Culture and Commutativity

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Abstract

The extent to which people can infer new mathematical concepts in the absence of cultural support is not clear. We test such learning with a simple math concept: additive commutativity. Experimental work with children in industrialized cultures suggests that cultural support is necessary, since children take time to learn commutativity and ultimately show signs of knowing it after entering school. However, children are at a disadvantage in learning because they are not yet cognitively mature. Moreover, they have only had a short time to experience the world and possibly learn principles like commutativity on their own. Unschooled adults, on the other hand, may be in a better position to have inferred commutativity on their own. We test indigneous Amazonians with variable levels of math cultural supports, and find that those with low cultural supports do *not* show signs of knowing additive commutativity.

Keywords: commutativity; cultural supports; arithmetic; education

Introduction

Most adults in WEIRD cultures (Henrich, Heine, & Norenzayan, 2010) understand principles that characterize our knowledge of arithmetic, such as commutativity. Formally, additive commutativity holds that natural numbers a and b always obey $a+b=b+a$. There is no firm consensus on when commutativity is acquired developmentally, but WEIRD children have been shown to possess an abstract understanding of the principle as early as 3 years old (Sophian, Harley, & Manos Martin, 1995), and a more complete understanding by around third grade (Haider et al., 2014). One question that remains is whether commutativity is learned only with strong cultural supports like formal education, or whether it could be deduced by learners autonomously.

We define autonomous learning as acquisition of knowledge through individual experience rather than cultural support like school and trade. For instance, commutativity might be learned based on simple observations of physical interactions in the world. Results from Petitto and Ginsburg (1982) are consistent with the possibility that commutativity may be autonomously inferred. The researchers recruited Dioula adults with no formal schooling. Participants were given a battery of math tasks, and succeeded on symbolic additive commutativity. However, this study had some limitations. Besides the fact participants were only asked one question on this concept, they were selected from a fairly uniform group (17 tailors and 3 cloth merchants), with a uniformly high level of market integration. Results from this population would not necessarily generalize to unschooled adults with low market exposure.

An alternative possibility is that substantial cultural math support is required to learn commutativity and related principles. Cultural support is social and could potentially include: formal schooling, specific language-based skills (e.g. the meanings of number words, exact arithmetic), and occupational experience (e.g. selling goods). These experiences have been individually implicated in general math achievement (Nguyen et al., 2016; Posner & Baroody, 1979; Boni, Jara-Ettinger, Sackstein, & Piantadosi, under review), and in commutativity mastery specifically (Baroody & Gannon, 1984; Baroody, 1987).

While these are helpful for commutativity acquisition, it is still unclear if they are necessary. We explore this question in the Tsimane', a farmer-forager population who reside in the Bolivian Amazon. Similar to the participants in Petitto and Ginsburg (1982), our participants are adults with little to no schooling. Unlike Petitto and Ginsburg (1982), we sampled participants from 9 different villages varying in their level of market integration, allowing us to test market exposure as a novel predictor of commutativity knowledge. Other cultural predictors we include are formal schooling, exact arithmetic knowledge, and number knowledge. Including formal schooling can help determine the relevance of a heavily structured learning environment. Arithmetic and number knowledge are specific mathematical abilities that may be necessary to scaffold commutativity knowledge. We also include age as a proxy for amount of time to autonomously experience quantities.

Our participant variability is an asset. Tsimane' living in separate villages have varying amounts of market exposure. They have attended different schools. Individual schools may emphasize dissimilar mathematical skills. Additionally, adult Tsimane's ages don't cleanly relate to their education. We are thus able to disentangle the effects of different predictors on commutativity knowledge. Ultimately, if it is possible to know commutativity without cultural support, our participants with limited math-related cultural support will succeed on commutativity, replicating the Dioula results. Additionally, age—a measure of possible time in which to experience quantities in the world—may be a positive predictor of commutativity knowledge. On the other hand, if commutativity

In J. Culbertson, A. Perfors, H. Rabagliati & V. Ramenzoni (Eds.), *Proceedings of the 44th Annual Conference of the Cognitive Science Society*.

requires support like substantial formal schooling or occupational experience, only participants with these experiences will succeed on commutativity.

