UNIVERSITY OF CALIFORNIA,
IRVINE

Student Participation and Agency in Mathematical Problem Posing

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Education

by

Priyanka Agarwal

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2019

DEDICATION

To my grandmothers and mother, for always fighting for what is right, their identity, and my freedom; and to my daughters, for helping me every day become a better person.

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ACKNOWLEDGMENTS

I am forever indebted to my village – the UCI community, mentors, family, and friends – without whom my Ph.D. journey would have been impossible. First and foremost, I would like to express my deepest appreciation to my committee of inspiring women scholars Rossella Santagata, Tesha Sengupta-Irving, Hosun Kang, Jacque Eccles, and Alessandra Pantano. Their unrelenting support and feedback on my dissertation proposal, study instruments, research analyses, and written drafts was crucial for my professional growth.

I am extremely grateful to Tesha for her unwavering belief in me and my work that enabled me to always put my best foot forward and has been the biggest motivator throughout my dissertation work. She has always been present to offer her constructive critique and advice on my research whether it was when writing together at her dining table, discussing analyses for a paper, or brainstorming ideas for dissertation research over many intra-state conference calls. Her invaluable insights about the interdisciplinary work we do as educational researchers and about our political and ethical responsibilities towards the students, teachers, and community we serve have shaped me into the person I am today.

The completion of my dissertation would not have been possible without the support and nurturing of my advisor Rossella. I cannot thank her enough for her enthusiasm and optimism towards my research. Her "big picture" insights on my multiple drafts have always pushed me to critically think about the implications of my work and have kept my research grounded in the immediate educational problems that it is meant to address. Her kind mentorship, her practical and professional advice, and the celebratory year-end parties have served as a reminder that people and the joy of working together always come first. I will always be indebted to her for her selfless sponsorship of me and my work.

Hosun, Jacque, and Alessandra have been unfailingly generous with their time and support throughout my dissertation work. Hosun's tireless advise as a committee member, as a course professor, and as a research lab mentor has taught me how to ask difficult research questions, be theoretically grounded, and strengthen the connections between theory, methods, and analyses. Jacque's ingenious suggestions and comments on the early drafts of my Ph.D. milestones, course papers, dissertation proposal, and job market application have been priceless. I still vividly remember when Hosun and Jacque had stopped by to patiently listen to my first year poster presentation and have since been invaluable contributors in my progress and dissertation completion. Alessandra's unparalleled support towards my research, her time as we brainstormed the design of the mathematics tasks over coffee, and her kind invitation to visit UCI Math Circle for initial piloting has been invaluable. Together my committee has inspired me to reach for the scholarly rigor they expect from the best, and motivated my desire to deliver that expectation.

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Special thanks to Elizabeth (Beth) van Es and Victoria Hand for their unrelenting sponsorship of my work, for always looking out for growth opportunities for me, and for inviting me to contribute towards the Co-ATTEND project. Beth's belief in my third-year milestone paper helped me persevere in getting it published, and for this, I will be forever grateful. Many thanks also to Thurston (Thad) Domina for his unparalleled support and advice on my second-year milestone paper and for letting me be a part of the EQUAL project. Collaborating with Elizabeth Mendoza, Carlos Sandoval, Miguel Abad, Jennifer Renick, Andrew Penner, Paul Hanselman, Marcela Reyes, Ryan Lewis, NaYoung Hwang, and Andrew McEachin through these projects have afforded me tremendous opportunities for scholarly growth.

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I thank my husband Anand for holding down the home fort as I increasingly devoted my time the last two months towards writing. Without his immense support, I would not have been able to take the tiny steps that have culminated towards finishing my doctorate. I thank my mother for her love, care, sacrifices, and for inspiring me to become who I am now. I thank my father for instilling in me community values and for showing me how to embrace forgiveness and humility—practices that are not only important for self-perseverance but for being able to contribute in and to the world.

I cannot close the acknowledgment without thanking my daughters, Anya and Mishika, who teach me daily what it means to love someone unconditionally in a world full of uncertainties. They have opened my heart in a way that never was. They are the reason why I write.

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EDUCATION

Ph.D., Education, Specialization: Learning, Teaching and Educational Improvement 2019
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M.S., Mathematics, Purdue University 2010

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PROFESSIONAL EXPERIENCE

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Business Analyst, GENPACT Analytics/ General Electric (GE)
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PUBLICATIONS

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Domina, T., McEachin, A., Hanselman, P., **Agarwal, P.**, Hwang, N., & Lewis, R. (2019). Beyond tracking and detracking: The dimensions of organizational differentiation in schools. *Sociology of Education*, 92(3), 293-322. https://doi.org/10.1177/0038040719851879

Sengupta-Irving, T. & **Agarwal, P.** (2017). Conceptualizing perseverance in problem solving as collective enterprise. *Mathematical Thinking and Learning*, 19(2), 115-138. https://doi.org/10.1080/10986065.2017.1295417

Domina, T., Lewis, R., **Agarwal, P.**, & Hanselman, P. (2015). Professional sense-makers: Instructional specialists in contemporary schooling. *Educational Researcher*, 44(6), 359-364. https://doi.org/10.3102/0013189X15601644

Conference Proceedings:

Agarwal, P. & Sengupta-Irving, T. (2018). Power in the digital age: A critical revision to productive disciplinary engagement (PDE). In Kay, J. and Luckin, R. (Eds.), *Rethinking Learning in the Digital Age: Making the Learning Sciences Count, Proceedings of the 13th International Conference of the Learning Sciences (ICLS) 2018,* Volume 1 (pp. 832-838). London, UK: International Society of the Learning Sciences.

K. K. Ng, **P. Agarwal**, N. Mullen, D. Du, and I. Pollak (2011). Comparison of Several Covariance Matrix Estimators for Portfolio Optimization. *Proceedings of IEEE International Conference on Acoustics, Speech and Signal Processing*, Prague, Czech Republic.

Manuscripts in Preparation:

Agarwal, P. (under review). Organizing for collective agency and risk-taking in mathematical problem posing.

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Agarwal, P. (in preparation). Characterizing shifts in student participation in mathematical problem-posing.

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Agarwal, P. (2019, February). Mathematical doubts and problem posing as a way to understand epistemic needs of students. Invited paper presented at College Preparatory Mathematics (CPM) Teacher Conference, San Francisco, CA.

Agarwal, P. (2018, December). Examining the role of mathematical problem posing for underserved youth’s learning. Invited colloquium presented at San Francisco State University, San Francisco, CA.

Agarwal, P. (2018, November). Problem posing pedagogy: processes of students’ mathematical doubts and wonderings. Invited paper presented at the California Educational Research Association (CERA) annual conference, Anaheim, CA.

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Agarwal, P. (2019, November). Dimensions of student doubts and mathematical problem posing. Brief research report to be presented at the annual conference of Psychology of Mathematics Education-North American Chapter, St. Louis, MO.

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Hand, V., van Es, E. A., Freeman, Q., **Agarwal, P.** (2018, June). Do you see what I see? Mathematics teacher noticing for equity. Poster presented at the 13th International Conference of Learning Sciences (ICLS), London Festival of Learning, London, UK.

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Domina, T., McEachin, A., **Agarwal, P.**, Hanselman, P., Hwang, N., & Lewis, R. (2016, April). Beyond tracking and detracking: The dimensions of organizational differentiation in schools. Paper presented at American Educational Research Association (AERA) annual meeting, Chicago, IL.

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Sengupta-Irving, T. & **Agarwal, P.** (2015, April). Productive struggle and the development of relational equity in a mathematics learning community. Paper presented at American Educational Research Association (AERA) annual meeting, Chicago, IL.

RESEARCH EXPERIENCE

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University of California, School of Education, Irvine, CA 2014

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Guest Lecture
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ABSTRACT OF THE DISSERTATION

Student Participation and Agency in Mathematical Problem-Posing

By

Priyanka Agarwal

Doctor of Philosophy in Education

University of California, Irvine, 2019

Associate Professor Rossella Santagata, Chair

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Problem-posing practices are considered important for nurturing students' inquiry, learning, and agency in mathematics. In this dissertation study[1], drawing on socially situated frameworks of learning, I designed and investigated a math curricular unit, centered on problem-posing (as against just problem-solving), in low-tracked eighth-grade classes in a school serving predominantly working-class Latinx neighborhoods. I partnered with a mathematics teacher to co-design and implement problem-posing-based lessons with the goals to (1) understand how students gain entry to the practice of problem-posing, (2) investigate how learning processes unfold over time in interaction with peers, materials, and the teacher, and (3) shed light on the ways in which students negotiate their agency and social risks of posing a math problem. Research data include video-based observations, student written-work, reflections, and classroom artifacts collected in two different settings of task-based paired interviews and a classroom-based teaching experiment.

[1] This dissertation study is supported by a California Education Research Association Research Partnership Grant, a College Preparatory Mathematics Dissertation Fellowship, and a University of California President's Dissertation Year Fellowship.

The study findings reach beyond the past focus on the cognitive aspects of problem-posing and its linkages with individual creativity and ability. Instead, they offer dimensions of mathematical doubts as a novel characterization to support expansive and agentic forms of student problem-posing (Chapter 1), an understanding of the sociocultural processes of productive posing (Chapter 2), and conceptualization of collective agency and risk-taking in problem-posing (Chapter 3). Together, the study findings elaborate on our understanding of how students become engaged problem-posers and the central role that student doubts and collective action play in supporting productive forms of posing. I also discuss how problem-posing practice is uniquely positioned to amplify capabilities, identities, and epistemic agency of students of color who get disproportionately sorted in remedial courses. The findings provide preliminary ideas of problem-posing pedagogy's potential to challenge the deeply-rooted deficit discourse of race, poverty, and ability in education. The findings have implications for the design and analysis of inquiry-oriented learning environments in both formal and informal settings.

INTRODUCTION

California is facing one of the biggest learning crises of our times—Latinx children now make up more than half of California's K-12 public school population (California Department of Education, 2019), yet continue to face large opportunity gaps in mathematics education including higher enrollment in segregated remedial classes and limited access to high-quality curriculum as compared to their white and Asian peers (e.g., Attewell & Domina, 2008; Dossey, McCrone, & Halvorsen, 2016; Fuller et al., 2019; Kalogrides & Loeb, 2013; Noguera, Pierce, & Ahram, 2015; Vasquez Heilig, Brown, & Brown, 2012; The Education Trust-West, 2017). This is even more problematic in under-resourced schools where structural practices and beliefs about how low-income students of color learn, shape a culture of pedagogy ("pedagogy of poverty") where learners have little or no agency to take up their curiosities for learning, and instead go through disconnected basic-skill development (e.g., Anyon, 1980, 1997; Donaldson, LeChasseur, & Mayer, 2016; McKinney & Frazier, 2008; Means & Knapp, 1991; Oakes, 1987, 1990; Reardon & Owens, 2014). Such pedagogical inequities ultimately result in achievement gaps as shown by lower rates of public-university admissions (only one-third of Latinx met the A-G requirements in mathematics in 2016) and college graduation (only 11% graduated with a college degree in 2015) (Mickelson, 2015; National Center for Education Statistics, 2017).

Current literature in mathematics education rebuts the "pedagogy of poverty" (Haberman, 1991; Ladson-Billings, 2014; Talbert-Johnson, 2004). Decades of research has found the facilitation of student authority, questioning, inquiry, and disciplinary discourse as features of effective pedagogy for students (Boaler & Staples, 2008; Hiebert et al., 1997; Nasir et al., 2011; NCTM, 2000, 2014; NRC, 2002, 2005; Smith & Stein, 2011; Yackel & Cobb, 1996; Yackel, Cobb, & Wood, 1991); a particular manifestation of which lies in problem-posing pedagogy.

Problem-posing is students' meaningful questioning, conjecturing, and thought-experimenting when faced with doubts about mathematically ambiguous situations. Problematizing the known facts, questioning why things are the way they are, and posing problems that nurture deeper understandings are important practices for learning. These practices are well-rooted in mathematics philosophy and education standards. The practice of problem-posing has also been recognized for its role in humanizing learning by empowering students to pursue problem spaces that are personally more satisfying. Although processes of mathematical problem solving and the rigor of mathematical tasks have received much attention in research, the role of problem-posing is still underrated in the work of math learners in schools, especially in low-tracked classrooms.

For some time, researchers have attempted to understand what problem-posing constitutes and its importance for enriching students' knowledge and experiences in mathematics. Problem-posing has been studied for its role in improving student learning and engagement (e.g., Cai, Moyer, Wang, Hwang, Nie, & Garber, 2013; Crespo & Sinclair, 2008; English, 1997, 1998, 2003; Lowrie, 2002; Singer, 2007); cultivating positive attitudes towards mathematics (e.g., Healy, 1993; Silverman, Winograd, and Strohauer, 1992; Whitin, 2004); and its relation to problem solving (e.g., Cai & Hwang, 2002; Koichu & Kontorovich, 2013; Singer & Voica, 2013). Additionally, studies have sought to understand differences in the quality of instruction by examining cross-national differences in student problem-posing capabilities (e.g., Cai, 1998, 2003). Findings over years of research support problem-posing as a vital practice that all school children should have access to. Given its importance, scholars have begun investigating the processes of how students pose mathematics problems and the design features of effective and productive problem-posing-based learning environments (e.g., Armstrong, 2013; Bonotto, 2013; Fiori & Selling, 2016), but this literature remains thin.

As such, there are four important shortcomings in this literature that the current study addresses: Firstly, most of the studies in the U.S. have been conducted in controlled experimental environments (with an exception of some large-scale classroom teaching experiments which have been done mostly outside the U.S.) and not within classroom ecologies as the current study will do. Secondly, while some studies focus on school-aged children—mostly high-performing children and mostly outside of the U.S.—a larger proportion of studies attend to adults (undergraduate students, and current or prospective teachers). In contrast, the current study was conducted in eighth-grade classrooms with twelve and thirteen-year-old students. QUASAR project (Silver, Smith, & Nelson, 1995)—the only large-scale research project conducted in the U.S. with middle-school students—does not explicitly investigate micro-level processes of student work in mathematical problem-posing as the current study does. Thirdly, current problem-posing literature fails to shed light on the socio-cultural processes of problem-posing, and the aspects of classroom learning environments that might support or limit student problem-posing. Current research agrees on the importance of creating effective learning communities where mathematical learning is not only a product of teaching and curriculum, but also of student mathematical discourse and collaboration (Boaler & Staples, 2008; Cohen & Lotan, 2014; Silver & Smith, 1996; Yackel, Cobb, & Wood, 1991). However, although current research provides coherent and well-accepted classroom dimensions (tasks, authority, accountability, and teacher's role) necessary for rigorous mathematical engagement, we need more work in teasing out attributes of these dimensions and a practice-based model relevant for problem-posing.

This dissertation study aims to fill this gap through the design, implementation, and analysis of an innovative math curricular unit centered on student problem-posing (in contrast to just problem-solving). I chose to conduct the study in the lowest-track classrooms in a predominantly working-class Latinx neighborhood school because this is a space where impacts of inequitable social and educational structures are the gravest (as discussed elsewhere in the introduction) but research thin. Additionally, since the research-practice partnerships have been consistently found to be effective in improving the day-to-day work of teachers and students by taking into consideration the variety of pressures they face, I partnered with a math teacher to co-develop and implement problem-posing lessons in his classrooms. My personal identity as an Asian-Indian immigrant who speaks English as a second language and my deep commitment to disrupt deficit models of learning and advance epistemic diversity and dignity for the learning of those who most often get systematically marginalized in education also shaped my decisions pertaining to the selection of the research site, design, and analytic methods in important ways. At the same time, the economic and social privileges I enjoy as a model-minority in the U.S. demanded that I learn anew and with an open mind and heart. Thus, this partnership was also done with the goal to allow me, as an early-career scholar, the experience of collaboratively designing and implementing a research study with a teacher to experience what it takes to build relationships with teachers and students in real-time, facing situations in complex school and classroom ecologies.

One of the goals of the project was to take students back to the world of wonderings and invite them to a space where they could question, conjecture, and tinker with mathematical ideas to construct their own mathematics problems for solving. This aligns with pedagogy oriented toward eliciting and supporting students' mathematical musings and giving students agency to solve their own mathematically rich problems. Alternatively, more typical experimental studies include an intervention curriculum which is evaluated using pre- and post-assessments or student surveys, but findings do not necessarily shed light on the intricate processes underlying observed phenomena, diverse participation pathways, and social and material aspects of student engagement within natural classroom environments. However, understanding how learning processes unfold over time in interaction with peers, materials, and the teacher is critical for designing productive learning environments. Methods of this study attempt to address this gap by collecting and analyzing micro-ethnographic data. Data corpus for the study comes from two different research settings: task-based paired interviews and a classroom-based teaching experiment. Research data include whole-class and small-group video-recordings, student written-work and reflections, and classroom and student-constructed artifacts. Multiple analytic methods including qualitative deductive and inductive coding (Saldana, 2015, 2016), constant comparative method (Glaser & Strauss, 1967), and interaction analysis (Bryanson, 2006) were used to gain an understanding of the phenomena using multiple lenses and angles.

The credibility and trustworthiness of findings was established using multiple strategies (Lincoln & Guba, 1985). Firstly, I established prolonged engagement in the field—the equivalent of forty-six working days at the school-site—collecting a range of micro-ethnographic data including interviews and classroom observations, and establishing rapport and relationships with the participants (the teacher and students), as well as non-participants, who allowed me access to the research site (the principal and staff members). Secondly, through the use of video-taped observations, journaling, memoing, keeping research logs of all activities, developing data collection chronology, and recording data analysis procedures, I have established an audit trail of all research activities (however, not yet reviewed in entirety by an external reviewer) (Lincoln & Guba, 1985). This was used for producing thick, rich descriptions of the setting, the participants, and the themes in the study. Thirdly, I used "validity-as-reflexive-accounting" (Altheide &

Johnson, 1994, p. 489) which meant that I returned to my data "over and over again to see if the constructs, categories, explanations, and interpretations" (Patton, 1980, p.339) I was making made sense. Fourthly, the meanings and the explanations that I was constructing were also reviewed by discussing them internally with the undergraduate research assistants in the initial phases of the analysis and then with the peers and colleagues not affiliated with the project during the advanced phases of the analysis. The peers and colleagues included fellow graduate students as well as early career and expert scholars within and outside my institution. Lastly, the data collected in the two different settings allowed me to triangulate the findings by systematically sorting through the data to find common and disconfirming evidence (Denzin, 1978; Miles & Huberman, 1994).

Throughout the process of designing the project, seeking approval through the institutional review board, collecting and storing data, and analyzing the findings, I adhered to ethical procedures and strategies for research with human subjects. I sought and received appropriate consents from all research participants. I protected the confidentiality of all data in both electronic (by password) and paper (by lock and key) formats throughout all stages of data collection and analysis.

In this three-study dissertation, I adopt the view that learning is a socially-situated activity, an ongoing process of becoming that occurs in constant interaction with social others (Lave, 1988). Jean Lave's (1988, 1997) foundational work on apprenticeship and situated learning that later culminated in her work on Communities of Practice with Etienne Wenger called Legitimate Peripheral Participation (LPP; Lave & Wenger, 1991) forms the mooring for the three studies reported here and address the following research goals: (1) understand the nature of legitimate peripheral participation in student problem-posing; (2) uncover how learning processes unfold over time in interaction with social others to allow shifts from the periphery towards full participation in the practice of problem-posing; and (3) explicate ways in which students negotiate agency and social risks of problem-posing. Each of the different research goals draws on a different facet of the same underlying theory. The theory of LPP takes a practice-centered approach and investigates learning as occurring in action and interaction and not simply in the heads of individuals.

More concretely, one of the facets of the LPP theory suggests that learning is a process of participation in communities of practice, which is at first legitimately peripheral. Legitimate peripherality is an inclusive way of belonging in a community, especially for newcomers, through limited participation in its practice. This limited participation enables an opening, a way-in, to gaining access to resources of the community and gradually increasing involvement in it. In chapter 1, I characterize what participation at a peripheral position looks like for students in the practice of problem-posing within the community of mathematical inquiry (Goos, 2004). I do that by conceptualizing problem-posing using the notion of mathematical doubts and ask: What math doubts emerge when students explore open unstructured artifacts in relation to problem-posing? How does a pedagogical context afford or shape the surfacing of doubts? Data from two different settings: task-based interviews (n=64) and classroom-based teaching experiment (n=57) was used. I use student written work and video-recorded small-group student interactions to identify critical moments when a mathematical doubt emerged, what led to it, and the student talk and activity that preceded and followed the emergence of the doubt. After analyzing for possible patterns, I find three dimensions of students' mathematical doubts that led to the student-posed problems—Pragmatic (finding purpose: what is it for?), Analytic (sensemaking: what is it?), and Transformative (questioning the established facts and reaching for new possibilities: why is it the way it is? How can it be changed?). I discuss why understanding these dimensions of students' doubts and how they shape problem-posing is essential to attend to the epistemic needs of students and to the ethical dimensions of problem-posing-based learning and instruction, and I examine how students enacting their doubts allow legitimate peripheral entry to the practice of problem-posing.

The second facet of the LPP theory focuses on how participation that is at first legitimately peripheral increases gradually in engagement and complexity towards more-intensive and full participation. Full participation does not refer to a location or a "closed domain of knowledge or collective practice for which there might be measurable degrees of 'acquisition' by newcomers" (p.36). Instead, it is "what partial participation is not, or not yet" (p. 37) and thus, emphasizes changing locations, perspectives, trajectories, identities, and forms of membership as newcomers move from peripheral to full participation. In chapter 2, I characterize the processes that enable students to engage in increasingly more sophisticated processes of problem-posing, explicate the nature of these processes and conditions that afford or constrain them. In particular, I ask: How do students shift from the periphery of their doubts to engage more fully in posing mathematical problems? By coding, constantly comparing what emerged, revising the codes, and thematizing the various phases of student problem-posing in the two settings of task-based interviews (n=32 pairs) and classroom experiment (n=7 four-member groups), I found three processes through which students gradually increase their participation in the practice of problem-posing. I call these processes assembling, casting, and carving and shaping. Further, by focusing on how students negotiated meaning-making when they worked together with their peers, I delineate shifts in participation and resources that structured or limited those shifts within each of the three processes. By doing so, I am able to draw out features of a learning environment that may enable student movement from peripheral to full participation in the practice of problem-posing.

The third facet of the LPP theory that I draw on is what Lave and Wenger (1991) identify as “the problem of access” (p.103). Lave and Wenger point out that “depending on the organization of access, legitimate peripherality can either promote or prevent legitimate participation” (p.103). It is assumed that by participating in the knowledge-producing practice of problem-posing and following their own lines of inquiry, students will gain access to exercise their agency in the discipline and their learning. However, surpassing the traditional authority of teachers and textbooks and negotiating status hierarchies in the social milieu of group work may not be a simple activity disconnected from their access and participation in the practice of problem-posing. Thus, for learners, especially minoritized learners, to truly exercise their agency in problem-posing may mean embracing social, cultural, and disciplinary risks. In chapter 3, I problematize the ease with which the literature talks about student agency in mathematical problem-posing and examine what is socially and disciplinarily at stake for students when doing this work. I ask: What are the social and disciplinary risks of problem-posing and how do students negotiate them to exercise agency in problem-posing? I find that when the problem-posing agency was investigated in relation to the social risks of posing, two sociocultural aspects of student activities emerged that allowed students to productively negotiate emerging risks: active listening and foregoing control over one’s own ideas to pursue collective goals. Analyses of small-groups reveal that the social and disciplinary risks of problem-posing were always present, but their negotiations were most productive when socially-constructed relations paved the way for individuals in the group to attain collective agency. Drawing on the tenets of Follet’s notion of “power-with” (as against “power-over”), I delineate the mechanisms through which the individual and collective agency were attained by a group of individuals.

Together, the dissertation findings reach beyond the past focus on the cognitive aspects of problem-posing and its relation with individual creativity and ability and contributes towards the practice and research on mathematical problem-posing in the following ways: first, a conceptual and analytical characterization of problem-posing that explicitly links it to mathematical doubts will prove useful for practitioners and researchers in finding ways and conditions that would best allow students to reveal and shape their nascent doubts towards mathematics problems instead of concealing them; second, characterization of processes and conditions that allow shifts in student participation in problem-posing will prove useful for the design and analysis of inquiry-based learning environments in diverse school and classroom settings; third, characterization of student problem-posing agency in relation to the social and disciplinary risks of posing will allow future investigations of pedagogical conditions that best support youth in negotiating risks and participating in agentic problem-posing, especially in reform-based collaborative learning environments.

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CHAPTER 1

Study 1. The Dimensions of Mathematical Doubts and its Relation and Affordances for Problem-Posing

Mathematics instruction in schools usually begins with a problem given by the teacher. But often in professional mathematics and in real life, problems are not given. Rather, they are formulated by people in their efforts to make sense of the situations that have troubled or perplexed them (Schon, 1979). It is often this troubling experience or a feeling of doubt or uncertainty that propels the need to know, act, and change something (Dewey, 1933). Turning this feeling of doubt into a coherent problem for solving is often a productive activity in mathematics and other disciplinary and social spaces (Freire, 1970; Polya, 1957). Yet, the nature of student doubts in relation to mathematical problem-posing in school mathematics remains undertheorized.

I define problem-posing as a practice that constitutes student questioning, conjecturing, and thought-experimenting when faced with doubts about mathematically ambiguous situations. This definition calls attention to the notion of *mathematical doubts* as a source of the posed problem and the conditions in which doubts are surfaced. I investigate the nature of mathematical doubts that young adolescents have about mathematically ambiguous situations or artifacts and the role that these doubts play for student problem-posing. Using two different research settings—task-based student interviews and a classroom teaching experiment—I ask: What math doubts emerge when students explore open unstructured artifacts in relation to problem-posing? How does a pedagogical context afford or shape the surfacing of doubts? These questions reflect an effort to understand what I refer to as students' *epistemic needs* about what students what to know and do, and their developing understanding of what can be known and done, and how to know it.

To be clear, not all doubts refer to what I call mathematical doubts for the purpose of this study. The distinction is central to what follows so I offer a few examples of contrast. Doubting their mathematical creativity, doubting a female peer's capability, or doubting the utility of mathematics in their lives could influence student learning. But these matters of doubt, as powerful as they may be, are not specifically what I take up in this paper as mathematical doubts. I present these examples for contrast to help clarify my focus instead on students' doubts about the mathematical features of a given object, their lingering uncertainties about the mathematical relationships of those features, perplexities about real-life applications of mathematical ideas evident in a situation, and so on. In short, student doubts inclusive of confusions, perplexities, musings, wonderings, and uncertainties that lead to questions of mathematical nature is considered in this study rather than doubts about self or peer's capability in mathematics.

The subsequent section offers a rationale and a theoretically informed discussion of mathematical doubts in relation to student problem-posing. A description of context, research design, and methodology follows. Findings about the nature of doubts and pedagogical context are presented next followed by the discussion and implications for research and practice. Throughout, I emphasize how the notion of mathematical doubt can help expand the ways we interpret, organize, and study learners' development as agentic problem-posers.

The Rationale for Conceptualizing Problem-Posing using Doubts

In defining problem-posing using the notion of mathematical doubts, I assume that encountering a math doubt is a precondition for posing a problem. This assumption while theoretical in nature is supported by many examples of mathematical work of students and mathematicians in the literature. Pierce (1974) considered "doubt" as a "starting of any question, no matter how small or great" (p.253). Similarly, Dewey argued that "The origin of thinking is some perplexity, confusion, or doubt" (Dewey, 1997, p. 12). In mathematics education research, Dillon (1988) referred to student questions that are "expressive of doubt, uncertainty, or perplexity" (p.199) as knowledge seeking tools that allow students to initiate a search for knowing. More recently, Engle & Conant (2002) advanced the idea of student problematizing using Hiebert et al.'s (1996) research who in turn was inspired by Dewey's theories (1996). Engle & Conant (2002) examined numerous classroom learning episodes and determined that when students participated in voicing and resolving their own "disciplinary uncertainties" with others they were more productively engaged in learning. A common theme across these ideas is the recognition that it is the unsettled, uneasy feeling of wanting to know more about something or of not having fully understood or resolved something, in short, our doubts, that drives questioning and inquiry.

Within the problem-posing literature, Whitin's (2004) action research with his fourth-grade students indicates that when students were given opportunities to explore open math situations and voice their observations of them, they were better positioned to ask problems that students themselves were interested in resolving. Later, Whitin (2006) also emphasized the importance of students' general observations and curiosities for their problem-posing. He notes that even when student statements seemed too obvious at first, they were expressive of what children were seeing in the given situation or task. Simple observations often provided rich connections to children's curiosities and the problems they posed later. For example, fourth-grade students in his class were exploring the perimeter of squares when some students noticed that for the given squares the perimeters were all even numbers. When students were invited to explore this observation they posed a related problem: Is the perimeter of squares always even? The posed problem allowed students to more fully explore the numerical and geometric properties of squares. Whitin reflected that asking generative questions such as "What do you notice?" and "What do you predict?" encouraged students to explore their observations and doubts from which they created meaningful math problems.

This research suggests that even basic observations of students, and I would add, perplexities verbalized in the form of statements or perceived non-mathematical, can act as resources for future inquiry. I further add that students' initial observations, doubts, and musings, including those that often get perceived as obvious, absurd, or redundant, may offer a window into students' ways of seeing the mathematical world around them. The research in this vein is still exploratory and in its early phases and, thus, thin and non-conclusive. Nevertheless, it provides strong theoretical ground to seek clarity on the nature of mathematical doubts that young adolescents have about the given situations or artifacts for problem-posing. Doubts give us the leverage to step outside the *certainty* that so strongly epitomizes the discipline of mathematics and instead investigate the confusions and uncertainties that might engender productive problem-posing.

Framing Doubts as Legitimate Peripheral Participation in Problem-Posing

Socially situated theories of learning posit that the development of higher-order cognitive processes is rooted in the socially situated experiences and context of all human activities (Lave, 1988; Lave & Wenger, 1991; Vygotsky, 1978). This view would suggest that capacities to participate in mathematical practices such as problem-posing are intricately linked to the historical experiences of the problem-posers and mobilized through interactions with social others in the context (people, objects, tools, artifacts). According to this framework, problem-posing can be viewed as a social practice that begins with a particular kind of experience. Lave & Wenger (1991) framed this initial experience as "legitimate peripheral participation". The participation is in a limited peripheral capacity because of newcomer's limited experiences in the community of practice and its activities. It is at the same time legitimate because it allows learners to access modes of practices and behaviors otherwise not available to them. Through legitimate peripheral participation, eventually and gradually, learners develop conceptual models, knowledge, and skills to build complex relational repertoires to participate more fully in the community.

Studies have underscored the role of cultural artifacts for providing students entry to problem-posing practices (Bonotto, 2013; English, Fox, & Watters, 2005; Fiori & Selling, 2016). Consider, for example, Fiori & Selling's (2016) examination of a summer school intervention with middle school students. Students freely explored various work stations that contained everyday objects (such as nuts, bolts, L-pipes, geoboards, patterned tiles, etc.) and contemplated the mathematical concepts that emerged from them. This usually began with students choosing a station and playfully exploring the given objects. Through this playful exploration students subsequently nominated ideas that reflected their aesthetic choices, and I add, their nascent perplexities, about the features of the given artifacts that were later used for creating more structured math problems. The authors found that similar to mathematicians' practices, students were engaged in the acts of discernment. Similarly, Bonotto (2013), in a study conducted with a diverse set of fifth-grade students in school and out-of-school settings in Italy, found that when students were given opportunity to explore semi-structured real-life situations (e.g., leaflets containing discount coupons for supermarkets and stores, pricing and membership options for amusement parks), they often brought to the fore a wide repertoire of their experiences outside of school to engage with it. This provided a stimulus for students to determine specific aspects of the situation and artifacts for problem generation.

A common finding across these studies is students locating a *problematic* aspect of the given artifact while drawing on their experiences and aesthetic preferences to create a math problem. The notion of "problematic" is closely tied to the idea of *doubts.* The dependence on personal experiences and preferences emphasize students' cognitive reach and need to personally connect with the objects, artifacts, and people in the context in which they are problem-posing. In capturing the relational process of initial doubts, problem-finding, and the situatedness of context, Dewey noted that "the starting point [of thought–action] is actually problematic, and that the problematic phase resides in some actual and specifiable situation" (Dewey, 1958, p. 67). For him, the crux of an inquiry and problem finding begins from a doubt—a problematic aspect of the situation. Doubts reflect an intricate connection with the one who doubts and the artifact or situation about which the doubt emerges. The focus on mathematical doubt in problem-posing, therefore, underscores the epistemic need to contemplate and engage with the social and physical world using mathematics as a disciplinary tool (Vygotsky, 1980).

