H^1}, \qquad n = 2. \tag{2.8}$$

Due to the first factor in the right of both estimates, we may without loss of generality take v to vanish in $B(0, 1/2)$. For (2.7) we integrate

$$2 \int_{1/2}^R x^{-1} v v_x dx = \int_{1/2}^R x^{-2} v^2 dx + R^{-1} v^2(R).$$

Using Cauchy-Schwarz the conclusion follows.

For (2.8) we argue in a similar fashion. We have

$$2\int_{B_R\setminus B_{1/2}} |x|^{-2}(\ln(2+|x|^2))^{-1} vx\nabla v dx = \int_{B_R\setminus B_{1/2}} (2+|x|^2)^{-1}(\ln(2+|x|^2))^{-2}v^2 dx + \int_{\partial B_R} |x|^{-1}(\ln(2+|x|^2))^{-1}v^2 d\sigma$$

and conclude again by Cauchy-Schwarz. The lemma is proved. □

On a related note, we include here another result which simplifies the type of local error terms we allow in the non-trapping case.

Lemma 2.1. *Let $n \geq 1$ and $R > 0$. Then for each $\varepsilon > 0$ there is $m_\varepsilon > 0$ and $c_\varepsilon > 0$ so that*

$$\|\langle x\rangle^{-\frac{3}{2}} u\|_{L^2} \leq \varepsilon \|u\|_X + c_\varepsilon \|S_{<m_\varepsilon} u\|_{L^2(\{|x|<R\})}. \tag{2.9}$$

Proof. Frequencies in u which are large enough can be estimated solely by the first term on the right. It remains to show that for large m we have

$$\|\langle x\rangle^{-\frac{3}{2}} S_{<m} u\|_{L^2} \leq \varepsilon \|S_{<m} u\|_X + c_{\varepsilon,m} \|S_{<m} u\|_{L^2(\{|x|<R\})}.$$

For large x the left hand side can also be estimated solely by the first term on the right. It remains to show that for large m, k we have

$$\|\langle x\rangle^{-\frac{3}{2}} \chi_{<k} S_{<m} u\|_{L^2} \leq \|\langle x\rangle^{-\frac{3}{2}} \chi_{>k} S_{<m} u\|_{L^2} + c_{k,m} \|S_{<m} u\|_{L^2(\{|x|<R\})}.$$

We argue by contradiction. Suppose this is false. Then there exists a sequence $u_j \in X$ so that

$$\|\langle x\rangle^{-\frac{3}{2}} \chi_{<k} S_{<m} u_j\|_{L^2} = 1, \qquad \|\langle x\rangle^{-\frac{3}{2}} \chi_{>k} S_{<m} u_j\|_{L^2} < 1, \qquad \|S_{<m} u_j\|_{L^2(\{|x|<R\})} \to 0.$$

The functions $\langle x\rangle^{-\frac{3}{2}} S_{<m} u_j$ are uniformly bounded in all Sobolev spaces $H^N(\mathbb{R}^n)$; therefore on a subsequence we have uniform convergence on compact sets,

$$S_{<m} u_j \to u.$$

Then the function u satisfies

$$\|\langle x\rangle^{-\frac{3}{2}} \chi_{<k} u\|_{L^2} = 1, \qquad \|\langle x\rangle^{-\frac{3}{2}} \chi_{>k} u\|_{L^2} < 1, \qquad \|u\|_{L^2(\{|x|<R\})} = 0.$$

But u is also frequency localized in $|\xi| < 2^{m+1}$ and is therefore analytic. Then the last condition above implies $u = 0$ which is a contradiction. □

2.3. **Paradifferential calculus.**

Here, we seek to frequency localize the coefficients of P. A similar argument is present in [28], where for solutions at frequency 2^k the coefficients are localized at frequency

$$|\xi| \ll 2^{k/2}\langle x\rangle^{-1/2}.$$

Such a strong localization was essential there in order to carry out the parametrix construction. Here we are able to keep the setup simpler and use a classical paradifferential construction, where for solutions at frequency 2^k the coefficients are localized at frequency below 2^k. For a fixed frequency scale 2^k, we set

$$a^{ij}_{(k)} = S_{<k-4} a^{ij},$$

and we define the associated mollified operators

$$A_{(k)} = D_i a^{ij}_{(k)} D_j.$$

It is easy to verify that the mollified coefficients $a^{ij}_{(k)}$ satisfy the bounds

$$\begin{aligned} &|\partial^\alpha(a^{ij}_{(k)} - I_n)| \lesssim \kappa_k(|x|)\langle x\rangle^{-|\alpha|}, \qquad |\alpha| \le 2, \quad k > 0 \\ &|\partial^\alpha(a^{ij}_{(k)} - I_n)| \lesssim \kappa_k(|x|)2^{|\alpha|k}\langle 2^k x\rangle^{-|\alpha|}, \quad |\alpha| \le 2, \quad k \le 0. \end{aligned} \tag{2.10}$$

The next proposition will be used to pass back and forth between $A_{(k)}$ and A. We first define

$$\tilde{A} = \sum_k A_{(k)} S_k.$$

Proposition 2.2. *Assume that the coefficients a^{ij} satisfy* (1.4)*, and that $b = 0$, $c = 0$. Then*

$$\sum_k 2^{-k} \|S_k(A - A_{(k)})u\|^2_{X'_k} \lesssim \kappa^2 \|u\|^2_X, \tag{2.11}$$

$$\|(A - \tilde{A})u\|_{X'} \lesssim \kappa \|u\|_X, \tag{2.12}$$

$$2^{-k} \|[A_{(k)}, S_k]u\|_{X'_k} \lesssim \kappa \|u\|_{X_k}. \tag{2.13}$$

Proof of Lemma 2.2: We begin by writing

$$S_k(A - A_{(k)}) = A^{med}_k + A^{high}_k$$

with

$$\begin{aligned} A^{med}_k &= \sum_{l=k-4}^{k+4} \sum_{m=-\infty}^{k+8} S_k D_i (S_l a^{ij}) D_j S_m \\ A^{high}_k &= \sum_{l>k+4} \sum_{m=l-4}^{l+4} S_k D_i (S_l a^{ij}) D_j S_m. \end{aligned}$$

For A^{med}_k we take $l = k \ge m$ for simplicity; then it suffices to establish the off-diagonal decay

$$\|S_k D_i (S_k a^{ij} D_j S_m v)\|_{X'_k} \lesssim \kappa 2^m \|S_m v\|_{X_m}. \tag{2.14}$$

If $k \ge m \ge 0$ then we have

$$\begin{aligned} \|S_k D_i (S_k a^{ij} D_j S_m v)\|_{X'_k} &\lesssim 2^k \|S_k a^{ij} D_j S_m v\|_{X'_k} \\ &\lesssim \kappa 2^{-k} \|\langle x\rangle^{-2} D_j S_m v\|_{X'_k} \\ &\lesssim \kappa 2^{-k} \|D_j S_m v\|_{X_m} \\ &\lesssim \kappa 2^{m-k} \|S_m v\|_{X_m}. \end{aligned}$$

If $k \geq 0 > m$ then we have two spatial scales to deal with, namely 1 and 2^{-m}. To separate them we use the cutoff function $\chi_{<-m}$. For contributions corresponding to large x we estimate

$$\begin{aligned}
\|S_k D_i(S_k a^{ij} \chi_{\geq -m} D_j S_m v)\|_{X'_k} &\lesssim 2^k \|S_k a^{ij} \chi_{\geq -m} D_j S_m v\|_{X'_k} \\
&\lesssim \kappa 2^{-k} \||x|^{-2} \chi_{\geq -m} D_j S_m v\|_{X'_k} \\
&\lesssim \kappa 2^{m-k} \|D_j S_m v\|_{X_m} \\
&\lesssim \kappa 2^{2m-k} \|S_m v\|_{X_m}.
\end{aligned}$$

For contributions corresponding to small x, we first note that by Bernstein's inequality, see (2.3), we have

$$\|D_j S_m v\|_{L_t^2 L_x^\infty(A_{\leq -m})} \leq 2^{\frac{n+1}{2}m} \|S_m v\|_{X_m}. \tag{2.15}$$

Then

$$\begin{aligned}
\|S_k D_i(S_k a^{ij} \chi_{<-m} D_j S_m v)\|_{X'_k} &\lesssim 2^k \|S_k a^{ij} \chi_{<-m} D_j S_m v\|_{X'_k} \\
&\lesssim 2^{-k} 2^{\frac{n+1}{2}m} \|\langle x\rangle^{-2} \chi_{<-m} \kappa(|x|)\|_{(X_k^0)'} \|S_m v\|_{X_m} \\
&\lesssim \kappa 2^{-k} 2^{\frac{n+1}{2}m} \max\{1, 2^{\frac{3-n}{2}m}\} \|S_m v\|_{X_m} \\
&\lesssim \kappa 2^{-k} \max\{2^{\frac{n+1}{2}m}, 2^{2m}\} \|S_m v\|_{X_m}
\end{aligned}$$

where $(X_k^0)'$ is the spatial part of the X'_k norm, i.e. $X'_k = L_t^2 (X_k^0)'$.

Finally if $0 > k \geq m$ then the spatial scales are 2^{-k} and 2^{-m}, and we separate them using the cutoff function $\chi_{<-m}$. The exterior part is exactly as in the previous case. For the interior part we use again (2.15) to compute

$$\begin{aligned}
\|S_k D_i(S_k a^{ij} \chi_{<-m} D_j S_m v)\|_{X'_k} &\lesssim 2^k \|S_k a^{ij} \chi_{<-m} D_j S_m v\|_{X'_k} \\
&\lesssim 2^k 2^{\frac{n+1}{2}m} \|\langle 2^k x\rangle^{-2} \chi_{<-m} \kappa(|x|)\|_{(X_k^0)'} \|S_m v\|_{X_m} \\
&\lesssim \kappa 2^k 2^{\frac{n+1}{2}m} \max\{2^{-\frac{n+1}{2}k}, 2^{-2k} 2^{\frac{3-n}{2}m}\} \|S_m v\|_{X_m} \\
&\lesssim \max\{2^{\frac{1-n}{2}k} 2^{\frac{n+1}{2}m}, 2^{-k} 2^{2m}\} \|S_m v\|_{X_m}.
\end{aligned}$$

Hence (2.14) is proved, which by summation yields the bound (2.11) for A_k^{med}. The bound for A_k^{high} follows from summation of (2.14) in a duality argument.

We note that in all cases there is some room to spare in the estimates. This shows that our hypothesis is too strong for this lemma. Indeed, one could prove it without using at all the bound on the second derivatives of the coefficients.

The bound (2.12) follows by duality from (2.11). The proof of (2.13), as in [28], follows from the $|\alpha| = 1$ case of (2.10). □

The next proposition allows us to treat lower order terms perturbatively in most of our results.

Proposition 2.3. *a) Assume that b, c satisfy* (1.5) *and* (1.6)*. Then*

$$\|(b^i D_i + D_i b^i + c)u\|_{\tilde{X}'} \lesssim \kappa \|u\|_{\tilde{X}}. \tag{2.16}$$

b) Assume that b satisfies (1.5) *and* $\operatorname{div} b = 0$*. Then*

$$\|(b^i D_i + D_i b^i)u\|_{X'} \lesssim \kappa \|u\|_X. \tag{2.17}$$

Proof. This proof parallels a similar argument in [28]. However in there only dimensions $n \geq 3$ are considered, and the bound (1.6) is stronger to include the full gradient of b. Thus we provide a complete proof here. We consider two cases, the first of which is similar to [28], while the second requires a new argument.

Case 1: The estimate (2.16) **for** $n \geq 3$ **and** (2.17) **for** $n = 1, 2$**.** The estimate for the c term is straightforward since, by (1.6),

$$\langle cu, v\rangle \lesssim \kappa \|\langle x\rangle^{-1} u\|_{L^2_{t,x}} \|\langle x\rangle^{-1} v\|_{L^2_{t,x}} \lesssim \kappa \|u\|_{\tilde{X}} \|v\|_{\tilde{X}}.$$

For the b term, we consider a paradifferential decomposition,

$$\begin{aligned}(b^i D_i + D_i b^i)u = &\sum_k (S_{<k} b^i D_i + D_i S_{<k} b^i) S_k u \\ &+ \sum_k (S_k b^i D_i + D_i S_k b^i) S_k u \ + \sum_k (S_{>k} b^i D_i + D_i S_{>k} b^i) S_k u.\end{aligned} \tag{2.18}$$

The frequency localization is preserved in the first term; therefore it suffices to verify that

$$\|(S_{<k} b^i D_i + D_i S_{<k} b^i) S_k u\|_{X'_k} \lesssim \kappa 2^k \|S_k u\|_{X_k}.$$

The derivative yields a factor of 2^k, and we are left with proving that

$$\|S_{<k} b^i v\|_{X'_k} \lesssim \kappa \|v\|_{X_k}.$$

This in turn follows from the pointwise bound

$$|S_{<k} b^i| \lesssim \begin{cases} \kappa_k(|x|)\langle x\rangle^{-1}, & k \geq 0, \\ \max\Big\{2^k \kappa_k(|x|)\langle 2^k x\rangle^{-1}, \kappa 2^k \langle 2^k x\rangle^{-2}\Big\}, & k < 0 \end{cases}$$

which is easy to obtain. The second term on the second line above is only needed in the worst case $n = 1$.

The remaining two terms in (2.18) are dual. Hence it suffices to consider the last one. We want the derivative to go to the low frequency; therefore we rewrite it in the form

$$\sum_k 2 S_{>k} b^i D_i S_k u - i S_{>k} \operatorname{div}\, b \ S_k u. \tag{2.19}$$

We consider the two terms separately. The second one occurs only in the case of (2.16) but the first one occurs also in (2.17). So we need to show that

$$\|\sum_k S_{>k} b^i D_i S_k u\|_{X'} \lesssim \kappa \|u\|_X.$$

This will follow from the dyadic estimates

$$\|S_m b^i S_k u\|_{X'_m} \lesssim \kappa \|S_k u\|_{X_k}, \qquad m > k.$$

Given the pointwise bound on $S_m b^i$, this reduces to

$$\|S_k u\|_{X_m} \lesssim \|S_k u\|_{X_k}.$$

For $|x| > \max\{2^{-k}, 1\}$ this is trivial. For smaller x we use (2.3), and the conclusion is obtained by a direct computation.