Method

We worked with the *Centro Boliviano de Investigacion y de Desarrollo Socio Integral* (CBIDSI), a local research center. CBIDSI recruited participants, coordinated logistics, provided us with native translators, and served as experts on Tsimane' culture.

In our study, we: collected demographic information, gave participants general mathematical tasks, and subsequently administered our main commutativity experiment.

Participants

We recruited 45 Tsimane' participants, aged 18-66 years old (female=23, male=22), from 9 different villages. These villages are all located in Beni, a department in the Bolivian Amazon. We specifically recruited Tsimane' with low levels of formal schooling (0-5 years).

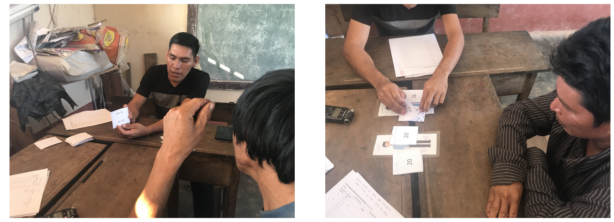
Prior to Experiment: Measuring Cultural Supports We first administered the Give-A-Number task, a test of number knowledge (Wynn, 1990). We placed a pile of 10 tokens on a table, and asked participants to grab different quantities of tokens, from 1 to 8. This was administered in number list order, and then in random order. Each participant's "knower-level" was then determined using a Bayesian inference model (Lee & Sarnecka, 2010). According to this model, subset-knowers (e.g. 1-,2-,3- and 4-knowers) are participants who only know a limited subset of number meanings, and CP-knowers are participants who succeed on the Give-N task for arbitrarily large "N."

Secondly, we measured arithmetic competence by giving participants twelve different addition problems. Problems were given visually and verbally, translated into Tsimane' by native Tsimane' speakers. Participants stated their answer verbally.

The main demographic measures relevant to this study we collected were years of formal education, village, and age. We considered age a proxy for amount of time in the world in which to possibly experience and manipulate natural quantities. Additionally, we obtained village-level market integration measurements from R. Godoy, CBIDSI, and Luz (2012). We selected travel time to nearest market in particular, as a good measure of market integration. We reasoned that that temporal proximity to a market (regardless of distance) would seem to make more frequent trips and participation in the market economy possible. Travel time from village to market is in hours.

Commutativity Experiment

The general design of our commutativity tasks was based on Baroody and Gannon (1984); Wilkins, Baroody, and Tiilikainen (2001). We administered two kinds of commutativity tasks: "symbolic" (first) and "word problem" (second).



(a) Symbolic task

(b) Word problem task

Figure 1: Setup for the symbolic and word problem tasks

Both problem sets included commuted experimental trials, and non-commuted control trials.

To start, we administered the Symbolic task (Figure 1a). We used 28 blank index cards. Each index card had a pair of sums on them, one on top of the other.

To start, participants were first given two practice trials (not included in our analyses): [2 + 1 ; 2 + 8] and [11 + 13 ; 11 + 11]. For each practice trial, participants were instructed to compute the result of the first pair, then the result of the second pair. Then, they were asked whether the pairs resulted in the same answer or a different answer. Participants were given feedback if they gave the wrong answer. Both of our practice trials had an answer of "different" because we did not want to introduce a "same" bias, which could lead us to over-estimate commutativity knowledge. After practice trials, participants were told they would be given further pairs, but quickly, and that they just had to say whether the pairs resulted in the same answer, a different answer, or it was not possible to know either way. The total time each card was presented was approximately 4 seconds.

In our analyses, if participants ever said an answer that was not exactly the right answer ("same" or "different" according to the problem type), such as "not possible to know" or a non-sequitur, these answers were counted as wrong. The pairs of sums (Table 1) were presented in random order. An example experimental trial would be: $12+14....14+12$. Different, the same, or not possible to know?" The trials were counterbalanced, so that on one trial, the question was "different, the same, or not possible to know" and on the next the question was "the same, different, or not possible to know." Participants were not given feedback on experimental trials.