An important theme in the above ideas is that the act of posing a new problem often begins at a periphery—when experiencing doubts, confusions, or musings about a given situation. Doubts often transcend prior knowledge and experiences, carrying the residue of unresolved questions, and prompt a desire to change or see anew; offering both a beginning for inquiry as well as opportunities for productive struggle in current activity (Abramovich, 2015; Hiebert et al.,1996). As such, the notion of doubts has a potential to help researchers see connections between students' past knowing and their future-making. Embracing student doubts may enable them to gain access to the practice of problem-posing by connecting what they know with what is still unresolved for them. The emphasis is on stimulating doubt and using it to make new problems for solving. I outline this conceptualization in Figure 1.1.

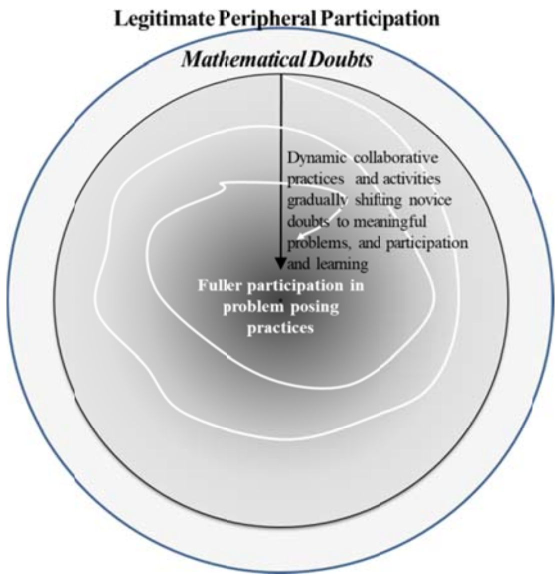

Figure 1.1. Voicing doubts as legitimate peripheral participation in the communities of mathematical inquiry

The outer circle in figure 1.1 represents the space where students' mathematical doubts are voiced. If taken up, students gain entry to the community of mathematical inquiry (Goos, 2004) wherein through shifting collaborative practices students mobilize and shape their doubts into structured mathematics problems for solving. Students voicing their mathematical doubts, seen as a legitimate form of peripheral participation, can allow educators and researchers to trace students' shifts toward productive forms of problem-posing more organically. The attention to students' mathematical doubts can provide an essential conceptual tie missing in the problem-posing literature that has for so long focused on the outcome of problem-posing rather than how students gain access to it.

The Politics of Educating Low-tracked students

Lave & Wenger (1991) remind us that:

> Any given attempt to analyze a form of learning through legitimate peripheral participation must involve analysis of the political and social organization of that form,

> its historical development, and the effects of both of these on sustained possibilities for learning. (p. 64)

The political and social organization of problem-posing as a practice for students' learning rests upon the political/social organization of mathematics education in general as well as the historical development of problem-posing as a teaching and learning practice in schools that takes its root in the work of professional mathematicians. Mathematics education holds a special place in the U.S. public schooling as a gatekeeper of educational and economic opportunity (Gutierrez, 2008, 2010; Ladson-Billings, 1997; Martin, 2009a, 2009b; Weissglass, 2002). Still, unequal access to quality mathematics education persists. Exclusion of lower-tracked and other academically marginalized students from quality mathematics education is widely documented, with remedial and basic skills predominating curricula and pedagogy in low-tracked classrooms. Tracking systems exacerbate these inequalities by systematically assigning higher proportions of minoritized students in lower-tracked classrooms (even after controlling for test scores), and allocating still fewer educational opportunities to them at the classroom level (Gamoran, 1992; Oakes 1992; Useem 1990). Mathematics is transmitted to students of color as a canon of procedures and facts and as instrumental means to reach goals defined elsewhere. We should, unfortunately, expect to find macro-level discourses enacted and drawn upon in the implementation of problem-posing-based lessons in low-tracked classrooms in schools predominantly serving minoritized students. Therefore, doubts as a conceptual notion also have a danger of getting loaded with a negative connotation, which often it does.

Doubts often get identified as expressing mistakes or misconceptions. Voicing doubts, in particular, holds implications for students of color, questioning by whom is often met with the interpretation that students are being disruptive or inattentive (Langer-Osuna, 2016; Philip, Olivares-Pasillas, & Rocha, 2016). Moreover, doubts by minoritized students that do not readily fit within the canon of school mathematics are more likely to be deemed irrelevant, ignored, or challenged (Godfrey & O'Connor, 1995; Jansen & Middleton, 2011). Unfortunately, this is also combined with learned helplessness over time in schools where learners learn to ignore their mathematical doubts, however unsettling they may be, to avoid "getting caught not knowing" (Varenne & McDermott, 1998). "Getting caught not knowing" may also be related to why students may avoid surfacing their nascent doubts about mathematical ideas in classrooms. For instance, Lowrie (2002) found that children as young as six years of age tended to ask traditional problems similar to what they had seen in classroom lessons instead of more open-ended problems. This tendency to see authority only in the teacher or textbook may also falsely favor students who may follow learned procedures to solve routine math problems over those who may diverge from expectations more often in order to make conceptual sense of the world around them or seek to understand the concepts in relation to their lives.

Doubts as a Way to Challenge Deficit Perspectives attached to Low-tracked Students

If the meaning-making in mathematics is a product of social construction, then surely problem-posing by students in schools must also be a result of their shifting practices in their classroom communities, however restrictive they may be. Problem-posing may also be more than as defined within the realm of formal cognitive systems and strategies and may even have a distinct characteristic from how mathematicians practice it. I argue that an explicit attention to doubts and musings that students experience in relation to problem-posing may be one way to identify its affordances for classroom communities of inquiry and provide students legitimate peripheral access to it. I hypothesize that a task that by-design treats student doubts as contributions to problem-posing will encourage students to verbalize and utilize them towards crafting problems that address students' own epistemic concerns as against teachers' preferences. Attention to mathematical doubts in this study may provide a window to understand students' ways of seeing the mathematical world around them (in this case, mathematically ambiguous artifacts) and as a way to disrupt the deficit perspectives linked to students who express them.

Methods

Research Setting and Design

The study was carried out in two settings at a middle school (VMS) that served students from predominantly working-class and immigrant Spanish-speaking families. The school serves over 98% Hispanic students, 40% students identified as English language learners and about 96% eligible for free or reduced meals. In the first phase, I conducted thirty-two task-based paired interviews (Houssart & Evens, 2011) with sixty-four students. Seventh and eighth-grade students from both honors and non-honors classes participated in the interviews. Parent consent forms were distributed to all students in all of the six seventh-grade classes at the school including honors and non-honors. There were two different teachers teaching these classes. Out of around 180 students in the six periods, sixty-four students agreed to participate in the interviews. Parental consent forms were also distributed to all students in two eighth-grade non-honors periods. Out of around sixty students, eighteen students agreed to participate and were interviewed. See Table 1.1 for further student frequency breakdown.

In the second phase, a teaching experiment (Cobb, 2000) was implemented in two eighth-grade class-periods at the same school. The lessons for the teaching experiment were designed in collaboration with a middle school teacher, Mr. R, who taught both the classes. Mr. R is a white-American and at the time of the study, he was in his second year of teaching eighth-grade non-honors mathematics classes at VMS, where he also previously taught seventh-grade courses for about ten years. Both classes were low-tracked math classes and students were enrolled in them because they had scored at less than the 50th percentile on the district benchmark math tests in their seventh-grade.

Table 1.1. Number of student participants

Phase I	**Task-based interviews (2016-17)**	7th-grade non-honors (low-tracked)	36
		7th-grade honors	10
		8th-grade non-honors (low-tracked)	18
		Total Phase I	**64**
Phase II	**Classroom-based teaching experiment (2017-18)**	8th-grade non-honors period 1 (low-tracked)	25
		8th-grade non-honors period 2 (low-tracked)	31
		Total Phase II	**56**

The study was conducted in two phases as a series of task-based interviews and a teaching experiment because of multiple reasons. Firstly, paired interviews with the students provided a simpler setting in which to familiarize myself as a researcher with the ways adolescents may take up problem-posing before implementing it in a messy ecology of classrooms. Secondly, implementing problem-posing tasks directly with students in a simpler setting of interviews allowed me to revise task activities, the language and structure of the tasks, and the design of the lessons to be implemented in the classrooms in order to make them more comprehensible to students. Thirdly, interviews, as compared to classroom-based teaching experiment, provided a different setting in which to study the nature of student doubts and problem-posing for the purposes of triangulating the findings.

The tasks students worked on during both the interview and the classroom phase were designed such that students could verbalize their doubts and musings prior to posing a problem.

This was essential for understanding the proposed relationship between doubts and problem-posing. The interviews and classroom lessons were designed to elicit, take up, and support students' mathematical doubts, thereby giving students agency to pose mathematically rich problems.

All the interviews and classroom teaching were videotaped. Students' written work was collected to capture their non-verbalized written ideas in the form of representations, procedures, and conceptual maps.

Phase I – Task-based Paired Interviews

The interview technique deployed was a variant of typical task-based clinical interviews, referred to as task-based paired interviews (Houssart & Evens, 2011; also used in Boaler, 1997 and Schoenfeld, 1985). Rather than interviewing individual students, I interviewed pairs of students together. Students were interviewed in pairs for a number of reasons. Firstly, paired interviews helped establish an atmosphere of ease and comfort by increasing the proportion of students-to-adult in the room and shifting the power imbalance at least numerically (Kellett & Ding, 2004). Secondly, asking students to work together during the interview helped simulate (to some extent) the social nature of classroom interactions and study its possible influence on student problem-posing. This understanding gained from the interviews also informed the participation structures for the classroom-based experiment. Thirdly, since it has been found that social processes of students' joint work are central and influential for student engagement and perseverance (Gresalfi, 2009; Sengupta-Irving & Agarwal, 2017; Zhang, Scardamalia, Reeve, & Messina, 2009), I hoped to better sustain student persistence and generate richer student responses by allowing paired students to engage with each other's ideas during the interview (Lewis, 1992).

Pairing students. The students were paired based on similar ability and dispositions towards mathematics (Cooper, 2003) and compatibility to work together (Highet, 2003) as recognized by their math teacher. This was done to promote comfort, confidence, and discussion between students during the interview. There were times when the ideal pairing was not possible due to scheduling conflicts. At those times, I leaned in favor of pairing students based on compatibility.

Interviewer's role. Before beginning the interview, I introduced myself by saying: "I am a former math teacher who now works with the teachers in understanding how students learn and in improving instruction. We, as teachers, agree that the best math teachers are those who learn from their students. Today, I have invited you here to talk about a few math tasks that I have designed for a teacher who will use them next year in his class for students to work on and learn from. I need your help because before I pass them on to the teacher I want to understand how students might think about these tasks and how they might do these tasks." The goal was to establish for students the need for their expertise and the contribution they are making by participating in the interview. It was explained to the students that the interviews are confidential and what students talk about and share with me will not be shared with their teachers, school staff, friends, or parents, but my research team may watch it to learn from them just the way I am learning from them right now. It was also told to them that at any time they can ask to leave if they wish so and their participation or non-participation will not impact their grades in any way. I then checked with the students to see if they still wanted to participate or if they had any questions for me, and I took their written consent.

My role as an interviewer was to listen, question, and facilitate student work and interactions, without delivering direct procedural instruction (Maher, 1998; Martino and Maher, 1999). The tone and phrasing of my questioning was non-evaluative and open-ended so that I could assure students that I was not assessing them, but instead, that I was trying to understand how they were approaching the task and what they were thinking about doing to answer it. For example, I would often say, “What you wrote/shared/did looks interesting/useful, can you say a little more about it?” I was also careful to facilitate interviews in a manner such that both students could have a fair chance to talk without one student dominating the interview; this ‘one student dominating’ is considered one of the downsides of paired interviews (Breakwell, 1990). I also facilitated peer interactions on those tasks during which they were asked to work together. I reminded students at various points throughout the interview that, “this is not a test or assessment and the tasks do not have any right or wrong answer. They can be answered and solved in multiple ways.” There were three parts to the interview, and before starting each part students were asked if they still wanted to continue the interview and if they felt comfortable. All students decided to continue except for two students who asked for a restroom break and returned after using the restroom.

The task. A growth pattern with three steps (see Figure 1.2) made up of blue snap blocks was used as an artifact given to the students for a problem-posing task during the interview. Growth pattern tasks are also called *generalizing problems* (Lee & Wheeler, 1987) or *geometric patterns* (NCTM 2000) and appear in many published curriculums such as Interactive Mathematics Project, College Preparatory Mathematics, and Mathematics Education Collaborative *patterning tasks* (see Parker, 2009). Figure 1.3 shows the setting of the interview. The artifact (a growth pattern made up of snap blocks) was placed on a sheet of white paper in the center. Students were also provided with extra snap blocks to use, along with graph paper, and colored markers, and they also had access to the whiteboard in the room. Many students took up the use of the whiteboard as a preferred option when working together during the interview. Students also used a workbook given to them for their written work. Typically, pattern problems are used to develop an understanding of variables, expressions, and generalizations. Pattern problems promote multiplicative thinking, algebraic reasoning, and functional relationships. Students look for relationships while progressing from one figure to the next regarding what is staying the same and what is changing.

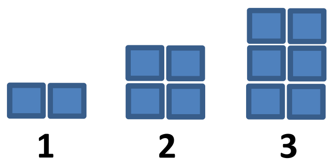

Figure 1.2. The growth pattern task for the task-based paired interviews

Figure 1.3. The setting of the task-based paired interviews

The interviews lasted for a range of thirty to fifty minutes. There were three parts to the interview (see Appendix A). In the first part, students were invited to individually "write down thoughts, observations, ideas, or questions" about the given set of objects. When students told me that they were done, I asked them to use their observations about the artifact to make and write a few interesting and challenging math problems for their friends to solve. The part of the task was done individually in their workbooks. When students were done writing the problems, I asked them to share their thoughts they had written and the math problems they had made. I then asked them questions to further understand their explanations and actions and also invited their partner to ask questions and reflect on what their peer shared.

In the second part of the interview, students were asked to problem solve by thinking about how the pattern will grow visually. They were asked the following series of typical questions to think/work together with their partners: (a) Find Case #4. (b) Find one more case. Any case you want. Fill in the Case# ___. (c) Find how many squares there will be in Case #100? (d) How many squares will be there in Case #N? Once again, they worked in the worksheet given to them and also verbally shared their work and solutions with their peers when working together and with me. I asked them questions to further understand their thinking and work.

In the third and the last part of the interview, students were again asked to problem-pose together with their partners but this time they were asked to specifically create some pattern problems. The task instructions included these instructions: “For his students' homework, Mr. Nunez wants to make up some interesting and challenging pattern problems. Help Mr. Nunez by writing as many pattern problems as you want to in the space below.”

Phase II – Classroom Teaching Experiment

For the teaching experiment phase of the study, I worked closely with Mr. R who co-designed and taught all the lessons in his two low-track (non-honors) eighth-grade math classrooms. Mr. R and the students did not have any history of participation in a teaching experiment and in problem-posing based teaching and learning. This meant as a researcher I had to carefully build the research momentum in the class and the relationships with the teacher and the students. I started by observing his classrooms with no videotaping. Gradually, I introduced video cameras starting with the whole-class camera first and then added a camera for one group at a time. This allowed me to answer students' questions about videotaping and help them become comfortable with the cameras before videotaping for the experiment.

Design and video-recording. The current study investigates a problem-posing lesson conducted over two days as a part of a ten-day-long unit in Mr. R's two class periods. All the students who consented to participate in the study formed a part of small-groups that were followed and videotaped. This included sixteen students in period 1 who formed four small groups and ten students in period 3 who formed three small groups. All groups were four-member groups except one group in period 3 that had two students. One wide-angle GoPro camera focused on the teacher and whole-class discussions. Three flip cameras with table microphones captured the small groups consisting of all the consented students.

Researcher's role. I was introduced to the students by Mr. R as a graduate student who has a project to complete for my higher education studies. He explained that I want to understand how students learn so I can help improve mathematics instruction and that I would like to learn from them. I then explained to students the various parts of the project, how confidentiality and anonymity will be maintained, and how I will protect the data that I collect in the classroom. The risks to participating in the study were also explained such as possible discomfort from getting videotaped and an option to quit their participation later on.

I acted as a participant-observer during lesson delivery. On his request, I assisted Mr. R in facilitating small-group discussions in the class. Also, Mr. R requested that I assist him when he initially modeled how to problem-pose. Mr. R and I took turns sharing our ideas, asked each other questions, and revised our ideas in light of the questions asked about the image that was used to model (see Appendix B) how to problem-pose before students worked on their own problem-posing task.

The task. On day 1, at the fortieth minute of the fifty-minute-long period after teacher modeling, the task was presented to the students for problem-posing and students continued working on it the next day. An image of a square with a border (see Figure 1.4) was used as a task-artifact given to the students for the problem-posing task. Known as the 'Border Problem' (Boaler & Humphreys, 2005), it is typically used as a pre-algebra task to develop knowledge of variables, equivalent expressions, and generalizations (Moss & Beatty, 2006). Students were given a worksheet with the image and prompt for the task (see Appendix C). Figure 1.5 shows the setting of the classroom: four students in each group worked together on the task.

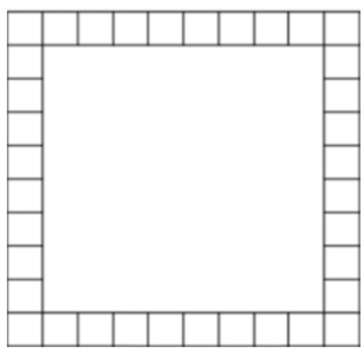

Figure 1.4. The border problem task for the classroom experiment

Figure 1.5. The setting of the classroom experiment

Similar to the interview, the image was used for student problem-posing instead of as a Border Problem. On day 1, students were asked to list what they notice and wonder about the image and to share it with other students in their groups. After six minutes of individual writing, Mr. R asked each group to share with the class one notice and one wonder. On day 2, instruction began with a recap and asked each group to also share a notice and a wonder with the whole class. Students were then asked to share, compare, contrast, and ask questions about each other notice and wonder list. Students after sharing their notice and wondering were asked to use their observations and questions from the 'notice and wonder list' and create some math problems together with others in their group. The class ended with students reflecting on what they liked about making their own problem and what they found challenging about it.

Data Analysis

The goal of the analysis was to investigate the nature of student doubts and what affords students surfacing their doubts. The data corpus for analysis consisted of approximately 22 video-hours from the interviews and 6 video-hours from the classroom, along with students' written work. Several times, while analyzing the video, I was able to go back to the students' written work to recreate and examine specifically what the student was referencing.

Data review. Prior to analysis, the author and two undergraduate research assistants digitized students' written work and watched all the interview and classroom videos. We followed Heath, Hindmarsh, & Luff's (2010) three stages of data preparation: a preliminary review, a substantive review, and an analytic review of the data corpus (see also Bryanson, 2006). During the preliminary review phase, we watched all the small-group videos. Whenever needed, we watched the whole-class video in reference to the small-group video. We cataloged the basic features such as the participants, the time-stamp of different activities within the episode, and wrote a simple description of the students' mathematical work, discussions, and nonverbal activities. We first reviewed 15% of the videos together, and, through discussions, solidified the perspective and approach to writing them before writing the rest. This yielded 32 content logs from the interviews and 14 content logs from the classroom teaching experiment.

During the substantive review phase, to find specific instances of the phenomenon (mathematical doubts), we first focused on the section of worksheets where students listed their noticing and wondering about the given artifact or image (interview part-1 and classroom day-1). Here we first separated the instances of student observations from those of doubts. Student observations were defined as students noticing obvious, visible, or surface-level features of the given artifact such as the number of blocks in the pattern problem or the shape of the given Border image. Any statements that were not about the obvious visible features of the artifact were once again reviewed. For this, we watched the part of the videos again where students shared and explained their statements of noticing and wonderings and refined our listing of student doubts.

Next, for the analytic review of data, we created logs of student doubts. We gathered what Heath, Hindmarsh, & Luff (2010) refer to as "candidate" instances of student problem-posing, and we followed the analytic methodology similar to Sengupta-Irving and Agarwal (2017). In doubt logs, we systematically recorded the instances of initial doubts and the explanations that transpired after an initial doubt's emergence in the form of verbal and nonverbal activities or the changing form of initial doubts. The doubt logs served as the principal data for the detailed analysis, including coding. We compared and contrasted activities within and across the settings to get a robust sense of the nature of student doubts, pedagogical affordances, and their relation to problem-posing.

Coding. The main purpose of coding the doubt logs was to determine if there are any patterns in the data that could more systematically help us understand the nature of students' doubts. The coding generally followed Saldana's (2015) "Four Ductions": deduction, abduction, induction, and retroduction to draw on what we know from the literature about the nature of mathematical thinking and questioning specific to the tasks but also to stay open to the new meanings emergent in the data. We conducted three iterations of coding.

In an initial examination of the data, we used open In Vivo coding. This round of coding helped in gathering the mathematical features of the tasks (growth pattern and Border image) given artifacts that students focused on and the different ways in which students perceived the given artifact. It did not, however, shed light on what students might have wanted to know about or do with the artifact. Moreover, we noticed that some students' doubts were not in the form of a question. Instead, they were mere statements such as "case number times two will give the total number of squares". Upon further scrutiny, we agreed that these statements were not basic observations that needed to be discounted from our analysis like we had discounted students' noticing of visible or obvious features. Rather, these statements were conjectural statements or speculations—attempts to generalize or see beyond what is visible and obvious. We concluded that students must have engaged in some tacit reasoning using the observable features of the artifact. Forming conjectures and proving or refuting them is a way for mathematicians to solve unresolved math doubts they have (Lakatos, 1963). Thus, questions and conjectures are both valid outcomes of attempts to resolve doubts. We, therefore, agreed to include conjectural statements as a part of data and went back to systematically code the list of student doubts as "question" or "conjecture".

In the second iteration, I looked at student wonderings through the open lens of what students might be doing or trying to know when posing a doubt. Emergent codes included codes such as recalling, generalizing, computing, historical grounding, finding relevance, and asking what-if, why, and how questions. In the third iteration, these codes were then thematized into four categories of doubt, which are described in the Findings section. To further confirm these themes and to extract the tacit conceptual meanings behind what students were doing or saying, I conducted open Process Coding (Saldana, 2016, p. 96). Still, these themes at this stage of coding remained rather interpretive and tentative, but they were confirmed once I conducted a more focused analysis of students' verbal explanations and activities using the videos.

Results - Research Question 1

The Three Dimensions of Mathematical Doubts

The analysis of student doubts revealed three broad dimensions: Pragmatic, Analytic, and Transformative (see Table 1.2). These three dimensions together illustrate, at a broad level, the nature of students' concerns, perplexities, and musings about the artifacts they were exploring (growth pattern and Border image). The three dimensions, seemingly distinct and disparate at first, come together to shape a powerful and robust system of student problem-posing. I will begin by describing the three dimensions of doubts, explain their relationships to the problems that get posed, present a detailed example of how doubts are surfaced and taken up, and lastly discuss the comparative results between the two settings. Then, in the discussion that follows, I consider the results from a broader, more interpretive perspective, examining the implications of the relationship between mathematical doubts and problem-posing.

Table 1.2. The dimensions of mathematical doubts

The dimensions of doubts and definitions	Sub dimensions
Pragmatic [Doubts about purpose, significance, or relevance. *What is it for?*]	General questioning related to the purpose of the artifact Focused questioning about the significance of a specific feature of the artifact General questioning related to the use of the artifact (for math/STEM or in general) Drawing on past/familiar information to determine/ conjecture the purpose of the new information
Analytic [Doubts about the non-obvious features of the artifact. Making sense of the non-obvious aspects: *What is it?*]	Extending the pattern Finding the rule Figuring a solution to a self-posed math problem
Transformative [Questioning the established facts and reaching for new possibilities. *Why is it the way it is? What if?*]	Questioning and transforming the shape and color Questioning and transforming the order and orientation Questioning and transforming the pattern or mathematical aspects

Pragmatic doubt. This dimension includes doubts about the purpose, significance, or relevance of the given artifact. The larger question that students appeared to ask was: *What is it for?* Students wanted to know why the given object or image was worth exploring in the first place, and they were questioning the significance of specific features of the object, its use in STEM, or its relationship to their lives. At other times, rather than questioning, students simply speculated or conjectured what the given artifact can be used for or what they think it could be. It was common for students to draw on some familiar aspects of the artifact to pose a pragmatic doubt such as the way it looked or something that it reminded them of. For instance, a student thought about a cage she needed to find a way out of upon seeing the snap cubes used for the growth pattern during the interview. Likewise, upon seeing the border image, students thought about it as a picture frame. On the one hand, such conjectural statements were not simple observations of the obvious visible features of the artifact. On the other hand, students' familiarizing or questioning the relevance of the given artifact by asking questions such as—What is this for? Why are they there? What is its purpose?—seemed too general to be considered mathematical doubts at first. Upon further analyses of how students explained their thinking and what transpired during their problem-posing, connections between pragmatic doubts and student-posed math problems became clearer. When students were voicing their pragmatic doubts, they were displaying their unique ways of gaining familiarity with the unfamiliar situation in a number of ways. At times students were drawing upon what they already knew or had experienced, at other times students were questioning the significance of features they were noticing in relation to ambiguous properties of the artifact, and in some instances, students were seeking connections between the artifact and its relation with mathematics/STEM in general.

I present three representative examples (Table 1.3) that display the nature of pragmatic doubts and how the voicing of them allowed students to consider varied aspects of the given artifact and create mathematics problems of different types ranging from word problems to problems requiring relational and geometric thinking.

Table 1.3. Representative examples of students explaining their pragmatic doubts

Talk/Act	Analytic Memo
Roberto: I was wondering what's the meaning of the cubes, like if they were just like food or just the regular cubes … like if they were just counting-cubes Priyanka: Okay. You said something-food, like if it's for food. Roberto: Like I was wondering what they will be using the cubes for, like for what problem. Either food, toys, or just regular cubes Priyanka: Hmm. Okay. So you think the cubes can represent food? Roberto: Yeah (with a proud hearty smile) Priyanka: Okay. Like how? Roberto: Like let's say-a KitKat-Maya has two pieces and if you get two KitKats, each of them will have four pieces.	- "meaning", "what they will be using the cubes for" = **General questioning related to the purpose of the artifact** ("cubes"). - Wonders if they are food, toys, or counting-cubes. - Attaches real-life meaning to the growth pattern (2,4,6) by thinking of 1 kitkat=2 pieces; 2 kitkat=4 pieces, and so on. - Also, prior to this discussion, he had written word problems such as "3 people … bring 12 pizza slices. If each brings a factor of 2 and Edwin brings the lowest and Chris gets the highest, how much pizza slices did Marko bring?"
Valeria: I wonder why there are numbers on the board next to the blocks. Priyanka: That's an interesting observation. Why do you think they are? Valeria: To know which is number one, number two, and number three? Ana: To see what levels they are in? Priyanka: What do you mean by that?	- "why"=**Focused questioning about the significance of a specific feature (case "numbers on the board") of the artifact**. - Reasons together with Valeria that case numbers are the same as the length of the block formation in that case. Ana further explains the explicit rule that connects the

Ana: Like they grow. They are all going by twos, they all multiply by two– (*Valeria nods as she listened to Ana and then interrupts her to add*-) Valeria: The sides? (*spoke hesitantly but clearly*; *moved her two fingers from one point to another in the air representing the side; Looked at Ana as if wanting Ana to confirm her*) Valeria: The sides on the border. (*when Ana did not seem to follow, Valeria gestured along the length (the larger side) of case# 3, which was 2x3 formation of the cubes, and then also gestured across the length of case#2*) #3 has 3 cubes going up, #2 has two cubes going up. The sides (*again looks at Ana*). Ana: (nodded in agreement) Yeah and they all multiply by two, like you have three and three (*gestures to show two columns of three in case #3*), so you multiply three by two to get six. (*gestured using her fingers to show how each case has two columns and therefore they are multiplying number of cubes in each column by 2 to get total number of cubes*)	case number to the total number of cubes in that case. - Moving from pragmatic to analytic space.
Elisa: Will we be using like any other sets of like cubes like trying to build something only using these - dimensions? Priyanka: What do you mean by that? 'By using these dimensions'? Elisa: Like how, you know, you have two (*points to case#1*) times two (*points at the second row in case#2*) which make those four cubes (*circles around case#2*) Are you going to like build something (*gestures as if stacking on top of case#2*) using these (*points at the cubes in case#2*) specific ones - dimensions? Priyanka: Is your question, if you can use these to build something else – Elisa: Yeah! Priyanka: - or do you want to extend it or to build over it? Elisa: To build over it, to build it on the top.	- "using"=**General questioning related to the use of the artifact** - Wonders if she can use it "to build something" - Also, prior to this discussion she had written word problems such as "family looking for 2 home that has a volume of 24 unit cube. Which dimension will they choose: 1x2x4; 2x2x3; 3x2x4" (drew 3D diagrams) - Moving from pragmatic to transformative space.

In each of the three episodes presented above, students were questioning the purpose, significance, or its use. Roberto was perplexed about the "meaning of the cubes" and what they might use the cubes for. He wondered, "What they will be using the cubes for, like for what problem?" and conjectured if the cubes can represent something from real-life, "food, toys", which is explained using the example of Kitkat. Prior to sharing his doubts out loud, he had created several other word problems using the reference of food and based on the given pattern (case1=2, case2=4, case3=6). In the second example, Valeria wondered why the numbers were placed next to the blocks and reasoned together with her partner that the numbers (case numbers) represent the length of the area of blocks in each case: “#3 has 3 cubes going up, #2 has two cubes going up”. Valeria’s pragmatic doubt about the significance of the artifact’s feature (case numbers) set the students up to shift towards the analytic space that later Valeria verbalized as the relation between the “sides” and the case number. Ana explained this as the relation between the number of cubes in each case and the case number: “They all multiply by two … so you multiply three [the case number] by two to get six [number of cubes in that case]”. Elisa wondered if she would be able to use the given dimensions of the blocks in each case to build towers on top by using more blocks. In one of her problems she drew 3-dimensional figures and assumed they were houses to create a word problem. Elisa’s pragmatic doubt about the artifact’s use allowed her to shift towards the transformative space (transforming the given artifact; see transformative doubt below). Later she tinkered with the blocks to think about the rate of change in the case of growing three-dimensional figures.

Analytic doubt. This dimension includes doubts about the non-obvious features or characteristics of the given artifact in relation to its visibly obvious aspects. Students were reaching beyond what they had noticed to make sense of the characteristics that were not apparent**.** Students were discerning the information or data that might be missing or not obvious or plain to their eyes. The larger question that students appeared to ask was: *What is it?* For instance, for the artifact in the interviews (growth pattern), students questioned or conjectured a general relation between the number of cubes and the case number (“Rule is n times 2” or “In each case the number of the case is multiplied by 2”); attempted to extend the pattern (“Why are there only three stages?” or “Will I be required to find the amount of cubes in a certain row?”); and figure out other mathematical properties of the artifact such as area, perimeter, rate of change, etc. For the border problem in the classroom, students conjectured, for example, different ways to figure out how many squares there were without counting each one or questioned, for example, 'How many triangles would be there if they added diagonals to each square?' In these questions or statements, it was not immediately clear if students were simply observing or wondering. I decided to code them as a doubt because they were not observations of simple features. Rather they were conjectures to a related tacit question about the mathematical relationships between the visible obvious features. For example, it can be visibly noticed that there are case numbers below the blocks and that each case has 2, 4, 6 blocks progressively—an observation. When students went beyond the simple observations to think about how those two observable features are related (case number times two gives the number of blocks) or conjectured other bigger cases with an attempt to extend the pattern, then it became apparent that students' doubts were about the relationships that were not immediately obvious among the visible or obvious features of the artifact.

Additionally, analytic doubts seemed inspired by students' prior knowledge or familiar math problems. In that sense, these doubts seemed a bit restrictive but at the same time reflected students' developing local authority of being able to ask themselves what teachers or textbooks have asked of them and to apply their knowledge in different or unknown situations. In contrast, there were a few cases where students' new and more open analytic doubts were shaped not due to their familiarity of the problems but as influenced by their pragmatic or transformative doubts (see for example the case of Valeria and Ana above). The analytic doubts set up students to create problems that would extend the given pattern, allow them to determine a generalized rule for the pattern, or ask and answer for other mathematical properties in relation to the artifact (such as area, perimeter, number of total squares, etc.).