It remains to consider the second term in (2.19), for which we want to show that in dimension $n \geq 3$

$$\| \sum_k S_{>k} \text{div } b \; S_k u \|_{\tilde{X}'} \lesssim \kappa \|u\|_{\tilde{X}}. \tag{2.20}$$

For this we establish again off-diagonal decay,

$$\|S_m \text{div } b \; S_k u\|_{X_m'} \lesssim \kappa (m-k) 2^k \|S_k u\|_{X_k}, \qquad m > k. \tag{2.21}$$

This follows from the pointwise bounds

$$|S_m \text{div } b| \leq \kappa 2^{2m} \langle 2^m x \rangle^{-2}, \qquad m < 0$$

$$|S_m \text{div } b| \leq \kappa \langle x \rangle^{-2}, \qquad m \geq 0.$$

We consider the worst case $0 > m > k$ and leave the rest for the reader. We use $\chi_{<-k}$ to separate small and large values of x. For large x we have

$$\|\chi_{>-k} S_m \text{div } b \; S_k u\|_{X_m'} \lesssim \kappa \| |x|^{-2} \chi_{>-k} \; S_k u\|_{X_k'} \lesssim \kappa 2^k \|S_k u\|_{X_k}.$$

For small x we use (2.3) instead,

$$\|\chi_{<-k} S_m \text{div } b \; S_k u\|_{X_m'} \lesssim \kappa 2^{2m} 2^{\frac{n-1}{2}k} \|\chi_{<-k} \langle 2^m x \rangle^{-2}\|_{(X_m^0)'} \|S_k u\|_{X_k} \lesssim \kappa 2^k \|S_k u\|_{X_k}.$$

The last computation above is accurate if $n \geq 4$. In dimension $n = 3$ we encounter a harmless additional logarithmic factor $|m-k|$. However if $n = 1, 2$ then the above off-diagonal decay can no longer be obtained.

Case 2: The estimate (2.16) **in dimension** $n = 1, 2$**.** The c term is again easy to deal with. We write the estimate for b in a symmetric way,

$$|\langle (b^i D_i + D_i b^i) u, v \rangle| \lesssim \kappa \|u\|_{\tilde{X}} \|v\|_{\tilde{X}}.$$

We use the decomposition in Section 2.2,

$$u = u^{in} + u^{out}, \qquad v = v^{in} + v^{out}.$$

We consider first the expression

$$\langle (b^i D_i + D_i b^i) u^{out}, v^{out} \rangle.$$

For this we can take advantage of the improved L^2 bound (2.4) to carry out the same computation as in dimension $n \geq 3$, establishing off-diagonal decay. Precisely, the difference arises in the proof of (2.21), whose replacement is

$$\|S_m \text{div } b \; (1 - T_k) S_k u\|_{X_m'} \lesssim \kappa (m-k) 2^k \|S_k u\|_{X_k}, \qquad m > k. \tag{2.22}$$

Consider now one of the cross terms,

$$\langle (b^i D_i + D_i b^i) u^{in}, v^{out} \rangle = \langle (2 b^i D_i - i \text{div} b) u^{in}, v^{out} \rangle.$$

The proof for the other cross term will follow similarly. For the div b term we use the L^2 bound for both u^{in} and v^{out}, as in the case of c. For the rest we use (2.6) and (2.5) to estimate

$$|\langle b^i D_i u^{in}, v^{out} \rangle| \lesssim \|u^{in}\|_{\dot{H}^1} \|b v^{out}\|_{L^2} \lesssim \|u\|_X \|v\|_X.$$

Finally, consider the last term

$$\langle (b^i D_i + D_i b^i) u^{in}, v^{in} \rangle.$$

In dimension $n = 1$, we can easily estimate it by

$$|\langle (b^i D_i + D_i b^i) u^{in}, v^{in} \rangle| \lesssim \|u^{in}\|_{\dot H^1} \|\langle x\rangle^{-1} v^{in}\|_{L^2} + \|v^{in}\|_{\dot H^1} \|\langle x\rangle^{-1} u^{in}\|_{L^2} \lesssim \|u\|_{\tilde X} \|v\|_{\tilde X}.$$

This argument fails for $n = 2$ due to the logarithmic factor in the L^2 weights. Instead we will take advantage of the spherical symmetry of both u^{in} and v^{in}.

In polar coordinates we write

$$b^i D_i = b^r D_r + r^{-1} b^\theta D_\theta$$

and

$$\text{div } b = \partial_r b^r + r^{-1} b^r + r^{-1} \partial_\theta b^\theta.$$

For a function $b(r, \theta)$, we denote $\bar b(r)$ its spherical average. By spherical symmetry, we compute

$$\langle b^i D_i u^{in}, v^{in} \rangle = \langle (b^r D_r + r^{-1} b^\theta D_\theta) u^{in}, v^{in} \rangle = \langle D_r u^{in}, \bar{b^r} v^{in} \rangle.$$

Then we can estimate

$$|\langle (b^i D_i + D_i b^i) u^{in}, v^{in} \rangle| \lesssim \|u^{in}\|_{\dot H^1} \|\bar{b^r} v^{in}\|_{L^2} + \|v^{in}\|_{\dot H^1} \|\bar{b^r} u^{in}\|_{L^2} \lesssim \|u\|_{\tilde X} \|v\|_{\tilde X}$$

provided we are able to establish the improved bound

$$|\bar{b^r}(r)| \lesssim \langle r\rangle^{-1} (\ln(2 + r))^{-1}. \tag{2.23}$$

For this we take spherical averages in the divergence equation to obtain

$$\partial_r \bar b^r + r^{-1} \bar b^r = \overline{\text{div } b}.$$

At infinity we have $b(r) = o(r^{-1})$. Integrating from infinity we obtain

$$\bar b^r(r) = \int_r^\infty \frac{s}{r}\, \overline{\text{div } b}(s) ds.$$

Hence

$$|\bar b^r(r)| \lesssim \int_r^\infty \frac{s}{r} (1 + s)^{-2} (\ln(2 + s))^{-2} ds$$

and (2.23) follows. □

3. Local smoothing estimates

In this section we prove our main local smoothing estimates, first in the exterior region and then in the non-trapping case.

3.1. **The high dimensional case $n \geq 3$: Proof of Theorem 1.3.** The proof uses energy estimates and the positive commutator method. This turns out to be rather delicate. The difficulty is that the trapping region acts essentially as a black box, where the energy is conserved but little else is known. Hence all the local smoothing information has to be estimated starting from infinity along rays of the Hamilton flow which are incoming either forward or backward in time.

We begin with the energy estimate. This is standard if the right hand side is in $L^1_t L^2_x$, but we would like to allow the right hand side to be in the dual smoothing space as well.

Proposition 3.1. *Let u solve the equation*

$$D_t + Au = f_1 + f_2, \qquad u(0) = u_0 \tag{3.1}$$

in the time interval $[0,T]$. Then we have

$$\|u\|^2_{L^\infty_t L^2_x} \lesssim \|u_0\|^2_{L^2} + \|f_1\|^2_{L^1_t L^2_x} + \|u\|_{\tilde{X}_e} \|f_2\|_{\tilde{X}'_e}. \tag{3.2}$$

Proof. The proof is straightforward. We compute

$$\frac{d}{dt}\frac{1}{2}\|u(t)\|^2_{L^2} = \Im\langle u, f_1 + f_2\rangle.$$

Hence for each $t \in [0,T]$ we have

$$\|u(t)\|^2_{L^2} \lesssim \|u(0)\|^2_{L^2} + \|u\|_{L^\infty_t L^2_x}\|f_1\|_{L^1_t L^2_x} + \|u\|_{\tilde{X}_e}\|f_2\|_{\tilde{X}'_e}.$$

We take the supremum over t on the left and use bootstrapping for the second term on the right. The conclusion follows. ☐

To prove (1.14) we need a complementary estimate, namely

$$\|\rho u\|^2_{\tilde{X}} \lesssim \|u\|^2_{L^\infty_t L^2_x} + \|f_1\|^2_{L^1_t L^2_x} + \|\rho f_2\|^2_{\tilde{X}'} + \|\langle x\rangle^{-2} u\|^2_{L^2_{t,x}}. \tag{3.3}$$

Given (3.2) and (3.3), the bound (1.14) is obtained by bootstrapping, with some careful balancing of constants.

It remains to prove (3.3). We will use a positive commutator method. We shall assume that $b = 0$ and $c = 0$. For a self-adjoint operator Q, we have

$$2\Im\langle Au, Qu\rangle = \langle Cu, u\rangle$$

where

$$C = i[A, Q].$$

As a consequence of this, we see that

$$\frac{d}{dt}\langle u, Qu\rangle = -2\Im\langle (D_t + A)u, Qu\rangle + \langle Cu, u\rangle.$$

Taking this into account, the estimate (3.3) is an immediate consequence of the following lemma.

Proposition 3.2. *There is a family $\mathcal{Q}$ of bounded self-adjoint operators Q_ρ with the following properties:*

(i) L^2 boundedness,

$$\|Q_\rho\|_{L^2 \to L^2} \lesssim 1$$

(ii) $\tilde{X}$ boundedness,

$$|\langle Q_\rho u, f\rangle| \lesssim \|\rho f\|_{\tilde{X}'} \|\rho u\|_{\tilde{X}}$$

(iii) Positive commutator,

$$\sup_{Q_\rho \in \mathcal{Q}} \langle Cu, u\rangle \geq c_1 \|\rho u\|_{\tilde{X}}^2 - c_2 \|\langle x\rangle^{-2} u\|_{L^2_{t,x}}^2.$$

We first note that the condition (ii) shows that $Q_\rho u$ is supported in $\{|x| > 2^M\}$ and depends only on the values of u in the same region. Hence for the purpose of this proof we can modify the operator A arbitrarily in the inner region $\{|x| < 2^M\}$. In particular we can improve the constant κ in (1.4) to the extent that (1.7) holds globally. Similarly, we can assume without any restriction in generality that $u = 0$ in $\{|x| < 2^M\}$.

Using (ii), we may argue similarly and assume that (1.8) and (1.9) hold globally if lower order terms are present. The estimate (2.16) then justifies neglecting the lower order terms in A. I.e., we may assume that $b = 0$, $c = 0$.

Proof. The main step in the proof of the proposition is to construct some frequency localized versions of the operator Q_ρ. Precisely, for each $k \in \mathbb{Z}$ we produce a family $\mathcal{Q}_k$ of operators Q_k, which we later use to construct Q_ρ. We consider two cases, depending on whether k is positive or negative.

We first introduce some variants of the spaces X_k. Let $k \in \mathbb{Z}$ and $k^- = \frac{|k|-k}{2}$ be its negative part. For any positive, slowly varying sequence $(\alpha_m)|_{m \geq k^-}$ with

$$\sum_{k \geq k^-} \alpha_j = 1, \qquad \alpha_{k^-} \approx 1,$$

we define the space $X_{k,\alpha}$ with norm

$$\|u\|_{X_{k,\alpha}}^2 = 2^{-k^-} \|u\|_{L^2(A_{\leq k^-})}^2 + \sum_{j > k^-} \alpha_j \| |x|^{-1/2} u\|_{L^2(A_j)}^2.$$

Then our low frequency result has the form

Lemma 3.3. *Let $n \geq 1$ and $k < 0$. Then for any slowly varying sequence (α_m) with $\alpha_{-k} \approx 1$ and $\sum_{m \geq -k} \alpha_m = 1$, there is a self-adjoint operator Q_k so that*

$$\|Q_k u\|_{L^2} \lesssim \|u\|_{L^2}, \tag{3.4}$$

$$\|Q_k u\|_{X_{k,\alpha}} \lesssim \|u\|_{X_{k,\alpha}}, \tag{3.5}$$

$$\langle C_k u, u\rangle \gtrsim 2^k \|u\|_{X_{k,\alpha}}^2, \qquad C_k = i[A_{(k)}, Q_k] \tag{3.6}$$

for all functions u frequency localized at frequency 2^k.

Proof. We argue exactly as in [28, Lemma 9]. The only difference is that here we work with the operator $A_{(k)}$ whose coefficients have less regularity, but this turns out to be nonessential.

We first increase the sequence (α_m) so that

$$\begin{cases} (\alpha_m) \text{ remains slowly varying,} \\ \alpha_m = 1 \text{ for } m \leq -k \\ \displaystyle\sum_{m>-k} \alpha_m \approx 1, \\ \kappa_m \leq \varepsilon \alpha_m \text{ for } m > -k. \end{cases} \tag{3.7}$$

To this slowly varying sequence we may associate a slowly varying function $\alpha(s)$ with

$$\alpha(s) \approx \alpha_m, \quad s \approx 2^{m+k}.$$

We construct an even smooth symbol ϕ of order -1 satisfying

$$\phi(s) \approx \langle s \rangle^{-1}, \quad s > 0 \tag{3.8}$$

$$\phi(s) + s\phi'(s) \approx \frac{\alpha(s)}{\langle s \rangle}, \quad s > 0. \tag{3.9}$$

We notice that the radial function $S_{<10}(D)\phi(|x|)$ satisfies the same estimates; therefore without any restriction in generality we assume that $\phi(|x|)$ is frequency localized in $|\xi| < 2^{10}$.