The Word Problem task followed the Symbolic task, and was meant to present problems in a slightly more naturalistic context. See Table 2 for the specific problems used, and Figure 1b for the physical setup. For each sum shown, we had two index cards, each representing one addend with an Arabic numeral. No plus signs were used in this task. Stapled underneath each card was another card, containing a dot cloud with a cardinality corresponding to the Arabic numeral above it. In the task, as we verbalized each number, we would open the index card to reveal the dot cloud underneath, in an effort to reinforce the number as a concrete quantity.

The story given to participants was that there are two men,

Table 1: Problems used for the symbolic task: 12 commuted, 16 non-commuted.

Commuted Pairs (Correct Answer: "Same")	**Non-Commuted Pairs** (Correct Answer: "Different")	**Non-Commuted Pairs** (Correct Answer: "Same")
12 + 14 14 + 12	10 + 12 20 + 20	13 + 13 13 + 13
13 + 16 16 + 13	12 + 12 15 + 12	16 + 11 16 + 11
14 + 16 16 + 14	12 + 15 12 + 20	11 + 18 11 + 18
15 + 13 13 + 15	13 + 11 10 + 11	14 + 14 14 + 14
16 + 12 12 + 16	13 + 15 13 + 10	
18 + 12 12 + 18	17 + 12 13 + 12	
12 + 15 15 + 12	12 + 18 20 + 18	
12 + 17 17 + 12	14 + 12 10 + 12	
13 + 12 12 + 13	14 + 15 11 + 15	
13 + 14 14 + 13	15 + 14 15 + 12	
15 + 14 14 + 15	15 + 15 11 + 15	
17 + 13 13 + 17	16 + 13 11 + 11	

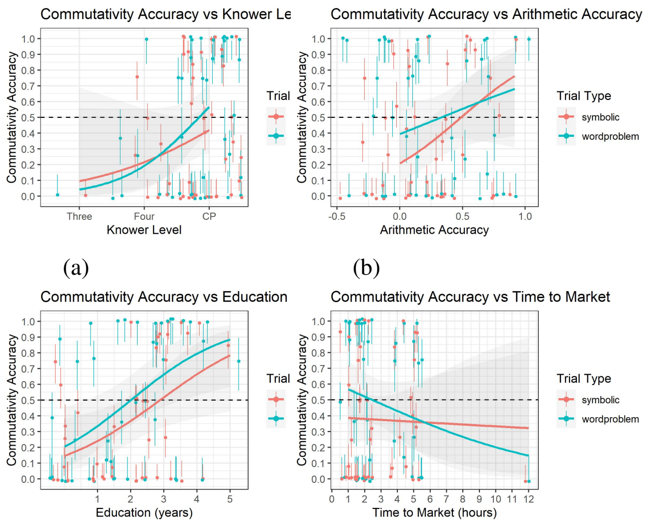
Table 2: Problems used for the word problem task: 8 commuted, 4 non-commuted.

Commuted Pairs (Correct Answer: "Same")	**Non-Commuted Pairs** (Correct Answer: "Different")
19 + 17 17 + 19	14 + 19 14 + 14
15 + 18 18 + 15	22 + 22 27 + 27
17 + 14 14 + 17	20 + 20 26 + 26
17 + 25 25 + 17	13 + 19 13 + 13
16 + 19 19 + 16	
18 + 14 14 + 18	
15 + 17 17 + 15	
23 + 18 18 + 23	

Juan and Pedro, represented by caricatures placed in front of participants. Juan was always on the participants' left, and Pedro was always on the participants' right. Each man was holding pebbles. One example of a problem given would be: "Juan has 19 pebbles in one hand, and 17 pebbles in the other hand. Pedro has 17 pebbles in one hand, and 19 pebbles in the other hand. Who has more? Does Juan have more? Does Pedro have more? Do they have the same amount? Or is it not possible to know?" As above, any answer that was not the correct one (whether it be an incorrect indication of cardinality or "not possible to know") was coded as wrong. As in the symbolic task, in the word problem task, participants were given a short time to answer each trial, so as to prevent outright computation. Each trial took about 8 seconds to present to participants. Approximate total testing time for the symbolic and word problem trials was 5-10 minutes. Trials were presented in random order.