Transformative doubts. This dimension included doubts that questioned/problematized the given features of the artifact to consider alternative or new possibilities. The bigger question that students seemed to be asking was: *why is it the way it is* and generated *what-if* scenarios. They seemed driven by their aesthetic preferences (shape, color, order, orientation, etc.) about the objects and the image that paved the way into their mathematical or purposive curiosities. When a student questioned why the given shape is a square and if any other shape could be used, it was not immediately clear what mathematical goals students had in mind but through tinkering a similar pattern with another shape they could explore the mathematical quandaries that arose with it. Students also problematized the color of the blocks in the pattern and used the red snap cubes to represent the negative space of the pattern or to turn the even-numbered pattern to an odd-numbered pattern (see exchanges with Andres in Table 1.4). At other times, students seemed interested in generating diverse possibilities to satisfy their need for adventure and playfulness. For example, the empty space in the Border image seemed to have provided the invitation for students to problematize it, but in the beginning, it was not clear towards what mathematical goals they were doing so. Afterwards, explanations, actions, and representations of students made it evident that students tinkered with the empty space and filled it with other shapes or more squares to reframe their transformative doubt "Why is the middle empty?" to ask analytic questions such as, "What other shapes and how many can the empty space be filled with?" There were also times when students were already thinking about a particular aspect of the artifact but extended their thinking in order to problematize it, transform it, and muse over the properties of the transformed objects or image. For example, consider Liliana's explanations in the below table:

Table 1.4. Representative examples of students explaining their transformative doubts

Talk/Gestures	Analytic Memo
Andres: I was wondering if you were to add the red cube, can it make it an odd number. Priyanka: Can you show me what you are thinking? Andres: So if for case number one since it is 2 cubes you will have three cubes which is an odd number. You keep adding up … so adding one more cubes to each of the case. So for each I added one more. [3,5,7 pattern] Priyanka: Okay. Okay. So. … why are you doing this? Andres: I was thinking because those are even (points to the given pattern) and red might be odd (*adds one red cube to the blue cubes in the given cases*).	- "add the red cube … make it an odd number" – **questioning and transforming the shape and color**. - He noticed that the given pattern was only made up of blue cubes when extra red cubes were also available. He added red cubes to represent odd numbers. Added one red cube to each case to turn the even pattern (2,4,6) to an odd pattern (3,5,7).
Liliana: I wondered if I move one cube will it change the whole problem? Priyanka: Hmm. Show me. How would you – Liliana: Like if … Priyanka: You can move it around or you can use the extra cubes there Liliana: Like If I move this one (*touches one snap cube in the block formation*), would it change the area or like- Priyanka: How will you move it? You can move it. Liliana: (*picks up the blocks and took one snap cube out from the 2x2 formation and attaches it on top of the other cube to make it 2 cubes and then one cube and then one cube*) Like this one and put it up there. Will it change the area, the perimeter? Will it still be the same?	- "move one cube … change the whole problem" – **questioning and transforming the mathematical aspects**. - Prior to talking about moving a cube elsewhere, Liliana had talked about what areas she sees in each case. - Wondered if she changes the shape by moving one piece in the same case to another position, would it change the area and perimeter.

Liliana wondered if moving one cube will change the whole problem. The "problem" referred to her earlier thinking about the areas of the figure in each case. Subsequently, she explained that she wondered if moving a cube to a different location would change the area or the perimeter (see Figure 1.6; Liliana moved the light blue cube in the first figure to transform it to the second figure).

Figure 1.6. Liliana's tinkering with the cubes

Table 1.5. The exhaustive list of examples for the three dimensions of doubts from the two research settings

CODING AND THEMATIZING FOR DIMENSIONS OF MATHEMATICAL DOUBTS			
RQ#1: What types of initial doubts emerge when students explore open unstructured situations?			
Dimensions	**Sub Codes (Process Coding)**	**Initial Student Doubts (Representative Examples)**	
		Interview	**Classroom**
Pragmatic [Doubts about purpose, significance, or relevance. *What is it for?*]	General questioning related to the purpose of the artifact	Im wondering what's the meaning of the cubes. Is it for food, toys or	What is the purpose of all this? What does it do?
		What is this for? Why are they there?	I wonder what we will do with it?
		How can it help me?	How is this making us learn?
	Focused questioning about the significance of a specific feature of the artifact	Why are the cubes going up by two?	
		Why are there numbers on the board next to the blocks?	
		Why does a hole penetrates the cube?; Why are some holes covered	
		How come there are two colors? What are the red ones for? What	
	General questioning related to the use of the artifact (for math/STEM or in general)	Do we have to calculate anything about the cubes?	I wonder if you can solve a problem with it
		I wonder if we are going to create or build something with these	I wonder if this would make a good quiz question
		How come it says 'case #1'? Does this mean we will be given a word	I wonder if you can make an equation out of the square
		I think they are tools that are going to help us with math	Can I use this as a tool?; I wonder if you can make something out of i
	Drawing on past/familiar information to determine/ conjecture the purpose of the new information	It kind of reminds me of like a cage you need to find a way out	Can you use it as a picture frame?
		The kind of remind me of a dice	It looks like a brick
		What are blocks from elementary school doing in a intermediate	Can I make a game out of it?
Analytic [Doubts about the features of the artifact. Making sense of the non-obvious aspects: *What is it?*]	Extending the pattern	I wonder if it will end; This pattern can definitely keep going by	I wonder if there is any pattern;
		Why are there only three stages?	How many patterns are there in the square?
		I wonder how many blocks would you have to add at #10; If to add another case their would be 8 cubes	I wonder what was the first figure; I wonder what would the 100 figure look like
		Will I be required to find the amount of cubes in a certain row	
	Finding the rule	The number of blocks up is the number on the case	How can you figure out how many squares there are without counting?
		Rule is n*2; That if you multiply the case number with two you get the number of blocks	I also noticed that you could add them by 8 on the each side. I wonder if you could multiply each side which is four sides in total by the number that we're multiplying by ?
		Do the blocks keep multiplying by 2 or do they switch?; Why are the blocks multiplied by 2 and not another number?	
	Figuring a solution to a self-posed math problem	Do we need to find out rate of change	How many squares are there in the border?; How many squares in total?
		I wonder if this would be like multiplying	36 squares surrounding one big square; 38 squares in total; There are 40 squares
		I think of the perimeter	How many square halfs are there
		I wonder the area	I wonder how many triangles are there
Transformative [Questioning the established facts and reaching for new possibilities. *Why is it the way it is? What if?*]	Questioning and transforming the shape and color	Why are they in a cube shape? Can they be in other shape?	Can it be another shape?
		Why are they only blue? Why not red blocks? If red was added it would may be an odd number.	How many different shapes can you make with it?; I wonder if I can break this to a different shape
		What would happen if added a red cube?	How many other shapes can be made with the same number of squares?
	Questioning and transforming the order and orientation	Does the order they are placed in matter?	If I rearrange the square what would happen?
		I am wondering what if the numbers go down instead of up?	I wonder if you can move it
		Each one is stacked upon others, why?; They are stacking up, not sideways.	
	Questioning and transforming the pattern or mathematical aspects	Why does it start of with two and not zero	Why is the middle empty?
		Can there be negatives or fractions in these kind of patterns?	I wonder how many squares it will have if I fill the inside; I wonder how many triangles we can do into the shape
			What shapes can make inside the square?; You can't put more shapes besides a square.

Mathematics Problems in Relation to Student Doubts

Tables 1.6 and 1.7 outlines the representative examples of student-generated math problems that had connections with students' initial doubts (as evident in talk or action) during the interviews and classroom experiment respectively. The tables do not include the doubts that were not taken up for making a mathematics problem. During interviews, 25 out of 32 pairs (78% of the student-pairs) and during the classroom experiment, 10 out of 14 (71% of the student-groups) took up their doubts (either individually or jointly) to create a math problem.

To elaborate on a few examples, we will start by considering a Roberto, who initially posed a *pragmatic* doubt during interview asking if the given cubes were for "food, toys, or just regular cubes," and who later created five math word problems, all of which included a reference to some food like pizza or chips. One of his multi-step word problems was as follows: "3 people Brian, Marco, and Chris bring 12 pizza slices. If each brings a factor of 2 and Brian brings the lowest and Chris get the highest, how much pizza slices did Marco bring?" This problem is related to several common core state standards including concepts learned in elementary school (finding factors and multiples and setting up numerical expressions) and early middle school (use of letters as placeholders and setting up of one variable simple equation).

Additionally, during a classroom experiment, consider, as an example, Diego, who drew on a pragmatic doubt of his peer, who expressed, "It looks like a picture frame." [Santiago] later discussed this idea for a word problem together with peers in his group: "Maybe it's a picture frame and we have to guess how big the picture frame could be. A girl- She buys a picture frame and she wants to put a picture in it but she doesn't know if she got the right size."

Table 1.6. Representative examples of those students-created doubts that were taken up for posing math problems during task-based interviews

MATHEMATICAL PROBLEM POSING AND RELATION TO DOUBTS (Interviews) RQ#2: How do students mathematize their initial doubts to pose meaningful math problems		
	Initial Student Doubts	**Math Problems**
Pragmatic Doubts	Im wondering what's the meaning of the cubes. Is it for food, toys or just regular cubes.	3 people Brian, Marco, and Chris brings 12 pizza slices. if each bring a factor of 2 and Brian brings the lowest and Chris gets the highest. How much pizza slices did Marco bring?
	I wonder what the meaning of the blocks are	Is this a right, acute, obtuse triangle? (made a triangle with base=4, height=2, side=6)?;
	Is this based on something?	Sussun gets paid $2.00 a day for babysitting. Within 3 days she already has $6.00. How much money will she have by the end of the week?
	Why are there numbers on the board next to the blocks?	What do the case numbers represent in the figures made of cubes?
	Do we have to calculate anything about the cubes?	y=6*2/4 ; (7*20/5)+(4*4/10)=?;
	I wonder if we are going to create or build something with these cubes.	What would be the unit rate of the given pattern (drew a 3D pattern with constant length but growing width and height);
	What are blocks from elementary school doing in a intermediate	Creates several arithmetic expressions for solving. E.g.,
Analytic Doubts	I wonder if it will end; This pattern can definitely keep going by	Solve for case# 7 in the given pattern.
	Why are there only three stages?	What would be the number of cubes in stage #15?
	Will I be required to find the amount of cubes in a certain row	A piece of bread falls to the floor, collecting 2 cells of bacteria a second. How much bacteria would that break have collected in a minute?
	The number of blocks up is the number on the case	What is case # for 20,418 cubes?
	Do the blocks keep multiplying by 2 or do they switch?; Why are the blocks multiplied by 2 and not another number?	(Makes a pattern with four hand going outside from the center) Determine the rule. (Answers: N*4 +1)
	Do we need to find out rate of change	What is in case#4? What is in case#8? What is the rate of change?
	I wonder if this would be like multiplying	If we have 250 cubes, what would we have to multiply by to get 1000 cubes?
	I think of the perimeter	What would the perimeter of all three be?
	I wonder the area	(Draws a 5 by 7 grid. x=5 and y=7) Asks x*y=? (Answers 5*7=35=number of squares)
Transformative Doubts	Why are they in a cube shape? Can they be in other shape?	(Makes a growth pattern linking the case number with the area of growing triangles made by putting square cubes together
	Why are they only blue? Why not red blocks? If red was added it would may be an odd number.	(Made a growth pattern with odd number of cubes in each case. If 3 cubes total, represnted two with blue and one with red. If 7 cubes total, represented 6 with blue and 1 with red, and so on so forth
	Each one is stacked upon others, why?; They are stacking up, not sideways.	(Makes a pattern that was stacked down rather than up and also sideways on the right rather than left.) Find the rule and how many there would be in case 10.
	Why does it start of with two and not zero	(Determine two cases going down. Answer that Case zero there would be no cubes, and in the case previous to it, there would be negative two)
	Can there be negatives or fractions in these kind of patterns?	(Assume red to be negative and blue to be positive. Makes a pattern where case -1,-2,-3 grow by 3, starting from 3; and case 1,2,3, grow by 4, starting from 4.; For #1, I can put 3 red cubes for negative. For #2, I can put none. For#3, I can add 3.

Within *analytic* doubts, a student who wondered, “Will we be required to find the amount of cubes in a certain row [case]?” created a word problem where the same question is asked indirectly: “A piece of bread falls to the floor, collecting 2 cells of bacteria a second. How much bacteria would that bread have collected in a minute?” The student is considering the case number to represent seconds and the number of blocks in each case to represent the growth of bacteria every second. The problem solver will have to first figure out the growth rule and then apply it to find the number of bacteria in the 60th stage (i.e., 60 seconds = 1 minute). This shows how student doubts were generative for creating mathematics problems.

Table 1.7. Representative examples of those students-created doubts that were taken up for posing math problems during classroom-based teaching experiment

MATHEMATICLA PROBLEM POSING AND RELATION TO DOUBTS (Classroom) RQ#2:		
Final Category	**Initial Student Doubts**	**Math Problems**
Pragmatic Doubts	It looks like a picture frame.	A girl- She buys a picture frame and she wants to put a picture in it but she doesn't know if she got the right size.
	What is the purpose of all of this? What is the square for?	Wendy wanted to make a box but she only had 40 cubes. How many more does she need to make a 1000 cubed box? How big will the box be three times from what is shown?
Analytic Doubts	I wonder if you could multiply each side which is four sides in total by the number that we're multiplying by ?	$32 = 8 \cdot x$
	How many squares are there?	How many squares are there without counting them?
	I wonder if you can make an equation out of the square	There is 10 on the side and 8 on the top and bottom. Use these numbers to make an equation
	I wonder if I could make a pattern.	If figure1=4, figure2=16, figure3=36. What will figure 6 look like?
Transformative Doubts	I wonder how many squares it will have if I fill the inside	If this [10x10 square] is figure 100, how much tiles would fig.1, 2, and 3 have and how would you find the answer?
	The middle of the square is empty. Can it filled with more squares?	Find ways to fill in the empty space with using 16 squares or less; How many squares can fit in the middle? How many shapes can we fit inside and what kind?; How many right triangles can you fit inside the square? How many triangles can be made in the center if each triangle is half the size of the small square?; How many squares of length 2 cm can fit in the center of the square?
	Can I move it around? Can it be another shape? Can you make other shapes with it?	Can the shape be transformed into other shape? Give me a proof. Give me some ideas. Give me your thoughts. Possible shapes: diamond, triangle, maze, rectangle and processes of transformation: by stretching the figure along the symmetrical lines, by adding lines diagonally or vertically, or by moving the axis.

Within *transformative* doubts, during the classroom experiment, a group of students shared the following transformative doubt, "Can they turn into more than two shapes?" and later shaped it into a more structured problem: "Can the shape be transformed into other shape? Give me a proof. Give me some ideas. Give me your thoughts." Dia in the group also asserted some conjectures about the possible shapes "diamond, triangle, maze, rectangle" (corrected for spellings) and processes of transforming like "by stretching the figure along the symmetrical lines, by adding lines diagonally or vertically, or by moving the axis."

During interviews, students had also problematized the given feature of the artifact such as why there is no case zero, if red cubes could represent odd numbers, and if negatives and fractions can be represented in the given pattern. These *transformative* doubts took the form of math problems where students extended, modified, or created a new pattern relating to the *what-if* scenario posed as doubts. For example, a pair of students who wondered: "Why are they in a cube shape? Can they be in any other shape?" created a triangular-shaped growth pattern (see figure 1.9), and by filling full and partial unit squares in each triangle, the pair of students attempted to solve for the area of the triangles using the number of cubes. Later they posed: "figure out the area of the tenth triangle"—which was essentially finding Case #10. During the classroom experiment, a group of students who shared a transformative doubt: "I wonder why the middle is empty," later created a math problem: "How to fill in the empty space using 16 squares or less?" that is a linear optimization problem with a linear system consisting of multiple variable equation along with a given constraint:

$$\alpha + 4\beta + 9\gamma + 16\delta + 25a + 36b + 49c + 64d = 64$$

Such that: $$\alpha + \beta + \gamma + \delta + a + b + c + d \leq 16 \quad \text{(using 16 squares or less)}$$

In sum, students during both the interviews and classroom experiment took up their initial doubts in a variety of ways to pose interesting math problems. This provides us a preliminary understanding of how doubts might inform students' making of their own mathematics problems.

Results - Research Question 2

Surfacing Doubts in Constant Dialogue with Artifacts and People

While students shared several thoughts, observations, and wonderings about the given artifact (snap-cube pattern and the Border image) in their private writing, surfacing those doubts publicly was not as natural for them. Most students were hesitant to share doubts that seemingly had nothing to do with mathematics such as pragmatic doubts. However, questioning from me and peers, engaging with each other's ideas, and tinkering with the objects allowed students to voice and further shape the meanings behind their nascent wonderings. While in most cases during the interview, questioning that surfaced and reshaped students' doubts was done by me (interviewer); however, during the classroom work, much of the questioning was done by the peers. Below, I expand upon the example of Ana and Valeria I presented earlier from the interview to show how student doubts surfaced in constant dialogue with the artifact and people Figure 1.7 presents a diagram of these relations.

Example: Ana and Valeria (interview). When invited to write down thoughts and questions about the given artifact, Ana seemed eager to begin and quickly wrote down her ideas while Valeria pondered for a long time before writing her ideas down. Ana and Valeria each wrote five thoughts about the artifact and made some math problems individually. At times, Valeria seemed to be uncertain about what to write, looked around the room, and exchanged quick glances and smiles with Ana while also stealing quick peeks at Ana's work.

Generating peripheral doubts. Ana initially raised three pragmatic doubts: 1) "Why are some holes covered and some are not?"; 2) "It kind of reminds me of like a cage you need to find a way out"; and 3) "This kind of remind me of a dice." But these doubts were never shared with others by Ana. Ana also asked a transformative doubt that she did share, and this doubt shaped later problem-solving: “Why are they in a cube shape can they be in other shape?” Ana explained that since this pattern will keep growing into “…much bigger rectangles they won’t turn into another shape.” When asked what she meant by “another shape”, she added, “They will still have four sides.” Valeria noticed that cubes were stacked going up and posed a transformative doubt about why that was so, but it was not shared with others. Valeria also posed two pragmatic doubts: “What are blocks from elementary school doing in the intermediate school and why are there numbers on the board next to the blocks?” These doubts were later discussed and formed a point of departure for their self-posed math problems. Valeria explained that she had used such cubes in elementary school for math and also sometimes “just for fun.” Ana chimed in and said that she had also used them in the elementary school for a project where they had to make a boat out of clay, put it in a bowl of water, and then they had to see how many cubes the boat could carry. While all the doubts raised by Ana and Valeria were either pragmatic or transformative, we will see in the next section how they shifted into the analytical space once students started exploring them for problem-posing.

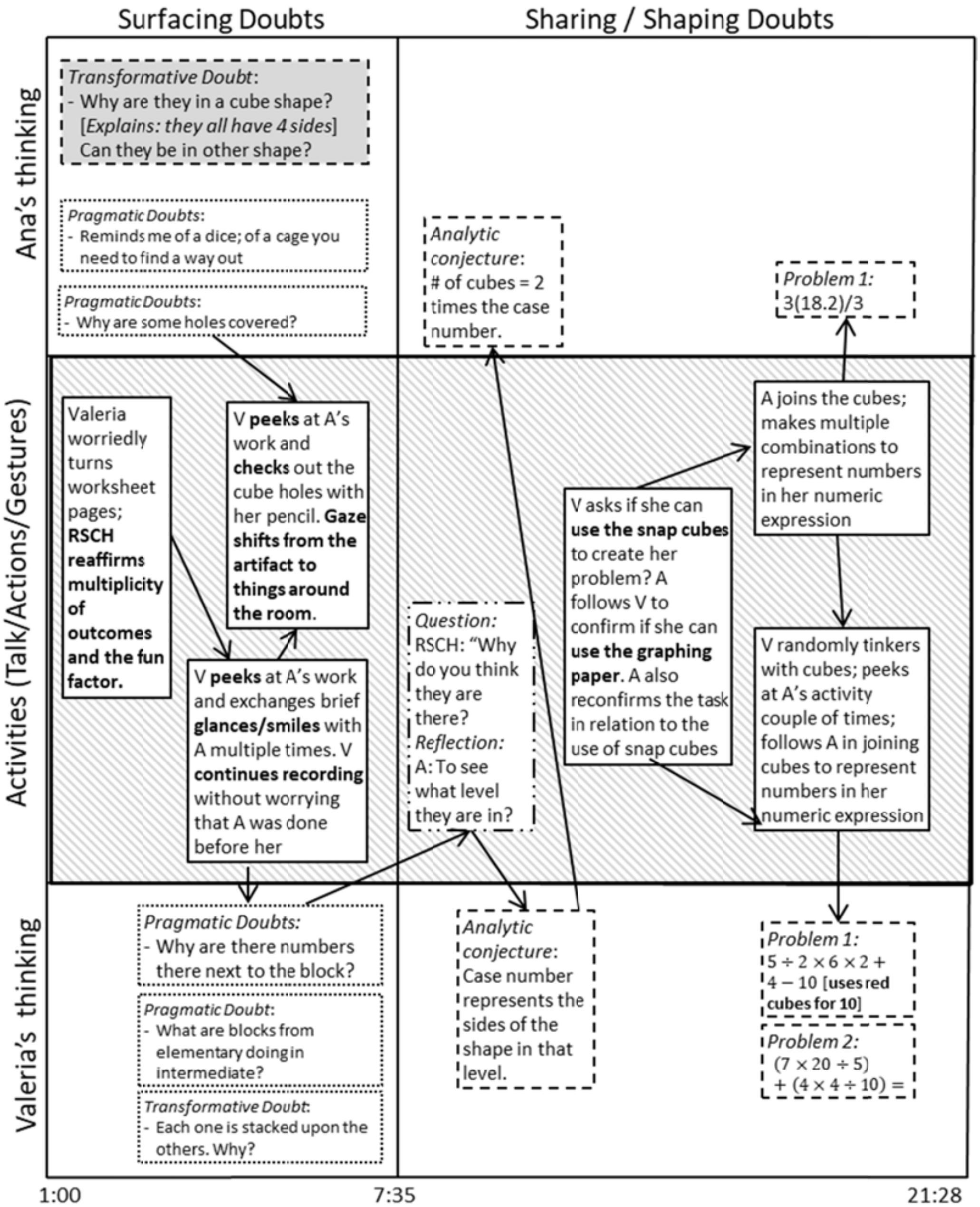

Figure 1.7. Movement of doubts in relation to talk and activities. Dotted boxes represent doubts, solid boxes represent activity, arrows represent the movement and links between students' joint activities and doubts

*Shaping doubts***.** When asked to pose some math problems using the observations and thoughts brainstormed earlier about the given artifact, both Ana and Valeria immediately started fiddling with the extra snap cubes given to them and started attaching them together. Initially, it seemed that they were attaching the cubes randomly, but soon Ana used these cube creations to represent a number for a numerical expression. For instance, Ana created the problem $3 \times (8 \times 2) \div 3$ and used a row of three cubes, eight cubes, and two cubes to represent the numbers in her numerical expression. Valeria seemed very distracted at this point and continued to attach cubes, and looked around the room blankly. When Valeria noticed what Ana was using her cube formations for, she also did the same to make her expression: $5 \div 2 \times 6 \times 2 + 4 - 10$. Such purposing of cubes to represent numbers in numeric expressions by students may have been a response to their pragmatic wondering about the ways they had seen cubes used in the elementary school and its relation in intermediate school. Valeria, however, further transformed and extended the meaning of the cubes. Recall that the given pattern had used only blue cubes. When creating her cube formations for the above expression, Valeria instead used blue cubes for all her numbers except for when she was subtracting 10 in the end, for which she used red color cubes. She represented negatives or subtractions by the red color. When asked how they would solve the expressions they were creating, both students elaborated the solution using PEMDAS order of operations.

Valeria shared another wondering (Line 46 below) out loud with Ana and me, and the following conversation occurred (by "numbers" in Line 46 she was referring to the case numbers written underneath each case in the pattern):

18:40	46	Valeria	I wonder why there are numbers on the board next to the blocks.
	47	Researcher	That's an interesting observation. Why do you think they are?
	48	Valeria	(hesitantly and slowly) To know which is number one, number two, and number three?
	49	Ana	To see what levels they are in?
	50	Researcher	(*to Alondra*) What do you mean by that?
	51	Ana	Like they grow. They are all going by twos, they all multiply by two– (*Valeria nods as she listened to Ana and then involuntarily interrupts Ana as if she realized something and wanted to sharet-*)
	52	Valeria	The sides? (*spoke hesitantly but clearly*; *moved her two fingers from one point to another in the air representing the side; Looked at Ana as if wanting Ana to confirm her*)

	53	Valeria	The sides on the border. (*when Ana did not seem to follow, Valeria gestured along the length (the larger side) of case# 3, which was 2x3 formation of the cubes, and then also gestured across the length of case#2*) #3 has 3 cubes going up, #2 has two cubes going up. The sides (*again looks at Ana*).
	54	Alondra	(nodded in agreement) Yeah and they all multiply by two, like you have three and three (*gestures to show two columns of three in case #3*), so you multiply three by two to get six. (*gestured using her fingers to show how each case has two columns and therefore they are multiplying number of cubes in each column by 2 to get total number of cubes*)

When I further pressed Valeria to explain what she was saying about the sides earlier, Valeria once again explained that she sees the case # in the sides of each case formation, gesturing along the sides for each case. Drawing on what Ana had explained, Valeria further illustrated using gestures how each case has two cubes at the bottom (as a width). Ana promptly added, "It's like multiplying, you multiply 2 by 3 [for case#3 to get the total number of cubes]."

What is significant about this exchange is the way a simple redirection ("Why do you think they are…?") of a surface-level doubt inquiring why the cubes had numbers below them opened up space for students to turn it into a meaningful exploration about the relationship between the case number and the pattern. We also notice that Valeria, who when working alone had seemed distracted and uninterested (taking long breaks to stare around the room or wanting to peek in Ana's workbook), seemed more focused and confident when sharing and bouncing ideas off of her partner. Ana and Valeria were not as clear in their explanations initially (Lines 48-49), but gradually as they took turns explaining and listening to each other, Valeria could see how the case number was related to the pattern visually ("the sides") and Ana could better articulate how the case number was related to the total number of cubes in each case visually ("they all multiply by two").

The self-posed inquiry about the explicit relationship in the given pattern was also mathematically productive for students. The mathematical concept at the heart of the growth-pattern problems is determining a non-recursive relationship between the case number and the total number of cubes in a case. Most students usually initially see the recursive relationship that both Ana and Valeria had also noticed – “going up by twos” or “each one is stacked upon others” (case N = two more than case N-1), but it does not help solve for larger case numbers such as case 1000 without first solving all the 999 cases that came previously. Students problematizing the recursive relationship (case N in terms of case N-1) on their own and instead discovering a functional relationship (case N in terms of N) is one of the learning goals of growth pattern problems. Determining a non-recursive relation in the pattern promotes functional thinking, algebraic reasoning and creates an opportunity for students to see why they might need symbols when representing and solving larger cases. Using the understanding that students gained by seeking resolution of the problem—what is the relationship between the case numbers and the growing pattern—Ana and Valeria later created a more general algebraic function for the given pattern (case N = Nx2) and as I describe in the next section, also might have been instrumental in their efforts to create a new pattern directly drawing on Ana’s musing about the given shape (see Figure 1.8).

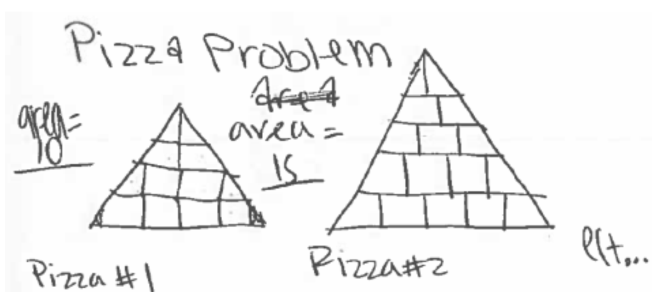

Figure 1.8. Ana and Valeria’s triangle pattern problem

Problem-posing. In the new pattern, Ana changed the shape to a triangle but still used squares to figure its area out. In shaping this idea further, Ana and Valeria also posed a new doubt for themselves: How do we use unit squares to find the area of a triangle? There was a reach to make the problem more challenging for the problem solver. Ana identified early on that since "triangle is not four shapes [she meant sides], it is three. So it is harder to find the area…" and recognized that the problem has a trick to it like it's a puzzle: "So it makes it comp-. So you may think that it is 14 but it is actually not." With the help of Valeria, Ana further expanded her figure to a pattern, although not very precisely drawn, the triangles were growing: both the base and height were increased by one unit square each time and, though not explicitly clear, it was assumed that the base is the same as the height of the triangles. Using this pattern, Ana created a problem: "Figure out the area of the tenth triangle." Together, Ana and Valeria not only attempted to figure out how to combine unit squares that were not full squares to determine the area of a specific triangle, they also created a problem that would require figuring out a functional rule to determine the area of a triangle using unit squares given the number of squares in its base. The self-posed problem they were working on required exploring and using triangle congruency properties when combining smaller shapes into full unit-squares (Grade 8 standards) and figuring a pattern of combining unit squares as the triangles grew. The problem could also provide an opportunity for students to compare different ways of finding the area of a triangle and could also be taken up to introduce very basic ideas of integral calculus.

Comparing the Two Settings: Task-based Interviews and Teaching Experiment

Table 1.5 illustrates the qualitative affinity in the kinds of doubts students posed across the two settings even though the tasks, the available resources, and the pedagogical contexts were very different for the two settings (read column 3 in comparison with column 4). In both the settings, students were perplexed about the purpose, significance and relevance of the given artifact in relation to their learning and its use in mathematics (e.g., "How can it help me?" during interview and "How is it making us learn?" in classroom; "I think they are tools that are going to help us in math," during interview and, "Can I use this as a tool?" in the classroom). Students were considering what the artifact reminds them of in order to gain familiarity with the unknown situation of both the tasks (e.g., "It reminds me of a dice" during the interview and "It looks like a brick" in the classroom). In both the settings, students were asking questions about extending the given pattern, solving simple arithmetic questions using the artifact ("area/perimeter" during the interview and "number of squares" in the classroom), and figuring out non-obvious rules (in relation to the case# in pattern task during the interview and in relation to the number of squares in the border task in the classroom). Students were also questioning, challenging, and transforming the given shape, color, orientation, and other features of the artifacts in both the settings.

However, quantitatively there were differences in the frequency of occurrence of doubts and type of doubts during the interviews as compared to the classroom experiment (see Table 1.8). First, during the task-based interviews, there were 17 students (out of 64) who did not pose any doubts. In the classrooms, each student posed at least one doubt. Moreover, during interviews, 64 students posed 98 total doubts—an average of 1.5 doubts per student. During the classroom experiment, 57 students posed 164 doubts—an average of 2.9 doubts per student (counting only non-repetitive doubts for each student in both interviews and classroom experiment). Second, during the interviews, 47% of the doubts posed by the students were pragmatic doubts as compared to only 16% in the classroom (see Figure 1.9). Additionally, during interviews, only 13% of the doubts were transformative as compared to 51% during the classroom experiment. While the percentage frequency of analytic doubts in both the settings remained comparable, pragmatic doubts were less prevalent during the classroom session while transformative were more prevalent. In the next section, I will discuss possible reasons for the differences in the frequencies of doubts and its dimensions between interview and classroom experiment.

Table 1.8. Descriptive statistics of doubts across two research settings

	Interview	Classroom
Total # of students	64	57
# of students who did not pose any doubts (i.e., posed only observations)	17	0
Total doubts posed	98	164
Average # of doubts per student	1.5	2.9

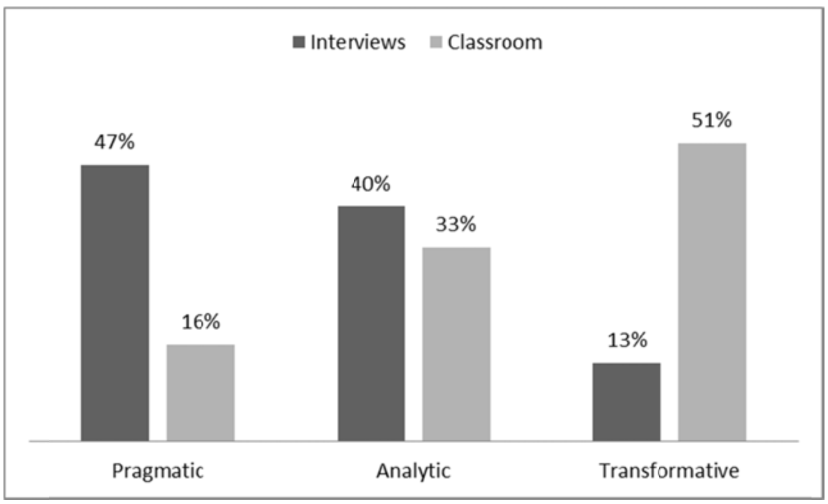

Figure 1.9. Percentage distribution by dimensions of doubts for the two research settings

Possible Reasons for the Quantitative Differences between the Two Settings

Prompt of the task. One of the reasons for a higher number of average doubts in classrooms may be the prompt of the task. During the interviews, students received the following prompt: “What do you notice or wonder about the objects in front of you? Write down thoughts, observations, ideas, or questions that come to your mind when you see these objects.” Students answered this prompt first. When they were done writing their thoughts (combined notice and wonder), I moved to the next task of problem-posing. In the classroom, students received this prompt first: "What do you notice?" and were given 3 minutes to write their observations. Then students received this prompt: "What do you wonder?" and were again given 3 minutes to write their wonderings. The separate time allotted to write wonders may have pushed students to specify at least one doubt and more doubts per student on an average.