We now define the self-adjoint multiplier

$$Q_k(x, D) = \delta(Dx\phi(2^k\delta|x|) + \phi(2^k\delta|x|)xD).$$

For small δ this takes frequency 2^k functions to frequency 2^k functions. The first property (3.4) follows immediately. The estimate (3.5) is also straightforward as the weight in the $X_{k,\alpha}$ norm is slowly varying on the dyadic scale. It remains to prove (3.6) for which we begin by computing the commutator

$$\begin{aligned} C_k = {} & 4\delta D_i \phi(2^k\delta|x|) a^{ij}_{(k)} D_j \\ & + 2^{k+1}\delta^2 \Big(Dx|x|^{-1}\phi'(2^k\delta|x|)x_i a^{ij}_{(k)} D_j + D_i a^{ij}_{(k)} x_j |x|^{-1}\phi'(2^k\delta|x|)xD \Big) \\ & - 2\delta D_i \phi(2^k\delta|x|)(x_l \partial_l a^{ij}_{(k)}) D_j + \partial_i (a^{ij}_{(k)} (\partial_j \partial (\delta x \phi(2^k\delta|x|)))). \end{aligned} \tag{3.10}$$

The positive contribution comes from the first two terms. Replacing $a^{ij}_{(k)}$ by the identity leaves us with the principal part

$$C^0_k = 4\delta D\phi(2^k\delta|x|)D + 4\delta D\frac{x}{|x|}2^k\delta|x|\phi'(2^k\delta|x|)\frac{x}{|x|}D$$

which by (3.9) satisfies

$$\langle C^0_k u, u\rangle \geq 4\delta\langle(\phi(2^k\delta|x|) + 2^k\delta|x|\phi'(2^k\delta|x|))\nabla u, \nabla u\rangle \gtrsim \delta 2^{2k}\Big\langle \frac{\alpha(2^k\delta|x|)}{\langle 2^k\delta x\rangle}u, u\Big\rangle.$$

Since $a^{ij}_{(k)}(x) - \delta^{ij} = O(\kappa_k(|x|))$, the error we produce by substituting $a^{ij}_{(k)}$ by the identity has size

$$\delta 2^{2k}\left\langle \frac{\kappa_k(|x|)}{\langle 2^k\delta x\rangle}u, u\right\rangle.$$

It remains to examine the last two terms in C_k. Using (2.10), we see that

$$|\delta\phi(2^k\delta|x|)(x_l\partial_l a^{ij}_{(k)})| \lesssim \frac{\delta\kappa_k(|x|)}{\langle 2^k\delta x\rangle}.$$

So, the third term yields an error similar to the above one.

Finally,

$$|\partial_i(a_{(k)}^{ij}(\partial_j\partial(\delta x\phi(2^k\delta|x|))))| \lesssim \frac{\delta^3 2^{2k}}{\langle 2^k\delta x\rangle^3} \lesssim \frac{\delta^3 2^{2k}\alpha(2^k\delta|x|)}{\langle 2^k\delta x\rangle},$$

which yields

$$\langle \partial_i(a_{(k)}^{ij}(\partial_j\partial\delta x\phi(2^k\delta|x|)))u, u\rangle \lesssim \delta^3 2^{2k}\Big\langle \frac{\alpha(2^k\delta|x|)}{\langle 2^k\delta x\rangle}u, u\Big\rangle.$$

Summing up, we have proved that

$$\langle C_k u, u\rangle \geq c_1\delta 2^{2k}\Big\langle \frac{\alpha(2^k\delta|x|)}{\langle 2^k\delta x\rangle}u, u\Big\rangle - c_2\delta^3 2^{2k}\Big\langle \frac{\alpha(2^k\delta|x|)}{\langle 2^k\delta x\rangle}u, u\Big\rangle - c_3\delta 2^{2k}\left\langle \frac{\kappa_k(|x|)}{\langle 2^k\delta x\rangle}u, u\right\rangle. \tag{3.11}$$

In order to absorb the second term into the first we need to know that δ is sufficiently small. This determines the choice of δ as a small universal constant. In order to absorb the third term into the first we use the last part of (3.7) and the fact that α is slowly varying on the dyadic scale to estimate

$$\kappa(|x|) \lesssim \varepsilon\alpha(2^k|x|) \lesssim \delta^{-1}\varepsilon\alpha(2^k\delta|x|).$$

Thus the third term is negligible if $\varepsilon \ll \delta$. This determines the choice of ε in (1.7), (1.8) and (1.9). $\square$

We continue with the result for high frequencies.

Lemma 3.4. *Let $n \geq 1$ and $k \geq 0$. Then for any sequence (α_m) with $\alpha_0 = 1$ and $\sum_{m\geq 0}\alpha_m = 1$ there is a self-adjoint operator Q_k so that*

$$\|Q_k u\|_{L^2} \lesssim \|u\|_{L^2}, \tag{3.12}$$

$$\|Q_k u\|_{X_{k,\alpha}} \lesssim \|u\|_{X_{k,\alpha}}, \tag{3.13}$$

$$\langle C_k u, u\rangle \gtrsim 2^k\|u\|^2_{X_{k,\alpha}}, \qquad C_k = i[A_{(k)}, Q_k], \tag{3.14}$$

$$2\Im\langle [A_{(k)}, \rho_{<k}]u, Q_k\rho_{<k}u\rangle \lesssim 2^{-k}\|\langle x\rangle^{-2}u\|^2_{L^2_{t,x}} \tag{3.15}$$

for all functions u frequency localized at frequency 2^k. Here, $\rho_{<k} = S_{<k-4}\rho$ where ρ is as in the definition of $\tilde{X}_e$.

Proof. We replace the sequence (α_m) by a larger one satisfying an analogue of (3.7), namely

$$\begin{cases} (\alpha_m) \text{ is slowly varying,} \\ \alpha_0 = 1 \\ \displaystyle\sum_{m\geq 0}\alpha_m \approx 1, \\ \kappa_m \leq \varepsilon\alpha_m \text{ for } m \geq 0, \end{cases} \tag{3.16}$$

and let α be a slowly varying function satisfying

$$\alpha(s) \approx \alpha_m, \qquad s \approx 2^m.$$

We construct ϕ as in the low frequency case so that (3.8) and (3.9) are satisfied. Then we set

$$Q_k = 2^{-k}\delta(D_i a^{ij}_{(k)} x_j \phi(\delta|x|) + \phi(\delta|x|) a^{ij}_{(k)} x_i D_j).$$

This choice is not very different from the one in the low frequency case. The metric a^{ij} is inserted in order to insure a crucial sign condition in the proof (3.15).

The first property, (3.12), is immediate from the properties of ϕ and (2.10). The bound (3.13) is also straightforward since the coefficients $a^{ij}_{(k)}$ are bounded.

Proof of (3.14)**:** In order to prove (3.14), we calculate (using the symmetry of a^{ij})

$$\begin{aligned} C_k =& \delta 2^{-k}\Big[2D_l a^{lm}_{(k)} \partial_m (a^{ij}_{(k)} x_j \phi(\delta|x|)) D_i + 2D_i \partial_l (a^{ij}_{(k)} x_j \phi(\delta|x|)) a^{lm}_{(k)} D_m \\ & - 2D_l \partial_i (a^{lm}_{(k)}) a^{ij}_{(k)} x_j \phi(\delta|x|) D_m - \partial_l (a^{lm}_{(k)} \partial_i \partial_m (a^{ij}_{(k)} x_j \phi(\delta|x|)))\Big]. \end{aligned} \tag{3.17}$$

The main positive contribution is obtained by substituting a by I_n in the first two terms,

$$\begin{aligned} C^0_k &= 2^{-k}\delta\left[2D_l \partial_l (x_i \phi(\delta|x|)) D_i + 2D_i \partial_l (x_i \phi(\delta|x|)) D_l\right] \\ &= 4\cdot 2^{-k}\delta\left[D\phi(\delta|x|)D + D\frac{x}{|x|}\delta|x|\phi'(\delta|x|)\frac{x}{|x|}D\right]. \end{aligned}$$

As in the low frequency case, this satisfies

$$\langle C^0_k u, u\rangle \gtrsim \delta 2^k \left\langle \frac{\alpha(\delta|x|)}{\langle\delta|x|\rangle} u, u\right\rangle$$

for any function u localized at frequency 2^k. The other contributions are shown to be smaller error terms. Consider for instance the error made by substituting $a^{ij}_{(k)}$ by I_n in the first term. By (2.10), we can estimate

$$|a^{lm}_{(k)}\partial_m(a^{ij}_{(k)} x_j \phi(\delta|x|)) - \delta^{lm}\partial_m(\delta^{ij} x_j \phi(\delta|x|))| \lesssim \frac{\kappa_k(|x|)}{\langle\delta x\rangle}$$

which contributes to $\langle C_k u, u\rangle$ an error of size

$$\delta 2^k \left\langle \frac{\kappa_k(|x|)}{\langle \delta x\rangle} u, u\right\rangle.$$

A similar contribution comes from the second term and the third term. Finally, for the last term in C we have

$$|\partial_l(a^{lm}_{(k)}\partial_i\partial_m(a^{ij}_{(k)} x_j \phi_k(\delta|x|)))| \lesssim \frac{2^k\kappa_k(|x|)}{\langle x\rangle\langle\delta x\rangle} + \frac{\delta^2}{\langle\delta x\rangle^3} \lesssim \frac{2^k\kappa_k(|x|)}{\langle\delta x\rangle} + \frac{\delta^2\alpha(\delta|x|)}{\langle\delta x\rangle}$$

which yields an error of size

$$\delta\left\langle\frac{\kappa(|x|)}{\langle\delta x\rangle}u, u\right\rangle + \delta^3 2^{-k}\left\langle\frac{\alpha(\delta|x|)}{\langle\delta x\rangle}u, u\right\rangle.$$

Summing up we have proved that

$$\langle C_k u, u\rangle \geq c_1\delta 2^k\left\langle\frac{\alpha(\delta|x|)}{\langle\delta x\rangle}u, u\right\rangle - c_2\delta^3 2^{-k}\left\langle\frac{\alpha(\delta|x|)}{\langle\delta x\rangle}u, u\right\rangle - c_3\delta 2^k\left\langle\frac{\kappa(|x|)}{\langle\delta x\rangle}u, u\right\rangle. \tag{3.18}$$

Choosing δ small enough (independently of (α_m) and k), the second term on the right is negligible compared to the first. Since α is slowly varying, by (3.16) the last term is also negligible provided that ε is sufficiently small. Hence (3.14) follows.

Proof of (3.15)**:** We denote by L the self-adjoint operator

$$L = x_i a^{ij}_{(k)} D_j + D_i a^{ij}_{(k)} x_j$$

and begin by calculating

$$\begin{aligned}\frac{1}{i}[A_{(k)}, \rho_{<k}] &= -D_i a^{ij}_{(k)} (\partial_j \rho_{<k}) - a^{ij}_{(k)} (\partial_i \rho_{<k}) D_j. \\ &= -|x|^{-1} \rho'_{<k} L + i x_i a^{ij}_{(k)} \partial_j (|x|^{-1} \rho'_{<k})\end{aligned}$$

and

$$2^k Q_k \rho_{<k} = \delta \rho_{<k} \phi(\delta |x|) L - i x_i a^{ij}_{(k)} (\rho_{<k} \partial_j \phi(\delta|x|) + 2\phi(\delta|x|) \partial_j \rho_{<k}).$$

Thus, after one integration by parts we obtain

$$2^k \Im \langle [A_{(k)}, \rho_{<k}] u, Q \rho_{<k} u\rangle = -\delta \int |x|^{-1} \rho'_{<k} \phi(\delta|x|) \rho_{<k} |Lu|^2 dxdt + \int V |u|^2 dxdt \tag{3.19}$$

where the scalar function V is given by

$$\begin{aligned} V = &(x_i a^{ij}_{(k)} \partial_j + \partial_i a^{ij}_{(k)} x_j) \left(\rho_{<k} \phi(\delta|x|) x_l a^{lm}_{(k)} \partial_m (|x|^{-1} \rho'_{<k}) \right) \\ &+ (x_i a^{ij}_{(k)} \partial_j + \partial_i a^{ij}_{(k)} x_j) \left(|x|^{-1} \rho'_{<k} x_l a^{lm}_{(k)} (\rho_{<k} \partial_m \phi(\delta|x|) + 2\phi(\delta|x|) \partial_m \rho_{<k}) \right) \\ &- \left(x_i a^{ij}_{(k)} \partial_j (|x|^{-1} \rho'_{<k}) \right) \left(x_l a^{lm}_{(k)} [\rho_{<k} \partial_m \phi(\delta|x|) + 2\phi(\delta|x|) \partial_m \rho_{<k}] \right). \end{aligned}$$

Morally speaking, the first term in (3.19) is negative and can be dropped. This is true modulo the tails that are introduced by the frequency cutoff which is applied to ρ. Since

$$|r^{-1}(\rho'(r) - \rho'_{<k}(r))| \lesssim 2^{-Nk} \langle r \rangle^{-N},$$

the error is estimated by

$$2^{-Nk} \| \langle x \rangle^{-2} u \|^2_{L^2_{t,x}}.$$

On the other hand the weight V is bounded and rapidly decreasing at infinity,

$$|V| \lesssim \langle x \rangle^{-N},$$

from which (3.15) follows. □

We now return to the proof of Proposition 3.2. We choose Q_ρ of the form

$$Q_\rho = \sum_{k=-\infty}^{\infty} \rho S_k Q_k S_k \rho$$

where for each k we have an L^2 bounded self-adjoint operator localized at frequency 2^k.

The L^2 boundedness of Q_ρ follows from the L^2 boundedness of Q_k, and the $\tilde{X}$ boundedness of Q_ρ follows from the $X_{k,\alpha}$ boundedness of Q_k after optimizing in α. It remains to consider the commutator C. We write

$$C = i \sum_k [A, \rho S_k Q_k S_k \rho].$$

We first replace A by $A_{(k)}$ and ρ by $\rho_{<k}$ for $k > 0$ and by 1 for $k < 0$. This generates error terms which we need to estimate.