Results

Overall, participants were more successful on non-commuted trials (mean accuracy: 76%) than on commuted trials (mean accuracy: 42%). We believe this may be because of a bias to answer "different." Some participants may have interpreted our question of "Are the *sums* the same?" as "Are the *ad-*



(a) (b)

(c) (d)

Figure 2: In these plots, dots represents individual participants, data is color-coded by trial type, and logistic curves are fitted to the data. The y-axes on each of these panels represents commutativity accuracy. Each x-axis represents a different demographic predictor. Qualitatively, all predictors have a positive relationship with commutativity accuracy, with the exception of time to market, which has no relationship with commutativity accuracy in the regression.

dends the same?" Also, the two practice trials we gave them before the task had an answer of "different." Note that, although there seems to be an overall bias to say "different," participants say "different" *more* (73%) for non-commuted trials (which mostly have a correct answer of "different"), and less (56%) for commuted trials (which always have a correct answer of "same"), suggesting that the bias comes on top of genuinely correct answers. In any case, participants who answered "same" on commuted trials might reasonably be said to know commutativity, because they avoided the prevalent "different" bias and answered correctly.

In the following results, we show how participants with low levels of each of the cultural supports we look at also have low accuracy in commutativity. In contrast, the non-cultural predictor of age (as a proxy for experience with quantities in the world) is *not* positively associated with commutativity accuracy—and in fact, is negatively associated.

Commutativity accuracy according to different predictors

Fig. 2 shows the relationship between different predictors (x-axis) and commutativity accuracy (y-axis). The horizontal dashed line represents chance performance on commutativity.[1] Each sub-plot shows a different predictor. Each point represents an individual person and points are jittered so as to prevent overlap. Error bars are binomial confidence intervals, set at 50% confidence to minimize visual cluttering. Two logistic curves are fit to the data, one per trial type (symbolic or word problem). The shadow around the curves represents the standard error (95% confidence) for each condition, respectively. All participants completed symbolic (red) and word (teal) problems.

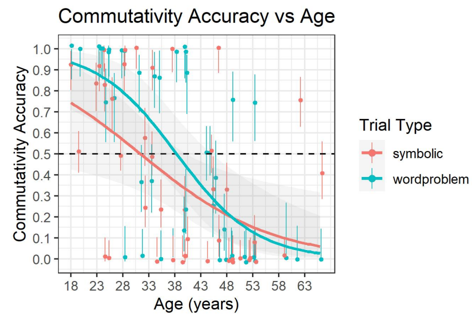


Figure 3: Commutativity accuracy vs. age. Each dot represents an individual, data is color-coded by trial type, and logistic curves are fitted to the data. As age increases, commutativity accuracy decreases.

Overall, in panels 2a, 2b, and 2c, there are increasing trends: people improve at commutativity with higher knower-level, arithmetic accuracy, and education, respectively. More interestingly, people are substantially below chance at the left side of the plots, when they have low values of these cultural support predictors. By and large, three- and four- knowers show low accuracy on commutativity (Panel 2a). People with low arithmetic accuracy tend to show low accuracy on commutativity as well (Panel 2b). People with low years of formal education also show low accuracy on commutativity (Panel 2c). Towards the right end of the panels, people approach ceiling on the commutativity task, suggesting the Tsimane' do learn commutativity under the right conditions (with sufficient cultural supports).

Panel 2d shows an overall flat to slightly decreasing trend: people get worse at commutativity the farther from the market they live. The individual furthest away from the market is below chance on commutativity, and there is a somewhat even split (leaning below) of people above and below chance on commutativity at distances closer to the market. This trend, however, is not statistically reliable (see below).