Teacher modeling. Another reason for a higher number of average doubts per student during the classroom experiment may have been the modeling of problem-posing that was done in the classrooms. For the classroom teaching experiment, the teacher and I co-designed a part of the lesson on the first day that included modeling of how to notice, wonder, and pose problems. For the modeling part of the lesson, a different image was presented and the teacher and I took turns sharing what we noticed and wondered about the image. After initial modeling by the teacher and me, students were invited to share their wonderings about the same image as a whole-class discussion allowing students to hear what their peers were thinking. Sharing and listening to the variety in doubts that can be raised about an image by adults and peers may have helped students appreciate the range of possibility and thus provided more ideas to draw upon.

Teacher modeling may have also been the reason why, in the classrooms, students asked fewer pragmatic doubts. When the teacher and I were modeling problem-posing, Dia asked about the image (see Appendix B): "How does this affect our lives?" Upon hearing the question, Mr. R, who was transitioning to the next task, said in a dismissive tone, "Excuse me. You should have asked that before. That's a classic question. What is it for and why, what's the point?" Later, when Mr. R reviewed the wonderings and math problems that the class together had created, he asked,

> Are there any of the problems that are not kind of math-worthy? Such as those that you probably won't ask in a math class, that you will not see in a math class, that are not that challenging, or they are just kind of complaining observations. (*Pause*) So I will start, I am looking at the what and why questions – like why are we looking at this and what's it for [Dia's question]—I don't see them as very interesting because it doesn't challenge my intellect.

Here we see an instance where it was established early on for students that doubts about the *purpose* of the artifact are not "interesting" and they do not "challenge … intellect". Such norm-setting around pragmatic doubts may have influenced students to avoid voicing their pragmatic doubts in the classroom. In contrast, during the interviews, it was left open for students to share any thought they have. Often during the interviews students were also reminded that "There is no right or wrong answer for such questions. So, anything you write or say is good. It is not a test." Such invitations and assurances might have encouraged students to write/say what occurred to them naturally about the artifact.

As such, the intensity with which the teacher pursued the doubts was diluted by his competing interest in the curriculum that oriented him towards the analytical doubts and problem-solving of those doubts. Thus, in a way, a pedagogy that is oriented towards the mathematics practice of problem-solving alone can obscure the attention needed towards doubts for problem posing.

The nature of the artifact. A reason for the differences in the nature of doubts between the two settings might have been a result of the differences in the nature of the given artifact. The artifact and its inherent properties may have stimulated more categorically specific ideas and doubts than others. In the interviews, the artifact was already set up to have three growing stages waiting to be extended, as is typical of such problems, and that's where students focused mostly. This inherent invitation to extend the pattern was further fueled by the availability of other extra blocks that students could use to find more figures in the pattern. The Border image, on the other hand, was presented to them as a printed image on a sheet of paper, it was self-contained, and as one student described, it was a "closed shape", i.e., it had a clear enclosure in the form of its border. This may have invited students' attention toward the border itself or the empty space that the border surrounded. Indeed, most student doubts focused around figuring out if there was any pattern that could be used to determine the number of squares in the border (analytic doubts) or were about the empty space, which in fact became a space of opportunity for them to play with, tinker, and transform (transformative doubts).

Discussion

By conceptualizing problem-posing using the notion of mathematical doubts, the study answered the following research questions: What mathematical doubts emerge when students explore open unstructured artifacts? How does a pedagogical context afford or shape the surfacing of doubts?

The study draws attention to the ways the initial doubts of students served to deepen their lines of mathematical inquiry through surfacing and shaping their doubts towards mathematics problems. Drawing on two distinct settings of task-based paired interviews (n=64) and a teaching experiment in two classrooms (n=57), the analysis revealed three dimensions that explicate the nature of students' mathematical doubts: pragmatic, analytic, and transformative. When raising a pragmatic doubt, students are concerned about knowing the purpose, significance, or relevance of the artifact in general or as related to its specific features. When raising an analytic doubt, students are making sense of the non-obvious aspects of the artifact in relation to their observations of it. When raising a transformative doubt, students are questioning the obvious facts and features of the artifact and are reaching for new possibilities by asking what-if questions such as what if it was in a different shape, what if I filled the empty part with other shapes, what if the pattern was decreasing, or what if the pattern represented three-dimensional figures, and so on. Together the three dimensions of doubt illustrate students' epistemic needs about what students seek to know and do in relation to mathematics when they explore mathematically ambiguous artifacts (chapter 2 further clarifies how students' epistemic needs change with the shifts in student participation in problem-posing). The problems that students created provide further evidence of the links between student doubts and problem-posing. Additionally, students were able to utilize their rudimentary doubts such as. 'What are the numbers for?, 'Why is the shape square?', or 'Why is the middle empty,' to create interesting mathematics problems that were at their grade-level or higher in terms of cognitive demand.

Additionally, the sociocultural approach to studying doubts in this study has allowed attention to the pedagogical context—task, artifact, and social interactions. The findings emphasize the constant dialogue with artifacts and peers as creating a context for doubts to be surfaced, shared, and taken up. This context then becomes a form of legitimate peripheral participation allowing students a pathway to problem-posing. Literature on group work in mathematics classrooms, however, suggests that engagement in communities of practice is not a power-neutral engagement and people's ideas do not always get taken up; and if they do, it is not always because of the merit of the ideas (Barron, 2003; Esmonde, 2009; Langer-Osuna, 2011, 2016). In the context of this study, it would, therefore, suggest that surfacing of doubts is also a function of how students position themselves or are positioned in relation to their peers or their ideas (Herbel-Eisenmann, Wagner, Johnson, Suh, & Figueras 2015). Thus, the doubt may become visible and gain traction only if it is first acknowledged by self and shared with social others *and* if it is seen worthy of further consideration by others. For example, in the Ana and Valeria example, Valeria's doubt, "I wonder why there are numbers on the board next to the blocks," at first might seem provocative to an adult. It holds risk of getting shunned or mocked because it is not clear what the student might want to understand by that basic existential question other than using it as an evasive maneuver. Her intention or need behind asking that question, however, was in stark contrast to what we might think at first. And when her observation about numbers was appreciated and invited for further exploration—"That's an interesting observation. Why do you think they are?" it opened up the required space for Valeria, and with her for Ana, to mobilize it towards revealing an important conceptual knot through relational thinking at the heart of growth pattern problems (see pages 24-28 for a full analysis).

Findings also suggest that by attending to mathematical doubts of students we are also attending to their epistemic needs about what they seek to know and do, and as will be discussed in the next chapter, students' changing perspectives on what can be known and done, and how to know it. Thus, creating conditions that would allow students to pursue their doubts is also an act of nurturing students' epistemic agency (Agarwal & Sengupta-Irving, 2019; Stroupe, 2014). Miller, Manz, Russ, Stroupe, & Berland define epistemic agency as "students being positioned with, perceiving, and acting on, opportunities to shape the knowledge building work in their classroom community". Though Miller et al., refer to it in relation to science education, scholars in mathematics education have also argued how nurturing students' epistemic agency influence students' relationship with the discipline and its learning (Gresalfi, Martin, Hand, & Greeno, 2009). Denying or ignoring them, however, would be an act of epistemic injustice. Literature on tracking and race suggests a great deal of denial of epistemic needs and of injustice that low-tracked students and students of color face in the classrooms (Donaldson, LeChasseur, & Mayer, 2016; McKinney & Frazier, 2008; Means & Knapp, 1991; Oakes, 1987, 1990; Reardon & Owens, 2014). By recognizing the inherent ethical nature of student doubts and the ethics of considering doubts as something that not only belongs, but provides a foundation in a mathematics class, this study opens up a space for critical dialogue about what a mathematics problem is, who can be considered a problem-poser, and why we need problem-posing as a practice for school children. These theoretical considerations justify a stronger foundation for researchers and educators from which to mobilize the various dimensions of student doubts for the study of mathematical problem-posing and problem-solving.

A nuanced clarity of student mathematical doubts in the current study offers a window into the varied ways in which students interpret, reimagine, enact, and re-create the mathematical world, when given agency to do so. This view can be used to assert the value in designing learning environments and curricula that focus on cultivating students' sense of how knowledge gets generated and who gets to shape it. According to Hiebert et al., the math practice of identifying new problems or making the given task problematic "provides an opportunity for students to 'recognize the inventiveness of their own practice' (Lave, Smith, & Butler, 1988, p. 69) and to see mathematics as an intellectual activity in which they can participate" (p.17). Emphasis on student doubts mobilizes multiple entry points and pathways for student problem-posing. It asserts that there is never *one* right problem to ask but many, sprouting from the students' own unique ways of interacting with and perceiving the open situation and making it problematic; and they may all be significant and meaningful (Brown & Walter, 2005) as long as they address students' unique epistemic needs and allow moving the doubts and their resolutions forward. Indeed, the variety of doubts and problems that students posed and solutions that they offered in the study make evident the diverse pathways of inquiry that such a process leads to. As such, classrooms that position student doubts as generative may better prepare students for mathematical problem-posing and its processes. Such classrooms may also be better positioned to challenge the norms that posit mathematical knowledge as a universal truth and mathematical creativity as a fixed trait of only some students.

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APPENDIX A

Task-based Paired Interview Workbook

A. What do you notice or wonder about the objects in front of you? Write down thoughts, observations, ideas, or questions that come to your mind when you see these objects.

B. Using this visual pattern and based on your observations, can you make a few math problems (interesting and/ or challenging) for your friends in the class to solve?

C. Based on your understanding, how do you think this pattern will grow visually? Can you draw and find a few more cases?

Case# **4**

Find one more case. Any case you want. Fill in the **Case# ___**

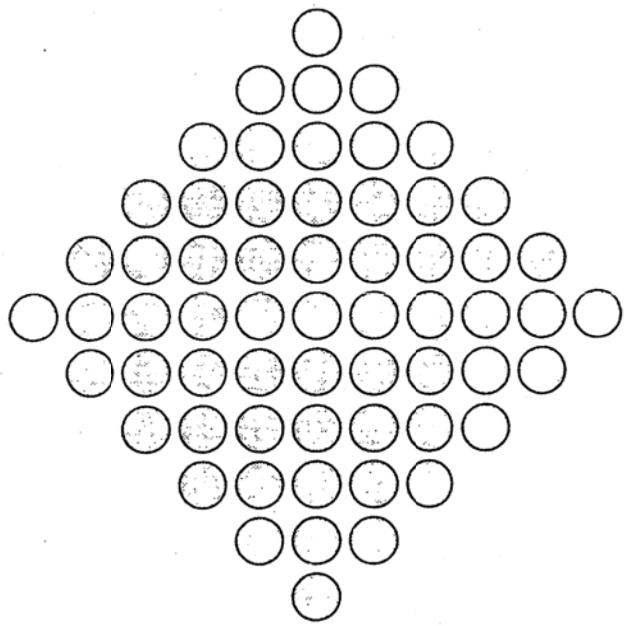
D. Together with your partner, find how many squares will be there in case #100?

E. How many squares will be there in case# N?

F. For his students' homework, Mr. Nunez wants to make up some interesting and challenging pattern problems. Help Mr. Nunez by writing as many pattern problems as you want to in the space below.

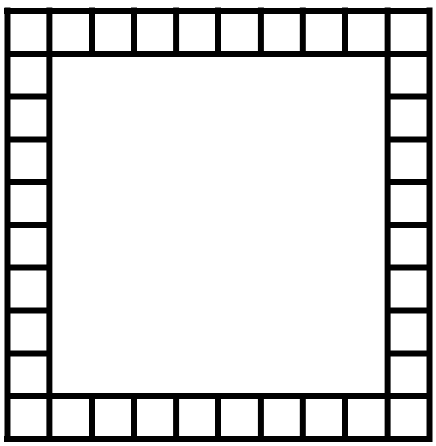
APPENDIX B

The image used for teacher modelling part of the task



APPENDIX C

Classroom-based Teaching Experiment Workbook



What all do I **wonder** about the given image?

What all do I **notice** about the given image?

<u>Using your group's wonder list, create an interesting math problem about the given image in your group.</u>

(Work with your group)

<u>Use this page to do any work related to the given tasks or to brainstorm ideas in your group.</u>

CHAPTER 2

Study 2. Characterizing Shifts in Student Participation in Mathematical Problem-Posing

The affordances of situating student problem-posing as a prominent feature of students' mathematical activity have been considered in terms of the potential for deeper engagement in mathematical inquiry and generativity. Problem-posing as a mode of learning is well-rooted in the philosophy of mathematics (Lakatos, 1963; Polya, 1954, 1957) and supported by research on inquiry-oriented mathematics classrooms (e.g., Boaler, 2002; Goos, 2004; Lampert, 1990; Moses & Cobb, 2001; Silver, Smith, & Nelson, 1995; Turner, Gutiérrez, Simic-Muller, & Díez-Palomar, 2009). Findings suggest that mathematics education should provide students with authentic experiences that characterize the activity of professional mathematicians in order to nurture students' agency. As such, problem-posing is seen as a means to engaging students as doers and creators of mathematics and not just knowers and consumers. Problem-posing has also been considered for its role in advancing student proficiency in problem-solving (e.g., Ball, 1993; Chazan & Ball, 1999; Staples, 2007) and as a way to facilitate students' deeper understanding of the standards-based mathematics content (Kilpatrick, 1987; NCTM, 2014; NRC, 2001; Silver, 1994). The National Research Council's (2001) *Adding it up* report identifies problem-posing as a "strategic competence," (p.124) and the National Council of Teachers of Mathematics's (2000) *Principles and Standards* consider it an important step that should precede students' problem-solving. Aspects of problem-posing have also been articulated in the Common Core math practice standards: "They make conjectures … consider analogous problems…try special cases and simpler forms of the original problem" (Common Core State Standards Initiative, 2010).

Given the push to include mathematical problem-posing as part of the curriculum (NCTM, 2000; NRC, 2005), researchers have begun theorizing designs for effective problem-posing-based learning environments (e.g., Lueng, 2013; Singer, Ellerton, & Cai, 2015). Making progress towards this goal includes understanding socio-mathematical processes through which students gain legitimate entry to the problem-posing activity and persevere in posing meaningful problems. Past empirical studies of mathematical problem-posing have tended to foreground the cognitive aspects of problem-posing and the characteristics of its outcome—the problem that gets posed (e.g., Cai & Hwang, 2002; Silver & Cai, 1996). This emphasis on cognitive strategies of posing and its outcome, although important, mystifies the act of posing by suggesting it to be a capacity of creative and high-achieving students (e.g., Ellerton, 1986; Leung, 1993; Leung & Silver, 1997). An over-examination of the cognitive aspects of problem-posing in the literature overlooks it as a basic practice that each and every student should have access to in their classrooms.

Understanding processes of problem-posing is even more urgent now due to the simultaneous press for collaborative learning in K-12 schooling (Boaler & Staples, 2008; Cohen & Lotan, 2014; Sherin, 2002; Silver & Smith, 1996). When students work collaboratively they engage with the content and also with each other, and are influenced by others' ways of thinking, doing, and learning (Sengupta-Irving, 2009, 2014). The social milieu of collaborative work requires students to negotiate more than just the content and overall learning environment. They also have to attend to the social dynamics of learning together (Barron, 2000, 2003; Cohen & Lotan, 2014; Esmonde & Langer-Osuna, 2013; Gresalfi, 2009). As scholars interested in student problem-posing, we must account for more than just the individual cognitive strategies of posing;

we must also attend to how problem-posing unfolds in action among students and influences each other's practices.

In chapter 1, I defined problem-posing as a practice that constitutes student questioning, conjecturing, and thought-experimenting when faced with doubts about mathematically ambiguous situations. I outlined the nature of students' doubts and displayed how they were connected to the problems students posed. I argued that when students have opportunities to voice, share and take up their nascent doubts then they also gain access to problem-posing. Thus, voicing doubts was seen as a legitimate form of peripheral participation in problem-posing. But peripherality without participation is not enough (Lave & Wenger, 1991) and it is important for us to understand what affords or shapes student participation in how they take up initial doubts to problem-pose. Thus, in this study, I ask: How do students shift from the periphery of their doubts to engage more fully in posing mathematical problems? The study focuses on understanding students' collaborative practices, the role of the given task and materials, and other resources that students need to more fully engage in problem-posing.

Literature Review

There are at least two related sub-strands within the problem-posing literature that I draw on to conceptualize the study of problem-posing processes and practices: 1) research characterizing aspects of good math problems and problem-posers and, 2) studies investigating how students become engaged problem-posers and what problem-posing entails.

Characteristics of Math Problems and Problem-posers

A fundamental question that scholars have attempted to answer in the last three decades is the nature of problems that students pose when specifically asked to do so and what that tells us about the learners. Both experimental and naturalistic studies have continued to provide strong evidence that, in general, students are able to generate mathematically solvable problems. Silver & Cai (1996) conducted a survey study of more than 500 ethnically and linguistically diverse sixth and seventh grade students from low-income communities as a part of the QUASAR project. Students had no prior instruction in problem-posing. Silver & Cai found that nearly 80% of the students generated at least one problem, 90% of which were mathematically solvable and several were syntactically and semantically complex. Using Silver & Cai's solvability and complexity criteria, Bonotto (2013) investigated problems posed by 71 fifth-grade students in two Italian schools. Although the problem-posing activity was more open and ill-structured than that used by Silver & Cai, it was found that almost all students were able to generate mathematically relevant problems and more than half of them were solvable. Even naturalistic classroom-based studies such as Armstrong (2013) and Fiori and Selling (2016) have concluded that students were, in general, able to pose solvable math problems without instruction.

In fact, a considerable proportion of past research on problem-posing, especially studies investigating student creativity, have applied Silver & Cai's (1996) criteria of solvability and complexity to assess student work (Bonotto & Dal Santo, 2015; Cai & Hwang, 2002; Ngah, Ismail, Tasir, Said, & Haruzuan, 2016). Problems were deemed solvable if they were appropriately structured in a question form and included the necessary information needed to be solved. Questions such as "Why didn't Jerome drive more?" or statements like "Jerome drove 50 miles" are examples of non-solvable problems. Problems were considered complex if they constituted a larger number of structural relations linguistically and were mathematically open-ended (i.e., allowed more than one solution to the problem).

Problem-posing has also been used as evidence of students' knowledge, problem-solving ability, and creativity, though often with mixed and conflicting results and interpretations (e.g., Van Harpen & Sriraman, 2013; Yaun & Sriraman, 2011). Using written assessments, various studies have investigated content knowledge, problem-solving, and problem-posing to find that those with higher mathematical ability often pose more coherent and complex problems (Cai & Hwang, 2002; Ellerton, 1986; Harel, Koichu, & Manaster, 2006; Harpen & Presmeg, 2013; Silver & Cai, 1996). For instance, Silver & Cai (1996) found that while many students were able to pose solvable problems, more successful problem solvers generated more mathematical problems and their problems were more mathematically complex as compared to less successful problem solvers. However, some studies of preservice teachers (Crespo, 2003; Leung, 1993; Leung & Silver, 1997) have been unable to establish a clear connection. In a comparative study of students in the U.S. and China, Van Harpen & Sriraman (2013) found contradictory links between problem-posing and student ability and instead suggested that while problem-posing may be related to basic knowledge and skills, it is not related to students' ability, especially ability to solve routine math problems as tested by written assessments.

I argue that when we characterize a problem based on solvability, complexity, and creativity alone—characteristics that center outcomes of problem-posing, we lose sight of students' doubts that might have impelled them to pose problems in the way they did and processes that might have further shaped them. If our goal of students generating their own problems is to nurture inquiry, agency, deeper engagement, and learning, then we need to shift our focus away from assessing problems in relation to the discrete categories of solvable/unsolvable, creative/not creative, complex/not complex to instead consider how problem-posing allows students to shape their lines of inquiry and extends agency to reason, argue, and resolve mathematical quandaries they self-identify. Posing and reposing a problem in a way that satisfies what students want to know and do in a particular situation, i.e., their epistemic needs[2], can make evident for students that math is not simply a curated form of facts and procedures but a tool that allows resolution of human perplexities and musings about the world in which they live. Understanding student problems in this way might help educators see how students view the mathematical world and what their unique ways of knowing and doing are.

Processes of Problem-Posing

Given the importance of problem-posing for nurturing students' mathematical inquiry and for broadening their view of themselves in relation to the discipline, another line of research has focused on understanding the process through which students who have not been formally taught "problem-posing" pose problems (e.g., Brown & Walter, 2005; Christou, Mousoulides, Pittalis, Pitta-Pantazi, Sriraman, 2005; English, 1998; 2014; Silver, Mamona-Downs, Leung, and Kenney, 1996). Findings in this vein are more varied and dependent on the nature of the problem-posing task and the setting, as well as on the scholars' subjective focus and theoretical underpinnings. For instance, while some scholars have used writing assessments in attending to the cognitive strategies students use (e.g., Brown & Walter, 2005, 2014; Cai & Cifarelli, 2005; Leung, 1993; Silver, Mamona-Downs, Leung, & Kenney, 1996), others have focused on student activities and discourses in naturalistic classroom environment (e.g., Armstrong, 2013; Bonotto, 2013; Fiori & Selling, 2016); yet others have focused on stages of problem-posing as it occurs before, during, or after problem-solving (Silver, 1994; see also Brown & Walter, 1983/2005; Singer & Voica, 2013). Across these varied approaches, research has consistently found problem-posing that occurs *during* problem-solving to be a recursive process involving posing, solving, reflecting, revising, and re-posing (Armstrong, 2013; Cai & Cifarelli, 2005; Leung,

[2] Epistemic need, defined in Chapter 1, refers to what students are drawn to know and do, and their developing understanding of what can be known and how to know it.

1993). Through recursive processes of problem-posing, students clarify and enhance their understanding of disciplinary concepts in solving the given problem Lakatos (1963).

Fiori & Selling (2016) studied problem-posing occurring *before* problem-solving to find a similar recursivity in how students pose, revise, and refine math problems with their peers. Additionally, they found problem-posing to be a non-linear intuitive process, especially in the absence of a structured task and objective. The focal classroom in Fiori & Selling's (2016) study was part of a five-week summer school class for middle school students entering sixth or seventh grade. Students were given freedom to generate mathematical ideas, choose between problems, and collaborate with peers. Most importantly, "mathematical judgments of taste were encouraged and supported, primarily by encouraging students to pose and evaluate mathematics problems" (p.221). In these summer-school classes, existing outside of the pressures of regular curricula, "students were encouraged to stay with a problem for as long as it was still attractive to them" (p.221). The thirty students came from diverse ethnic, class, and first-language backgrounds. The room had physical stations consisting of everyday objects such as L-pipes, nuts, bolts, SET game cards, geoboards, dice, snap cubes, pattern blocks, pine cones, and colored beads of various shapes and sizes. Students were encouraged to explore various stations, think about the mathematics that emerged from the setting, use the given objects to search for interesting problems, and switch stations if they were not inspired. The teacher helped students explore mathematical ideas in the stations and helped refine student-posed-problems. By following a sample of students through observations (field-notes) and interviews, the authors theorized that students were impelled to make discerning choices when nominating, combining, and balancing ideas similar to how professional mathematicians do (Fiori, 2007). More specifically, students made discerning choices when nominating ideas from myriad possibilities, when trying ideas out, or when abandoning ideas that seemed unproductive or chaotic, until they eventually found the one promising problem. Students combined ideas when juxtaposing multiple students' ideas in order to gain new perspectives on old concepts. Additionally, students favored ideas that were simple yet which provided potential to expose deep relationships.

Understanding the processes of problem-posing, such as nominating, combining, and balancing ideas are important for advancing research on the design of posing-based learning environments. However, since these processes were derived from the work of only a few selected students, it is not clear if there were any differences in paths students take in finding a problem. Moreover, the study by Fiori & Selling (2016) took place in a summer school where students had an abundance of time to work on activities and teachers did not face standards-based and administrative pressure. Thus, it is not clear if these processes will also hold within the complex ecology of typical classrooms during the school year. Indeed, problem-posing under the conditions of more typical schooling raises new avenues of investigation into the recursive processes of problem-posing, how students marshal social setting, materials and one another's thinking to pose problems, and what role the teacher plays in relation to problem-posing with students.

Research Goals

I argue that to better support students in problem-posing and to engage them in becoming agentic problem-posers, it is crucial to (1) understand the trajectories of participation that unfold as students move from the periphery toward fuller participation and (2) to clarify the nature of collaborative activities that allow students to shift their practices and thinking through and in problem-posing. More specifically, I ask: How do students shift from the periphery of their doubts to engage more fully in posing mathematical problems? What is defined as a mathematical problem in this study is a departure from how it has been defined in the past by using the criteria of solvability, creativity, and complexity. A *problem* in the study is defined as one that allows students to resolve their initial mathematical doubts for which they do not already have a resolution. This conceptualization contrasts with prior studies because it ties problem-posing to student doubts—mathematical ideas that are yet unresolved for students but within reach (Hiebert & Grouws, 2007; NCTM, 2014). The problem as created by the students may not be complex, well-structured (i.e., solvable), or creative from the perspective of teachers or researchers, but as long as it is meaningful for the students in moving their doubts forward toward resolution, it is considered valuable.

Conceptual Framework: Shifts in Participation

Broadly considering learning as fundamentally a socially situated activity, I draw from a framework within sociocultural theories of learning, namely, legitimate peripheral participation (LPP; Lave & Wenger, 1991). They describe LPP as a socially situated process by which newcomers gradually move toward fuller participation in a given community's activities by interacting with other community members. The focus of the LPP model is on activities, practices, and processes of knowing, which are especially relevant to understanding the processes of problem-posing; the focus is not, as seen in prior research, on simply its outcomes. Lave and Wenger view learning as a *shift in participation* within a community of practice. The argument is that people learn more effectively through participating in the praxis of the community, rather than by first learning procedures and then applying them. In the present study, legitimate peripheral participation is conceptualized as a position where students surface their doubts (including perplexities, musings, wonderings, uncertainties, and conjectures; see Chapter 1) about mathematically ambiguous artifacts as a way-in to the practice of problem-posing.

Using Goos (2004), I conceptualize the community of practice in which students are being apprenticed as *communities of mathematical inquiry*. In the learning communities of inquiry, students do not rely on the teacher's unquestioned authority, rather they are "expected to propose and defend mathematical ideas and conjectures and to respond thoughtfully to the mathematical arguments of their peers" (Goos, 2004, p. 259). Within this learning community, the idea of novice-expert is coordinated around the model of the collaborative zone of proximal development (collaborative ZPD) where students make this shift towards more sophisticated practices with the assistance of peers of comparable expertise and the teacher (Forman & McPhail, 1993; Goos, Galbraith, & Renshaw, 2002). While peers overall may be of comparable expertise, they may hold distributed expertise in various parts of the activities, thus providing impetus to each other towards shifts in participation. For example, there may be differences in students' fluency in facilitating group talk, using representations and tools to move the reasoning forward, in generating multiple dimensions of doubts, in their forms of questioning, and so on and so forth. Fuller participation in the practice of problem-posing within this community involves students persevering in participating in recursive processes of posing and solving in order to shape and seek resolution to their initial nascent doubts that are not yet resolved.

The ideas of *peripheral legitimacy* and *situated practices* of a community are central to making sense of newcomer's shifts in participation (Wenger, 1998). Peripherality suggests "an opening, a way of gaining access to sources for understanding through growing involvement" (Lave & Wenger, 1991, p.37). Considering peripheral participation as legitimate allows newcomers access to a safe space where they can make mistakes, explore freely, observe other members, engage in the community practices, gain support, and gradually extend and expand their practices. Thus, peripherality also suggests that an individual's position within a community changes over time, i.e., participation is understood through a temporal focus. Additionally, critical to Lave and Wenger's analysis is their recognition of *multiplicity in participation*. An individual's behavior cannot be recognized without other community members and the situated practices of the community. Students cannot fully participate if they are not recognized as such by other students and the teacher, no matter what their skills or knowledge. Students might, in fact, be competent mathematically, but these capacities are not relevant if they are not practiced and demonstrated within the community activities. According to this theory, skills and knowledge are understood in relation to the process of becoming a kind of person, in relationship with community members and activities. And so there "may well be no such simple thing as 'central participation' in a community of practice" (p. 35), Participation can only be seen as shifting towards more full participation through possibilities of multiple pathways and situated negotiations. Wenger (1998) says a community of practice exists when "people are engaged in actions whose meanings they negotiate with one another" (p.73). Thus, there is a sense that individuals can *belong* in multiple ways within the situated practices of the community.

In the study, the temporality of participation is captured in gradual and dynamic shifts in how students take up their doubts for problem-posing and how they revise and refine the initially formulated problems towards more meaningful ones over time. Recall that problems are meaningful if they allow students to move their doubts forward toward resolution. The situated focus in the study emerges in the analysis of the small group of students and in how these smaller communities, within the larger community of the classroom construct meanings of their interactions, roles, and norms over time in relation to the task, tools, artifacts, and the teacher. The study follows students' activities during one 50-minute long class period. Thus, while it is assumed that students will make certain shifts toward fuller participation, it is not expected that students will come to fully internalize their problem-posing processes or become fully agentic problem-posers by the end of a single class period because that, of course, requires engagement in the practice and activities of the community over longer periods of time. The goal is to understand the nature of those shifts over a fifty-minute long activity knowing that with more time and support students may eventually extend their participation toward fuller participation. Lastly, while the study investigates shifts in participation, it does not have the capability to investigate shifts in identities, which is another dimension of learning described by Lave & Wenger (1991) and takes much longer to change. Figure 2.1 outlines the conceptual framework. Emphasizing peripheral, temporal, and situative aspects of participation make visible the nature of shifts in participation that students make even at the peripheral position, pathways they may take over time, and barriers they may face.

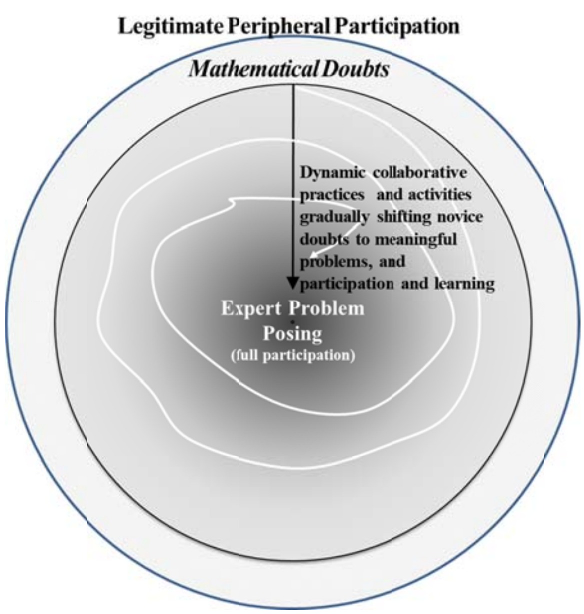


Figure 2.1. Conceptual framework of legitimate peripheral participation in problem-posing

Finally, Lave (1988) and Lave and Wenger (1991) recognize the role of *structuring resources* that "shape the process and content of learning possibilities and apprentices' changing perspectives on what is known and done." I characterize students' perspectives on "what is known and done" as their epistemic needs and something that impels them to surface their doubts. When they surface their doubts and take them up for posing problems, they begin to shift from the periphery to becoming active members of the community. As their participation shifts in the community so do their "perspectives on what is known and done" (p.91). Structuring resources to accommodate the changing nature of students' epistemic is crucial. Authors identify sponsorship by the master/experts, developing relations between community-members, characteristics of the division of labor, learning norms that unfold, epistemological role of the artifacts in the context, and discourse (for newcomers purpose is not to simply "learn *from* talk" but "learn *to* talk") (p.109) as key resources that allow or constrain the shifts in participation.