If $k < 0$ then these error terms are estimated as follows. We first want to substitute A by $A_{(k)}$, and as such, we see errors of the form

$$|\langle [A,\rho]u, \sum_{k<0} S_k Q_k S_k \rho u\rangle| + |\sum_{k<0} \langle (A - A_{(k)})\rho u, S_k Q_k S_k \rho u\rangle|. \tag{3.20}$$

For the first term, we use (3.5) (after optimizing in α)

$$\begin{aligned}
|\langle [A,\rho]u, \sum_{k<0} S_k Q_k S_k \rho u\rangle| &\lesssim |\langle -2iD_i a^{ij}(\partial_j \rho)u + \partial_j((\partial_i \rho)a^{ij})u, \sum_{k<0} S_k Q_k S_k \rho u\rangle| \\
&\lesssim \|u\|_{L^2_{t,x}(2^M<|x|<2^{M+1})} \| \sum_{k<0} S_k Q_k S_k \rho u\|_{L^2_{t,x}(\{2^M<|x|<2^{M+1}\})} \\
&\lesssim \|\langle x\rangle^{-2} u\|_{L^2_{t,x}} \|\sum_{k<0} S_k Q_k S_k \rho u\|_X \\
&\lesssim \|\langle x\rangle^{-2} u\|_{L^2_{t,x}} \|\rho u\|_{\tilde{X}}.
\end{aligned}$$

For the second term in (3.20), we use (2.11) and (3.5) to see that

$$\begin{aligned}
|\sum_{k<0} \langle (A - A_{(k)})\rho u, S_k Q_k S_k \rho u\rangle| &\lesssim \Big(\sum 2^{-k} \|S_k (A - A_{(k)})\rho u\|^2_{X'_k}\Big)^{\frac{1}{2}} \|\rho u\|_X \\
&\lesssim \varepsilon \|\rho u\|^2_{\tilde{X}}.
\end{aligned}$$

For the remaining errors, we use the fact that $A_{(k)}$ preserves localizations at frequency 2^k combined with (2.10), and (3.4) to see that

$$\begin{aligned}
|\sum_{k<0} \langle A_{(k)}(1-\rho)u, S_k Q_k S_k \rho u\rangle| &\lesssim \|\langle x\rangle^{-2} u\|_{L^2_{t,x}} \|\langle x\rangle^2 \sum_{k<0} S_k Q_k S_k A_{(k)}(1-\rho)u\|_{L^2_{t,x}} \\
&\lesssim \|\langle x\rangle^{-2} u\|_{L^2_{t,x}} \|(1-\rho)u\|_{L^2_{t,x}}
\end{aligned}$$

and respectively,

$$\begin{aligned}
|\sum_{k<0} \langle A_{(k)}u, S_k Q_k S_k (1-\rho)u\rangle| &\lesssim \|\langle x\rangle^{-2} u\|_{L^2_{t,x}} \|\langle x\rangle^2 A_{(k)} \sum_{k<0} S_k Q_k S_k (1-\rho)u\|_{L^2_{t,x}} \\
&\lesssim \|\langle x\rangle^{-2} u\|_{L^2_{t,x}} \|(1-\rho)u\|_{L^2_{t,x}}.
\end{aligned}$$

In both formulas above the last step is achieved by commuting the x^2 factor to the right, where it is absorbed by the $(1-\rho)$ factor. The two possible commutators may yield an extra 2^{-2k} factor, which is compensated for by the two derivatives in $A_{(k)}$.

On the other hand if $k \geq 0$ then we have the bound

$$|\rho - \rho_{<k}| \lesssim 2^{-Nk} \langle x\rangle^{-N}.$$

This estimate clearly provides summability in k, and the control for the correction terms similar to the above ones follows from analogous arguments. The terms, e.g., of the form $\|(1-\rho)u\|_{L^2_{t,x}}$ are simply replaced by $\|\langle x\rangle^{-2} u\|_{L^2_{t,x}}$.

Hence we are left with the modified commutator

$$\tilde{C} = i\sum_{k<0} [A_{(k)}, S_k Q_k S_k] + i \sum_{k\geq 0} [A_{(k)}, \rho_{<k} S_k Q_k S_k \rho_{<k}]$$

where all terms are now frequency localized. The first term is rewritten in the form

$$i[A_{(k)}, S_k Q_k S_k] = i[A_{(k)}, S_k] Q_k S_k + i S_k Q_k [A_{(k)}, S_k] + S_k C_k S_k.$$

For the first two terms we use the commutator estimate (2.13) and the X_k boundedness of Q_k (3.5). We can, thus, bound the corresponding inner products by

$$\varepsilon\|\rho u\|_{\tilde{X}}^2 + \varepsilon\|(1-\rho)u\|_{L^2_{t,x}}^2.$$

For the third term, we shall use (3.6).

Next we consider the high frequency terms in C,

$$[A_{(k)}, \rho_{<k} S_k Q_k S_k \rho_{<k}] = \rho_{<k}[A_{(k)}, S_k Q_k S_k]\rho_{<k} + [A_{(k)}, \rho_{<k}]S_k Q_k S_k \rho_{<k} + \rho_{<k} S_k Q_k S_k [A_{(k)}, \rho_{<k}].$$

The first term is treated as above but using (3.14) instead. For the remaining two terms we commute both outside factors inside. This yields a main contribution which is estimated by (3.15),

$$2\Im\langle [A_{(k)}, \rho_{<k}]S_k u, Q_k \rho_{<k} S_k u\rangle \lesssim 2^{-k}\|\langle x\rangle^{-2} S_k u\|_{L^2_{t,x}}^2.$$

The remaining terms involve an extra commutation which kills the remaining derivative in $A_{(k)}$. Also $\rho_{<k}$ is differentiated, which yields rapid decay at infinity. Hence we can bound them by

$$\|\langle x\rangle^{-2} S_k u\|_{L^2_{t,x}}^2.$$

Summing up, we have proved that

$$\langle Cu, u\rangle \geq c_1 \left(\sum_{k<0} 2^k \|S_k u\|_{X_{k.\alpha(k)}}^2 + \sum_{k>0} 2^k \|S_k \rho_{<k} u\|_{X_{k,\alpha(k)}}^2\right) - c_2\left(\|\langle x\rangle^{-2} u\|_{L^2_{t,x}}^2 + \varepsilon\|\rho u\|_{\tilde{X}}^2\right)$$

where for each k we have used a different α denoted by $\alpha(k)$. Optimizing with respect to all choices of $\alpha(k)$ we obtain

$$\langle Cu, u\rangle \geq c_1 \left(\sum_{k<0} 2^k \|S_k u\|_{X_k}^2 + \sum_{k>0} 2^k \|S_k \rho_{<k} u\|_{X_k}^2\right) - c_2\left(\|\langle x\rangle^{-2} u\|_{L^2_{t,x}}^2 + \varepsilon\|\rho u\|_{\tilde{X}}^2\right)$$

which for ε sufficiently small yields part (iii) of the proposition. □

3.2. The non-resonant low dimensional case $n = 1, 2$: Proof of Theorem 1.3.

Almost all the arguments in the high dimensional case apply also in low dimension. The only difference arises in part (ii) of Proposition 3.2. Since the multiplication by ρ is bounded in both $\tilde{X}$ and $\tilde{X}'$, the property (ii) reduces to proving that

$$\sum_{k=-\infty}^{\infty} S_k Q_k S_k : \tilde{X} \to \tilde{X}.$$

In dimension $n \geq 3$ the $\tilde{X}$ norm is described in terms of the X_k norms of its dyadic pieces, and the above property follows from the X_k boundedness of Q_k at frequency 2^k.

However, in dimension $n = 1, 2$ the $\tilde{X}$ norm also has a weighted L^2 component. The high frequency part $k \geq 0$ of the above sum causes no difficulty, but the low frequency part does. We do know that

$$\sum_{k=-\infty}^{0} S_k Q_k S_k : X \to X.$$

Therefore, due to Lemma 1.2, it would remain to prove that

$$\Big\| \sum_{k=-\infty}^{0} S_k Q_k S_k u \Big\|_{L^2_{t,x}(\{|x|\leq 1\})} \lesssim \|u\|_{\tilde{X}}.$$

Unfortunately, the operators $S_k Q_k S_k$ act on the 2^{-k} spatial scale; therefore without any additional cancellation there is no reason to expect a good control of the output in a bounded region. The aim of the next few paragraphs is to replace the above low frequency sum by a closely related expression which exhibits the desired cancellation property.

First of all, it is convenient to replace the discrete parameter k by a continuous one σ. The operators S_σ are defined in the same way as S_k by scaling. Let ϕ_k be the functions in Lemma 3.3. The functions ϕ_σ are defined from ϕ_k using a partition of unity on the unit scale in σ. The normalization we need is very simple, namely $\phi_k(0) = 1$, which leads to $\phi_\sigma(0) = 1$. The operators Q_σ are defined in a similar way. Then it is natural to substitute

$$\sum_{k=-\infty}^{0} S_k Q_k S_k \to \int_{-\infty}^{0} S_\sigma Q_\sigma S_\sigma d\sigma$$

and all the estimates for the second sum carry over identically from the discrete sum.

However, the desired cancellation is still not present in the second sum. To obtain that we consider a spherically symmetric Schwartz function ϕ^0 localized at frequency $\ll 1$ with $\phi^0(0) = 1$. Then we write ϕ_σ in the form

$$\phi_\sigma(x) = \phi^0(x) + x^2 \psi_\sigma(x).$$

The modified self-adjoint operators $\tilde{Q}_\sigma$ are defined as

$$\tilde{Q}_\sigma = S_\sigma Q_{\sigma,\phi^0} S_\sigma + 2^{2\sigma} \delta^2 x S_\sigma Q_{\sigma,\psi_\sigma} S_\sigma x$$

where, as in Lemma 3.3, we set

$$Q_{\sigma,\phi} = \delta(Dx\phi(2^\sigma \delta|x|) + \phi(2^\sigma \delta|x|)xD).$$

We claim that the conclusion of Proposition 3.2 is valid with the operator Q defined as

$$Q_\rho = \rho Q \rho, \qquad Q = \int_{-\infty}^{0} \tilde{Q}_\sigma d\sigma + \sum_{k=0}^{\infty} S_k Q_k S_k. \tag{3.21}$$

The family $\mathcal{Q}$ is obtained as before by allowing the choice of the functions ϕ_k to depend on the slowly varying sequences $(\alpha_j^\sigma)_{j\in\mathbb{N}}$ which are chosen independently[2] for different k.

[2]In effect, without any restriction in generality, one may also assume that α_j^σ is also slowly varying with respect to σ

There is no change in part (i) of Proposition 3.2. For part (ii) we need to prove that

$$\|Qu\|_{\tilde{X}} \lesssim \|u\|_{\tilde{X}}. \tag{3.22}$$

The high frequencies are estimated directly from the X norm; therefore we have to consider the integral term in Q and show that

$$\left\| \int_{-\infty}^{0} \tilde{Q}_\sigma u \, d\sigma \right\|_{\tilde{X}} \lesssim \|u\|_{\tilde{X}}.$$

The X component of the $\tilde{X}$ norm is easily estimated by Littlewood-Paley theory, so due to Lemma 1.2, it would remain to prove the local L^2 bound

$$\left\| \int_{-\infty}^{0} \tilde{Q}_\sigma u d\sigma \right\|_{L^2_{t,x}(\{|x|\leq 1\})} \lesssim \|u\|_{\tilde{X}}. \tag{3.23}$$

We can neglect the time variable in the sequel. We have the L^2 bound

$$\|\tilde{Q}_\sigma u\|_{X_\sigma} \lesssim \|S_\sigma u\|_{X_\sigma}$$

which leads to

$$\|\nabla \tilde{Q}_\sigma u\|_{X_\sigma} \lesssim 2^\sigma \|S_\sigma u\|_{X_\sigma}$$

and the corresponding pointwise bound

$$\|\nabla \tilde{Q}_\sigma u\|_{L^\infty(A_{<-\sigma})} \lesssim 2^{\frac{n+1}{2}\sigma} \|S_\sigma u\|_{X_\sigma}$$

which establishes the convergence and the bound for the corresponding integral

$$\left\| \int_{-\infty}^{0} \nabla \tilde{Q}_\sigma u \, d\sigma \right\|_{L^\infty(A_{<0})} \lesssim \|S_{\leq 0} u\|_X.$$

Hence in order to prove (3.23) it remains to establish a similar bound for the integral at $x = 0$. Assume first that $u \in L^2$, which arguing as above guarantees the uniform convergence of the integral. Denoting by K_σ the kernel of S_σ we have

$$\begin{aligned}
(\tilde{Q}_\sigma u)(0) &= (S_\sigma Q_{\sigma,\phi^0} S_\sigma) u(0) \\
&= \langle K_\sigma, Q_{\sigma,\phi^0} S_\sigma u \rangle = \langle Q_{\sigma,\phi^0} K_\sigma, S_\sigma u \rangle \\
&= \int Q_{\sigma,\phi^0}(x, D_x) K_\sigma(x) \int K_\sigma(x-y) u(y) dy dx \\
&= (S^1_\sigma u)(0)
\end{aligned}$$

where S^1_σ is the frequency localized multiplier with spherically symmetric Schwartz kernel

$$K^1_\sigma = Q_{\sigma,\phi^0}(x, D_x) K_\sigma * K_\sigma.$$

Due to the frequency localization we can define

$$S^1_{<0} = \int_{-\infty}^{0} S^1_\sigma d\sigma.$$

The punch line is that by construction the operators S^1_σ have the same kernel up to the appropriate rescaling. This implies that the symbols of $S^1_{<0}$ are constant for $|\xi| \leq 2^{-4}$.

Hence both the symbols and the kernels $K^1_{<0}$ of $S^1_{<0}$ are Schwartz functions which coincide modulo rescaling. Hence for all functions $u \in L^2$ we have

$$\int_{-\infty}^0 \tilde{Q}_\sigma u(0) d\sigma = \langle K^1_{<0}, u \rangle$$

which leads to the estimate

$$\left| \int_{-\infty}^0 (\tilde{Q}_\sigma u)(0) d\sigma \right| \lesssim \|u\|_{\tilde{X}}.$$

This completes the proof of the estimate (3.23) for all $u \in L^2$, and, by density, shows that the integral

$$\int_{-\infty}^0 \tilde{Q}_\sigma d\sigma$$

has a unique bounded extension to $\tilde{X}$.

