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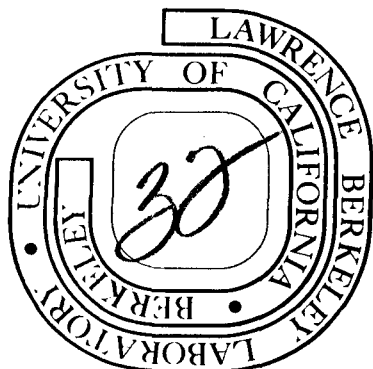
J. D. Meiss and N. R. Pereira

September 1977

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INTERNAL WAVE SOLITONS*

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September 1977

Abstract

A numerical solution of a nonlinear integro-differential equation for internal waves shows soliton behavior. Two and three Lorentzian solitons with different velocities pass through one another unscathed. A Lorentzian initial condition with larger than soliton amplitude decays into solitons, with velocities predicted by the five conservation laws.

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Waves within a stratified fluid (internal waves) are described by a nonlinear integro-differential equation first derived by Benjamin¹ and Davis²; they also found the solitary wave solutions. Their existence was experimentally confirmed at the same time.² The solitary wave is mathematically remarkable because it is not a hyperbolic function like most known solitons, but instead has Lorentzian shape (viz. Eq. (3)).

Ono³ recently reconsidered the internal wave Eq. (1), which we will call the Benjamin-Ono equation. He assumes that the solitary waves are in fact solitons, calculates which initial conditions could decay into only solitons, and thereby suggests exact solvability.⁴⁻⁶

Although the Benjamin-Ono equation is very similar to the Korteweg-de Vries equation, it has defied many attempts at solution by an analogous inverse scattering technique. The Hilbert transform in the dispersive term excludes, for instance, incorporation in the AKNS scheme, which only allows indefinite integrals. The fact that no higher conservation laws had been found also seemed to indicate lack of exact solvability.

It is therefore interesting that our numerical computations show soliton-like behaviour, and that we subsequently found two additional invariants. The present note reports our results.

Internal waves occur in many physical situations as discussed by Benjamin.¹ The appropriate linear dispersion relation is $\omega(k) = k(1 - |k|)$. The corresponding nonlinear evolution equation becomes the Benjamin-Ono equation

$$u_t + 6 u u_x + H u_{xx} = 0. \quad (1)$$

Here H is the Hilbert transform defined by

$$[Hf](x) \equiv \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(z)}{z-x} dz. \quad (2)$$

Solitary waves are Lorentzians:

$$u_s(\zeta) = \frac{2}{3} \frac{V}{(V\zeta)^2 + 1}. \quad (3)$$

where $\zeta \equiv x - Vt$. Note that the amplitude and width depend on the single parameter V , which is the velocity.

Our numerical work indicates that Benjamin-Ono solitary waves pass through each other on collision, which makes them solitons in the usual sense.⁶

For our two soliton collisions the final state is the same as would be for no interaction between the solitons, except for the phase shifts. The phase shift for the larger soliton is always positive and for the smaller is negative. When the ratio of the two velocities approaches unity the phase shifts become larger, but it is difficult to quantify the results since the soliton interaction falls off slowly with distance.

A three-soliton collision with velocities 2.5, 2.0, and .7 is presented in Fig. 1. The solitons are separated initially as far as is practical in the computations, but some overlap is unavoidable. To minimize the overlap the trailing portion of the larger soliton is put on the far right in the graph, which is permissible since our system is periodic.

Figure 1(b) shows the solitons during the interaction. The solitons do not overlap completely during the collision. Figure 1(c) gives the final state of three solitons with the original velocities, and the

correct relation between velocity, width and amplitude. The arrows indicate the positions the solitons would have if they did not interact. The shifts in position, or phase shifts, are qualitatively similar to those of the Korteweg de Vries equation.^{4,5}

Exact solvability of a soliton equation is probably equivalent to the existence of an infinite set of independent conserved quantities.^{6,7} The Benjamin-Ono equation has the evident polynomial invariants³

$$I_1 = \int u dx, \quad I_2 = \int \frac{1}{2} u^2 dx, \quad I_3 = \int \frac{1}{3} u^3 + \frac{1}{6} u H u_x dx.$$

With the Hilbert transform of a product of two functions $f(x)$ and $g(x)$, $H(fg) = H[H(f)H(g)] + H(f)g + fH(g)$, one can verify the invariance of

$$I_4 = \int \frac{1}{4} u^4 + \frac{1}{4} u^2 H u_x + \frac{1}{18} (u_x)^2 dx, \quad (4a)$$

$$I_5 = \int \frac{1}{5} u^5 + \frac{2}{9} u^3 H u_x + \frac{1}{6} u^2 H(u u_x) + \frac{1}{18} u (H u_x)^2 + \frac{1}{6} u (u_x)^2 + \frac{1}{54} u_x H u_{xx} dx. \quad (4b)$$

For a soliton, these invariants have the values $I_1 = 2\pi/3$, $I_2 = 2\pi V/9$, $I_3 = \pi V^2/18$, and $I_4 = \pi V^3/81$, and $I_5 = \pi V^4/1944$.