Methods

Study Context

The study was conducted in two low-tracked math classrooms at the Valley Middle School (VMS; pseudonym) in one of the largest school districts in California primarily serving working-class and immigrant Spanish-speaking families. The school district serves over 95% Hispanic students, 41% students identified as English language learners and about 90% of the population is eligible for free or reduced meals.

Teacher selection

During my preliminary work in the school-district in 2016-17, I observed nine seventh and eighth-grade math teachers' instruction and interviewed ten eighth-grade math teachers (as part of another larger study[3]). Among the classrooms observed, Mr. R was the only teacher who implemented elements of group work and active learning in his teaching. He willingly shared with me his pedagogical thinking, his knowledge of reform-based teaching principles such as flexible groupings—gained when pursuing his Masters in Teaching degree—and his past work advocating for de-tracked classrooms. For these reasons, and because the design of my study includes students working with peers in small-groups, I invited Mr. R to participate in the study to be implemented in the next school year, 2017-18. Mr. R is a white-American and, at that time, in his second year of teaching non-honors mathematics classes in eighth-grade at VMS. He also previously taught seventh-grade courses for about ten years.

I observed Mr. R's classrooms three more times at different time points (for a total of four days) during the 2016-17 school-year and video-recorded one class over two days (with parental permission). At the beginning of the next school year, prior to the problem-posing intervention, I observed two class periods over nine days that were co-selected for the experiment (six were video-recorded with parental permission). These observations across different time points and school-year prior to the experiment were intended to help me understand Mr. R's typical instruction and the nature of learning communities he nurtured.

Mr. R's typical week followed the following structure: On Mondays, a quiz was given testing last week's material. Tuesdays were reserved for either starting a new unit or introducing a new method. Mr. R solved a few easy problems to show how the method is used. On Wednesdays, students would solve similar problems with the teacher as a whole class routine. On Thursdays and Fridays students would solve more problems, including perhaps more challenging multi-step problems on the same topic in their groups and present their work to the

[3] The larger study was funded by Spencer Foundation with PI Dr. Thurston Domina. As a part of the study, I conducted ten teacher interviews at the district. The interviews focused on understanding teachers' beliefs about ability-based curriculum and instructional differentiation, their perceptions of student needs and resources.

class either verbally or on the class board. As the week progressed, problems would go from easy to a little more difficult. The tasks were traditional in nature and did not demand high cognitive engagement. Math discussions were surface-level and did not go deeper into the concepts. While Mr. R often engaged in questioning students when problem-solving, they were always related to the procedures as opposed to conceptual connections. Moreover, I did not notice any occurrences of student problem-posing during my preliminary visits to his classrooms, and student questioning was minimal at the most.

So, while Mr. R reflected during interviews that he believes all students are capable learners, had knowledge of reform-based instruction and often encouraged students to share their ideas and solutions; his pedagogical practices and participation structures were limited in allowing students to take up authority and accountability towards their learning. For instance, students in his class always sat in groups of four, but the time allocated for students to work with peers was only 1-3 minutes long. Additionally, there were no norms that were co-constructed with the students about their accountability to peers and their thinking. During the short group work-time students largely solved problems individually and check their answers with peers, their problem-solving activity was not collaborative.

Settings and Participants

The data sets are drawn from two separate data collection phases at VMS constituting two very different settings: Task-based paired interviews of seventh and eighth-grade students with the author at the end of the school-year 2016-2017, and a teaching experiment in two eighth-grade non-honors (low-tracked) classes taught by their teacher Mr. R at the beginning of the school year 2017-2018. These settings are summarized in Table 2.1 (for additional details about settings and tasks see Chapter 1).

Table 2.1. Summary of the two research settings

	Task-based paired interview (Houssart & Evens, 2011)	**Classroom-based teaching experiment** (Cobb, 2000)
School	Valley Middle School	Valley Middle School
Month/ Year	June 2017 (end of 2016-17 school-year)	September 2017 (beginning of 2017-2018 school-year)
Grade	7th-nonhonors (36), 7th-honors (10), 8th-nonhonors (18)	8th-nonhonors (2 periods with 25 and 31 students respectively)
Total # of students	64	26 videotaped (56 total)
Group formation	2 students interviewed together	4 students in a group
Student interaction videotaping	Yes (all)	Yes (4 groups of 4 students each in period 1. 2 groups of 4 students each and 1 group of 2 students in period 2 using separate cameras and mics)
Sessions/ Length	32 interview sessions (1 per pair), each 35-50 minutes long	Instruction over 50 minutes long period
Data for analysis	27 video-hours; written work; pictures of student work and activities	6 video-hours; written work; pictures of student work and activities; audio-recorded student interviews
Task artifact	Three stages of a growth pattern made using snap cubes were displayed on the desk for students to use for problem-posing (Figure; See Appendix A for task worksheet)	An image of a square with unit squares on its border (referred as Border image) was printed on the worksheet shared with the students and displayed on the class smart-board to use for problem-posing (Figure; See Appendix C for task worksheet)
Task participation structures	1. General talk to make students feel comfortable 2. Individual quiet writing of observations and wonderings 3. Individual creation and writing of math problems 4. Verbal sharing of doubts and problems with clarifying questions from the researcher and the peer to understand student thinking 5. Group problem solving: find case 4, case 100, case N (with clarifying questions from the researcher) 6. Group problem-posing 7. Sharing of the problems with the researcher including questioning from the researcher to understand student thinking	**Day 1:** 1. A whole-class discussion where the teacher modeled how to notice and wonder and work with peers (assisted by the researcher acting as teacher's work partner) using a sample image (Appendix B) 2. Individual quiet writing of student notice and wonder using Border image (6 minutes) 3. Whole-class sharing of doubts (1 doubt and 1 student per group) **Day 2:** 4. Whole-class sharing of doubts as a reminder to previous class (1 doubt and 1 student per group) 5. Group sharing of doubts with clarifying questions from group peers (10 minutes) 6. Group problem-posing (15-20 minutes) 7. Whole-class student reflections
Tools	Worksheet with written task and space for students to work on (see Appendix A), graphing paper, extra snap cubes, colored markers, room white-board	Worksheet with written task and space for students to work on (see Appendix C), graphing paper, calculator, ruler, protractor, colored markers, A4-sized desk white-board

Participant Selection

Task-based paired interviews. The task-based interviews were conducted at the end of the 2016-17 school year. Parent consent forms were distributed to students in all six seventh-grade classes at VMS (including four non-honors and two honors), a total of 180 students. Forty-six students (36 non-honors and 10 honors) agreed to participate and all were interviewed. Parent consent forms were also distributed to all students in two eighth-grade non-honors classes taught by Mr. R consisting of 60 students. 18 students agreed to participate and were all interviewed.

Classroom-based teaching experiment. The teaching experiment was conducted at the beginning of the 2017-18 school year. Parental consent forms were distributed in two of Mr. R's classes (period 1 and period 3). These classes were selected by Mr. R because there was a preparation period between them that Mr. R thought would give him time to reflect and revise the lesson if needed. Both classes were low-tracked math classes, which meant all students enrolled in them had scored less than the 50^{th} percentile on the district benchmark math tests in their seventh-grade year. All students in the two periods were Hispanic. About 37% of students were designated ELLs and 56% re-designated English fluent. 94% of the students were eligible for free or reduced lunch.

In period 1, 16 out of 25 students and in period 3, 10 out of 31 students agreed to participate. All students who agreed to participate were videotaped (n=26). Written work was collected from all participating and non-participating students in both the periods and all students received the same instruction and tasks. Students participating in the study were grouped together forming four small-groups of four students in period 1 and three small groups in period 3 (two groups with four students each and one group with two students). Cameras that were recording student-groups were placed such that they captured only the students participating in the study and caused the least disruption in the class. Camera microphones were attached to the desks.

Analytic Methods

The study seeks to answer: How do students shift from the periphery of their rudimentary legitimate doubts to engage more fully in posing meaningful mathematical problems? A micro-ethnographic (qualitative) approach was employed to gain an in-depth and holistic understanding of shifts in learners' practices and organization of their problem-posing activities that allowed for those shifts. The data included videotaped observations, fine-grained analysis of the collaborative process, written-work (including all drafts and final outcomes of student problem-posing), and student reflections for both settings.

Following the qualitative research tradition, I aimed for "thick descriptions" (Geertz, 1973; Lincoln & Guba, 1985) of the student practices and activities, while also attempting to identify general trends and significant patterns among them (Miles, Huberman, Saldaña, 2013). Achieving this goal required comparing and contrasting groups of students within each setting, as well as juxtaposing emerging patterns of one setting with the other setting. The goal was to capture multiplicity in student practices and activities with an eye to general patterns they might be following and also account for deviant and disconfirming data about problem-posing processes.

Prior to analysis, I, along with two undergraduate research assistants, watched all videos from the two settings (paired interviews and classroom experiment) and created content logs for each video. The content logs documented general activities such as the type of activity, discussion, peer collaboration, and time spent on various sub-tasks within the given problem-posing task. A more specific content sheet was created that documented only students' written work, i.e., students' mathematical thoughts on (1) what they notice/ wonder, (2) initial problem-posing, and (3) later problem-posing. I then re-read the content logs and re-watched each paired interview to expand upon the content sheet generated earlier. In the expanded content sheet, I added what students did and said in relation to their written work (see Table 2.2 for a sample from the expanded content sheet). The analysis followed two stages as described below.

Table 2.2. Example of content sheet

Video Name	**Pair #**	**Student Name (pseudonym)**	**Notice/Wonder**	**Initial problem-posing (I)**	**Later problem-posing (II)**
24_6.13	**Pair 24**	Alan	Wonder if it involves math; pattern going 2,4,6; tools to help with math (*reasoned that cubes remind him of counting blocks when asked to explain his thinking*); patterns that can help people understand things; wonder if I will use them	If x red cubes and y blue cubes, how many in total; Jim x, Jacob, y, Alex z, how many Jim has more than Jacob; want to buy 3 cubes, have $3. 12 cubes cost 1.50, how much money needed to buy 3 cubes;	Made a staircase pattern, how many are in total (*Dulce: "it is a pattern of adding by 1, so if each strip is a case, case 7 would be 7 blocks, what would case 32 be? It would be 32 times 1 because the base is 1 and height 32"*)
24_6.13		Dulce	A pattern; wonder if it will end, add 2 each time, looks like a math problem seen before, wonder if it's leading somewhere; it is going forever (*when asked to explain reasoned that it can go forever in any direction; will be subtraction if it was going the other direction, also mentioned that instead of subtracting you can keep divide as well to create smaller squares in the other direction*)	find case 7; what if you multiply by 1 or 3 instead of 2; add up by 5 instead of 2, find case 4; find case# with 14 cubes	blue: $2, red $3, case#7 has 2 red and 7 blue, how much money spent?
26_6.13_H	**Pair 26**	Daniel	See cubes; same color; 2 blocks added each time; why are their numbers underneath the blocks?; why adds 2 each time (*when asked why he thinks they are added he reasons that it is because it is a pattern; when further asked about what pattern he sees, he says blocks going up and can also go down and they will be negative, uses red blocks to denote negatives in the pattern*)	2x+2 (*explains:2 is two cubes, x is number beneath and 2 is two cubes added; it can get the number of cubes it can get in each figure; revises to say the formula will tell how many cubes will come up after*), 2x+1, 2x+x, etc…	(*Made a pattern in which change is alternating between 4 and 3. case1=3, case2=3+4, case3=3+4+3, case4=3+4+3+4, etc.*) Find the rule (*to solve it, Michell suggested they have 4+3=7x100=700 and then will add 10 to it, so 710)*;
26_6.13_H		Michell	See blue blocks; 2 blocks added each time; number on the bottom; cubed shaped; white paper	2x2+2 [*explains: 2 (case 1) x 2 = 4 (case2) and add 2 to get 6 (case 3)*]; 2(2)+2; 6-2-2; 2square+2	Made a 3s pattern first together with Daniel (*then Daniel thought it would be more challenging to alternate like above*)

In the first stage of data analysis, I wanted to understand how students' participation was shifting from voicing doubts (a position of legitimate peripheral participation) to posing more meaningful problems, and what practices underpinned these shifts. Data analysis was primarily inductive: categories and themes emerged mainly from the collected data, and preliminary hypotheses about the problem-posing processes were grounded in what students were doing and saying (Marshall & Rossman, 2016; Hatch, 2002; Strauss & Corbin, 1998). I started by analyzing the paired interviews that helped me generate patterns of practices and shifts. These themes were then confirmed as well as revised by analyzing the classroom data. The episodes were analyzed according to the patterns of participation as well as mathematical qualities of participation. The patterns and mathematical qualities of participation are interlinked such that changes in participation gives rise to shifts in mathematical thinking and shifting mathematical thinking shapes new participation forms (Gresalfi, Martin, Hand, & Greeno, 2009).

I analyzed the paired-interview data using a constant comparison method (Glaser & Strauss, 1967), where the unit of analysis was the pair of students. I began by reviewing each column in the content sheet and coded the three columns for existence of doubts, type of doubts (chapter 1 findings used as coding schema), type of initial and later math problem created, if there existed any relation between the doubts and the problems posed as explained (saying) and explored (doing) by the students, the extent to which the students collaborated and used the given tools, and a very brief memo of shifts in participation from doubt to the problem-posed (see Table 2.3). Next, based on these codes, I grouped each pair into various categories by sorting, constantly comparing, and re-categorizing until saturation in category adjustments had occurred. For this, I printed the content sheet and cut out the strips for each pair, laid the strips out on the table, and constantly compared, contrasted, sorted, moved, and re-sorted the pairs on the basis of the codes that were generated earlier. The analytic process led to the understanding of problem-posing practices (as will be described in the findings section).

Table 2.3. Example of content-sheet coding

Pair #	Student Name (anonymized)	Notice/Wonder	Doubt Type (Findings from Chapter 1; P=Pragmatic; A=Analytic; T=Transformative)	Initial problem-posing (I)	Code (I)	Later problem-posing (II)	Code (II)	Analytic Memo
Pair 24	Alan	Wonder if it involves math; pattern going 2,4,6; tools to help with math (*reasoned that cubes remind him of counting blocks when asked to explain his thinking*); patterns that can help people understand things; wonder if I will use them	P	If x red cubes and y blue cubes, how many in total; Jim x, Jacob, y, Alex z, how many Jim has more than Jacob; want to buy 3 cubes, have $3. 12 cubes cost 1.50, how much money needed to buy 3 cubes;	simple addition problems; attaching real-life meaning to the cubes	made a staircase pattern, how many are in total (Karina: it is a pattern of adding by 1, so if each strip is a case, case 7 would be 7 blocks, what would case 32 be? It would be 32 times 1 because the base is 1 and height 32);	addition problem	P doubts --> related blocks to numbers and then nominated various ideas using numbers in various mathematical real-life situations --> shifted to think about staircase pattern. Did not use the blocks, largely worked on the worksheet. Reflected thinking on each other's ideas when asked but did not work together
	Dulce	A pattern; wonder if it will end, add 2 each time, looks like a math problem seen before, wonder if it's leading somewhere; it is going forever …	A, T	find case 7; what if you multiply by 1 or 3 instead of 2; add up by 5 instead of 2, find case 4; find case# with 14 cubes	figuring rule of the pattern; changing the pattern (what-if);	blue: $2, red $3, case#7 has 2 red and 7 blue, how much money spent?	figuring rules of different patterns. using red and blue blocks to create a bi-variate equation and attaching real-life meaning to the cubes	
Pair 26	Daniel	See cubes; same color; 2 blocks added each time; why are their numbers underneath the blocks?; why adds 2 each time …	P, A, T	2x+2 (*explains:2 is two cubes, x is number beneath and 2 is two cubes added; it can get the number of cubes it can gt in each figure; revises to say the formula will tell how many cubes will come up after*), 2x+1, 2x+x, etc…	algebraic expression [f(n)=2(n-1)+2]	made a pattern in which change is alternating between 4 and 3. case1=3, case2=3+4, case3=3+4+3, case4=3+4+3+4, etc. Find the rule (*to solve Michell suggested they have 4+3=7x100=700 and then will add 10 to it, so 710*);	made a challenging pattern-figuring rule of the pattern	P, A doubt --> unique algebraic Expression --> 3s pattern --> more challenging pattern (could not solve). Used blocks to tinker, used the whiteboard to work ideas out. Engaged peers in ideas and worked together.
	Michell	See blue blocks; 2 blocks added each time; number on the bottom; cubed shaped; white paper	0	2x2+2 [*explains: 2 (case 1) x 2 = 4 (case2) and add 2 to get 6 (case 3)*]; 2(2)+2; 6-2-2; 2square+2	numeric expression	made a 3s pattern first (*then Daniel thought it would be more challenging to alternate like above*)	made a 3s pattern-figuring rule of the pattern	

The content sheet from the classroom data was then analyzed using the same process of coding as for interviews but categorization was done by asking if a group's practice followed one of the themes found from interviews or not. One of the themes was absent and one new theme emerged after analyzing the classroom data. Since the participation structures were more dynamic in the small-group setting in the classroom, there were times when students or groups of students did not neatly fit any one theme but were somewhere in between the two themes sharing partial characteristics of both.

In the second stage of the data analysis, I wanted to more clearly understand how students organized themselves in relation to the task, tools, rules, roles, peers, teacher and/or the researcher that afforded or constrained access to certain practices and shifts in those practices. Interaction analysis as an analytic method was used (Bryanson, 2006). First, I carefully selected six distinct cases from the paired-interviews—two cases from each practice theme that had emerged after the first-stage analysis. Each of the seven small-groups from the classroom setting was selected as cases for analysis. All 13 cases (6 from interview and 7 from the classroom) were transcribed capturing utterances, expressions, gestures, tones of voice, etc. Analytic memos were created to more carefully document the role of the tools, norms, and roles in allowing (or not) subjects to reconstruct objects of problem-posing and in influencing their outcomes.

Findings

The analyses of shifts in students' problem-posing processes revealed three patterns across thirty-two pairs of students in interviews and seven groups of students in classrooms. The changes were observed in how students shifted their thinking from nascent doubts to more meaningful problems in their mathematical and social practices. The three posing practices that emerged were:

1. *Assembling* their observations or doubts about the given task artifact to ask basic arithmetic or geometric problems that students could easily resolve
2. *Casting* known problems using a pre-existing mold (i.e., using a problem seen before)
3. *Carving* out promising ideas from multiple nascent doubts and *shaping* them into mathematical problems that are yet unresolved for students.

The three themes formed a constitutive set of practices of posing. Casting required students to also assemble their observations and doubts, and students who carved and shaped their doubts also initially created initial problems by assembling and casting.

1. Assembling

Most students during interviews and the classroom experiment began problem-posing by assembling their observations and/or doubts to create basic arithmetic or geometric problems that they could easily solve. For instance, during interviews students posed various numeric expressions as problems such as 2/6x4, 6x4x2, or 6-4=2. When asked why they chose those numbers for their problems, students explained they saw those numbers in the arrangement of the given pattern and used them for making math questions. Other students created numeric expressions unrelated to the given pattern such as $(7 \times 20 + 5) + (4 \div 4 \times 20 \times 100) - 10$ while tinkering with the extra snap cubes they were given. They explained that the blocks reminded them of the elementary school where they used them for counting or solving math questions. When asked how easy or difficult the problems will be for a seventh-grader to solve, most students said they would be easy, and students were easily able to solve them themselves.

Example: group 4. In the classrooms, Group 4 in period 1 and Group 7 in period 3 made the shifts required from surfacing to assembling their doubts in a manner whereby they could create a very basic math problem. Here, I present Group 4's talk and activities as an example of how those shifts occurred while drawing out salient aspects that afforded the shifts. Group 4 consisted of two girls and two boys: Ana, Dia, Juan, and Santiago. Girls sat next to each other and opposite the two boys. In the past, students in the group had difficulty working together and had remained off-task for a large chunk of several class-periods. When Mr. R invited the class to share their notice/wonder list with each other in their groups, group 4 students squabbled over who should start first and over the placement of the desk whiteboard—Dia and Santiago were pushing Ana to share first in a teasing manner, Ana was ignoring them, and Juan was looking around the classroom.

I, as a participant-observer and facilitator for group discussions (as requested by Mr. R), approached the group and attempted to help them initiate some simple turn-taking. I invited them by saying: "Take turns to share your ideas" … What do you have, Dia?" followed by more invitations to each member to share observations about the given image that were not already shared by a peer. By emphasizing not to share what has been shared by a peer, I was tapping into participation structures that I was expecting to nurture for students, namely, that they listen to each other, make connections of what they hear with their own ideas and doubts to draw out similarities, differences, and clarifying questions, and voice their own doubts so others could do the same. From the perspective of the problem-posing practice, the goal was for students to co-construct a repertoire of doubts that they could later draw on, investigate, refine, and reshape together for the creation of a math problem. The participation structure was already built-in to the task by-design and included in Mr. R's verbal instruction of it. I was simply re-asserting it for students.

> Now, in your groups, you're going to share your ideas that you wrote down on your page: your notices and wonder list. And, that way, everybody's up to speed with what

everybody on their team has. Be sure that you are discussing ideas that seem similar, identify any ideas that are different. If you have questions about what people meant, this is the time to ask them, and be sure to clarify all your ideas so everybody knows what you are talking about. Your list will be used for something in about 5 minutes. So be sure you know what your lists are and what they mean. You have 5 minutes to talk it over. (@19:50)

Students using the Border image shared:

Dia	It's a square, it's empty and it's 38 squares.
Ana	There are tiny squares surrounding this empty square. And then all the corners are ninety degrees.
Santiago	It looks like a picture frame.
Juan	There are ninety-degree angles right here.

I then invited students to share their wonders: "So make sure you share what you wondered and discuss what ideas are same and what are different." Ana initiated and soon students were elaborating and asking each other questions without much support from me.

			Talk and action	**Analytic Memo**
1	21:40	Ana	What is the pattern and what is the relationship?	*Analytic doubts*
2		Santiago	(*while looking down*) I don't know buddy.	
3		Dia	(*while looking blankly towards the desk*) The relationship is that they are all squares.	*Conjecture*
4		Ms. P	Relationship between what? So, if you want to find a relationship between A and B, what is that A and B? What is the relationship between?	*Clarifying question*
5	21:56	Ana	I don't want to do that one (*looks at her worksheet again*). Can they turn into more than two shapes? (*looks at Ms. P as if for approval)*	 *Transformative doubt*
6		Ms. P	Umm-hmm (*affirmative*) Share it with your team members and discuss it. (*walks away*)	
7	22:06	Santiago	(*to Ana:)* Yeah. It could. It could. You could just turn it around (*gestures a rotation with his hand*)	*Probably in relation to his own wonderings he had recorded on his sheet: "Can you make other shapes with it? What other shapes? How many different shapes?"*
8		Ana	Yeah but that's just one shape (*showing one finger*).	*Dia and Ana had recorded the rotation that Santiago referred to as something she noticed rather than wondered about: "There is a rhombus/diamond if rotated … 45 degrees". This idea was also modeled by Mr. R and me in the*

				previous class.
9		Santiago	That's right. That's one (*points towards the original shape*). And that's two (*rotates the shape and points towards it*).	
10		Ana	(*To Santiago:*) Can it be more than two shapes? (*in a bit of frustration*) Could it?	
11		Santiago	Yes. Yes.	
12		Ana	That's what I am asking.	
13	22:45	Santiago	It could. It could. If you cut it right here in half (*points towards the center of the shape*), it could [*unintelligible*].	

In the brief one-minute episode above, we notice students wondering about the pattern, relationships, and shapes. We also see Ana ignoring the question about "relationships" and instead nominating another wondering (lines 4, 5). Maybe she did not know how to elaborate or maybe she felt threatened by the question from the researcher. But soon her wondering about making different shapes took off probably because both Santiago and Dia had, on the previous day, individually wondered about the shape themselves. Rather than sharing his own wondering, Santiago engaged with Ana's by making conjectures about what it might mean to "turn it into more than two shapes": turning it around or cutting it? Later, Dia took initiative to further elaborate this idea with more conjectures. She wrote on the desk whiteboard (see figure 2.2): "Can the shape be transformed into other shape? Give me a proof. Give me some ideas. Give me your thoughts." Later she added more conjectures on her worksheet about stretching the figure along the symmetrical lines, by adding lines diagonally or vertically, or by moving the axis.

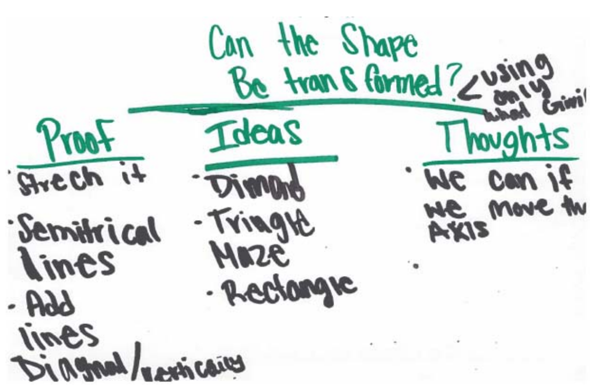


Figure 2.2. Dia's problem

Dia seemed eager to discuss her problem with a peer as she looked at Juan and elaborated: "If we stretch it- could we like stretch it? (*gestures stretching something with hands*)". In her question is implicit both an attempt to conjecture what this stretching might look like ("if we stretch it-") and uncertainty about it ("could we stretch it?"). Her question, however, was not taken up by others, but instead, Santiago asked Dia to show him what she had written. He transitioned to write a similar idea on his worksheet without further discussing it: "Try to make the shape presented in more than 2 shapes. Proof. Ideas. Thoughts." It was clear Santiago was directly drawing on his prior discussion with Ana about "turn it around" to present it as another shape, as well as Dia's form of the question ("Proof. Ideas. Thoughts."), He was combining ideas to create a structured problem in a manner similar to how Dia had, but he did not engage with Dia or Ana in further discussing the problem. In parallel, Juan asked me for clarification on the task and I asked him to use his and other's wonderings to think of a problem. Ana, by this time, had abandoned her transformative doubt about shapes and had instead taken up her analytic doubt about patterns, "Find me a pattern. Use it to find out how many squares would be in the empty part of the square [if it] would be filled." Ana, seeking an expert's approval on her problem, asked me to look at it, but in an attempt to create more discussion among peers, I invited Ana to share her problem with others:

178		Priyanka	(*To Ana*) Tell him (*pointing to Juan*). Tell him what you have.
179		Ana	(*to Priyanka*) Finding a pattern -
180		Priyanka	Tell him.
181	41:33	Ana	Why?
182		Priyanka	Because you are a team.
183		Ana	But we all have different questions.
184		Priyanka	That's fine.
185		Dia	(*in Spanish to Ana*)
186		Priyanka	If you have different questions-
187		Dia	(*to Ana*) That's why we are in a group.
188		Priyanka	If you have different questions. Chose one that you all find -
189		Ana	(*fast and loudly starts reading from her worksheet*) Find me a pattern. Use it to find out how many squares would be in the empty part of the square, would be filled.
190		Priyanka	Okay.

191		Priyanka	(*to all*) Ana is sharing her question. Do you have any questions on her question?
192		Juan	(*politely*) Actually, that sounds good.
193	41:52	Priyanka	(*To Juan*) Share with her what you have
194		Juan	I am thinking we can make a triangle-
195		Priyanka	Show it to her.
196			(*Juan shows his sheet to Ana that has a triangle with a border. Ana looks at it but does not say anything.)*
197		Juan	We can just tell them- cover your triangles with squares too.

Ana seemed reluctant to share her problem with others and seemed to question the participation structure of creating a problem together (line 181) and her role in it (line 183). According to her, it was impossible to maintain the participation structure of working together due to the fact that they were all interested in "different questions". Still, she reluctantly shared her idea with Juan on my insistence. Then, I invited Juan to share his idea. In response, Juan assembled the ideas he had been hearing about changing the shape and filling inside the empty part to craft an idea for a problem: "I am thinking we can make a triangle- We can just tell them cover your triangles with squares too." Juan did not get any responses. It is interesting to note that when Juan tinkered with his idea by drawing it out on his sheet (figure 2.3), he instead filled it with curved lines. Fitting squares inside the angled space of a triangle would have posed a struggle (although a productive struggle), but this struggle was not verbalized. In the end, all four students submitted their individual problems rather than a common group problem.

Figure 2.3. Juan's triangle

Summary: assembling. Recall that we had defined shifts in participation as students moving from surfacing doubts to creating problems that allow students to resolve their initial mathematical doubts for which they do not already have a resolution. In analyzing the above example, I will highlight three aspects, namely, the processes through which students posed their problems, shifts in student participation, and the nature of structuring resources that allowed or constrained the shifts.

The problem students created was largely the result of assembling each other's observations and doubts. Santiago and Dia assembled what Ana has asked about creating another shape with what Mr. R and I had modeled the day before about the rotation. Juan assembled Santiago's conjecture about creating triangles with Ana's doubt about filling the shape in with more squares. Ana herself assembled her doubt about the empty space in the middle with Mr. R's modeling of seeing a pattern (rule) to count the number of squares to in fact cast her problem from the mold that Mr. R had provided. Thus, while it was largely the acts of assembling that helped them create their problems, there also seemed some early movement towards casting.

As students surfaced and shared their own and assembled each other's doubts, we notice some shifts in student participation in problem-posing over time: 1) Dia and Santiago shifted from their initial unwillingness to share their doubts or take up their doubts for problem-posing (by asking Ana to do so), to building upon their own and peers' doubts to assemble mathematics problems; 2) Juan shifted from being a silent listener with doubts held privately, to assembling peers' doubts aloud in order to create a thoughtful mathematical quandary; and 3) Ana shifted from sharing to sifting through her own doubts (relationship to shape to the empty space in the square), to casting her problem using the mold provided by Mr. R.

Structuring-resources: artifacts, participation-structure, and student discourse. The shifts in student participation were supported by structuring resources such as the following: the image as a task artifact; the participation structure of the task along with my interventions reasserting the norms; and student discourse. Doubts students surfaced in their individual writing, and drew on for problem-posing later, were immanent in the artifact although each student had their own unique ways of perceiving them. By reasserting the participation structure, I played the role of a master/expert who sponsored space for students in which to acknowledge their doubts and voice them. Student discourse further structured how the doubts were shared and how students connected each other's ideas; this discourse acted as a resource for their later assembling for creating mathematics problems.

Structuring-constraints: individual work. However, since there were no attempts to further solve and reshape the problems, it is not clear if students could resolve the doubts they had. On the contrary, there is some evidence that they could not (Dia's failed attempts to think through how to stretch the figure, Juan and Ana's problems that were not discussed further) with the exception of Santiago who had created a problem around his already resolved doubt (turning it to make another shape or cutting it diagonally to make a triangle). What did Juan mean by wanting to cover the triangle with squares? Of what size and how many? Why did he not fit squares in the triangle in his representation? What was Dia thinking when she thought about stretching or about symmetric lines or the axis for transforming the figure? What mathematical quandaries do such problems allow students to resolve? Peer discussion around some of these ideas, questioning by peers, and/or attempts to solve their own problems might have allowed students to further refine, shape, or repose their problems given more time or if supported by the adults in the room. Moreover, though students shared and listened to each other, the kind of division of labor they created (individualistic work) did not allow them to question or revise their ideas. Assembling observations and doubts, thus, formed an initial limited but legitimate form of engagement in problem-posing.

2. Casting

Several students shifted from assembling their doubts towards casting, which represents their shifting participation in problem-posing as I will discuss next. For casting, students abandoned their pragmatic and transformative doubts in favor of analytic doubts that were cast using the pre-existing mold of traditional problems they had seen in the classroom or textbook that teachers would find worthy. As an illustration from the task-based interviews, a student reasoned with his peer: "I was thinking we can do what Mr. [A] sometimes does in class for more challenging patterns." For instance, during interviews, after seeing the given pattern (number of cubes = 2n, for case n) a student wondered if the pattern will keep going up (analytic doubt) and when asked to make some math problems, she asked: "What is case #57?" The student attempted to determine the rule and solve the problem. This initial exploration of the given pattern and its rule allowed her to later make a pattern of her own but of a similar nature—a pattern that had a factor different than 2 (i.e., number of cubes = 3n or 5n and so on)—and she asked to find a bigger case and a rule for it. This way of creating a problem, i.e., casting using a pre-existing mold was evident in several students' work activity. Some students explained to me or their peers that by asking to solve for a much bigger case number (such as case# 20,418), they were creating a more challenging problem.