It remains to prove part (iii) of Proposition 3.2. If $\tilde{Q}_\sigma$ is replaced by $S_\sigma Q_\sigma S_\sigma$ then the high dimensional argument applies by simply replacing sums with integrals. Hence it remains to estimate the difference. Commuting we obtain

$$\tilde{Q}_\sigma - S_\sigma Q_\sigma S_\sigma = i\delta^2 2^{2\sigma} \left(S'_\sigma Q_{\sigma,\psi} x S_\sigma - S_\sigma x Q_{\sigma,\psi} S'_\sigma - S'_\sigma Q_{\sigma,\psi} S'_\sigma(D) \right).$$

Commuting again to take advantage of the cancellation between the first two terms, by semiclassical pdo calculus we can write

$$\tilde{Q}_\sigma - S_\sigma Q_\sigma S_\sigma = \delta^2 R_\sigma(2^\sigma \delta x, 2^{-\sigma} D)$$

where the symbol $r_\sigma(y,\eta)$ is localized in $\{|\eta| \approx 1\}$ and satisfies

$$|\partial_y^\alpha \partial_\eta^\beta r_\sigma(y,\eta)| \leq c_{\alpha\beta} \langle y \rangle^{-2}.$$

This implies the bound

$$\|(\tilde{Q}_\sigma - S_\sigma Q_\sigma S_\sigma) u\|_{X'_\sigma} \lesssim \delta^2 2^{-\sigma} \|S_\sigma u\|_{X_\sigma}.$$

Therefore without any commuting we obtain

$$|\langle [\tilde{Q}_\sigma - S_\sigma Q_\sigma S_\sigma, A_{(\sigma)}] u, u \rangle| \lesssim \delta^2 \|u\|^2_X.$$

This error is negligible since, as one can note in the proofs of Lemmas 3.3, 3.4, the constant c_1 in (iii) has size $c_1 = O(\delta)$.

3.3. **The resonant low dimensional case $n = 1, 2$: Proof of 1.4.** The proof follows the same outline as in the non-resonant case, with minor modifications. The energy estimate (3.2) is now replaced by

$$\|u\|^2_{L^\infty_t L^2_x} \lesssim \|u_0\|^2_{L^2} + \|f_1\|^2_{L^1_t L^2_x} + \|u\|_{X_e} \|f_2\|_{X'_e}. \tag{3.24}$$

Instead of the exterior smoothing estimate (3.3), we need to prove

$$\|T_\rho u\|^2_X \lesssim \|u\|^2_{L^\infty_t L^2_x} + \|f_1\|^2_{L^1_t L^2_x} + \|T_\rho f_2\|^2_{X'} + \|\langle x \rangle^{-2}(u - u_\rho)\|^2_{L^2_{t,x}}. \tag{3.25}$$

The estimate (1.15) then follows from the previous two estimates as well as (2.9).

The lower order terms will still be negligible. Indeed, letting $B = 2b^i D_i$, we have

$$T_\rho B u = B T_\rho u - (B\rho)(u - u_\rho) + (1-\rho) \left(\int (1-\rho) dx \right)^{-1} \int (B\rho)(u - u_\rho) dx.$$

Therefore by (2.17), we obtain

$$\|T_\rho Bu\|_{X'} \lesssim \varepsilon \|u\|_{X_e},$$

which combined with the X boundedness of our multiplier below shows that the lower order terms can be neglected.

The estimate (3.25) follows from

Proposition 3.5. *There is a family $\mathcal{Q}_{res}$ of bounded self-adjoint operators Q_{res} with the following properties:*

(i) L^2 boundedness,

$$\|Q_{res}\|_{L^2\to L^2} \lesssim 1,$$

(ii) X boundedness,

$$|\langle Q_{res}u, f\rangle| \lesssim \|T_\rho f\|_{X'} \|T_\rho u\|_X,$$

(iii) Positive commutator,

$$\sup_{Q_{res}\in\mathcal{Q}_{res}} \langle Cu, u\rangle \geq c_1 \|T_\rho u\|_X^2 - c_2 \|\langle x\rangle^{-2}(u-u_\rho)\|^2_{L^2_{t,x}}.$$

Proof. We construct Q_{res} as in the non-resonant case but with the modified truncation operator

$$Q_{res}u = T_\rho Q T_\rho.$$

with Q given by (3.21).

The properties (i) and (ii) are straightforward. For (iii) we note that

$$S_k T_\rho u = S_k \rho(u-u_\rho)$$

while

$$T_\rho Au = \rho Au + c(1-\rho)\int (1-\rho)A(u-u_\rho)dx = \rho A(u-u_\rho) - c(1-\rho)\int (u-u_\rho)A\rho dx.$$

Hence we can express the bilinear form $\langle Au, Q_{res}u\rangle$ in terms of the operator Q_ρ in the nonresonant case

$$\langle Au, Q_{res}u\rangle = \langle A(u-u_\rho), Q_\rho(u-u_\rho)\rangle - c\int (u-u_\rho)A\rho dx \ \langle (1-\rho), QT_\rho u\rangle$$

which implies that

$$\langle C_{res}u, u\rangle = \langle C(u-u_\rho), u-u_\rho\rangle + c\Im \int (u-u_\rho)A\rho dx \ \langle (1-\rho), QT_\rho u\rangle.$$

Hence we can apply part (iii) of Proposition 3.2 and (3.22) to obtain the desired conclusion. □

3.4. **Non-trapping metrics: Proof of Theorem 1.11.** This requires some modifications of the previous argument. First of all, instead of the energy estimate (3.2), we need a straightforward modification of it, namely

$$\|u\|^2_{L^\infty_t L^2_x} \lesssim \|u_0\|^2_{L^2} + \|f_1\|^2_{L^1_t L^2_x} + \|u\|_{\tilde{X}} \|f_2\|_{\tilde{X}'}. \tag{3.26}$$

We still need the exterior local smoothing estimate (3.3). However, now we can complement it with an interior estimate, namely

$$\|(1-\rho)u\|^2_{\tilde{X}} \lesssim \|u\|^2_{L^\infty_t L^2_x} + \|f_1\|^2_{L^1_t L^2_x} + \|\rho u\|^2_{\tilde{X}} + \|(1-\rho)f_2\|^2_{\tilde{X}'} + \|(1-\rho)u\|^2_{L^2_{t,x}}. \tag{3.27}$$

The conclusion of Theorem 1.11 is obtained by combining the three estimates (3.26), (3.3) and (3.27).

It remains to prove (3.27). This is obtained by applying to the function $v = (1-\rho)u$ the local bound

Proposition 3.6. *Assume that the coefficients a^{ij}, b^i, c are real and satisfy* (1.4), (1.5), *and* (1.6). *Moreover, assume that the metric a^{ij} is non-trapping. Let v be a function supported in $\{|x| \leq 2^{M+1}\}$ which solves the equation*

$$(D_t + A)v = g_1 + g_2, \qquad v(0) = v_0 \tag{3.28}$$

in the time interval $[0,T]$. Then we have

$$\|v\|^2_{L^2_t H^{\frac{1}{2}}_x} \lesssim \|v\|^2_{L^\infty_t L^2_x} + \|g_1\|^2_{L^1_t L^2_x} + \|g_2\|^2_{L^2_t H^{-\frac{1}{2}}_x} + \|v\|^2_{L^2_{t,x}}. \tag{3.29}$$

Proof. We use again the multiplier method. The following lemma tells us how to choose an appropriate multiplier.

Proposition 3.7. *Assume that the coefficients a^{ij} satisfy* (1.4). *Moreover, we assume that the Hamiltonian vector field H_a permits no trapped geodesics. Then there exists a smooth, time-independent, real-valued symbol $q \in S^0_{hom}$ so that*

$$H_a q \gtrsim |\xi|, \qquad \text{in } \{|x| \leq 2^{M+1}\}.$$

This proposition is essentially from [8], if a^{ij} were smooth. See also Lemma 1 of [23], which includes some discussion of the limited regularity.

Working in the Weyl calculus and using this multiplier Q, we compute

$$\frac{d}{dt}\langle v, Qv\rangle = -2\Im\langle (D_t + A)v, Qv\rangle + i\langle [A,Q]v, v\rangle$$

which after time integration yields

$$\langle i[A,Q]v, v\rangle = \langle v, Qv\rangle|_0^T + 2\Im\langle g_1 + g_2, Qv\rangle.$$

For the second term on the right, we apply Cauchy-Schwarz and use the L^2 and $H^{\frac{1}{2}}$ boundedness of Q to obtain

$$|\langle (D_t + A)v, Qv\rangle| \lesssim \|v\|^2_{L^\infty_t L^2_x} + \|g_1\|^2_{L^1_t L^2_x} + \|g_2\|_{L^2_t H^{-\frac{1}{2}}_x} \|v\|_{L^2_t H^{\frac{1}{2}}_x}.$$

Hence

$$\langle i[A,Q]v, v\rangle \lesssim \|v\|^2_{L^\infty_t L^2_x} + \|g_1\|^2_{L^1_t L^2_x} + \|g_2\|_{L^2_t H^{-\frac{1}{2}}_x} \|v\|_{L^2_t H^{\frac{1}{2}}_x}.$$

Then it remains to prove the positive commutator bound

$$\langle i[A,Q]v,v\rangle \geq c_1\|v\|^2_{L^2_t H^{\frac{1}{2}}_x} - c_2\|v\|^2_{L^2_{t,x}}. \tag{3.30}$$

The positive contribution comes from the second order terms in P. Precisely, we have

$$i[D_i a^{ij} D_j, Q(x,D)] = Op(H_a q) + O(1)_{L^2\to L^2}.$$

The first symbol is positive, and we can obtain a bound from below by Gårding's inequality. The first order term yields an L^2 bounded commutator, and the zero order term is L^2 bounded by itself.

Here, we remind the reader that we are not working with classical smooth symbols but instead with symbols of limited regularity, and we refer the interested reader to the discussion in Taylor [29, p. 45] for further details on these otherwise classical results. □

3.5. **Non-trapping metrics: Proof of Theorem 1.12.** The argument is similar to the above one, with some obvious modifications. Instead of (3.26) we have

$$\|u\|^2_{L^\infty_t L^2_x} \lesssim \|u_0\|^2_{L^2} + \|f_1\|^2_{L^1_t L^2_x} + \|u\|_X\|f_2\|_{X'} \tag{3.31}$$

while (3.27) is replaced by

$$\begin{aligned}\|(1-\rho)(u-u_\rho)\|^2_X \lesssim \|u\|^2_{L^\infty_t L^2_x} + \|f_1\|^2_{L^1_t L^2_x} + \|\rho(u-u_\rho)\|^2_X + \|f_2\|^2_{X'}\\ + \|\langle x\rangle^{-2}(u-u_\rho)\|^2_{L^2_{t,x}}.\end{aligned} \tag{3.32}$$

The conclusion of Theorem 1.12 is obtained by combining the estimates (3.31), (3.25) and (3.32) and applying (2.9) to reduce the error terms to the form presented in (1.23).

It remains to prove (3.32). We first compute

$$\begin{aligned}D_t u_\rho &= \Big(\int (1-\rho)\,dx\Big)^{-1}\Big[\langle (D_t+A)u,(1-\rho)\rangle - \langle Au,(1-\rho)\rangle\Big]\\ &= \Big(\int (1-\rho)\,dx\Big)^{-1}\Big[\langle f_1+f_2,(1-\rho)\rangle - \langle u-u_\rho, A(1-\rho)\rangle\Big].\end{aligned}$$

The function $v=(1-\rho)(u-u_\rho)$ solves

$$\begin{aligned}Pv = (1-\rho)(f_1+f_2) - (1-\rho)\Big(\int (1-\rho)\,dx\Big)^{-1}\Big[\langle f_1+f_2,(1-\rho)\rangle - \langle u-u_\rho, A(1-\rho)\rangle\Big]\\ + [A,(1-\rho)](u-u_\rho).\end{aligned}$$

Then we apply (3.29) to v to obtain

$$\begin{aligned}\|v\|^2_{L^2_t H^{\frac{1}{2}}_x} &\lesssim \|v\|^2_{L^\infty_t L^2_x} + \|(1-\rho)f_1\|^2_{L^1_t L^2_x} + \|[A,(1-\rho)](u-u_\rho)\|_{L^2_t H^{-\frac{1}{2}}_x}\\ &\qquad + \|(1-\rho)f_2\|^2_{L^2_t H^{-\frac{1}{2}}_x} + \|\langle x\rangle^{-2}(u-u_\rho)\|^2_{L^2_{t,x}}\\ &\lesssim \|u\|^2_{L^\infty_t L^2_x} + \|f_1\|^2_{L^1_t L^2_x} + \|\rho(u-u_\rho)\|^2_X + \|f_2\|^2_{X'} + \|\langle x\rangle^{-2}(u-u_\rho)\|^2_{L^2_{t,x}}\end{aligned}$$

and (3.32) follows.

4. Time independent nontrapping metrics

The aim of this section is to prove Theorems 1.19,1.21. Thus we work with a nontrapping, self-adjoint operator A whose coefficients are time independent. We prove Theorem 1.19 in detail, and then outline the modifications which are needed for Theorem 1.21.

4.1. **Proof of Theorem 1.19.** Here we shall provide the details for the $n \neq 2$ case. The general case follows with the obvious logarithmic adjustments to the $\tilde{X}$ spaces in $n = 2$.

We break the proof into steps.

Step 1: Without any restriction, we assume that $u_0 = 0$ and that u is the forward solution to (1.1). Nonzero initial data u_0 can be easily added in via a TT^* argument.

Step 2: We add a damping term to the equation

$$(D_t + A - i\varepsilon)u_\varepsilon = f$$

in order to insure global square integrability of the solution u_ε. Applying our nontrapping estimate (1.22) we have

$$\|u_\varepsilon\|_{\tilde{X}} \lesssim \|f\|_{\tilde{X}'} + \|u_\varepsilon\|_{L^2_{t,x}(\mathbb{R}\times B(0,2R))}. \tag{4.1}$$

We want to eliminate the second term on the right (when we add P_c on the left).