One possibility for commutativity acquisition we outlined in the introduction is that amount of time to experience quantities in the world might translate to commutativity understanding. One way to measure this is by using a proxy for exposure to sets in the real world: age. Contrary to this hypothesis, in Fig 3, younger participants (x-axis) have higher commutativity accuracy (y-axis) than older participants, in both the symbolic and word problem tasks. The negative relationship between age and commutativity accuracy is likely because the oldest Tsimane' have had the least exposure to school. Kempf and Kempf (2018) chronicle the gradual intro-

[1] We choose a chance line of 50%, representing the expected performance if participants guessed randomly.

duction of formal schooling to the Tsimane', starting in the 1960s and gradually increasing until today (which explains why younger Tsimane', born more recently, are exposed to more formal education than their predecessors were).

Overall, Figures 2 and 3 suggest participants may perform slightly better on word problems than on symbolic problems, although the trend is not entirely clear, and in some cases participants perform better on symbolic problems.

Table 3 shows a logistic regression that statistically evaluates the trends in Figures 2 and 3, and allows us to test each predictor while controlling for the others. The regression predicted accuracy on commutativity problems according to the predictors seen in Table 3.

The regression's intercept shows that subset-knowers are below 50% commutativity accuracy ($\beta = -2.04, t = -7.87, p < 0.01$). There is also a reliable, positive association between CP-knower status ($\beta = 0.61, t = 2.68, p < 0.05$) and commutativity accuracy. This shows that when the other predictors are taken into account, number knowledge is uniquely important to commutativity knowledge. Arithmetic accuracy is also uniquely important to commutativity knowledge: it is a reliable predictor even when education (and the other predictors in that table) are taken into account ($\beta = 0.86, t = 2.42, p < 0.05$). Furthermore, there is a reliable effect of formal education on commutativity accuracy ($\beta = 0.26, t = 3.44, p < 0.01$). Overall each of these predictors is uniquely predictive even when the other predictors are taken into account in the regression. These findings are consistent with the theory that commutativity cannot be learned with low cultural support.

While we expected that inhabitants of villages close to a market town might succeed on commutativity, we found that the time to nearest market coefficient has no reliable association with the commutativity accuracy outcome ($\beta = 0.03, t = 0.78, p > 0.05$). If time to market is actually a valid proxy for market integration, this implies that the extent to which Tsimane' participate in the market is not related to their knowledge of commutativity. This is somewhat surprising, given that participation in the market requires mathematical transactions during sales and purchases. It is also possible that time to market is simply a noisy measure, or a measure whose variance is already accounted for by other factors.

The trial type coefficient shows that people perform reliably better on word problems than on symbolic problems ($\beta = 0.63, t = 3.81, p < 0.01$). Symbolic problems require knowledge of the concept of formal sums and/or the plus sign notation. Even if someone has been taught these elements, it may be more difficult to understand math concepts in this abstract format as opposed to a word problem format tethered to a concrete scenario (and not using any formal notation).

Finally, the regression shows that age has a reliable *negative* association with commutativity accuracy ($\beta = -0.97, t = -9.55, p < 0.01$). This suggests that more absolute time to theoretically experience quantities and quantity manipulations in the world does not enhance commutativity knowledge. As mentioned above, this is likely due to the fact that older Tsimane' tend to have less schooling. This further highlights the importance of formal education for commutativity.