When pragmatic or transformative doubts were present, students most often took them up for nominating initial ideas for their problems but later abandoned them to instead pose traditional problems following their analytic doubts. For example, students who wondered "what is the meaning of blocks" and "is it for food or toys or just regular cubes" went on to nominate various ideas for word problems while attaching real-life meanings to the blocks and the given pattern. For example, one of the problems she asked was: "3 people Edwin, Marko, and Chris bring 12 pizza slices. If each brings a factor of 2 and Edwin brings the lowest and Chris gets the highest, how much pizza slices did Marko bring?" (commas added for clarity). This is an interesting algebraic problem directly related to the given pattern as well as her pragmatic doubt about blocks representing something like food or toys. The eighth-grade students who were learning Pythagorean Theorem in their class at the time of the interview thought about the 3 cases of the pattern as 3 sides of a triangle and wanted to figure out if a triangle with sides as 2, 4, and 6 units will be a right triangle. There were also a few students who noticed the colors of the given blocks and conjectured using red cubes for negatives or odd numbers in the pattern and explained how the pattern can grow in the other direction that will require entering the negative space. It seemed students desired to see beyond just the abstract pattern and to instead attach a concrete meaning to them. By creating word problems using the given pattern, students were displaying their connection to real-world scenarios or engaging in the transfer of concepts from one situation to the other. But most often, students abandoned their initial problems that were inspired by their pragmatic or transformative doubts. Rather than shaping their word problems into more challenging problems, students created regular pattern problems using the given pattern as a mold inspired by their analytic doubts.

During the classroom experiment, Group 6 in period 1 and Group 8 and 9 in period 3 shifted from voicing to assembling to casting their doubts in a manner whereby they could create a molded math problem that either the teacher modeled for them or that they might have seen elsewhere. Below, I present group 6 as an example of this casting as a process of posing.

Example: group 6. Mr. R asked all students to share what they had individually noticed and wondered about the given image with their peers in their groups and to compare, contrast, and clarify each other's ideas by asking questions. As a response to this, Juan, in Group 6, performatively shared his observations and wonderings. He cleared his throat in a dramatic manner, grabbed the digital audio recorder and started reciting his notice/wonder list in a rhythmic tone, gently rocking his body as he spoke. He looked at his sheet and the recorder the whole time while he spoke. Juan's performative sharing set the tone for how the other two students recited their list (the fourth member was absent that day). While taking turns, they shared their written thoughts without any pauses or questioning:

Juan:	I wonder why there is a circle in the middle. (*Jaime laughs*) I wonder why there is a square in the middle. I wonder why they did put squares in the middle. I wonder why they didn't put more squares. I wonder why they put the middle so blank. Why there is a square in the middle- Why there is no square in the middle? Why is it only this shape? Why are the sides so square? (*passes the recorder to Edwin who passed it to Jaime*) Come on Jaime!
Jaime:	What I notice is that it's an empty square with nothing in it. (*in a conversational tone shifting his glance from his sheet to both his peers while speaking.* Edwin adds: "Oh shit!" *Jaime laughs.* Juan adds: "We can see that.") The sides are all squares. If we flip it like this (*shows by moving the sheet*) it's a diamond - (*Jaime promptly handed the recorder to Edwin without sharing his wonderings.)*
Edwin:	(*reads while looking down on his sheet the whole time he spoke*) I notice that it's a square. I notice that inside the square is empty. I notice that all sides are same. I notice that there are 40 squares. Why is it a big square? Why is it empty too? Why does it have all the same sides the same? I wonder what we are going to do with the square. I wonder why I did that and why we do that.

Right after this, Mr. R approached the group: "So, you guys sharing your list with each other? [Edwin: "Yeah"]. Did you add anything to your list from what other people said? [Edwin: "No"]. So? That way you can better understand everybody. Make sure you guys are coming up and sharing your list and taking notes on what everybody brings to the table and does anybody have any questions about the list" (@22:54). Mr. R was attempting to sponsor a space for students where students not only share their ideas passively, but engage with them by recognizing what doubts they might have in common and new doubts that peers share, and asking questions to clarify what peers mean. Jaime and Juan immediately picked up their pens and started writing something. But as soon as Mr. R left, Juan and Edwin engaged in off-task conversations about the girls they have dated or would like to date while Jaime sat quietly. Next, Mr. R introduced the next task in which he invited students to create "interesting and challenging" math problems drawing on joint wonderings. Students in group 6 remained off-task for another ten minutes before I approached the group and asked: "What do you have so far?" Edwin replied: "Nothing." Following utterances occurred:

70	32:21	Priyanka	You have some great wondering list, right (*points to each of their sheets*)? All of you. (*students nodded*) Take hints from there. You already have problems there. You all had some great ideas yesterday. Turn them into a math problem. (*walks away*)
71	32:21	Juan	(*Turned to Edwin immediately:*) Let's do: how many squares are in the figure and how can we find out. By killing Jaime or Eduardo (*smilingly*). (*they all laugh*)
72	32:44		(*Off-task conversations about Jaime's pin he was wearing.*)
73	34:10	Juan	Ooh, we need to work!
74		Edwin	(*To Juan:*) You make the question. He [Jaime] answers and I read.
75		Juan	(*To Jaime:*) Hey, hurry up (*unintelligible*) (*bossy tone*)
76		Edwin	(*To Juan:*) You need to make the question. He does it and I present it … because seat number 2 presents
77			(*Juan wrote on the sheet: "How many squares are there without counting them?" and passed it to Jaime to solve.*)

In the above interactions, I suggested they consult their wondering list to gain hints for creating their problem. Juan promptly offered an idea for their problem without consulting his wonderings (line 71). None of the students had initially surfaced this doubt about how many squares were there in the figure with the exception of Edwin who already had conjectured that there were 40 squares. They were in fact largely puzzled by the empty space in the middle. When Juan wrote the problem on the sheet he was more specific: "How many squares are there without counting them?" The phrasing of the question—"without counting" suggests that Juan's problem was inspired by the problem Mr. R had modeled the previous day using a different image. During teacher modeling the previous day, Mr. R had repeatedly emphasized how he wanted to use a pattern to determine the number of circles in the image. He had spent extra time (40 minutes instead of co-planned 15 minutes) explaining why adding the condition "without counting" made the problem "more interesting" and less "insulting" for him. I had written in my journal: "I was disappointed that he spent so much time on modeling and on emphasizing the problem that is usually anyhow asked of the border problem. Will this affect how students problem-pose tomorrow when the border image is given to them?" (reflective journal, 9/25/2017). It seems Juan had favored the problem that according to Mr. R is an interesting problem over his own musings about the image.

Thereafter, Edwin's recommendation to divide the labor (line 74) (Jaime solving the problem and Edwin presenting it), made it more difficult for students to think about alternative problems—it set up the activity to be organized around Juan's problem (molded after Mr. R's problem). Jaime immediately solved the problem and declared there would be 36 squares. Later when I returned to the group and inquired about the problem they had made, Jaime read the problem Juan had written. The following conversations emerged:

109	39:00	Jaime	How many squares are there without counting them.
110		Priyanka	How many squares are there without counting them.
111		Juan	(to Jaime) But you counted them.
112		Jaime	Yeah. So what?
113		Juan	(to Jaime) without counting them.
114		Jaime	What do you mean?
115		Juan	Without!
116		Jaime	Ah! (*raises his spectacles above his head and rubs his eyes and face and lightly smiles*)
117		Priyanka	First of all, do you like that question? (*all nodded*) Is that clear what you will have to do for it? (*All students said yes unanimously*) Okay, so see if you can solve it without counting. (*started walking away*)
118		Juan	Jaime! Solve it without counting! (*in a friendly encouraging tone*)
119		Priyanka	(*While walking away and in response to Juan's urge to Jaime to solve it, Priyanka circled around all of them suggesting they should work together and then took the sheet of paper Jaime was writing on and put it in the center of the table in an attempt to urge for joint work and ownership)* Take this sheet. Talk about it how to solve it.
120	39:57-41:25		(*joint problem solving ensued*)

In the above conversations, Juan himself problematized that the answer was not enough and they needed to determine a way to know the number of squares "without counting" (lines 113), although he expected Jaime to bear the responsibility: "But you counted them," (line 111) and "Jaime! Solve it without counting!" (line 118). After I urged them to work together, Jaime reasoned with Juan and Edwin that there are ten squares on each side and eight squares on the top and bottom (using the given image and his gestures), and therefore a total of 36. At this point, Edwin became passionately involved and contested that there is no eight, but "They are all ten. They are all ten. Look!" (@40:00) making a connection with his prior observation that there were 40 squares. He reasoned that each side has ten squares and ten times four makes forty. Juan took the lead in carefully going through each step of adding the two sides with the top and bottom squares (what Jaime had suggested) with Jaime's help who was acting as a calculator, and reasoned and recorded each step.

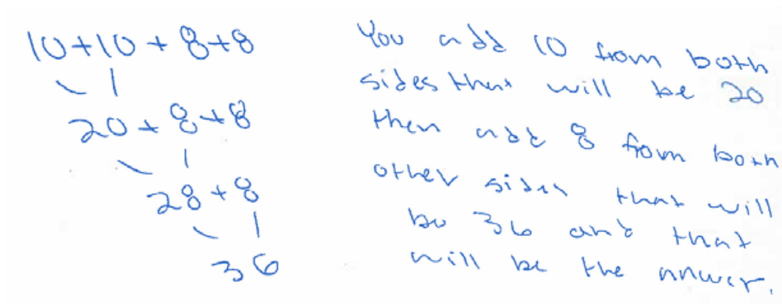


Figure 2.4. Juan's solution

Later, when Mr. R stopped by to check their progress and read their written work, he said, "This is good! Interesting. So, (*pauses to think*) Okay (*nods*). Okay (*walks away*)" (@43:40). Mr. R was happy to see the problem and detailed solution students had put together. When, at the end of the period, students were asked to reflect on the problem-posing experience, Edwin confidently shared with the whole class that he liked that, "we argued. Arguing!" When asked what they found most challenging, Juan whispered to Edwin who then shared, "finding the answer."

Summary: casting. Students created a problem that seemed similar to the process of metalworking called casting. In casting, liquid material is usually poured into a mold to give it the desired shape. Similarly, students used Mr. R's problem as a mold to cast their problem in relation to the given image while abandoning their own doubts (as evident from students' tacit refusal to refer to their wondering list when I invited them to). Overall, through the use of the mold available to them, students shifted from passively sharing their doubts (and a state of no-posing while off-task) to creating a mathematics problem and later also solving it.

It is important to note that even though students casted their problem using a teacher-given mold, the desire to solve it was their own (as evident in Juan's tone and focused questioning in lines 111-115) and was dialectically constitutive. By dialectically constitutive, I mean, the desire arose as a mutually constitutive set of discourses between the participants. A simple re-read of their problem by me created some unclear conditions that made Juan dwell on the very phrase "without counting" that he had earlier simply borrowed from Mr. R's problem. Their earlier division of labor and the fact that Jaime was solving the problem and not Juan, might have made it easier for Juan to challenge its solution. Even Edwin's conflict with Jaime's answer created conditions that made it necessary for them to take the problem up for a detailed investigation and resulted in them coming together to determine the solution. Edwin's confident tone when sharing with the class that he liked "arguing" about the solution evidence some shift in his disposition as if he felt good about arguing for his solution and gaining an opportunity to hear peers' explanations and justifications. Collective argumentation in the team had not been observed in the past five classes that were videotaped and instead the four students (including the absent student) were largely seen working individually. In the past, they came together to only check their answers or engage in off-task conversation. More importantly, the attempt to solve the problem allowed them to pose a new doubt: should they multiply the number of squares on one side by four to get the answer?

Structuring-resources: recursivity in problem-posing. While the initial problem was casted using Mr. R's problem, attempts to solve it acted as a structuring-resource that enabled a shift in students' participation from casting to posing a new doubt. This is in line with past research that has also found problem-posing to be a recursive process of posing, solving, reflecting, revising, and re-posing (Armstrong, 2013; Cai & Cifarelli, 2005; Fiori & Selling, 2016; Lakatos, 1963/2015; Leung, 1993). Other structuring-resources, as has already been pointed out, were teacher's modeling, teacher's problem and the given image as artifacts, my intervention, and student discourse in the form of their collective argumentation to reach a resolution.

Structuring-constraint: fixed division of labor. I want to point out the role of division of labor as a possible structuring-constraint. After Juan suggested his casted problem to others, Edwin's suggestion of rigid group roles and Jaime and Juan's take up of Edwin's suggestion curbed any possibility of further interrogation of Juan's problem or a possibility of other problems. After having solved the problem, students once again had the opportunity to transition towards reposing other problems possibly related to their own initial doubts. However, neither did students initiate this transition themselves nor did adults scaffold it for them or make them accountable for it through questioning or provision of tools.

3. Carving and Shaping

A few students moved beyond assembling and casting problems. Students posed several initial doubts of various kinds including pragmatic, analytic, and transformative. Students, however, did not abandon their pragmatic and transformative doubts and their initial problems connected to those doubts like the students did for casting. The problems were instead carefully carved by solving, tinkering, revising, and shaping. Just like a sculptor gradually scrapes away or carves out the unwanted pieces from clay or wood to shape a form, the students carefully carved out their problem by tinkering with multiple possibilities as related to their initial doubts, attempting to solve them, abandoning those that did not seem promising or were too easy or too difficult, and refining and shaping the promising ones. After exploring their initial doubts and ideas for problems, students nominated a promising problem and further shaped it in order to make it more challenging through collaborative tinkering, questioning, and revising.

During interviews, when working with their partners, students would suggest a revision to the problem they had created individually or jointly and would explain that the revision would make the problem more "difficult" or challenging. For instance, a pair of students first jointly created a simple pattern going up by three: 3, 3+3, 3+3+3, …. One of the students then suggested that rather than adding three at each step, they should alternate between adding three and four: 3, 3+4, 3+4+3, 3+4+3+4, and so on. He explained, "that would be more difficult". Sometimes, students would make revisions to their initial problems to pose a what-if question almost in a playful adventurous manner. As an example, a pair of students first explored problems involving arithmetic operations such as "3(18.2)/3". Later, when asked to pose more problems, Julia asked, "Does it have to be cube related?" Julia had initially posed a transformative doubt: "Why are they in a cube shape? Can they be in other shape?" Later, together with her partner, she constructed a pattern problem involving triangles. The students then tried to find the area of the growing triangles in the pattern by filling unit squares inside it. She explained to her partner that this would make it "complicated and tricky" because some squares are not full squares and they will have to combine the split squares in order to find the area. Thus, there seemed to be a conscious effort to shape a problem in a way that would add a challenge, a fun factor, or puzzlement to it.

During the classroom experiment, Group 1 had begun moving toward the process of carving after their initial assembling and casting, and group 3 effectively carved and shaped a meaningful problem for themselves. Here, I present both the cases, in brief, to highlight the structuring-constraints in group 1 and structuring-resources in group 3 that constrained or afforded shifts in participation.

Example: group 1. Two students had nominated two different problems. Erick had observed eight squares (minus the two corners) on each side and he reasoned with Bryan that he wanted to use this fact to create an equation that could be solved for a missing value: "$32 = 8 \cdot x$". Arturo, on the other hand, had proposed a problem about a growth pattern relating to the given image. He assumed that the given figure is figure 100 and asked: "if this is figure 100, how much tiles would fig.1, 2, and 3 have and how would you find the answer?" Later when Mr. R pushed Arturo to revisit his wondering list in order to think about other possibilities for the problem, Arturo argued, "All right. So the wonder list, I wrote … 'How many squares would it all be together if space would fit.' Since it's a 10 by 10, I'm saying it's a 100." Thus, for Arturo, figure 100 represented a square with 100-unit squares—something he proposed in direct relation to the doubt he had recorded on his worksheet: "Why is there space in the middle?" Interestingly, he had assembled it with the growth pattern problems that they had solved in the class two weeks back before the problem-posing intervention. When the question arose regarding which of the two problems to use as their group problem, Arturo argued, "Mine has more explanation though. His just has an equation." Bryan and Juan, on the other hand, argued that Erick's problem is "better" and "faster". Arturo continued asking aloud why Erick has taken 32 as total squares while he has 100, but before he could resolve this uncertainty he got distracted by a continued push from Eric and Juan to choose Erick's problem. Subsequently, Eric (who is labeled as an ELL student and is often teased for his spoken English) justified his choice, "I don't know how to write it [Arturo's word problem]." When Arturo attempted to start solving the problem, maybe to further clarify his ideas, he asked Mr. R:

591	40:50	Arturo	(*to Mr. R*:) So, we have to answer it, right?
592		Mr. R	You don't. The assignment was not to solve it. The assignment was to come up with a question for a good math problem.

In the end, everyone except Arturo wrote down Erick's equation (without posing a question about it) as their problem even after Arturo's continued attempts to further explain his problem to others and demand clarity on Erick's problem.

Summary: group 1. In this example, students were collaboratively assembling doubts and casting problems using simple equation problems and growth pattern problems as the molds they had taken from their prior school experiences. I argue that dialogues about (1) whose problem to choose? and (2) should the problem be solved? were potential structuring-resources in supporting students' shift towards carving that did not fully happen. Firstly, in attempting to decide which problem to choose, students were also negotiating and naming the objective for problem-posing, something that was absent in assembling and casting. For Arturo, the problem should have explanations and should make sense. For Bryan and Juan, the problem should be fast to solve and easy to write. Erick was undecided and he wrote both the problems on his submitted worksheet. Secondly, Arturo's reach to solve the problem shows unease on his part regarding not being able to fully understand the problem as he compared his 100 with Erick's 32; he seemed confused about what those numbers represented. Arturo's need to solve the problem was dialogically constituted as he reasoned what the two problems and the numbers in them meant. Peers did not engage in reasoning with him and later it was unknowingly culled in an effort to keep student on task by Mr. R. In the next example, I highlight, although briefly, how recursivity in problem-posing combined with fluid division of labor and their more purposeful efforts to understand the objective of their problem-posing allowed students to carve and shape a meaningful problem.

Example: group 3. Students began by sharing their observations and doubts with each other. Collectively, they decided to record their ideas on a sheet of paper rather than simply share them verbally—something they were not instructed to do by the teacher. This led them to build a collective repertoire of all potential doubts, a jointly-constructed artifact they utilized later for problem-posing. While sharing each other's observations and doubts, Leo assumed the role of a recorder, asking questions for clarity and comparison across ideas and refining ideas when recording. Thus, while assembling their ideas, students actively shared and built on each other's thinking. When invited to create problems by Mr. R, Leo inquired of his peers, "but how are we gonna do that?" This resulted in students re-reading the task instructions still displayed on the whiteboard (an activity initiated by Diego) and reaching a shared verdict: they would use their wondering statements to pose math problems.

Students nominated ideas for their problems in direct relation to the collective repertoire of doubts they had. Students not only drew on their collective doubts but also their classmates' doubts that had been shared during whole-class discussions, such as Santiago's statement that the figure looked like a picture frame. In this way, through collective questioning and shared understanding around what it means to problem-pose, they were widening their field of doubts as potential ideas for their problem-posing.

As students nominated problems, they were also tinkering with the representations, figuring the measures by counting as well as by using rulers, determining the answers using patterns, as well as checking them using calculators. As they moved from assembling their doubts towards the problem posing space, they first casted traditional problems like, "How big will the box be three times from what is shown?" and, "How many squares are in total?" Interestingly, they shifted into more analytical behaviors as soon as they solved them, fluidly entering new problem-posing space towards other possibilities in relation to their initial doubts. Like other groups of students, sharing and building on their doubts for assembling, casting familiar problems, collective argumentation, questioning, and recursive processes of posing and solving afforded shifts in student participation in problem posing.

Students throughout their twenty-minute-long problem-posing spree fluidly divided the labor among them. At various points in time, they were all participating in sharing their ideas to the collective pool of doubts; they were all problem-posers (except for Jorge) and problem-solvers. They were all taking up a leadership role in order to keep each other accountable by asking questions, and they were all providing the necessary support to each other by expanding upon shared ideas. Leo started in the role of a follower, but soon took up facilitating and recording, and finally took up the role of what it seemed like a captain. Diego seemingly began as a captain but moved into the role of resource-manager and supporter as Jesus and Leo started taking up more responsibilities. Jesus and Jorge started out as listeners, but Jorge took up the roles of contributor and accountability manager, and Jesus acted as a contributor and supporter.

Remarkably, what allowed this group to refine and shape their problem into a meaningful one was their negotiation of the objectives of problem-posing. After several rounds of tinkering, solving, and reposing, Leo had refined the earlier problem, "How many squares can we fit in the middle?" to, "Fill in the empty space with using less than 16 squares or 16." Leo had a vague idea in his mind and he struggled to phrase it at first, but after some tinkering, drawing, and dialogue with Diego (while Jorge and Jesus keenly listened), Leo was able to finally phrase it appropriately. He pondered aloud if the problem was too complicated—"I guess it would be complicated at first"—but added, "That's the point isn't it? To make it complicated?" Students agreed with Leo, although Leo still revised the problem to "Fill in the empty space with using only 16 squares," in order to make it less complicated. At this point, Jorge immediately questioned Leo, "Wouldn't that tell them the answer?" Leo reasoned back, "No, because they have to figure out how to make it into 16." Listening to Leo and Jorge, Diego then revised the problem by saying, "Okay, so not how many but how to make them."

Summary: carving and shaping. The final problem students posed—'How to fill the empty space using 16 or fewer squares'—is a multi-solution linear optimization problem. Turning their nascent doubts into a meaningful problem required students to assemble, cast, carve, and shape a problem in a dynamic way. First, students carefully carved out a promising idea out of their multiple doubts by posing, solving, and reposing while assuming different roles at different times to fluidly divide the labor. Finally, they shaped their final problem by dialogically setting a clear objective—the problem needs to be complicated, but not too complicated, and it needs to make sense and not give away the solution. Thus, recursive processes of posing, a fluid division of labor, and discursively setting the objective of problem-posing enabled shifts in student participation from assembling to casting, and finally, to carving and shaping.

The carvers (group 3 and a pair of students in the task-based interviews) were unexpectedly fluent at problem posing, to such a degree that it suggests there may be differences between carvers and other students in terms of their mathematical knowledge, social competencies, and/or peer relationships in their groups. However, comparison of all those things between carvers, casters, and assemblers did not reveal any clear patterns. If anything, the particular group of carvers in the setting, in fact, had the lowest test scores, had not previously worked together prior to eighth-grade, and were not friends (as revealed during post-intervention interviews). They also have had the same amount of time in their group as other student groups had in the class.

Given these facts, what are we to make of their fluency? Firstly, the facts suggest that carvers did not necessarily integrate prior school-taught mathematics knowledge or practices when problem-posing that might have set them up to have an advantage over assemblers and casters. Secondly, the situational specificity of the problem-posing activity must be examined for its affordances. The following variables allow us to see the specifications of the situation and consider its affordances: 1) the setting, where individuals came together to organize the learning activity in relation to each other; 2) the structured-resources—such as recursive processes, co-deciding objectives for problem-posing, materials and artifacts those given and those self-constructed, etc.—were discursively being shaped and reshaped, thus enabling shifts in participation, and; 3) student-roles and their division of labor as fluid and emergent in-the-moment, rather than being fixed or individual. These variables together can be seen as shaping the processes of carving in problem posing.

Discussion

The study investigated the following research question: How do students shift from the periphery of their doubts to engage more fully in posing mathematical problems? To answer this question, I first analyzed thirty-two paired task-based interviews using a constant-comparative method of analysis to determine three increasingly sophisticated processes through which students posed problems. Classroom data was then analyzed to confirm, disconfirm, and combine the old and new themes specific to the classroom. This process led me to reveal three processes of problem-posing that students seemed to follow: assembling, casting, and carving and shaping. In the second stage of the data analysis, I conducted interaction analysis using six carefully selected pairs from the interviews and all the seven small-groups from the classrooms to shed light on the conditions that structured the activity and shifts in student participation.

Assembling, Casting, and Carving and Shaping form three temporal processes that allow students to make shifts in their participation. The directionality is from peripheral to central as they engage in surfacing their doubts (peripheral), progress to problem-posing, and move toward fuller participation in problem-posing (central). These processes capture the possibility that the participation of an individual or a group of students in problem-posing practices can change over time as they shift from assembling → casting ↔ carving ↔ shaping. The variations between the different groups of students and the ways in which their shifting collaborative participation informed their decision-making in the activity imply that there may not be just better or worse problem-posers or more or less successful realizations of some basic creative competence. Rather, there appear to be qualitatively different processes of posing that are situatively constituted and constructed. Relations among persons, their activities, contexts, are all implicated in success --and failure –rather than in merely their cognitive strategies. By defining the classroom community as a community of mathematical inquiry, success was characterized as students moving their doubts forward toward resolutions by posing a meaningful problem. Thus, a meaningful problem was seen as a problem that allows students to resolve their initial mathematical doubts for which they do not already have a resolution. The problem, as posed, was considered meaningful from the point of view of students and how it provided opportunities for their initial doubts to be resolved; therefore, the meaningfulness of the problem arose in-situ and was revealed in the dialectical analysis of student activities.

One of the goals of the study was to explore implications for the design of learning environments that support student learning through problem-posing. Conditions that afforded or constrained students' shifting participation were evident in the analysis of student activity. I will discuss them next.

What Affords Shifts in Participation in Problem-Posing?

What might have afforded some students and not others to make the necessary shifts in participation? Below I highlight four aspects of students' collaborative activity that allowed students to make temporal shifts in their practices.

Artifacts. Artifacts, given and student-constructed, were crucial in enabling students to surface their doubts (as found in chapter 1 and also confirmed in the present study). Artifacts were also important in shifting student participation and thinking forward towards fuller participation. It was in the acts of student tinkering with the given artifact and tools that students analyzed and transformed the various properties of the squares. Student-constructed artifacts (collective doubt list, Juan's triangle drawing, Dia's problem on the desk whiteboard, etc.) structured transfer of ideas from one person to another as well as allowed for individuals to transform their own ideas through representation and tinkering (Juan's conjecture to fill the triangle with squares transformed into filling the triangle with curved lines).

Student discourse. Sharing and building on each other's mathematical doubts provided opportunities for students to co-construct a shared relational space for problem-posing and legitimate peripheral participation. Collective argumentation through questioning and justifying supported their attempts to refine and revise the posed-problems.

Recursivity in problem-posing. Reflecting on the problem posed, solving it, and re-posing it creates a space for movement of doubts towards more meaningful problems. Though problem-solving during problem-posing was not an end in itself, it was a structuring resource for achieving resolutions to the doubts posed earlier and for new or related doubts to emerge. Solving during problem-posing did not take form through the formal prescription of a problem-solving activity usually followed in classrooms. Therefore, if one of the purposes of communities of learning in mathematics is to sustain focus on student-led inquiry, this finding demands a different conception of problem-solving *during* problem-posing, serving an epistemic need for resolution of mathematical doubts, rather than serving as a display of procedural connectivity or knowledge. Additionally, this finding supports our emphasis on *resolution* over solution. Using theories of practice and drawing on her several years of research in widely diverse settings, Lave (1988) points out that problem-solving activity involving problems arising out of people's subjective experiences of dilemmas—and I add doubts—"often leads to more or less enduring resolutions rather than precise solutions" (p.124). By "more or less enduring resolutions," she means resolutions being constitutive of transformations that the student-posed doubts are going through as a result of the recursive processes of problem-posing and solving activity.

Student-driven problem-posing objectives. Setting student-led objectives for problem-posing may allow students to own the problem-posing activity. While a mathematician's goal when posing problems is to innovate and create new knowledge, this goal may or may not align with why students may be drawn to ask a mathematics problem. Current research in problem-posing has not yet shed light on how students set their objectives for why they are posing problems and toward what ends, i.e., what they would like to seek out of posing a problem. Findings in the current study suggest that students' reflective questions such as, "How are we supposed to do that? [problem-posing]", "What would make the problem more challenging but not too complicated?", "Who are we posing the problem for?", or "Which problem makes more sense?" allows students to decide and set the purpose and objectives behind the activity. This may allow students more buy-in in the processes of problem-posing and to persevere. In classrooms, students may find themselves negotiating conflicting objectives concerning why they were being asked to pose their own problems and why they want to. Their motivations may be related to finishing the class assignment on the one hand and resolving their doubts on the other. The first dictates, "Do your work fast and make the teacher happy," and the other suggests, "Take as much time, with as much help from others as needed, to resolve unresolved perplexities."

Fluid roles and division of labor. As seen in the presentation of the four groups, the ways in which students organized their roles in relation to their group influenced opportunities for shifts in participation. There were three types of division of labor—individual, fixed, and fluid. Individual and fixed work-roles allowed students to share and listen to each other but did not give enough momentum to question and revise their ideas. This work-role was suited for assembling and might also be suitable for casting. Fluid work-roles, by contrast, allowed students to effectively share-and-question and pose-and-solve without curbing what individuals might be able to contribute to the group thinking. Additionally, fluid work-roles may allow students to seek or establish discourse or support from within the group that moves their individual thinking and participation forward.

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CHAPTER 3

Study 3. Organizing for Collective Agency and Risk-taking in Mathematical Problem Posing

Mathematical problem-posing legitimizes learners to pursue their own lines of inquiry and actively participate in mathematical knowledge-building practices (Silver, 1994; NCTM, 2000). Problem-posing also encourages learners to connect school mathematics to their ways of perceiving and acting upon the world (e.g., Freire, 1970; Silver & Shapiro, 1992; Turner, 2003). It is believed that intentional constructive initiatives that are expected of learners in problem-posing may propel students' agentic engagement in mathematics. It does not, however, come without embracing fear and risk-taking by students (Sinclair, 2004). This study aims to trouble the ease with which the literature talks about student agency in problem-posing and examines what is socially and disciplinarily at stake for learners to do this work. Brown and Walter (1993) write, "it frequently takes not only an intellectual tour de force but emotional courage as well in order to pose a problem in a way that reconstrues what the culture at large has found acceptable" (p.xiii). For problem-posers, "reconstruing what a culture at large has found" may mean reframing the purpose of mathematical objects, questioning the validity of mathematical rules, and modifying the given assumptions to discover new patterns. It may also mean negotiating one's social position and cultural beliefs about who has authority to pose a problem, what is an acceptable math problem, and who is a good problem poser.

The instructional culture in the U.S. public-schools is still to a large extent incompatible with volition and independence that problem-posing demands of learners (McKinney & Frazier, 2008). Surpassing the traditional authority of teachers and textbooks in deciding what counts as a good problem could be daunting for learners. This is even more problematic for minoritized students for whom raising a question by drawing on diverse ways of knowing often means getting positioned as disruptive, silly, or foolish (Agarwal & Sengupta-Irving, 2019; Philip, Olivares-Pasillas, & Rocha, 2016; Warren & Roseberry, 2011). Furthermore, in collaborative classrooms, sharing emergent ideas or musings with others may mean negotiating tensions of getting one's ideas evaluated or critiqued by others (Esmonde, 2009). For learners, especially minoritized learners, to truly exercise their agency in problem-posing may mean embracing social, cultural, and disciplinary risks. Thus, the agentic value of problem-posing cannot be fully characterized without also investigating the situative risks it accompanies.

Using a micro ethnographic case study approach (Yin, 2003), I investigate the interplay of agency and risks in problem-posing. By closely following the interactions of a purposefully selected group of students, I delineate mechanisms through which students negotiated the risks—historically and situatively present—and exercised their agency to create a safe and productive space for collective problem-posing. These interactions took place in a low-tracked 8th-grade math class constituting all Latina/o students.

Theoretical Framework

Agency in Problem-Posing

Sociocultural literature identifies agency as a "transformative capacity" (Giddens, 1979, p. 88) taking form through agents' active work upon the world (through either active action or resistance) (Booker & Goldman, 2016; Engestrom, 2011a; Holland & Lave, 2009). In mathematics learning, agentic work takes place when students mobilize resources and take "risks to venture beyond a stipulated situation to explore and further develop a set of ideas" (Powell, 2004, p. 45; see also Pickering, 1995). Improvising heuristics to address a particular task, developing new means to organize learning activities, breaking away from normative practices or frames to generate new concepts are all examples of actions that Powell describes as *venturing* *beyond a stipulated situation*. In contrast, passively conforming to rituals without understanding the larger goals and following procedures without knowing why, personify actions that lack agency. However, breaking away from old to generate new (either by acting or resisting) just for the sake of it does not reflect true agency; agentic actions have purposeful intentions and goals.

Problem-posing pedagogy, by the very nature of practices it invites learners to engage in—such as doubting, questioning, conjecturing, refuting, and so on—authorize learners to improvise, to revive unresolved mathematical quandaries, to reframe previously asked problems, and to mathematize everyday social situations (Armstrong, 2013; Brown & Walter, 2005; Stylianides & Ball, 2008). Problem-posing inculcates epistemic openness and enables learners to undertake mathematical issues that they are genuinely interested in resolving (e.g., Engle & Conant, 2002).