Step 3: We want to take a Fourier transform in time and use Plancherel's theorem. For this we need to work with Hilbert spaces. These are defined using the structure introduced in the previous section. We denote by α a family of positive sequences $(\alpha(k)_j)_{j\geq k^-}$ which have sum 1 for each k and by $\mathcal{A}$ the collection of such sequences. For $\alpha \in \mathcal{A}$ we define the Hilbert space $\tilde{X}_\alpha$ with norm

$$\|u\|^2_{\tilde{X}_\alpha} = \sum_k 2^k \|S_k u\|^2_{X_{k,\alpha(k)}} + \|\langle x\rangle^{-1} u\|^2_{L^2_{t,x}}$$

as well as its dual $\tilde{X}'_\alpha$. Since

$$\|u\|_{\tilde{X}} \approx \sup_{\alpha\in\mathcal{A}} \|u\|_{\tilde{X}_\alpha}, \qquad \|u\|_{\tilde{X}'} \approx \inf_{\alpha\in\mathcal{A}} \|u\|_{\tilde{X}'_\alpha}$$

we can rewrite (4.1) in the equivalent form

$$\|u_\varepsilon\|_{\tilde{X}_\alpha} \lesssim \|f\|_{\tilde{X}'_\beta} + \|u_\varepsilon\|_{L^2_{t,x}(\mathbb{R}\times B(0,2R))}, \qquad \alpha, \beta \in \mathcal{A}.$$

We denote by X^0_α the spatial version of X_α, i.e. $X_\alpha = L^2_t X^0_\alpha$. Then we take a time Fourier transform, and by Plunderer this is equivalent to

$$\|\hat{u}_\varepsilon\|_{L^2_\tau \tilde{X}^0_\alpha} \lesssim \|\hat{f}\|_{L^2_\tau(\tilde{X}^0_\beta)'} + \|\hat{u}_\varepsilon\|_{L^2_{\tau,x}(\mathbb{R}\times B(0,2R))}.$$

This is in turn equivalent to the fixed τ bound

$$\|\hat{u}_\varepsilon(\tau)\|_{\tilde{X}^0_\alpha} \lesssim \|\hat{f}(\tau)\|_{(\tilde{X}^0_\beta)'} + \|\hat{u}_\varepsilon(\tau)\|_{L^2(B(0,2R))},$$

which we rewrite in the form

$$\|v\|_{\tilde{X}^0_\alpha} \lesssim \|(A - \tau - i\varepsilon)v\|_{(\tilde{X}^0_\beta)'} + \|v\|_{L^2(B(0,2R))},$$

or, optimizing with respect to $\alpha, \beta \in \mathcal{A}$,

$$\|v\|_{\tilde{X}^0} \lesssim \|(A-\tau-i\varepsilon)v\|_{(\tilde{X}^0)'} + \|v\|_{L^2(B(0,2R))}. \tag{4.2}$$

A similar computation shows that the estimate that we want to prove, namely (1.26) with $u_0 = 0$, can be rewritten in the equivalent form

$$\|P_c v\|_{\tilde{X}^0} \lesssim \|(A-\tau-i\varepsilon)v\|_{(\tilde{X}^0)'} \tag{4.3}$$

uniformly with respect to $\tau \in \mathbb{R}$, $\varepsilon > 0$.

Step 4: When $|\tau|$ is large, (4.3) follows from (4.2) combined with the elliptic bound

$$\tau^{1/4}\|v\|_{L^2(B(0,2R))} \lesssim \|v\|_{\tilde{X}^0} + \|(A-\tau-i\varepsilon)v\|_{(\tilde{X}^0)'}. \tag{4.4}$$

To prove this we replace v by $w = (1-\rho)v$ and rewrite it in the form

$$\tau^{1/4}\|w\|_{L^2} \lesssim \|w\|_{H^{\frac{1}{2}}} + \|(A-\tau-i\varepsilon)w\|_{H^{-\frac{1}{2}}}$$

for w with compact support. Since

$$\tau\|w\|_{H^{-\frac{3}{2}}} \lesssim \|(A-\tau-i\varepsilon)w\|_{H^{-\frac{3}{2}}} + \|Aw\|_{H^{-\frac{3}{2}}} \lesssim \|(A-\tau-i\varepsilon)w\|_{H^{-\frac{1}{2}}} + \|w\|_{H^{\frac{1}{2}}},$$

the bound (4.4) follows by interpolation.

Step 5: For τ in a bounded set we argue by contradiction. If (4.3) does not hold uniformly then we find sequences

$$\varepsilon_n \to 0, \qquad \tau_n \to \tau,$$

and $v_n \in \tilde{X}^0$ with $P_c v_n = v_n$ and

$$\|(A-\tau_n-i\varepsilon_n)v_n\|_{(\tilde{X}^0)'} \to 0, \quad \|v_n\|_{L^2(B(0,2R))} = 1.$$

On a subsequence we have

$$v_n \to v \quad \text{weakly* in} \quad \tilde{X}^0.$$

Since $\tilde{X}^0 \subset H^{\frac{1}{2}}_{loc}$, on a subsequence we have the strong convergence

$$v_n \to v \qquad \text{in } L^2_{loc}.$$

Hence we have produced a function v with

$$v \in \tilde{X}^0, \qquad P_c v = v, \qquad (A-\tau)v = 0, \qquad \|v\|_{L^2(B(0,2R))} = 1. \tag{4.5}$$

Depending on the sign of τ we consider three cases.

Step 6: If $\tau < 0$ then, using the bound (2.16) for the lower order terms in A, we obtain

$$\|D_i a^{ij} D_j v - \tau v\|_{(\tilde{X}^0)'} \lesssim \|v\|_{\tilde{X}^0}.$$

Then

$$\|v\|^2_{\tilde{X}^0} \gtrsim \langle v, D_i a^{ij} D_j v - \tau v\rangle \gtrsim \|v\|^2_{H^1},$$

and therefore $v \in L^2$ is an eigenfunction. This contradicts the relation $P_c v = v$.

Step 7: If $\tau = 0$ then there is either a zero eigenvalue or a zero resonance, both of which are excluded by hypothesis.

Step 8: It remains to consider the most difficult case $\tau > 0$. Here the properties (4.5) of v are no longer sufficient to obtain a contradiction. Instead we will establish an additional property of v, namely that v satisfies an outgoing radiation condition. In order to state this, we need an additional regularity property for v. We define the space $\tilde{X}^0_{med}$ with norm

$$\|v\|_{\tilde{X}^0_{med}} = \|v\|_{L^2(D_0)} + \|\nabla v\|_{L^2(D_0)} + \sup_{j>0} \||x|^{-\frac{1}{2}} v\|_{L^2(D_j)} + \||x|^{-\frac{1}{2}} \nabla v\|_{L^2(D_j)}$$

which coincides with the $\tilde{X}^0$ norm for intermediate frequencies but improves it at both low and high frequencies. Then we claim that $v \in \tilde{X}^0_{med}$. More precisely, we will prove the elliptic bound

$$\|v\|_{\tilde{X}^0_{med}} \lesssim \|v\|_{\tilde{X}^0} + \|(A - \tau - i\varepsilon)v\|_{(\tilde{X}^0)'}, \qquad 0 < \tau_0 < \tau < \tau_1 \tag{4.6}$$

with implicit constants which may depend on the thresholds τ_0, τ_1.

Now we define the closed subspace $\tilde{X}^0_{out}$ of $\tilde{X}^0$,

$$\tilde{X}^0_{out} = \{v \in \tilde{X}^0_{med} \,:\, \lim_{j\to\infty} \|r^{-1/2}(\partial_r - i\tau^{1/2})v\|_{L^2(D_j)} = 0\},$$

and also claim that v has the additional property

$$v \in \tilde{X}^0_{out}. \tag{4.7}$$

In other words this implies that v is a resonance contained inside the continuous spectrum.

We postpone the proof of (4.6) and (4.7) and conclude first our proof by contradiction, by showing that there are no resonances inside the continuous spectrum. Such results are known, see for instance [1], but perhaps not in the degree of generality we need here. In any case, for the sake of completeness, we provide a full proof.

Let χ be a smooth spherically symmetric increasing bump function χ with $\chi(r) \equiv 0$ for $r < 1/2$ and $\chi(r) \equiv 1$ for $r > 2$. Since A is self-adjoint, for large j we commute

$$\begin{aligned}
0 &= \frac{i}{2}\langle [A, \chi(2^{-j}r)]v, v\rangle \\
&= \Im \left\langle 2^{-j}\chi'(2^{-j}r)\Big(\frac{x_i a^{ij}}{r}\partial_j - i\tau^{1/2}\Big)v, v\right\rangle + 2^{-j}\tau^{1/2}\langle \chi'(2^{-j}r)v, v\rangle \\
&\qquad + 2^{-j}\Big\langle b^i \frac{x_i}{r}\chi'(2^{-j}r)v, v\Big\rangle.
\end{aligned}$$

Using the Schwarz inequality, (1.8), and the outgoing radiation condition, we conclude that

$$\lim_{j\to\infty} \|r^{-1/2} v\|_{L^2(D_j)} = 0 \tag{4.8}$$

which shows that v has better decay at infinity. We note that this is the only use we make of the radiation condition. From this, by elliptic theory, we also obtain a similar decay for the gradient,

$$\lim_{j\to\infty} \|r^{-1/2}\nabla v\|_{L^2(D_j)} = 0. \tag{4.9}$$

To conclude we use (4.8) and (4.9) to show that in effect $v \in L^2$; i.e. v is an eigenvalue. Then by the results of [16] v must be 0. Here, we shall again use a positive commutator argument. The multiplier we use is the operator Q_k, for some $k \le 0$, in Lemma 3.3 but where for simplicity we set $\delta = 1$. We have

$$0 = -2\Im\langle Q_k v, (A - \tau)v\rangle = \langle C_k v, v\rangle - 2\Im\langle Q_k v, (b^j D_j + D_j b^j + c)v\rangle$$

where

$$C_k = i[D_l a^{lm} D_m, Q_k].$$

The expression of the operator C_k is exactly as in the formula (3.10) but with unmollified coefficients a^{ij}. The main contribution C_k^0 is estimated as there by

$$\langle C_k^0 v, v\rangle \gtrsim \left\langle \frac{\alpha(2^k|x|)}{\langle 2^k x\rangle}\nabla v, \nabla v\right\rangle,$$

while the error terms are bounded by

$$\left\langle \frac{\kappa(|x|)}{\langle 2^k x\rangle}\nabla v, \nabla v\right\rangle$$

respectively

$$\left\langle \langle x\rangle^{-2} v, v\right\rangle.$$

The expression $\Im\langle Q_k v, (b^j D_j + D_j b^j + c)v\rangle$ can also be included in the two error terms. Thus we obtain

$$\left\langle \frac{\alpha(2^k|x|)}{\langle 2^k x\rangle}\nabla v, \nabla v\right\rangle \lesssim \left\langle \frac{\kappa(|x|)}{\langle 2^k x\rangle}\nabla v, \nabla v\right\rangle + \left\langle \langle x\rangle^{-2} v, v\right\rangle.$$

For $|x| > 2^M$ we have, by (3.7),

$$\kappa(x) \lesssim \varepsilon\alpha(2^k x);$$

therefore the first term on the right is essentially negligible. We obtain

$$\int \frac{\alpha(2^k|x|)}{\langle 2^k x\rangle}|\nabla v|^2 dx \lesssim \int_{D_{<M}} |\nabla v|^2 dx + \int \langle x\rangle^{-2}|v|^2 dx.$$

At the same time we have

$$0 = \left\langle \frac{\alpha(2^k|x|)}{\langle 2^k x\rangle} v, (A-\tau)v\right\rangle,$$

which after an integration by parts yields

$$\tau\int \frac{\alpha(2^k|x|)}{\langle 2^k x\rangle}|v|^2 dx \lesssim \int \frac{\alpha(2^k|x|)}{\langle 2^k x\rangle}|\nabla v|^2 dx + \int \langle x\rangle^{-2}|v|^2 dx.$$

Combining the two relations we obtain

$$\int \frac{\alpha(2^k|x|)}{\langle 2^k x\rangle}(|\nabla v|^2 + |v|^2) dx \lesssim \int_{D_{<M}} |\nabla v|^2 dx + \int \langle x\rangle^{-2}|v|^2 dx.$$

Finally we let $k \to -\infty$ to obtain

$$\int |\nabla v|^2 + |v|^2 dx \lesssim \int_{D_{<M}} |\nabla v|^2 dx + \int \langle x\rangle^{-2}|v|^2 dx < \infty$$

which shows that $v \in L^2$.

We note that (4.8) and (4.9) are not used in any quantitative way but serve only to justify the previous computations. More precisely, one can introduce in the computation a cutoff outside a large enough ball and then pass to the limit.

It remains to prove (4.6) and (4.7).

Step 9: Here we prove (4.6). We begin with the bounds on v. This is trivial for the high frequencies of v,

$$\|S_{>0}v\|_{X_0^0} \lesssim \|v\|_{\tilde{X}^0}.$$

To estimate the low frequencies, we compute

$$(\tau + i\varepsilon)S_{<0}v = S_{<0}Av - S_{<0}(A - \tau - i\varepsilon)v.$$

Writing A in the generic form

$$A = D^2a + Db + c,$$

we have

$$\begin{aligned} \|S_{<0}v\|_{X_0^0} &\lesssim \|S_{<0}D^2av\|_{X_0^0} + \|S_{<0}bv\|_{X_0^0} + \|S_{<0}cv\|_{X_0^0} + \|S_{<0}(A - \tau - i\varepsilon)v\|_{X_0^0} \\ &\lesssim \|av\|_{X^0} + \|bv\|_{X_0^0} + \|cv\|_{X_0^0} + \|(A - \tau - i\varepsilon)v\|_{(\tilde{X}^0)'} \\ &\lesssim \|v\|_{X^0} + \|\langle x\rangle^{-1}v\|_{L^2} + \|(A - \tau - i\varepsilon)v\|_{(\tilde{X}^0)'}. \end{aligned}$$

Once we control $\|v\|_{X_0^0}$, we can also obtain control of $\|\nabla v\|_{X_0^0}$ by a straightforward elliptic estimate.