Discussion

These results inform how we should think about psychological representations of mathematical concepts like commutativity. In principle, the way humans structure our mathematical systems could provide a model of how concepts operate in the human mind. For example, in some versions of Peano arithmetic, commutativity is axiomatic. Our results contribute to literature showing that, in contrast to these systems, people's natural mental concepts of arithmetic are not commutative, at least not automatically so. In Haider et al. (2014), school children only exhibited a mature understanding of commutativity by the time they were in third grade (Haider et al., 2014). In theory, the reason they did not understand commutativity earlier could be that they simply had not lived and experienced the world long enough to autonomously infer it. However in our current research, even by adulthood, commutativity is *not* autonomously acquired. This strengthens the argument that it may not be a natural, axiomatic ability to acquire. Perhaps this is not so surprising: commutativity does not hold for subtraction and division; moreover, in the real world, order often matters (Rips, Bloomfield, & Asmuth, 2008). Picture, for instance, forming a pile of fragile and heavy objects. If you start with the fragile objects, you will form a very different pile than if you started with the heavy items. Interestingly, these findings coincide with human intuitions on the relevance of commutativity. In Rips and Thompson (2014), college students were asked to rate mathematical properties by how essential they are to number systems (e.g. natural numbers, integers). The mean rated importance given to commutativity was a moderate 4.14 on a scale of 1-8 (1: "need not be a part of any number system" ; 8: "must be part of every number system")[2]. All this highlights the question: how *is* commutativity acquired? One possibility is that people may discover the commutativity principle through repeated exposure to addition problems, in which they repeatedly notice that $a + b = b + a$ (Baroody, Wilkins, & Tiilikainen, 2003; Baroody & Gannon, 1984).

The fact that commutativity is not axiomatic does not preclude other mathematical abilities from being axiomatic. In the study where commutativity received moderate scores for its importance to number systems (Rips & Thompson, 2014), in general, students weighed mathematical relations (e.g. associativity, trichotomy, etc.) and operations (e.g. addition, subtraction, etc.) as more relevant to defining number systems than the presence or absence of specific elements (e.g. presence of a first element, presence of successors, etc.). The authors argue that this favors a structuralist theory, whereby complex mathematical concepts are derived from a pre-existing scaffold of more primitive concepts.

[2]Note that commutativity was rated 4.83 upon repetition in a different group from the same participant pool.

Table 3: Output for logistic regression: Commutativity problem accuracy $\sim$ Education + Arithmetic Accuracy + Knower Level (CP or Subset) + Time to Market + Age + Trial Type (Word Problem or Symbolic). The predictors are continuous, except for Knower Level and Trial Type, which are dummy-coded. Age is z-scored, because we tested a limited range of ages: adults only.

	Estimate	95% CI lower	95% CI upper	SE	t	p
Intercept	-2.04	-2.56	-1.54	0.26	-7.87	0.00
Education	0.26	0.11	0.41	0.08	3.44	0.00
Arithmetic Accuracy	0.86	0.16	1.56	0.35	2.42	0.02
Knower Level: CP	0.61	0.17	1.07	0.23	2.68	0.01
Time to Market	0.03	-0.05	0.11	0.04	0.78	0.43
Age (z-scored)	-0.97	-1.17	-0.77	0.10	-9.55	0.00
Trial Type: Word Problem	0.63	0.31	0.96	0.17	3.81	0.00

Our study also allowed us to examine how commutativity knowledge relates to market integration. In prior literature, Dioula adults with no schooling were accurate on commutativity (Petitto & Ginsburg, 1982). We believed this to be because these participants had uniformly high market experience. Interestingly, in our population, market experience—low or high—made no difference in commutativity scores. This difference may be because Dioula participants have an overall higher market integration than Tsimane' participants. Also, our village-level measure of market integration (travel time to nearest market) may not be fine-grained enough, a limitation of our study.

Another limitation of our research is that we define commutativity in a narrow sense: success on abstract symbolic and concrete word problems. Our symbolic problems rely on knowledge of verbal number words and/or written numerals. Our word problems include dot cloud representations as an additional cue, but these representations are still pictorial. It is possible, therefore, that these task demands led us to underestimate the commutativity knowledge of some participants. A task using concrete objects may be better calibrated to remote Tsimane' culture.

Conclusion

We tested whether it is possible to infer commutativity—a seemingly simple math skill—purely by experiencing quantities in the world, or whether cultural support is necessary. By and large, our participants with low amounts of cultural support (exact number knowledge, arithmetic, and formal schooling) did not succeed on our commutativity tasks. Moreover, age (our intended proxy for experience with quantities in the world) did *not* confer an advantage in commutativity understanding. This suggests that commutativity is not an axiomatic psychological concept.

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