Agency as Mediated and Situative

Wertsch (1998) has emphasized the pivotal role of mediational means (cultural artifacts, tools, signs, language, and social relations) that shape cultural activity and human agency. According to this theory, the inclusion of mediational means in human activities does not just facilitate but fundamentally transform human agency in important ways. This conceptualization emphasizes a unit of analysis where agents are the irreducible aggregate of individuals acting together-with-mediational-means (Wertsch, Tulviste, and Hagstrom, 1993, p. 341).

Empirical studies in problem-posing have delineated how epistemic and cultural sensibilities of learners—together with materials, tools and interactions—structure students' problem-posing. For instance, Sweden, Gade, and Blomqvist (2015) found that Grade 4 students in Sweden appropriated *cultural signs and meanings* they were exposed to in the problems they created. Students alluded to references such as USA presidential elections and FBI agents in their problems. Instances of humor, morality, and human emotion were also evident in student-created problems. Authors found that cultural appropriation by students in their problems was mediated by the available material and cultural tools (vocabulary handed out on slips of paper and the use of native language) and classroom interactions in important ways. In a similar vein, in an investigation of middle school students' math engagement in a USA-based summer school, Fiori & Selling (2016) highlight the role of students' *aesthetic choices* about everyday tools and materials and peer interactions in nominating ideas for new problems. In their investigations of dynamic representational technologies, de Freitas and Sinclair (2014) found that it was the dynamic geometry environments and motion detectors that invited certain creative acts of learners. They reconceive mathematical agency as material, embodied, and distributed; and argue that even though individual(s) may still own intentionality and choice, the possibilities for making certain choices are also to a large extent shaped by the static properties of artifacts being explored (see also Martin, 2019).

The underlying premise is that agency resides in all human beings, but it is exercised differently in different contexts with varying consequences (Engestrom, 1999b; Gresalfi, Martin, Hand, and Greeno, 2009). In other words, while each child holds the capacity to problem-pose, the problem as posed is a mediated reflection of possibilities that arise when learners come together to pose and re-pose by negotiating ideas, social relations, constraints, resources, artifacts, and tools.

Agency and Risks

Ahearn (2001) points out that agency is not "ontologically prior" to a context, i.e., not a fixed attribute that people carry along with them, but it arises from "the social, political, and cultural dynamics of a specific place and time" (p.113). Since agentic actions have capacity to “break away” from dominant norms, prevalent epistemic concepts, and limiting boundaries to create something new; they always carry a risk of getting rejected, ignored, or mocked by the social others and for individuals to risk “getting caught not knowing” (McDermott, 1995, p.14) or perhaps disruptive (e.g., Warren & Rosebery, 2011). Farther away the action is from what has come to be considered normative; the higher is the social risk of taking such an action because it involves experimenting with uncertain and unproven scenarios. This risk is also unfairly higher for people who already hold minoritized positions in a given group, community, or social system, such as schools, irrespective of the nature of their actions (Gutierrez & Rogoff, 2003). Thus, even if an individual agency is authorized within a community, such as in problem-posing based learning environments, individuals may not exercise their agency because of the perceived risks of “local contentious struggles” (Holland & Lave, 2001) that may inappropriately outweigh the innovative capacities of those actions. Multiple mediated actions of individuals are always imbued with issues of power that arise due to the differently positioned histories of people, artifacts, communities, and institutions. Power dynamics give rise to the local contentious struggles or simply, perceived risks of tensions arising in the activity. Individuals may act passively and minimize the risks or take active agentic actions and face the risks. Past studies in problem-posing have not investigated questions such as: How do groups perform and negotiate multiple mediated agencies in the presence of social and disciplinary risks within a collaborative learning environment to productively problem-pose? I argue that the interplay of agency and risks comes into focus in the analysis of histories of people and groups; of emerging norms, actions, and relationships of differently located participants in groups; and of the historically institutionalized struggles that are implicated in the local activities.

Methods

Study Context

The study was conducted in two low-track 8th-grade mathematics classes at Valley Middle School (VMS; pseudonym) in one of the largest school district in California. The school district serves over 95% Hispanic students, 41% of students identified as English language learners and about 90% eligible for free or reduced meals. The district also has a history of academic tracking. Top students (based on standardized testing) attend fundamental schools and the rest attend regular neighborhood schools. Within neighborhood schools, students are sorted by ability in honors and non-honors courses. VMS, the neighborhood school, placed its top 50th percentile students in math honors and the rest in non-honors math classes, arguably the lowest track within the district. The Assistant Principal, previously a math teacher herself, explained during an interview that "since students in their school struggle with critical thinking skills, teachers focus on getting students ahead at least on fluency through memorization and procedural practices." As such, non-honors math courses in VMS constituted a big focus on building students' math basic skills and fluency over cultivating conceptual understanding—decisions that were rooted in deficit assumptions about what these kids are or are not capable of doing.

At VMS, only about 11% of the 6-8th-grade students met or exceeded the State standards in 2016-17 and 2017-18 (California Assessment of Student Performance & Progress, California Department of Education) as compared to the district average of 25% and the State average of 37%. The two classrooms in which the teaching experiment was conducted comprised of all Hispanic students.

Instructional Context

Mr. R who taught both the classes is white-American and was in his third year teaching non-honors 8th-grade math at VMS, where he also previously taught 7th grade courses for about ten years. Mr. R typically used traditional tasks and traditional instructional style in his teaching with some elements of group work and active learning (as observed prior to the problem-posing intervention in 38 class-periods over fifteen days across six different classes and two school-years). The typical week followed the following structure: the teacher introduces a new procedure by solving some problems; students solve a similar problem using the method taught; the difficulty of the problems assigned increases over subsequent days during the week; students take a quiz on the material learned during the week. Mr. R sincerely intended students to understand why they were doing what they were doing (as reflected during a pre-intervention interview), but the pedagogical decisions made about tasks, participation structures, and classroom resources, did not allow him to operationalize this goal (as observed).

I did not notice any occurrences of problem-posing during my pre-intervention visits to his classrooms and student questioning was minimal at the most. Though students were never discouraged for having questions or different ideas (and in fact, Mr. R encouraged such initiatives); there were no set norms, participation structures, or attempts to invite all students to do so explicitly. Students lacked opportunities to share their disciplinary uncertainties or they were too quickly redirected or assimilated to the teacher's ways of understanding. Students were seated in groups of four but group interactions were minimal, largely because students were often given three minutes or less to solve and report on each problem. During this short time, students could work individually but could not find time to organize group interactions.

The Problem-Posing Task

One of the aims to introduce problem-posing tasks in Mr. R's classroom was to engage students in mathematical practices of questioning and conjecturing through supporting students in their take up of wonderings about mathematically unstructured situations for posing and solving. The tasks and lessons were designed in close collaboration with Mr. R. One of the lesson included students getting an artifact/image (Figure 3.1) for problem-posing. Brown and Walter (1993) argued that even the dullest of pictures can "excite new interest" and can result in "many rich and interesting questions" (pp.307-308). This image is commonly used in the Border problem for teaching algebraic expressions and generalization (Boaler & Humphrey, 2005). The task usually asks: Describe a strategy for determining the total number of 1"x1" squares in the border of the 10" x 10" square, without counting each one. We, however, used it for students to generate their own math problems.

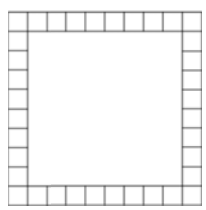

Figure 3.1. The 10x10 Border image

Data Sources

Data for this study comes from two 50- minute long class periods. Four small-groups in one class period and two small-groups in the other class period were consented (parental) and videotaped using a flip camera and a table mic. In the problem-posing task, students were asked to (1) first individually write what they noticed and wondered about the given image (6 minutes); (2) then, to share their observations and wonderings with their group-peers while comparing, contrasting and clarifying each other's ideas; and (3) lastly, to create an interesting and challenging math problem with their group-peers using the wondering ideas they had just discussed (15-18 minutes).

I considered each of the seven student groups as a separate case for analysis (Yin, 2013), choosing individuals acting together-with-mediational-means as a unit of analysis. By explicitly focusing on a collection of individuals connected through the activity and mediational means rather than individuals as separate entities, I was able to capture the rich and varied ways in which groups of students actively negotiate the risks that arise in their local situated contexts to engage in problem-posing together. Data included: videotaped observations (utterances, gestures, expressions, body positioning, tones of voice) and a collection of the teacher-provided and student-designed artifacts (students' final written products, their scratch work on the worksheets, their use of classroom tools and artifacts such as the given image, rulers, calculators, etc.). Total of five 5 video-hours of data was analyzed.

Researcher's Role

I was positioned as a teacher assistant and a participant-observer in the class. Upon Mr. R's request, I helped him model problem-posing. For this, we chose another image and took turns sharing what we noticed and wondered about it while asking questions and explaining our thinking. This occurred before students were given the Border image for their own problem-posing. Also upon Mr. R's request, I helped him facilitate class group discussions.

Analytic Approach

Data analyses were conducted with the following question in mind: How do groups negotiate the perceived risks of local contentious struggles in-situ and perform multiple mediated agencies to create a safe and productive space for collective problem-posing? I used a multi-step, multi-phase recursive process of analysis (Castanheira, Green, & Yaeger, 2009). This analytic approach proceeds as a series of cycles during which data is parsed (videotapes watched, transcripts read), preliminary understanding is gained and represented (through tables, charts, and analytic memos), and new questions are posed. First, together with two undergraduate research-assistants, I took a rough overview of all six small-group videos (and when needed whole-class videos) and made content logs, that is, a time-indexed list of topics that we identified together through team discussions (Miles, Huberman, & Saldana, 2013). Second, to understand the trajectory of ideas, we created another set of more detailed content logs where we documented students' initial wonderings about the given image (see Table 3.1 as an example), when which ideas got taken up or abandoned; and traced shifts in ideas from nascent wonderings to the fully crafted problem (see Figure 3.2 as an example).

Table 3.1. Initial nascent wonderings of students in Group 3

Doubt→ Student	**A** # of squares/ patterns	**B** Filling squares in the middle	**C** Filling shapes in the middle	**D** Breaking the given square
Jorge:		I wonder why the middle is empty.		**I wonder if we can break this down into different shape.**
Leo:	What is the total number of squares?	Can it be filled with more squares?		
Diego:	How many patterns/ squares in the image?	How many squares fit in the middle?	**How many different shapes you can fit inside the square?**	
Jesus:		How many squares fit in the middle?	**You can't put different shapes beside a square. (*Reframed by Leo: If we used something besides square would it make a difference?)***	

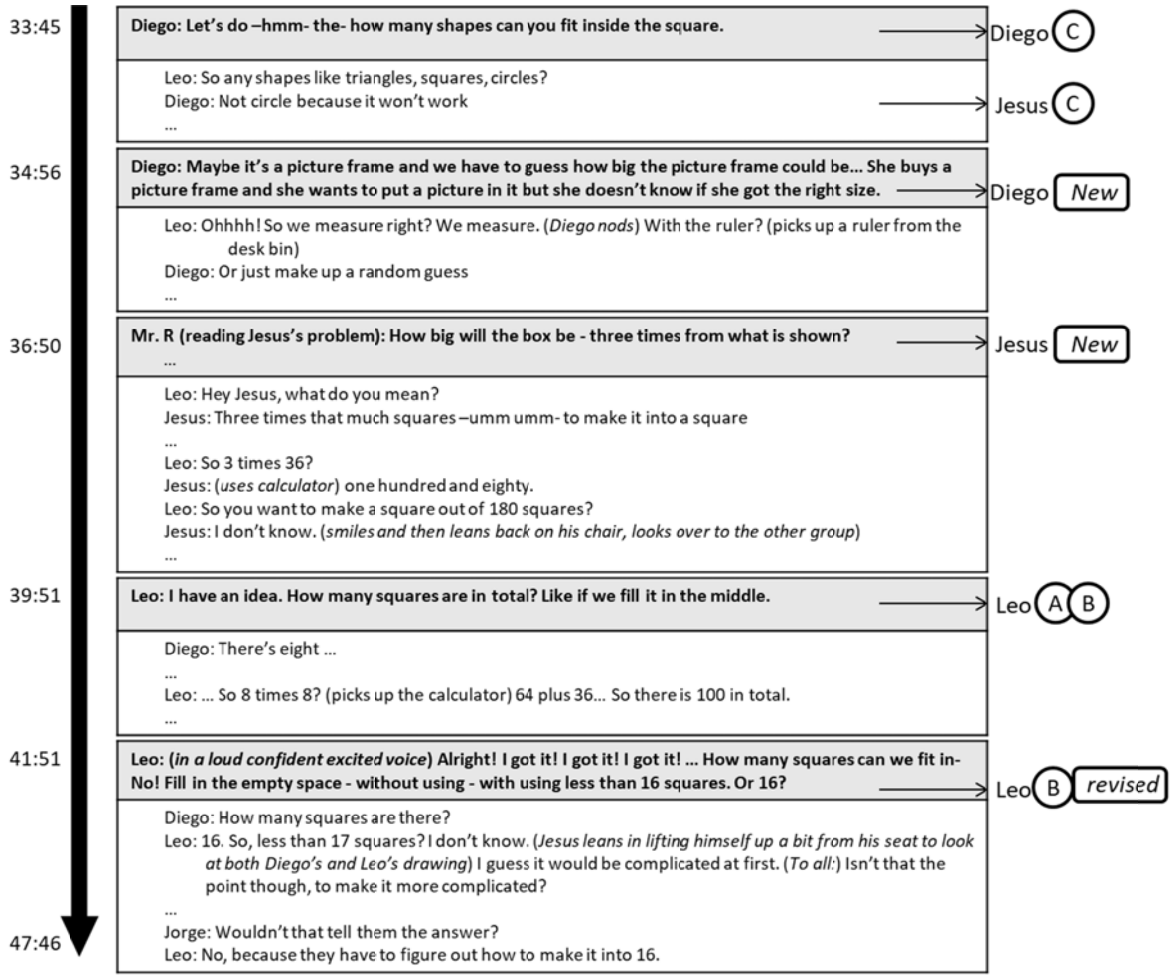

Figure 3.2. Tracing shifts from wonderings to the final math problem for Group 3

Third, to understand more specifically the nature of local contentious struggles in each small-group, I watched the full videos once again to understand the group dynamics and the role that individuals and mediational-means played towards nominating and shifting the ideas towards the final collective problem. At this stage, I created full transcripts of each small-group video documenting utterances and cross-talk as well as any obvious gestures, expressions, body movements, and tones of voice (Ochs & Schieffelin, 1989). I analyzed the transcripts, treating analysis of each transcript as a separate case for analysis, to determine critical moments (Maher, 2002) when a possible local tension emerged in each group and to understand the nature of those struggles vis-à-vis the local group-context. I noticed that while all groups faced the same tensions, their intensity and how they were negotiated varied. Table 3.2 outlines the tensions that emerged.

Table 3.2. Local contentious struggles in problem-posing

How to organize for problem-posing and collective activity?	• How do we share ideas with others? • Who will do what? Who will initiate?
How to make sense of the multitude of possibilities for a problem?	• How do we make sense of each other's noticing and wondering ideas? • Who is invited to explain and who is questioned, and by whom?
What does it mean to make a math problem?	• What does it mean to make a math problem? • Who will do what? How do we all contribute to the collective posing?
Which and whose problem is a good math problem?	• Whose idea should be used for the problem? Who are we making the problem for? • How do we decide which problem is a good math problem?

In the fourth phase of the analysis, I conducted interaction analysis (Bryanson, 2006) to trace how students negotiated the perceived risks of struggles in relation to the goals of the activity. For this, I chunked the transcripts by struggles (from start of one struggle to the start of the next struggle) and created analytic memos describing the actions and behaviors of the students during the period of the struggle (initiative to share an idea, decision to ignore peers' question, asking a question, listening, refusing to share, etc.) and their influence on how the struggle was negotiated, agency exercised, and idea nominated or revised.

For instance, when faced with the social risk of getting questioned upon sharing ideas with others, students drew distance from their peers and reverted to working individually as against engaging with the peer's question or critique (Group 4). In these groups, students gained little opportunity to effectively revise their wonderings into a meaningful problem. In other groups, students resisted sharing their own wondering and instead preferred to follow the lead of the student perceived as better in math (Group 7). When faced with the risk of getting mocked for the incorrectness of the English in their written idea, students found creative ways to abandon their own lines of inquiry to instead pursue a problem that included fewest words possible, describing it as “easy to write” problem (Group 1). When the given artifact failed to inspire them, students drifted to instead wonder about the multitude of other objects in the classroom, their social lives, or popular culture (Group 7). For these groups of students, the given problem-posing task remained a mere classroom assignment even when the students were creatively curious about other things. When faced with the disciplinary risk of figuring what constitutes a good math problem or how to make it, students diverged from their own wonderings and instead either mimicked the problem Mr. R had modeled as “math-worthy” (Group 6) or those recently learned or seen in the textbooks (Group 1, Group 8).

By comparing and contrasting what was emerging in each group, I found that one group in particular (Group 3) negotiated the risks in a way that individual agencies of the students gave way for the emergence of collective agency and productive problem-posing. In particular, in this group, the struggles of problem-posing (see Table 3.2) were more often verbalized and acted upon using the material and ideational tools, disciplinary practices, and peer interactions as compared to the other groups.

To further delineate the mechanisms of the focal phenomenon of the study, in the final phase, I conducted a deeper interaction analysis of only Group 3. For this, I conducted a frame by frame video analysis for each video chunk, capturing in detail the subtle social cues, gestures, expressions, body movements, and tones of voice of students. This also involved sometimes listening to only the audio with no picture and at other times watching only the video with no sound. The goal was to understand how students positioned themselves and others in relation to their peers, materials, and the activity and how did it influence agency/ risk interplay. Below I present the case analysis of group 3. Figure 3.3 represents the seating arrangement of students in group 3.

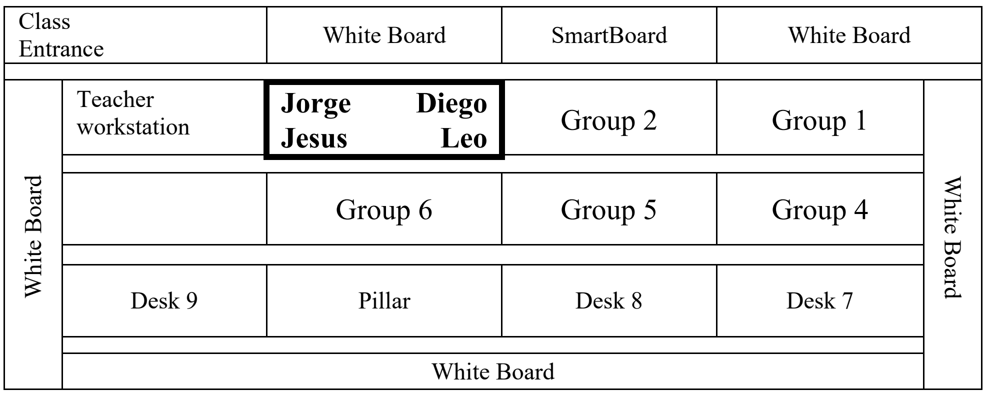

Figure 3.3. The seating arrangement of Group 3 in the first period

Group 3 Interactions Organized by Local Contentious Struggles

Struggle 1: How to organize for problem-posing and collective activity?

Mr. R opened up the activity by saying:

> Now in your groups … share your ideas that you wrote down on your page of about your notices and wonder list. ... Be sure that you are discussing ideas that seem similar, identify any ideas that are different. If you have questions about what people meant, this is the time to ask them, and be sure to clarify all your ideas so everybody knows what you are talking about. (Line# 158, @20:00-20:25)

Right after, Jorge leaned in gently towards Jesus, and Jesus and Leo both towards Diego, their gaze shifted from the teacher to the class board and to each other. Diego initiated the conversation by suggesting they use the whiteboard (a small A4-sized board on their desk) to make their list. Before jumping on to write on the whiteboard, Leo confirmed what they needed to do (@20:45) Diego re-read the task instructions from the class board out loud and summarized: "We are supposed to ask about the wonders and notice" (line 170; @21:12). Leo pulled out the desk board (line 168). While Diego read the instructions out loud, others were focused on listening and reading with him. Throughout this episode Jorge was sitting upright, his hands down somewhere near his body (not on the desk), his body still, and his gaze moving gently from the board to Jesus to Leo to Diego and then to his own worksheet. Jorge seemed tensed from his body posture, somewhat passive about initiating any action or talk himself, but at the same time attentive and ready to follow others. Jesus's hands were neatly folded on the desk above his worksheet as he leaned in to listen to Diego. Jesus was fiddling with his fingers and was looking around at the other groups as if impatient or anxious, but his gaze kept returning to his peers implying that he might have been listening and waiting for others to initiate. In these subtle ways students seemed to be just beginning to organize themselves in relation to the task, to their peers, and the resources available to them.

Leo directly asked Jesus: "Do you want to say your wo- umm, your notice first?" (line 171; @21:17). Jesus leaned forward, slightly raising himself up from his seat, picked up his worksheet, then looked at the class board as if reading the task instructions again, and finally settled back down on his seat. He looked down at his sheet intently and silently reading what he had written but did not say anything. Diego, Leo, and Jorge patiently waited for him. Jesus's silence for 20 long seconds as he fiddled with his worksheet suggested some reluctance, hesitancy, fear, or qualm about sharing or sharing first. Soon, instead of obliging Leo by sharing, Jesus in an unsure manner asked a question: "Wait. But what are we writing on?" (line 173,

@21:40). Jesus wanted to reconfirm how they were organizing their ideas. Diego suggested they write on a paper and Leo voluntarily assumed the charge to write. Jesus's deflection of Leo's invitation to share by asking how they will document it pushed the group to write and create a shared repository (Figure 3.4) of their collective ideas that organized students' problem-posing later on in important ways. It also gave a chance for Leo to invite his peers once more but seemingly with more openness and less intimidation: "What did you guys write down, for notice?" (line 178). An invitation that Jorge promptly accepted by sharing his observations about the image. Soon everyone shared their notice list one-by-one in a synchronous effortless manner, pausing only to clarify and ask questions.

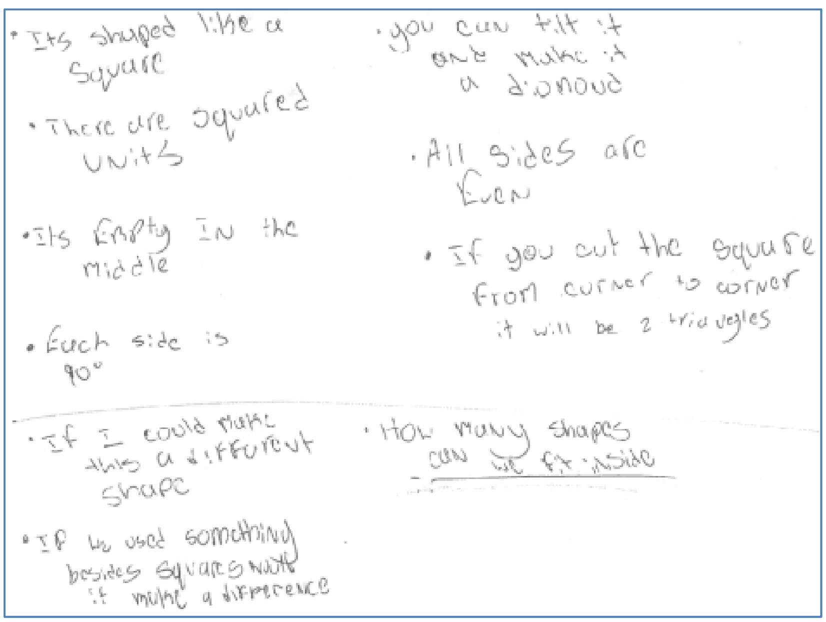

Figure 3.4. Student-created repository of collective ideas

At one point, Leo said that he did not know how to spell "diamond" when writing Diego's observation but quickly moved on. Leo also made repeated attempts to ensure he had not missed anybody's idea ("Is there more?" "What else?" "Is that it?) that allowed peers to continue contributing their observations. Jesus who had initially seemed reluctant to share was now responding by adding more observations, some of which were not there on his original list and were added more spontaneously (as in line 198).

197	24:05	Leo	What else?
198		Jesus	Two triangles (*gestures slicing the given square*). Nah. I don't know.
199		Diego	If you cut the square in two halves, it will make two triangles? Is that what you meant, Jesus?
200		Jesus	Hmm?
201		Diego	If you cut the square in two halves, it will make two triangles? (*Jesus looked elsewhere*)
202		Leo	What is that called? Diagonally? (*gestures slicing the given square*)
203		Diego	I think it is diagonal, not sure. Horizontal?
204		Leo	I'll write- cut the square corner to corner? (*writes it down*)
205	24:57	Leo	What else?

When Jesus hesitated about his new observation (line 198), Diego reframed it for him (line 199). Leo expressed uncertainty about a mathematical vocabulary (line 202), but found a way to express it together with Diego (lines 202-204). Although Jorge was quiet, he was actively listening as evident from his body positioning and gaze towards what Leo was writing.

Struggle 2: What does it mean to make a math problem?

Next, Mr. R invited the class to use their group ideas they just discussed to create math problems that are "mathematically challenging and kind of interesting" (@25:30)" after which students in the group 3 stayed quiet, separate, and a bit distracted. Slowly, after about 35 seconds, they turned their focus to their worksheets and the image—quietly and individually. Leo started writing or drawing something on the given image. He was probably counting the number of squares in the image. Diego picked up a calculator from the desk bin, fiddled with his worksheet a bit, and then wrote something on it. Jesus also seemed to be reading the instructions on the worksheet and pondering over the image silently. Jorge was sitting quietly but upright, hands down on his lap, gaze shifting on and off from Leo to Leo's worksheet and to the mainboard, as if waiting for a peer to initiate the group activity assigned to them. Finally, Leo leaned in a bit towards Diego and asked: "What's the question?" as in what's the task. Diego responded: “We have to make a math problem.” Immediately, in a concerned but calm voice, Leo asked, “but how are we gonna do that?” In other words, Leo was asking: what does it mean to make a math problem. In response to Leo’s question, Diego looked at the instructions again and said: “We have to take it from our wonder list to make it” (lines 224-227; @31:59). Leo immediately turned his attention to creating a group list of wonders like they had done for the notice list.

Struggle 3: How to make sense of the multitude of possibilities for a problem?

Leo once again called on Jesus to share his wondering and once again Jesus displayed reluctance as if unsure or fearful about sharing his idea without first knowing what others had. This seemed strange since Jesus had displayed more confidence about sharing towards the end of the last episode. Leo promptly moved to Jorge who once again obliged by sharing and Jesus finally mustered the courage to share after Jorge and Diego had shared.

236		Jorge	I wonder if we can break this to a different shape. (*Jesus and Diego quietly listen*)
237	32:36	Diego	How many different shapes can you fit inside the square? (*Leo writes it down; Jorge quietly listens; Jesus looks down at his sheet as if getting ready to share next*)
238	32:53	Jesus	(*Leans in, raising himself up slightly from his chair and speaks a bit hesitantly but without any prompt from Leo*) Hey you can put - You can write, you can’t put different shapes beside a square. (*Jorge and Diego quietly listen*)
239	32:57	Leo	What do you mean? (*looks up at Jesus in a sharp direct manner*)
240			(*Jesus looks away, fiddling with his fingers as if nervous. Does not respond.*)
241	33:06	Leo	(*to Jesus:*) What do you mean different shape? (*Jorge and Diego quietly listen*)
242			(*Jesus leans in towards Leo but does not say anything*)
243	33:11	Leo	(*to Jesus:*) The shapes inside the square? (*Jorge and Diego quietly listen*)
244		Jesus	(*Leans in more, but speaks while looking away instead of towards his peers, slowly in a dim voice*) Instead of squares like triangles-
245		Diego	(*yawns, rubs his forehead*) -triangles, rectangles. Different shapes! (*Leo writes down: “if we used something besides square would it make a difference?”)*

Mathematically, we notice that there was mathematical imagery that students were beginning to weave together, which is the idea of “different shapes”. Jorge wondered if the given image could be *split* into different shapes. Diego wondered if different shapes *could fit inside* the given square. Jesus conjectured that shapes other than the square *cannot fit inside* the given square.

The wondering and sense-making space students were entering (line 239), however, did not come without risks—authority to share with others comes with accountability to explain and reason with others. Leo asked Jesus what he meant by different shapes. Maybe Leo noticed the contradiction between Diego and Leo's wondering and wanted to clarify that. Maybe Leo wanted more clarification about why Jesus thought no other shape could fit inside. Jesus, who so far had been reluctant or shy, but had not shown any clear signs of nervousness was starting to fiddle, look away, as if nervous (lines 240, 242, 244). When at last he took up the courage to explain, in a low voice and slow manner, he was supported by Diego, although Diego's intervention also hindered Jesus from finishing his explanation. It remained unclear what Jesus meant and what Leo was trying to understand. Jesus went back to writing on his sheet. Their collective doubt about "different shapes" could not get fully discussed and resolved at this point in time. However, Leo seemed to have made sense of it by rephrasing Jesus's rejection of different shapes to a more open what-if scenario: "*if we used something besides square would it make a difference".* At this point, almost impatiently, Diego suggested, "Let's do, how many shapes can you fit inside the square" for their math problem (line 246, @33:45) and the students exited the wondering space to shift towards the problem-posing space.

Struggle 4: Which and whose problem is a good math problem?

Figure 3.2 outlines all the math problems and the kind of discussions students had about them over the next ten minutes. For the first three minutes, Diego and Leo explored the initial two suggestions that Diego offered for the problem by tinkering to fit different shapes in the square (Figure 3.5) and by measuring the given square using a ruler. During this time, Jesus was writing his own problem quietly. Jorge was looking restless as evident from his distracted body movements as if not knowing how to join the conversation.

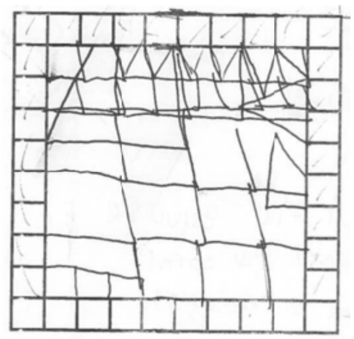

Figure 3.5. Leo's tinkering to fit different shapes in the image

Mr. R intervened to check on their progress, he noticed Jesus' math problem that he was writing quietly and read it out loud: "How big will the box be – three times from what is shown". Mr. R asked what Jesus meant by it. Jorge, Diego, and Leo stopped doing their work and attentively listened to Jesus who without any hesitation explained, "three times the width… Three times the amount of squares that this figure has". Mr. R asked, "Well. Box made out of squares or box made out of cubes?" and then before anyone could answer, he asked students to "massage and figure out how exactly to say it," so it is clear to everybody and then left the group. Leo seemed interested in Jesus's problem and promptly asked Jesus what he meant in a compassionate gentle manner. When Jesus explained, "Three times that much squares –umm umm- to make it into a square", Leo right away calculated the total number of squares in the border as 36 and together with Jesus figured out three times that. Leo asked Jesus: "So you want to make a square out of 108 squares? Jesus said, "I don't know" smilingly and looked elsewhere as if he needed more time to think or did not know how to explain his thinking or maybe the problem seemed too easy to him after having solved it. In any case, before students could think about it more, Diego impatiently offered to go back to his idea of the picture frame problem: "Should we do the picture frame one?" probably sensing Jesus abandoning his idea. At this time, Leo declared he has another idea: "how many squares are in total? If we fill it in the middle."

Instead of fitting different shapes, as Diego had suggested initially, Leo was suggesting they fit only squares, probably inspired by his own original wonderings (see Table 3.1) combined with Jesus's initial conjecture about fitting only squares inside it.

Leo and Diego started counting the squares when the researcher intervened and noticed Jesus's written problem. To ensure students were including everybody in their discussions, she checked if they had discussed his problem and asked Jesus if he would like to explain it again. Jesus in a confident calm tone (but with a slight hint of irritation at having been asked to explain his problem again) turned down the offer by shifting the focus to the new problem: "We are doing a different one." Leo concernedly and compassionately asked Jesus, "You want to do that one? You want to do that one?" Leo's tone suggested his readiness to re-consider Jesus's problem given that Jesus was willing to discuss it. Jesus shook his head. Leo and Diego reverted to solving how many squares they can fit in the middle and quickly figured the answer would be 64 unit squares by multiplying eight by eight, which was the number of squares on each side in the empty space.

Leo started tinkering on his sheet again when he declared with a voice filled with excitement, "Alright! I got it! I got it! I got it! Both Diego and Jorge immediately turned their gaze towards Leo. Leo added "How to fill it in without using- less than, umm, let's see (draws each one and counts) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 -", but stopped midway as if unsure and for a few minutes kept thinking about it by himself while others waited for him. At this moment, Jorge who had seemed distracted lifted his chin up and kept his eyes fixed on what Leo was drawing. Diego and Jesus also intently looked at Leo's tinkering with his drawing. Leo continued thinking but looked stuck and his attention shifted to Diego's drawing who was filling the empty space with unit squares (Figure 3.6).