Step 10: Here we prove the outgoing radiation condition (4.7) for v. This is obtained from similar outgoing radiation conditions for the functions v_n. However, v_n only converges to v in a weak sense. Hence we need to produce some uniform estimates for v_n which will survive in the limit.

$$\begin{aligned} &\|r^{-\frac{1}{2}}(D_r - \tau^{\frac{1}{2}})u\|^2_{L^2(D_j)} \\ &\lesssim \sum_{k=0}^{\infty} 2^{-\delta(k-j)^-} \Big(\|\langle r\rangle^{\frac{1}{2}}(A - \tau - i\varepsilon)u\|_{L^2(D_k)} \|\langle r\rangle^{-\frac{1}{2}}(u, \nabla u)\|_{L^2(D_k)} \\ &\qquad + \kappa_k \|r^{-\frac{1}{2}}(u, \nabla u)\|^2_{L^2(D_k)} \Big). \end{aligned} \tag{4.10}$$

In other words, there is decay when $k < j$. Applying to v_n, in the weak limit we obtain

$$\|r^{-\frac{1}{2}}(D_r - \tau^{\frac{1}{2}})v\|^2_{L^2(D_j)} \lesssim \sum_{k=0}^{\infty} 2^{-\delta(k-j)^-}\kappa_k$$

which implies (4.7).

The lower order terms in A can be treated perturbatively in (4.10). I.e. they can be included in the right hand side. Hence without any restriction in generality we assume that

$$A = D_i a^{ij} D_j.$$

We use again a positive commutator method. The multiplier is the self-adjoint operator

$$Q = b(R)\Big(\frac{x_i a^{ij}}{R}D_j - \tau^{\frac{1}{2}}\Big) + \Big(D_j \frac{a^{ij}x_i}{R} - \tau^{\frac{1}{2}}\Big)b(R), \qquad R^2 = x_i a^{ij} x_j$$

where the coefficient $b(R)$ is smooth, increasing and satisfies

$$b(R) \approx \begin{cases} 1 & R > 2^{j+2} \\ (2^{-j}R)^{\delta}, & 1 < R < 2^{j+2} \end{cases}$$

with δ a small parameter. We write

$$-2\Im\langle Qu, (A - \tau - i\varepsilon)u\rangle = \langle i[A, Q]u, u\rangle - 2\varepsilon\langle Qu, u\rangle. \tag{4.11}$$

We expect to get the main positive contribution from the first term on the right. The second term on the right on the other hand is essentially negative definite due to the fact that its symbol is negative on the characteristic set of $A - \tau$. Finally, the term on the left is bounded simply by Cauchy-Schwarz.

To shorten the notations, in the sequel we denote by E error terms of the form

$$E = DO(b(R)r^{-1}\kappa(|x|))D + O(b(R)r^{-1}\kappa(|x|)).$$

Such terms occur whenever a^{ij} is either differentiated or replaced by the identity and are easily estimated in terms of the right hand side of (4.10).

We evaluate the commutator $i[A, Q]$. A similar computation was already carried out in (3.17), which we reuse with $k = 0$, $\delta = 1$ and $\phi(r) = b(R)/R$. We obtain

$$\begin{aligned} i[A,Q] &= 4D\frac{b(R)}{R}D + 4Dx\left(\frac{b'(R)}{R^2} - \frac{b(R)}{R^3}\right)xD - 2\tau^{\frac{1}{2}}\left(\frac{b'(R)}{R}xD + Dx\frac{b'(R)}{R}\right) + E \\ &= 2D\left(2\frac{b(R)}{R} - b'(R)\right)D - 2Dx\left(2\frac{b(R)}{R^3} - \frac{b'(R)}{R^2}\right)xD \\ &\qquad + b'(R)(A-\tau) + (A-\tau)b'(R) + 2\left(Dx - \tau^{\frac{1}{2}}r\right)\frac{b'(R)}{rR}\left(xD - r\tau^{\frac{1}{2}}\right) + E. \end{aligned}$$

Our choice of b insures that the coefficient in the first two terms is positive,

$$2\frac{b(R)}{R} - b'(R) \geq 0 \qquad R > 1.$$

Hence we obtain

$$\langle i[A,Q]u,u\rangle \gtrsim 2\langle b'(R)(D_r - \tau^{\frac{1}{2}})u, (D_r - \tau^{\frac{1}{2}})u\rangle + 2\Re\langle (A - \tau - i\varepsilon)u, b'(R)u\rangle + \langle Eu, u\rangle$$

where we have inserted a harmless ε term.

It remains to evaluate the second term on the right in (4.11). We have

$$\begin{aligned} \tau^{\frac{1}{2}}Q = &-\Big(D_k\frac{x_l a^{kl}}{R} - \tau^{1/2}\Big)b(R)\Big(\frac{x_i a^{ij}}{R}D_j - \tau^{1/2}\Big) + \frac{b(R)}{2}(A-\tau) + (A-\tau)\frac{b(R)}{2} \\ &- \Big(D_i - D_l\frac{a^{lk}x_k x_i}{R^2}\Big)a^{ij}b(R)\Big(D_j - \frac{x_j x_m a^{mn}}{R^2}D_n\Big) - \frac{1}{2}(Ab(R)). \end{aligned}$$

The first and third terms are negative while the last term can be included in E. Hence we obtain

$$\tau^{\frac{1}{2}}\langle Qu, u\rangle \leq \Re\langle b(R)u, (A - \tau - i\varepsilon)u\rangle + \langle Eu, u\rangle.$$

Returning to (4.11), we insert the bounds for the two terms on the right to obtain

$$\langle b'(R)(D_r - \tau^{\frac{1}{2}})u, (D_r - \tau^{\frac{1}{2}})u\rangle \lesssim \Re\langle (A - \tau - i\varepsilon)u, (2b'(R) + \varepsilon\tau^{-\frac{1}{2}}b(R) + iQ)u\rangle + \langle Eu, u\rangle.$$

In the region D_j, we have $b' \approx 2^{-j} \approx r^{-1}$; therefore (4.10) follows.

4.2. **Proof of Theorem 1.21.** We proceed as in the nonresonant case. The bound (4.1) is replaced by

$$\|u_\varepsilon\|_X \lesssim \|f\|_{X'} + \|u_\varepsilon - u_{\varepsilon\rho}\|_{L^2_{t,x}(\mathbb{R}\times B(0,2R))}. \tag{4.12}$$

Using Plancherel as in Step 3, this is equivalent to the spatial bound

$$\|v\|_{X^0} \lesssim \|(A - \tau - i\varepsilon)v\|_{(X^0)'} + \|v - v_\rho\|_{L^2(B(0,2R))} \tag{4.13}$$

where X^0 is the fixed time counterpart of X. On the other hand the estimate that we want to prove, namely (1.28) with $u_0 = 0$, has the equivalent form

$$\|v\|_{X^0} \lesssim \|(A - \tau - i\varepsilon)v\|_{(X^0)'} \tag{4.14}$$

uniformly with respect to $\tau \in \mathbb{R}$, $\varepsilon > 0$.

For τ away from 0 we can easily bound the local average of v. We have

$$(\tau + i\varepsilon)v_\rho = (Av)_\rho - ((A - \tau - i\varepsilon)v)_\rho.$$

Therefore, by Cauchy-Schwarz,

$$\tau|v_\rho| \lesssim \|(A - \tau - i\varepsilon)v\|_{(X^0)'} + \|v\|_{L^2(B(0,2R))}.$$

Hence we are able to bound v in $\tilde{X}^0$ as well,

$$\|v\|_{\tilde{X}^0} \lesssim \|(A - \tau - i\varepsilon)v\|_{(X^0)'} + \|v\|_{L^2(B(0,2R))}, \qquad |\tau| > \tau_0. \tag{4.15}$$

Consequently, the argument for large τ rests unchanged.

Consider now the proof by contradiction.

In the case $\tau < 0$, we use the bound (2.17) instead of (2.16) for the lower order terms and show that v is an eigenvalue. However, by the maximum principle, there can be no negative eigenvalue for A.

The case $\tau = 0$ is the interesting one. Then v satisfies

$$v \in X, \qquad Av = 0, \qquad \|v - v_\rho\|_{L^2(B(0,2R))} = 1.$$

Hence v is a zero generalized eigenvalue; therefore it must be constant. But this contradicts the last relation.

Finally, due to (4.15), the case $\tau > 0$ is identical to the nonresonant case.

4.3. **Proof of Remark 1.23.** If $Av = 0$ then from

$$0 = \langle A(v - v_{D_j}), \chi_{<j}(v - v_{D_j})\rangle$$

and integration by parts, we obtain

$$\int_{D_{<j}} |\nabla v|^2\, dx \lesssim \int_{D_j} |x|^{-2}|v - v_{D_j}|^2 dx.$$

The right hand side is square summable with respect to j; therefore it decays as $j \to \infty$. We conclude that $\nabla v = 0$, and therefore v is constant.

5. Strichartz estimates

In this section we combine the smoothing estimates of the preceding sections with the long-time parametrix construction of [28] to obtain the Strichartz estimates of Theorems 1.5, 1.6, 1.13, 1.14, 1.20, 1.22. We begin by recalling the relevant results of [28]. A first result asserts that full Strichartz/local smoothing estimates hold under a smallness assumptions on κ in (1.4).

Theorem 5.1 ([28]). *Assume that the coefficients a^{ij} satisfy* (1.4) *with κ sufficiently small and $b = 0$, $c = 0$. Then for any Strichartz pairs (p_1, q_1), (p_2, q_2), the solution u to* (1.1) *satisfies*

$$\|u\|_{L_t^{p_1} L_x^{q_1} \cap X} \lesssim \|u_0\|_{L^2} + \|f\|_{L_t^{p_2'} L_x^{q_2'} + X'}. \tag{5.1}$$

For large κ, which is the case we are interested in here, it is shown that

Theorem 5.2 ([28]). *Assume that the coefficients a^{ij} satisfy* (1.4) *and $b = 0$, $c = 0$. Then there is a parametrix $K = \sum_k K_k S_k$ for $D_t + A$ with each K_k localized at frequency 2^k so that the following properties hold:*

(i) *For any Strichartz pairs (p_1, q_1) and (p_2, q_2) we have*

$$\|K_k S_k f\|_{L_t^{p_1} L_x^{q_1} \cap X_k} \lesssim \|S_k f\|_{L_t^{p_2'} L_x^{q_2'}} \tag{5.2}$$

and

$$\|Kf\|_{L_t^{p_1} L_x^{q_1} \cap X} \lesssim \|f\|_{L_t^{p_2'} L_x^{q_2'}}. \tag{5.3}$$

(ii) *For any Strichartz pair (p, q), we have*

$$\|((D_t + A)K - I)f\|_{X'} \lesssim \|f\|_{L_t^{p'} L_x^{q'}}. \tag{5.4}$$

As a consequence of this, it is also proved in [28] that

Theorem 5.3 ([28]). *Assume that the coefficients a^{ij} satisfy* (1.4) *and $b = 0$, $c = 0$. Then for any Strichartz pair (p, q), we have*

$$\|u\|_{L_t^p L_x^q} \lesssim \|u\|_{X \cap L_t^\infty L_x^2} + \|Pu\|_{X'}. \tag{5.5}$$

These are slight modifications of the results in [28] as our assumption (1.4) is not scale invariant and as such we have modified the definitions of $\tilde{X}_k$ and $A_{(k)}$ slightly. Scale invariance, however, was only assumed in [28] as a convenience, and the modifications that are necessary to adapt the proofs to the current setting are straightforward.

The above results are suitable for the high dimension $n \geq 3$ and for the low dimensional resonant case. However, for the low dimensional nonresonant case, we need a modified formulation of the last two theorems.

Theorem 5.4. *Assume that the coefficients a^{ij} satisfy* (1.4) *and $b = 0$, $c = 0$. There there is a parametrix K for $D_t + A$ with the following properties:*

(i) *For any Strichartz pairs (p_1, q_1) and (p_2, q_2) we have*

$$\|Kf\|_{L_t^{p_1} L_x^{q_1} \cap \tilde{X}} \lesssim \|f\|_{L_t^{p_2'} L_x^{q_2'}}. \tag{5.6}$$

(ii) *For any Strichartz pair (p, q),*

$$\|((D_t + A)K - I)f\|_{\tilde{X}'} \lesssim \|f\|_{L_t^{p'} L_x^{q'}}. \tag{5.7}$$

As a consequence of this, by the same duality argument as in [28], we obtain

Theorem 5.5. *Assume that the coefficients a^{ij} satisfy* (1.4) *and $b = 0$, $c = 0$. Then for any Strichartz pair (p, q), we have*

$$\|u\|_{L^p_t L^q_x} \lesssim \|u\|_{\tilde{X} \cap L^\infty_t L^2_x} + \|Pu\|_{\tilde{X}'}. \tag{5.8}$$

Proof of Theorem 5.4. The conclusion of the theorem follows by replacing the parametrix K with $(1-T)K + R$, where T and R are linear operators which are translation invariant in t and have the following properties:

$$\|(1-T)u\|_{\tilde{X}} \lesssim \|u\|_X, \tag{5.9}$$

$$\|(1-T)Kf\|_{L^{p_1}_t L^{q_1}_x} \lesssim \|f\|_{L^{p_2'}_t L^{q_2'}_x}, \tag{5.10}$$

$$\|ARf\|_{X'} + \|Rf\|_{\tilde{X} \cap L^{p_1}_t L^{q_1}_x} \lesssim \|f\|_{L^{p_2'}_t L^{q_2'}_x}, \tag{5.11}$$

$$\|ATu\|_{X'} + \|TAu\|_{X'} \lesssim \|u\|_X, \tag{5.12}$$

$$\|(T - D_t R)f\|_{\tilde{X}'} \lesssim \|f\|_{L^{p'}_t L^{q'}_x}. \tag{5.13}$$

We seek T, R of the form

$$Tu = \sum_{k=-\infty}^{0} T_k S_k u, \qquad Rf = \sum_{k=-\infty}^{0} R_k S_k f$$

where the operators T_k, R_k are localized at frequency 2^k, respectively $\geq 2^k$ and are defined by

$$T_k = u(t,0)\phi_k, \qquad R_k f = \phi_0(x) D_t^{-1} S^t_{>0} f(t,0) - \sum_{j=k}^{-1} (\phi_{j+1}(x) - \phi_j(x)) D_t^{-1} S^t_{>2j} f(t,0)$$

with $\phi_k(x) = \phi(2^k x)$ and

$$\phi(0) = 1, \qquad \text{supp } \hat{\phi} \subset \{|\xi| \in [1/2, 2]\}.$$

Notice that $Tu = u^{in}$ with u^{in} as in Section 2.2. As such, the bound (5.9) follows directly from (2.2) and (2.5). The bound (5.10) follows similarly using a Bernstein bound, Littlewood-Paley theory, and (5.2). For (5.12) we use Proposition 2.2 to replace A by $\sum A_{(k)} S_k$. Then we use the spatial localization coming from T, (2.3), and the two derivatives gain from $A_{(k)}$.

We consider now the X bounds in (5.11). For the second term in the left of (5.11), using Bernstein's inequality twice yields

$$\begin{aligned}
\left\|(\phi_{j+1}(x) - \phi_j(x)) D_t^{-1} S^t_{>2j}(S_k f)(t,0)\right\|_{X_j} &\lesssim 2^{\frac{2-n}{2}j} 2^{2j(-1+\frac{1}{p_2'}-\frac{1}{2})} \|S_k f(t,0)\|_{L^{p_2'}_t} \\
&\lesssim 2^{\frac{2-n}{2}j} 2^{2j(-1+\frac{1}{p_2'}-\frac{1}{2})} 2^{\frac{n}{q_2'}k} \|S_k f\|_{L^{p_2'}_t L^{q_2'}_x} \\
&= 2^{\frac{n}{q_2'}(k-j)} \|S_k f\|_{L^{p_2'}_t L^{q_2'}_x}.
\end{aligned}$$

The $j = 0$ term in R_k is estimated in a similar fashion. Summing with respect to $k \leq j \leq 0$ we use the off-diagonal decay to obtain

$$\|Rf\|_X \lesssim \left(\sum_{j=-\infty}^{0} \left(\sum_{k=-\infty}^{j} 2^{\frac{n}{q_2'}(k-j)} \|S_k f\|_{L_t^{p_2'} L_x^{q_2'}} \right)^2 \right)^{\frac{1}{2}}$$
$$\lesssim \left(\sum_{k=-\infty}^{0} \|S_k f\|^2_{L_t^{p_2'} L_x^{q_2'}} \right)^{\frac{1}{2}}.$$

The bound X bound for the second term in the left of (5.11) then follows from Littlewood-Paley theory. The $L_t^{p_1} L_x^{q_1}$ estimate follows from similar applications of Bernstein estimates and Littlewood-Paley theory.

For the first term in the left of (5.11), we may apply Proposition 2.2 to again replace A by $\sum A_{(k)} S_k$. As the derivatives in $A_{(k)}$ yield a 2^{2k} factor, the estimate for the first term in (5.11) follows from a very similar argument.

In order to complete the proof of (5.11), we examine the L^2 part of the $\tilde{X}$ norm. We may first apply (1.10) and (1.2) to reduce the problem to the bound

$$\| \sum_{k<0} R_k S_k f \|_{L^2_{t,x}(\{|x|\leq 1\})} \lesssim \|f\|_{L_t^{p_2'} L_x^{q_2'}}$$

in dimensions $n = 1, 2$. Here we use the fact that $\phi_{j+1}(0) - \phi_j(0) = 0$. Using this gain in a fashion similar to that from Section 2.2, we have

$$\|\phi_{j+1} - \phi_j\|_{L^2(\{|x|\leq 1\})} \lesssim 2^j.$$

Thus, arguing as above,

$$\|R_k S_k f\|_{L^2(\{|x|\leq 1\})} \lesssim \sum_{j\geq k} 2^j 2^{2j(-1+\frac{1}{p_2'}-\frac{1}{2})} 2^{\frac{n}{q_2'}k} \|S_k f\|_{L_t^{p_2'} L_x^{q_2'}}$$
$$\lesssim 2^{\frac{n}{2}k} \|S_k f\|_{L_t^{p_2'} L_x^{q_2'}}.$$

This can clearly be summed to yield the desired bound.

It remains to prove (5.13). For this we will show the bound

$$\|\langle x\rangle (T - D_t R) f\|_{L^2} \lesssim \|f\|_{L_t^{p'} L_x^{q'}}. \tag{5.14}$$

We have

$$(T - D_t R)f = -\sum_{k<0} \left(\phi_0 S^t_{\leq 0}(S_k f)(t,0) + \sum_{j=k}^{-1} (\phi_{j+1} - \phi_j) S^t_{\leq 2j}(S_k f)(t,0) \right).$$

Arguing as above we obtain

$$\|(\phi_{j+1} - \phi_j) S^t_{\leq 2j}(S_k f)(t,0)\|_{L^2} \lesssim 2^j 2^{\frac{n}{q_2'}(k-j)} \|S_k f\|_{L_t^{p_2'} L_x^{q_2'}}$$

respectively

$$\|x(\phi_{j+1} - \phi_j) S^t_{\leq 2j}(S_k f)(t,0)\|_{L^2} \lesssim 2^{\frac{n}{q_2'}(k-j)} \|S_k f\|_{L_t^{p_2'} L_x^{q_2'}}$$

and similarly for the $j = 0$ term. Then (5.14) is obtained by summation using the off-diagonal decay and Littlewood-Paley theory. □

Theorems 5.4, 5.5 will allow us to derive Theorems 1.5, 1.13, 1.20 from Theorems 1.3, 1.11, 1.19. Similarly, Theorems 5.1, 5.3 will allow us to derive Theorems 1.6, 1.14, 1.22 from Theorems 1.4, 1.12, 1.21.

5.1. **Proof of Theorems 1.13, 1.20, 1.14, 1.22.** The four proofs are almost identical, so we discuss only the first theorem. Suppose the function u solves

$$Pu = f + g, \qquad f \in \tilde{X}', \quad g \in L_t^{p_2'} L_x^{q_2'}$$

with initial data

$$u(0) = u_0.$$

We let K be the parametrix of Theorem 5.4 and denote

$$v = u - Kg.$$

Then

$$Pv = f + g - PKg, \qquad v(0) = u(0) - Kg(0).$$

Using the bounds (2.16), (5.6), and (5.7), we obtain

$$\|v(0)\|_{L^2} + \|Pv\|_{\tilde{X}'} \lesssim \|u(0)\|_{L^2} + \|f\|_{\tilde{X}'} + \|g\|_{L_t^{p_2'} L_x^{q_2'}}.$$

Then Theorem 1.11 gives

$$\|v\|_{L_t^\infty L_x^2 \cap \tilde{X}} + \|Pv\|_{\tilde{X}'} \lesssim \|u(0)\|_{L^2} + \|f\|_{\tilde{X}'} + \|g\|_{L_t^{p_2'} L_x^{q_2'}} + \|v\|_{L^2_{t,x}(A_{<2R})}.$$

Hence by (2.16) and Theorem 5.5 it follows that

$$\|v\|_{L_t^\infty L_x^2 \cap \tilde{X}} + \|v\|_{L_t^{p_1} L_x^{p_2}} \lesssim \|u(0)\|_{L^2} + \|f\|_{\tilde{X}'} + \|g\|_{L_t^{p_2'} L_x^{q_2'}} + \|v\|_{L^2_{t,x}(A_{<2R})}.$$

Using again (5.7) we return to u to obtain

$$\|u\|_{L_t^\infty L_x^2 \cap \tilde{X}} + \|u\|_{L_t^{p_1} L_x^{p_2}} \lesssim \|u(0)\|_{L^2} + \|f\|_{\tilde{X}'} + \|g\|_{L_t^{p_2'} L_x^{q_2'}} + \|u\|_{L^2_{t,x}(A_{<2R})}$$

concluding the proof of the Theorem.

5.2. **Proof of Theorem 1.5.** Suppose the function u solves

$$Pu = f + \rho g, \qquad f \in \tilde{X}_e', \quad g \in L_t^{p_2'} L_x^{q_2'}$$

with initial data

$$u(0) = u_0.$$

We consider two additional spherically symmetric cutoff functions ρ_1 and ρ_2 supported in $\{|x| > 2^M\}$ so that $\rho_2 = 1$ in the support of ρ_1 and $\rho_1 = 1$ in the support of ρ.

Let K be the parametrix of Theorem 5.4 and denote

$$v = u - \rho_1 K \rho g.$$

Then

$$Pv = f + \rho_2(\rho_1(\rho g - PK\rho g) - [P, \rho_1]K\rho g), \qquad v(0) = u(0) - \rho_1 K\rho g(0).$$

Using the bounds (2.16), (5.6), and (5.7), we obtain

$$\|v(0)\|_{L^2} + \|Pv\|_{\tilde{X}'_{e2}} \lesssim \|u(0)\|_{L^2} + \|f\|_{\tilde{X}'_e} + \|g\|_{L_t^{p'_2} L_x^{q'_2}}$$

where $\tilde{X}'_{e2}$ is similar to $\tilde{X}'_e$ but with ρ replaced by ρ_2. Then we can apply Theorem 1.3 to v to obtain

$$\|v\|_{L_t^\infty L_x^2 \cap \tilde{X}_e} + \|Pv\|_{\tilde{X}'_{e2}} \lesssim \|u(0)\|_{L^2} + \|f\|_{\tilde{X}'_e} + \|g\|_{L_t^{p'_2} L_x^{q'_2}} + \|v\|_{L^2_{t,x}(|x|\leq 2^{M+1})}.$$

We truncate v with ρ and compute

$$P\rho v = [P, \rho]v + \rho P v.$$

Then we can estimate

$$\|v\|_{L_t^\infty L_x^2} + \|\rho v\|_{\tilde{X}} + \|P(\rho v)\|_{\tilde{X}'} \lesssim \|u(0)\|_{L^2} + \|f\|_{\tilde{X}'_e} + \|g\|_{L_t^{p'_2} L_x^{q'_2}} + \|v\|_{L^2_{t,x}(|x|\leq 2^{M+1})}.$$

Hence by (2.16) and Theorem 5.5 applied to ρv, we obtain

$$\|v\|_{L_t^\infty L_x^2} + \|\rho v\|_{\tilde{X} \cap L_t^{p_1} L_x^{q_1}} \lesssim \|u(0)\|_{L^2} + \|f\|_{\tilde{X}'_e} + \|g\|_{L_t^{p'_2} L_x^{q'_2}} + \|v\|_{L^2_{t,x}(|x|\leq 2^{M+1})}.$$

Finally, we use (5.6) to return to u and obtain

$$\|u\|_{L_t^\infty L_x^2} + \|\rho u\|_{\tilde{X} \cap L_t^{p_1} L_x^{q_1}} \lesssim \|u(0)\|_{L^2} + \|f\|_{\tilde{X}'_e} + \|g\|_{L_t^{p'_2} L_x^{q'_2}} + \|u\|_{L^2_{t,x}(|x|\leq 2^{M+1})},$$

concluding the proof of the Theorem.

5.3. **Proof of Theorem 1.6.** The argument is similar to the one above. The chief difference is that we can no longer use the truncations by ρ, ρ_1, ρ_2 and instead we use the modified truncation operators such as T_ρ.

Suppose the function u solves

$$Pu = f + \rho g, \qquad f \in X'_e, \quad g \in L_t^{p'_2} L_x^{q'_2}$$

with initial data

$$u(0) = u_0.$$

We let K be the parametrix of Theorem 5.1 and denote

$$v = u - T_{\rho_1} K \rho g$$

Then we can write

$$Pv = f + T_{\rho_2}(T_{\rho_1}(\rho g - PK\rho g) - [P, T_{\rho_1}]K\rho g), \qquad v(0) = u(0) - T_{\rho_1} K \rho g(0).$$

Here we compute the commutator

$$[A, T_{\rho_1}]w = A\rho_1(w - w_{\rho_1}) - \rho_1 A(w - w_{\rho_1}) - (1-\rho)(Aw)_{\rho_1} = [A, \rho_1](w - w_{\rho_1}) - (1-\rho)(Aw)_{\rho_1}.$$

Also we have

$$(Aw)_{\rho_1} = c_\rho \int (1 - \rho_1) A(w - w_{\rho_1}) dx = -c_\rho \int (w - w_{\rho_1}) A\rho_1 dx.$$

Then using the bounds (2.17), (5.3), and (5.4), we obtain

$$\|v(0)\|_{L^2} + \|Pv\|_{X'_{e2}} \lesssim \|u(0)\|_{L^2} + \|f\|_{X'_e} + \|g\|_{L_t^{p'_2} L_x^{q'_2}}.$$

By Theorem 1.3 for v we get

$$\|v\|_{L_t^\infty L_x^2 \cap X_e} + \|Pv\|_{X'_{e2}} \lesssim \|u(0)\|_{L^2} + \|f\|_{X'_e} + \|g\|_{L_t^{p'_2} L_x^{q'_2}} + \|(1-\rho)(v - v_\rho)\|_{L^2_{t,x}}.$$

We truncate v with T_ρ and compute as above the commutator $[P, T_\rho]$. Then we estimate
$$\|v\|_{L_t^\infty L_x^2} + \|T_\rho v\|_X + \|P(T_\rho v)\|_{X'} \lesssim \|u(0)\|_{L^2} + \|f\|_{X_e'} + \|g\|_{L_t^{p_2'} L_x^{q_2'}} + \|(1-\rho)(v-v_\rho)\|_{L^2_{t,x}}.$$
Hence by (2.17) and Theorem 5.3 applied to $T_\rho v$, we obtain
$$\|v\|_{L_t^\infty L_x^2} + \|T_\rho v\|_{X \cap L_t^{p_1} L_x^{q_1}} \lesssim \|u(0)\|_{L^2} + \|f\|_{X_e'} + \|g\|_{L_t^{p_2'} L_x^{q_2'}} + \|(1-\rho)(v-v_\rho)\|_{L^2_{t,x}}.$$
Finally, we use (5.3) to return to u and obtain
$$\|u\|_{L_t^\infty L_x^2} + \|T_\rho u\|_{X \cap L_t^{p_1} L_x^{q_1}} \lesssim \|u(0)\|_{L^2} + \|f\|_{X_e'} + \|g\|_{L_t^{p_2'} L_x^{q_2'}} + \|(1-\rho)(u-u_\rho)\|_{L^2_{t,x}},$$
concluding the proof of the Theorem.

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