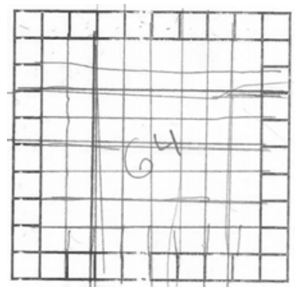

Figure 3.6. Diego fitting 1x1 squares in the empty space

Leo seemed to be in deep thought as if struggling to clearly see his idea. He looked over the desk and then his peers as if searching for something or needing some help. Jorge was still intently watching Leo and Diego and picked up his pencil for the first time during the entire period as if convinced they now have their final problem. Leo started drawing 2x2 squares inside the given square (Figure 3.7).

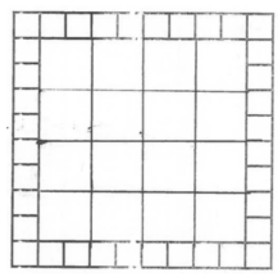

Figure 3.7. Leo fitting 2x2 squares in the empty space

Leo looked up and asked:

	44:51	Leo	How do we say that? How many squares, umm-
		Diego	How many squares can we fit in the middle?
		Leo	Like that? (*pointing to Diego's figure 3.6*)
		Diego	No, not like that. Like that! (*pointing to Leo's figure 3.7*)
		Leo	How many squares can we fit in- No! Fill in the empty space - without using - with using less than 16 squares. Or 16?
		Diego	How many squares are there?
	45:45	Leo	16. So, less than 17 squares? I don't know. (*Leo looked at Jesus and Jorge as if seeking comments. Jesus leans in to look at both Diego's and Leo's drawing*) (*To all:*) I guess it would be complicated at first, but- (*To all:)* That's the point isn't it? To make it complicated? (*shifting his gaze from Jesus and Jorge to Diego and then back to Jorge*)

		Diego or Jorge	Yeah. (*Jesus also nodded*)
		Leo	Right? Okay, yeah.

Leo's problem: "fill in the empty space with using less than 16 squares" – was about finding ways to fit 16 or fewer squares in the given Border image. He thought the problem would be complicated at first, but also thought that was the point of the task and asked for his peers' thoughts. Once agreed, they all started writing the problem and after thinking a bit Leo declared that it should be "with only using 16 squares." Jorge looked up at Leo, then looked down as if thinking; looked up confusedly again at Diego and Leo and finally asked:

	47:24	Jorge	Wait what was it?
		Leo	Fill in the empty space with using only 16 squares.
	47:30	Jorge	Wouldn't that tell them the answer?
		Leo	No, because they have to figure out how to make it into 16.
		Diego	Okay, so not how many but how to make them?
	47:46	Leo	Yeah

Jorge, who had been quiet throughout and had only talked when asked by Leo to share his notice and wonder list, confidently argues that the problem includes the answer, which should not be the case. Leo clarified that the problem is asking *how* to fill in the sixteen squares as against how many. Everybody agreed and wrote down their problem.

Analysis and Findings

Finding 1: Active listening, to negotiate risks and mediate agency in problem-posing

Initiating something, especially publicly, in an unfamiliar context of a new task and with people with whom one may not yet have developed a relational trust or collaborative norms could be risky (Boaler, 2008). People may have a conflicting understanding of the task (Voigt, 1994); reluctance to share due to perceived status hierarchies or uncertainties about how to work together (Langer-Osuna, 2018). Student interactions so far suggest that such risks existed (e.g., Jesus's reluctance to share; Jorge's silence; Leo's fumbling with spelling and math vocabulary, etc.) but were mitigated by how students used signs, language, artifacts, and tools as mediational means to organize themselves. Problem-posing is fundamentally a practice reserved for mathematicians or high-achieving students. Taking up their own lines of inquiry as a practice is a departure from what minoritized low-tracked youth are normatively asked to do in schools (McKinney & Frazier, 2008) and unto itself a risky endeavor. As students quietly pondered over the image they were taking this risk head-on. While Mr. R had given students specific instructions about the task, students did not follow them mindlessly. Instead, they first prepared themselves using the materials and artefacts available to them (desk board, instructions still displayed on the class board, their individual notice and wonder list, the given image) to gain a shared understanding of what they have to do and how they will do it evidencing agentic engagement that led to their co-constructed repository of collective ideas. Leo's question, "how are we gonna do that?" and its collective resolution (we have to use our wondering) is critical for how students later take up problem-posing as an activity that they owned rather than as an assignment they were supposed to simply finish. Through these agentic moves, students could find a starting point, an anchor that they could hold on to stay afloat to make the transcending open mission of problem-posing a little less overwhelming.

Students' agentic engagement in classrooms is often characterized by their verbal initiatives: asking questions, expressing preferences and opinions, sharing ideas, and so on so forth (Reeve, 2013), or by their actions mediated by material tools (de Freitas and Sinclair, 2012), and as also evidenced in our case. However, seen from the perspective of *risks*, a lens that is absent in research on agency, another important concept that mediated agency emerged. To advance research on student agency vis-à-vis risks, I highlight and conceptualize the notion of *Active Listening* as a mediational mean for structuring agency in problem-posing, and define it as students' active attention to the nonverbal *signs* in their local context (along with verbal utterances). The role of nonverbal *signs* in how individuals exercise their agency is not new in sociocultural theories of agency (Wertsch, Tulvist, & Hagstrom, 1993), but the role of individuals actively attending (listening) to nonverbal signs of social others is not something that is foregrounded in the empirical studies of student agency or mathematical learning.

The non-verbal signs were subtle social cues such as leaning in, staying silent, fiddling with their sheets, re-reading the instructions, looking in to read peers' work, and individual writing. Students' leaning-in as a participation sign suggested a symbolic willingness to come together to work and listen. Deciding to stay silent even when invited to share implied a peer's need for more time to either recollect ideas or courage to speak. Fiddling with the sheets and re-reading instructions conveyed their efforts to anchor themselves vis-à-vis the task and their peers. Most importantly, individual writing as against verbal sharing (Leo's writing of collective ideas and Jesus's quiet writing of his problem) prompted students to wait and listen to each other, and to compare, clarify, and question each other's ideas without getting restrained by how they might have initially felt about sharing (Jesus's hesitation) and what they might not know (spelling and math vocabulary). In Jesus's case, it also allowed the opportunity to express his ideas that the teacher and researcher could bring to the attention of his peers that otherwise might have gotten lost.

Active Listening (watching, attending, looking, etc.) to these signs enabled students to shift their actions—by restructuring the activity (writing vs. verbal sharing), asking clarifying questions (what do you mean by that), or inviting peers less intimidatingly—that in turn allowed students to discursively build shared understanding about the task and a sense of team trust and solidarity. Later when Jesus hesitated about his new idea, Diego stepped in to revoice it for him.

When Leo was not sure about a spelling or a mathematical vocabulary, neither was he afraid to admit and ask for help (he might have known others were listening with empathy) nor did others question his choice or ability to be the recorder for the team (they might have known Leo was listening to their ideas).

Finding 2: Foregoing control over one's own ideas to pursue collective imaginations

In students' problem-posing, we notice how students' ideas converged towards the empty space in the given image. In fact, if we revisit their initial wonderings (see Table 3.1, column B), we see that all four students were somewhat inspired by the empty space. Jorge had asked, "I wonder why the middle is empty" and the other three wanted to fill it up with squares. Their wonderings, which arguably were still very rudimentary in nature, gave rise to a challenging and interesting problem often found in the field of linear optimization—find ways to fill the empty space with using only 16 or less than 16 or 17 squares. All possible solutions of the student-posed problem can be found by solving this linear system (by either manual trial and error method or a computer program):

$$\alpha + 4\beta + 9\gamma + 16\delta + 25a + 36b + 49c + 64d = 64$$

Such that: $\alpha + \beta + \gamma + \delta + a + b + c + d = 16$ (using only 16 squares)

Or such that $\alpha + \beta + \gamma + \delta + a + b + c + d \leq 16$ (using less than 17 squares)

Or such that $\alpha + \beta + \gamma + \delta + a + b + c + d < 16$ (using less than 16 squares)

The big question students faced was what and whose problem to choose as they promptly moved from one problem to another while tinkering, solving, and revising them. There were several factors at stake for students to consider. Here I highlight two elements that are most directly related to the focus of inquiry. First, in deciding whose problem and what constitutes a good problem, students were "taking risks to venture beyond a stipulated situation"—the stipulated situation or norm of Mr. R's typical instruction where students ' assume the role of passive listeners and followers and solve the given problems rather than judging them. They were also venturing beyond the stipulated norm of problem-posing that was set by the teacher prior to student problem-posing. When modeling problem-posing using another image, Mr. R had asked students: “Are there any of the problems that are not kind of math-worthy … that you probably won’t ask in a math class, that you will not see in a math class?” He added, “I am looking at the, what and why questions—like why are we looking at this and what’s it for—I don’t see them as very interesting because it doesn’t challenge my intellect.” After the student problem-posing class, Mr. R reflected to me a disappointment about the kinds of initial wonderings that students had shared. In relation to the wondering about the empty space, Mr. R irritatingly said, “They were obsessed with this empty space… what's the empty space for? Making soup? …mathematical fact had very little to do with the question.” He reads one of the student-posed problems:

Mr. R	“*Finding a pattern used to find out how many squares there would be if the empty part of the square would be filled.”* They're obsessed with the empty part.
Priyanka	Does that irritate you?
Mr. R	I think it's limiting. That part's limiting because there's more to the shape than the empty part. … Everything zeroes on the center and the focus, and nobody's looking at exactly what's going on around it per se.

By “looking at exactly what’s going on around it”, Mr. R was referring to the border problem that is typically asked of this image: Determine the total number of squares in the border of the given square, without counting each one.

Second, human’s conception of what is “good” (in this case, of students) is influenced by what is considered “good” from the perspective of authoritative or dominant others, but it is also continually constructed, deconstructed, and reconstructed in discourse with each other and the mediational means available in a situated context (Holland & Lave, 2009). Students, in the current context, discursively and together with the tools, signs, and practices available to them determined which and whose problem to choose. Students followed their creative urges about the image (the empty space), figured how “complicated” the problem needs to be, and stayed accountable to the mathematical rigor and their peers. An important emergent concept that sheds light on how students achieved this clarity was what I define as *Foregoing control over one’s own ideas to pursue collective imaginations.*

Abandoning an individual creative urge in favor of other possibilities posed by others may not be simple. Students posing a group problem have a personal as well as a collective stake in the problem. In group 3, students explored five problems within a short duration of ten minutes, abandoning and choosing another problem fluently and quickly. Abandoning a problem seems to have depended on how much stake individual student wanted to maintain in a given problem, how well were they able to account for its sense-making, and the risk they perceived of working apart than working together towards finishing the task. Diego, for instance, pushed others to go back to his picture frame problem the first time, but he let it go when students started exploring something else the second time. Jesus attempted to explain his problem twice and tried solving it with Leo but let it go when others in the group thought it was time to move to something else. Even after exploring four problems posed by his peers and himself, which were promising in their own way, Leo still posed a fifth problem to probably satisfy his deeper intuition about what makes for a good problem and what is within their reach. Diego’s abandoning his way of filling the empty space (Figure 3.6) to consider Leo’s way of filling it (Figure 3.7) also represent the same tacit awareness and willingness to favor what is good for the team rather than staying attached to his own idea. Jorge must have also had an unstoppable urge to ensure their problem makes mathematical sense when he took up the courage to ask a critical question—“Wouldn’t that tell them the answer?” after staying silent for the whole episode.

Being able to abandon an idea to move to another was in part a function of discursively understanding what represents a "good" problem for them. A problem that is too open-ended (like how many and which different shapes can be filled in) with an astronomical large number of possible solutions may seem daunting and students may struggle to even begin when there is no entry point in sight. For instance, Leo initially tinkered with the given square haphazardly to fit different shapes in it and later suggested they think about fitting only squares. A problem that is too easy to solve, on the other hand, may not satisfy students' need for productive struggle. For instance, every time students solved the problem too easily, they abandoned the idea and moved to explore other possibilities. In the final problem, knowing at least one possible solution (i.e., filling the square with sixteen 2x2 squares) allowed students entry to the problem and balanced the creative challenge with approachability (Fiori & Selling, 2016). They could also tinker with the problem's constraint—using only 16 squares or less than 17—to revise it for approachability.

Finally, through actively listening to nonverbal signs (such as looking at peer's scratch drawings to understand their thinking) and by asking questions to push for peers' thinking, students maintained accountability to the disciplinary rigor. Consider Jorge's question to Leo: "wouldn't that tell them the answer?" and Leo's explanation: "No, because they have to figure out how to make it into 16." Such peer questioning and sensemaking allowed the team to revise the problem from "How much squares would fit in the blank space with using 16 squares" to a more meaningful problem "How would you fill in the empty space with only using 16 squares".

To summarize, negotiating the perceived risks of deciding which and whose problem is a good problem was an act of seeing one's actions as part of a greater whole—bringing into harmony students' own creative sparks (their wonderings); the improvisations that the given image invited (empty space as an opportunity to problematize and transform into something new); the manipulations that the tools afforded; and the accountability that the discipline and peers demanded.

Discussion

Using interaction analysis of a purposefully selected case, I described the role that active listening and foregoing control over one's ideas to pursue collective imaginings played in how students negotiated situative risks to exercise their agency. To further understand the findings in relation to agency in problem-posing, I draw on the concept of Follett's (1941) *power-with*. Follett distinguished two forms of power: power-with and power-over. She described power-with as a generative capacity; jointly achieved by integrating ideas and desires of people for shared objectives through collective action. In theorizing power-with in this way, Follett also contended the common fallacies that equity is achieved either as a result of equal redistribution of power among all or through negotiated compromise, i.e., by giving up a part of one's agency. Power-with or collective agency, instead, bestows opportunities for individuals to grow their capacities to act in ways that are transformative for them as well as for the social others with whom and for whom the action is happening.

Drawing on the lens of *power-with*, I forward the notion of collective agency in problem-posing as the ability to see individual actions as mediated-parts of a greater whole enabling collective risk-taking that is both transformative towards problem-posing and towards shifting individual student agency in problem-posing. By individual actions as mediated, I mean utilizing mediational means (tools and signs) to initiate, develop, and sustain a sense of assurance in peers' actions. When students tinkered with the shape using material means and tools, they could shape their tacit ideas into tangible artifacts for sharing and exploring with others. When students discussed their problems using ideational signs of active listening and questioning, they could abandon and revise their ideas with more ease to achieve disciplinary and social accountability. By individual actions as parts of a greater whole, I mean foregoing control over one's own ideas in order to pursue the calling of collective imaginations and epistemic accountability. As discussed above, students were willing to weave each other's ideas together to create a meaningful and promising group problem. Collective gain as a goal allowed students to more effectively decide when to share, when to listen, when to question, and when to abandon or revise an idea. The collective agency was a culmination of shifts in individual agency and risk-taking to achieve a collective gain that was arguably greater than the sum of individual gains (Bandura, 2001).

Problem-posing demands suspending certainty—the sense of false certainty that comes with following *prescribed* procedures to solve the *given* problems—a mainstay of passive learning. Collective agency in problem-posing, in contrast, nurtures embracing uncertainty embedded in the transcendent mission of posing anew. Posing anew by re-creating norms of collective sharing and sense-making to negotiate tensions inherent in dividing the labor, addressing peer critiques, managing differences of opinions, and mobilizing the resources. Posing anew by collectively problematizing what constitutes a math-worthy problem to negotiate tensions inherent in wringing something useful out of the swirl of ideas owned by different individuals.

Conclusion

Often in mathematics class, students are given the final canonized problem and its solution that mathematicians across different civilizations and cultures have developed over centuries, and little time is spent discussing how that problem came to be or allowing them to construct their own. Nurturing student agency in problem-posing involves a shift from valuing only what students pose to emphasizing students' participation in addressing the following epistemic tensions: (i) How do we organize our collective wonderings of unstructured mathematical situations? (ii) How do we make sense of our wonderings for problem posing? (iii) How do we collectively revise our collective wonderings to pose a meaningful math problem? and (iv) How do we decide what constitutes a good math problem? In sum, I argue to support development of learning environments in which students through mediated-collective actions, create a sense of assurance in each other and in the process of posing; so they could forego the fear of collective sharing and sense-making in favor of social courage; so they could trade the feelings of frustration of uncertainty with those of epistemic humility in the knowledge-producing process of problem-posing. Thus cultivating greater agency-driven moves and inculcating an ethical risk-taking behavior.

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CONCLUSION

Often in mathematics classrooms, students find themselves faced with the final, canonical form of mathematical knowledge that people the world over have constructed over many years. Even in the problem-posing curriculum, the expectation is for students to mimic the practices of mathematicians. The findings of this dissertation research, however, would imply that implementing problem-posing in classrooms will involve us learning how to value students' own unique ways of seeing and manipulating the mathematical world they are a part of. Adolescents are notorious for thinking big, bigger than what may be possible, and indeed, we see this nature in their need to transform what was given to them. Additionally, it was found that students were constantly seeking to know the purpose and significance of the mathematical objects as evident in their pragmatic doubts. We also observed how such pragmatic and transformative doubts were taken up by students in interesting and productive ways in order to craft meaningful math problems.

Though the student-posed problems has been a key area of investigation in mathematics education research, the evidence gleaned in the three chapters provides grounds for the conclusion that the sole characteristics of the final problem posed and students' cognitive skills of posing do not have a correspondingly broad role to play in the analysis of *how* problem-posing nurtures student inquiry and agency. Instead, the shifts in student inquiry rested on student *activities* in problem-posing that included voicing, assembling, casting, carving, shaping, and reshaping doubts and their resolutions in constant dialogue with people and materials (chapter 1 and 2). Students' epistemic needs—what students were drawn to know—was in dialectic relation to what they did, said, and heard. As Ana quickly moves from wanting to know about the "relationships" to "if it can become more than two shapes" to "find a pattern and use it to find how many squares can be filled", her perspectives on what she wants to know, what can be known, and how to know it was also changing. Further, the changing perspectives were dialectically related to what was being done and said by her, her peers and the adults present in those moments.

Furthermore, the shifts in student agency rested on how student activities responded to the social and disciplinary risks of problem-posing. Students recognized and verbalized the struggles of how to share and make sense of the ideas, who should initiate, do, explain, and question; how to decide what problem-posing means and which problem to choose; and finally, whose problem to choose and who we are making the problem for. Students negotiated the risks underlying the emerging struggles by actively listening, noticing, and sensing the verbal and non-verbal signs available to them in their immediate social world. They also negotiated the risks by foregoing control over one's own ideas when the situation demanded so. The focus remained on a social co-construction of how doubts were shaped, what doubts were worth shaping and resolving, and toward what ends they were shaping and resolving the doubts.

Given these findings, as we seek to understand their implications and to answer the broader question of how problem-posing allows students to nurture their mathematical inquiry and agency, it might be better to conceive of problem-posing as, what Lave (1988) called, "transformation of ideas and relations". And if problem-posing is about the transformation of ideas and relations, then we must also ask under what conditions the transformation occurs. As gleaned from the findings of the three chapters, the answer is not easily answerable in general terms because of the situated nature of the shifts in student practices and the structuring resources that afford those shifts as evidenced in the findings. Lave & Wenger (1991) point out that "participation is always based on situated negotiation and renegotiation of meaning in the world" (p.51). However, drawing on past literature, I will attempt to answer this question below and identify the limitations of the current study and areas of future research.

The Nature of the Task

The prompt and phrasing of the task influenced whether students surfaced their doubts or not and to what extent. When left open to write their thoughts down, students voiced fewer wonderings as compared to when specifically asked to write wonderings down. If the goal in problem-posing classrooms is to allow students to surface their doubts, wonderings, and musings, then teachers and researchers must pay close attention to the nature of the task and the language-in-use.

Limitation and future research. What alternative methods or tasks would allow learners to voice and acknowledge their mathematical doubts, curiosities, and diversity of perspectives? In the current study, students were asked to simply write their thoughts (in the interview) and wonderings (in the classroom) individually about an abstract artifact. I doubt if this kind of prompt is the most ideal or the only way to surface students' doubts. More studies will allow educators a range of possibilities to use in their classroom for problem-posing. For instance, studies have recorded the important role of Family Forest Walks (Marin & Bang, 2018) and Photovoice (Harper, 2017; Latz, 2017) through which students are enabled to not only identify disciplinary curiosities but they are also enabled to draw on their cultural experiences and identities in order to surface uncertainties that are directly relevant in their immediate environment, explore cultural and epistemic diversity of the discipline, and take up sociopolitical controversies as linked to the mathematical content (Agarwal & Sengupta-Irving, 2019). Culturally-relevant spaces may prove more effective in creating opportunities for students to make visible their ideological terrain of thoughts in relation to mathematics, thus engendering a sense of belonging for students.

The Nature of the Artifact

As per the results in chapter 1, during interviews, students asked more pragmatic doubts as compared to transformative, and vice-versa during the classroom experiment. The cause of this difference is not immediately clear, and there may be more than one factor. In chapter 1, I raised a possibility that the differences in the nature of student doubts across settings may arise due to the differences in the nature of the given artifact. Lave & Wenger (1991) have emphasized the epistemological role of artifacts in the context of the social organization of knowledge:

> Knowledge within a community of practice and ways of perceiving and manipulating objects characteristic of communities of practices are encoded in artifacts in ways that can be more or less revealing. Moreover, the activity system and the social world of which an artifact is part are reflected in multiple ways in its design and use and can become further "fields of transparency", just as they can remain opaque. (p. 102)

During interviews, students were shown a growth pattern made up of snap-cubes that many students were already familiar with. Thus, the artifact was in a way a reflection of historical know-how for the students and this know-how influenced how students perceived and manipulated it— that is, by extending the same pattern to find bigger figures instead of changing the pattern itself. As such, knowledge within the VMS community of students about the growth pattern was reflected in the artifact and was used overwhelmingly to manipulate it. During the classroom experiment, students were shown a printed image of a square with unit squares on its border. In the absence of students' prior familiarity with the image—where the properties of the border is used for mathematical manipulations, the other intrinsic properties of the image (such as the empty space surrounded by a border) and students' familiarity of the image outside of mathematics classroom (such as that it looks like a picture frame) took precedence in informing students' ways of perceiving and manipulating it.

Limitation and future research. We need more research that will investigate the nature of the artifacts and the nature of doubts that are revealed. For instance, I wonder if, instead of an abstract artifact, students were given an unstructured real-life situation (such as amusement park leaflets; see Bonotto, 2013) or everyday objects (such as nuts, bolts, pinecones, etc.; see Fiori & Selling, 2016), then would they still raise pragmatic, analytic, and transformative doubts, or would other dimensions of doubts emerge? I further ask if opportunities such as Family Forest Walks and Photovoice may provide space for students to self-select artifacts and if self-selecting artifacts might be more generative for their mathematical problem-posing? This kind of research, in particular, has implications for engaging minoritized students in problem-posing.

Student Discourse and Fluid Work-Roles

The importance of student discourse and work-roles in influencing shifts in student participation in collaborative problem-posing is not surprising news. Years of research has consistently shown that when students have opportunities to question and argue (as opposed to limiting them to only share and listen) they are better positioned to engage in exercising their authority to take up their own lines of inquiry (Engle & Conant, 2002; Forman & Ford, 2014; Manz, 2015; Smith & Stein, 2011; Yackel & Cobb, 1996). For example, Engle & Conant (2002) found that it was not enough for students to only exercise their authority to share their ideas. To move their ideas forward in a productive way, students also had to be accountable to each other's thinking by arguing and by providing and expecting justifications.

Research also finds that student discourse depends on the co-constructed participation structures, and is not just a result of what teachers do in the classrooms. For example, even within a single activity, students were found to vary in the work-roles (e.g., explainer/listener, instructor/facilitator) and collective work practices (e.g. individualistic, collaborative, instructive) they assume (Esmonde, 2009; Sawyer, Frey & Brown, 2013; Wood, 2013). Esmonde (2009) further found that the groups that were found to produce inequitable opportunities to learn used individualistic/instructing work practices more often than collaborative practices, and positioned peers based on their mathematical competence as experts and novices. I see similar patterns in my findings. Groups that remained peripheral in their participation in problem-posing (assembling and casting) employed the individualistic approach and fixed work-roles instead of more fluid and collaborative work-roles (carving and shaping). Individualistic approaches and fixed work-roles have been associated with western culture and affluent communities that thrive on competition (Mejia-Arauz, Rogoff, dexter, & Najafi, 2007; Mejia-Arauz, Rogoff, Dayton, & Henne-Ochoa, 2018). In contrast, children from Indigenous-heritage communities and immigrant communities from Mexico have been found to follow collaborative practices more fluidly and fitting within their holistic worldview that appears to be common in those communities. However, since this work draws on children working outside of school in leisure activities such as paper origami folding, it is unclear to what extent findings apply when children work together in the formal environments of schools.

Limitation and future research. Setting up effective norms for student discourse and division of labor is not an easy ask of teachers, especially in collaborative classroom settings where participation is structured mostly by the interactions among students in the absence of the teacher. Status hierarchies between students, the degree of trust students have in each other, and the respectful versus competitive ways in which mathematical argumentation can occur are some reasons norm-setting can get curbed in classrooms. Moreover, teachers' own beliefs about whose authority gets recognized and who can question in a mathematics classroom that invariably impacts the participation of minoritized students has implications for improving student discourse in an equitable way. Chapter 3 provides some understanding of micro-processes that are effective for students organizing their argumentation, but its implications for teachers and teaching practices are unclear and require more research. For instance, it would be important to study what teaching moves and practices allow students to develop increasing accountability from "inside-out" (Engle, 2012). That is, how do students shift from being accountable to self, to safe peers, to challenging peers, to teachers, and finally to external experts in the field. And how do students account for and seek accountability towards diverse epistemologies and forms of cultural knowledge? Problem-posing-based learning spaces are ripe for allowing students to take up their epistemic and cultural agency (Agarwal & Sengupta-Irving, 2019).

Small data in this study lack capability to put forward a strong claim for a link between fluid work-roles/division-of-labor and more sophisticated processes of problem-posing, and we need studies that can explore this hypothesis using larger and more diverse data-sets. There is also more research needed to explicate instructional techniques and norms that allow students to shift from fixed to more fluid divisions of labor. This shift is not straightforward because, on one hand, fixed work-roles can promote engagement of non-working group members, but at the same time can stifle movement if any member is not able to contribute in a role assigned to her.

Recursivity in Problem-Posing

Recursively posing, solving, reflecting, revising, and reposing was crucial for students to shift to increasingly more sophisticated problem-posing processes. It was earlier discussed in chapter 2 that this finding (that also aligns with the past research) has an important implication for how we relate problem-solving with problem-posing. I argued for a different conception of problem-solving *during* problem-posing as an epistemic need for resolution of mathematical doubts rather than a display of procedural connectivity or knowledge. As explained, the emphasis is on *resolution* over solution. By studying adult grocery shoppers and weight-watchers, Lave (1988) found that when engaging with problems in everyday settings, people do not look for precise solutions of arithmetic quandaries rather they look for "more or less enduring resolutions". While this may not be ideal for the teaching of mathematical problem-solving in schools where we expect students to advance their conceptual and procedural knowledge precisely and rigorously, it may be ideal for advancing the processes of problem-posing where the purpose of solving helps mobilize the doubts and the search for a meaningful problem.

Student-driven problem-posing objectives

Students setting their own objectives for problem-posing about what kind of problem to pose, who to pose the problem for and toward what ends, as I see, is probably the most important aspect of the practice of mathematical problem-posing by school-children and most distinct from the how mathematicians might pose a problem. While mathematicians pose the problem with an aim to generate new mathematical knowledge, that is not the explicit goal for school-children learning mathematics (this is not to say that they cannot). Therefore, while problem-posing is an intrinsically motivating activity for mathematicians, we do not know if that would hold for school-children being asked to pose. Therefore, I argue that it may be better to rather understand what school-children might want to know and do mathematically when given the opportunity to freely choose—what I refer to as students' epistemic needs.

As described in chapter 2, students set problem-posing objectives using a variety of scales ranging from a problem that would allow resolving initial doubts and is complicated, but not too complicated, to a problem that is easiest to write with fewer words. This implies that ways in which students decide their problem-posing objectives are not free of, and indeed, rather deeply linked to, identities that students co-construct for themselves within a space. Let me expand upon this idea using an example of group 1: Eric, who wanted to choose Erick's problem because it was easier to write, is also often mocked for his English by his peers. At the beginning of the episode, when Eric enthusiastically shared his wondering with others, he mispronounced "tails" for tiles. He wondered how many tiles are there but instead said how many "tails" are there. This became a central focal point for the rest of the period as peers mocked Eric throughout the rest of the period, although in a friendly playful tone. I present this example to contend that students in a mathematics classroom, especially those historically marginalized, have to do *more* than mathematics when they are being asked to do mathematics. Their mathematical problem-posing is in constant dialectic with identity negotiation and social and cultural positioning. Social, cultural, and political perplexities that are often present in the social milieu of classroom ecologies also become important objectives that students must negotiate and resolve as a way of belonging in that space. For Eric, resolving uncertainties of the use of English as a second language in a social and competitive space of a mathematics classroom took precedence over the mathematical objective of sense-making that his peer Arturo was pushing for.

Limitations and future research. I suggest that problem spaces where minoritized students can converge their sociopolitical uncertainties with the mathematical ones may offer a radical solution for advancing mathematical problem-posing (Agarwal & Sengupta-Irving, 2019). Literature offers a plethora of examples, however, not always directly in relation to problem-posing. This includes work on funds of knowledge (Moll, Amanti, Neff, & Gonzalez, 1992), culturally-relevant pedagogy (Ladson-Billings, 1995), and more recently, after-school maker-spaces (Barton, Tan, & Greenburg, 2016; Vossoughi, Hooper, & Escudé, 2016). What might this literature have to offer to the design and analysis of problem-posing in school spaces?

Teacher Modelling

While teacher modeling may have helped students voice more doubts (as compared to when the modeling was not done in interviews), it also culled surfacing of particular types of doubts that the teacher thought were not favorable for mathematics learning. In a similar vein, while teacher modeling may have influenced a group of students abandoning their own doubts to favor the problem that the teacher had molded for them, the same teacher modeling did not have the same influence on the other groups of students in the same class. Why that might be? The answer depends on the nature of who within the student-group initiated the activity and how that initiation was taken up by others and co-constructed. In group 6, Juan voluntarily shared his ideas passively and performatively, and in the absence of other students' questioning his way of sharing, it became the norm for how others shared, which later influenced their casting of a teacher-modeled problem. In group 4, Dia sarcastically invited Ana to share her ideas first but Ana ignored and rejected her invitation. Neither Dia nor Ana changed their ways of organizing the talk. Dia kept mocking Ana and Ana kept ignoring Dia in order to do her work individually, which later hampered their questioning each other and further shaping of their doubts and problems. Likewise, in group 3, Leo invited Jesus to share his ideas first. Leo's invitation was rejected by Jesus at first who kept silent. After a pause, Jesus responded, not by sharing but by instead asking a question: "what are we writing on?" Jesus's question, that geared toward the *organization* of the activity, changed the ways in which students participated in problem-posing from thereon and as they took up more sophisticated processes of carving and shaping the doubts. It, and other such instances of meaning-negotiation, also changed the ways students participated *with each other* to problem-pose through active listening and pursuing collective goals, thereby giving rise to increasing collective agency in their problem-posing.

Limitations and future research. Several unanswered questions arise from the above discussion. For example, can teacher modeling be done in a way that allows a space for students to voice their doubts while drawing on the epistemic diversity of the discipline without solely favoring the mainstay of mathematical inquiry like analytic doubts? Is it possible to promote collective agency in problem-posing in mathematics classrooms? We need more empirical studies investigating these specific practices and their mechanisms using counterfactuals and comparison groups.

To conclude, I would start from the beginning. I began my investigation by drawing on a theoretical foundation that suggests mathematical doubts are important for student-led inquiry. Thus, I defined problem-posing using the notion of doubts. However, I invite alternative characterizations of problem-posing so we could attend to the ecological validity of these notions further. The conceptualization of problem-posing using doubts as a source is consistent with the current literature within the philosophy of mathematics and social cognitive theories of learning that emphasize intuitions, feelings, and expressions for mathematics learning. However, we may still not have fully captured the complexities of practices in and out-of-school that marginalized youth may have to offer. Future studies in this regard will advance the study of problem-posing.

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