As suggested by Ono, we investigate the initial condition

$$u(x) = A \frac{2V/3}{(Vx)^2 + 1} = A u_s(x). \quad (5)$$

Assuming that the final state consists of nothing but N solitons, we find restrictions on their velocities from the conservation laws. The initial condition (5) with amplitude factor A gives $I_1 = 2A\pi/3$, whereas for N separated solitons $I_1 = 2N\pi/3$. Thus only when $A = N$ can the initial conditions (5) evolve into N solitons. The invariants give four equations for the final velocities of the solitons. When $A = 2$ we find

from these four equations two final velocities:

$$\begin{aligned}V_1 &= (2 + \sqrt{2}) V = 3.41 V, \\V_2 &= (2 - \sqrt{2}) V = 0.59 V.\end{aligned}\tag{6}$$

The numerical results in Fig. 2 confirm that for $A=2$, the final state consists of only two solitons with the predicted velocities for $V = 2/3$. For $A=1.8$, however, the final state is a single soliton with velocity $2.9 V$, plus a small bump of slowly decaying amplitude. For $A=2.5$ the final state contains two solitons with velocities $4.7 V$ and $1.3 V$, plus a small dispersing tail.

For the amplitude factor $A=3$ the conservation laws give for the velocities of the final solitons $V_1 = 6.29 V$, $V_2 = 2.29 V$ and $V_3 = 0.42 V$. The final state in the computation shows indeed three solitons. The velocity of the smallest can not be determined accurately, but the larger two have velocities $V_1 = 6.24 V$ and $V_2 = 2.26 V$.

Some computations with the Benjamin-Ono equation have started with a "sech²" initial condition. These evolve into a number of solitons that depends on the first invariant, but now $u(x)$ can become negative, and an oscillatory tail may appear.

The higher conservation laws in conjunction with our computations show that the Benjamin-Ono equation should possess an exact two- and three-soliton solution. We hope to report on the exact solvability in a future paper.

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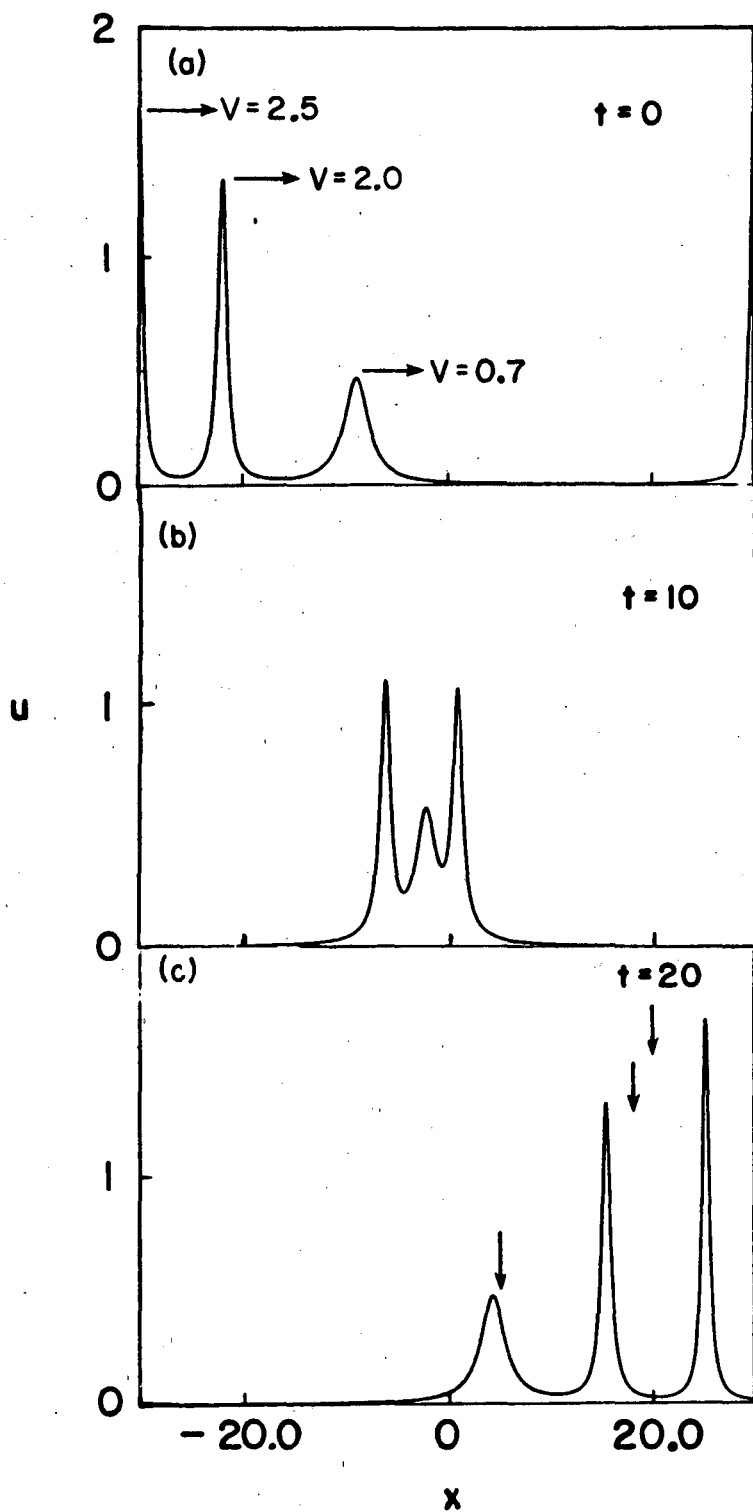
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Figure Legends

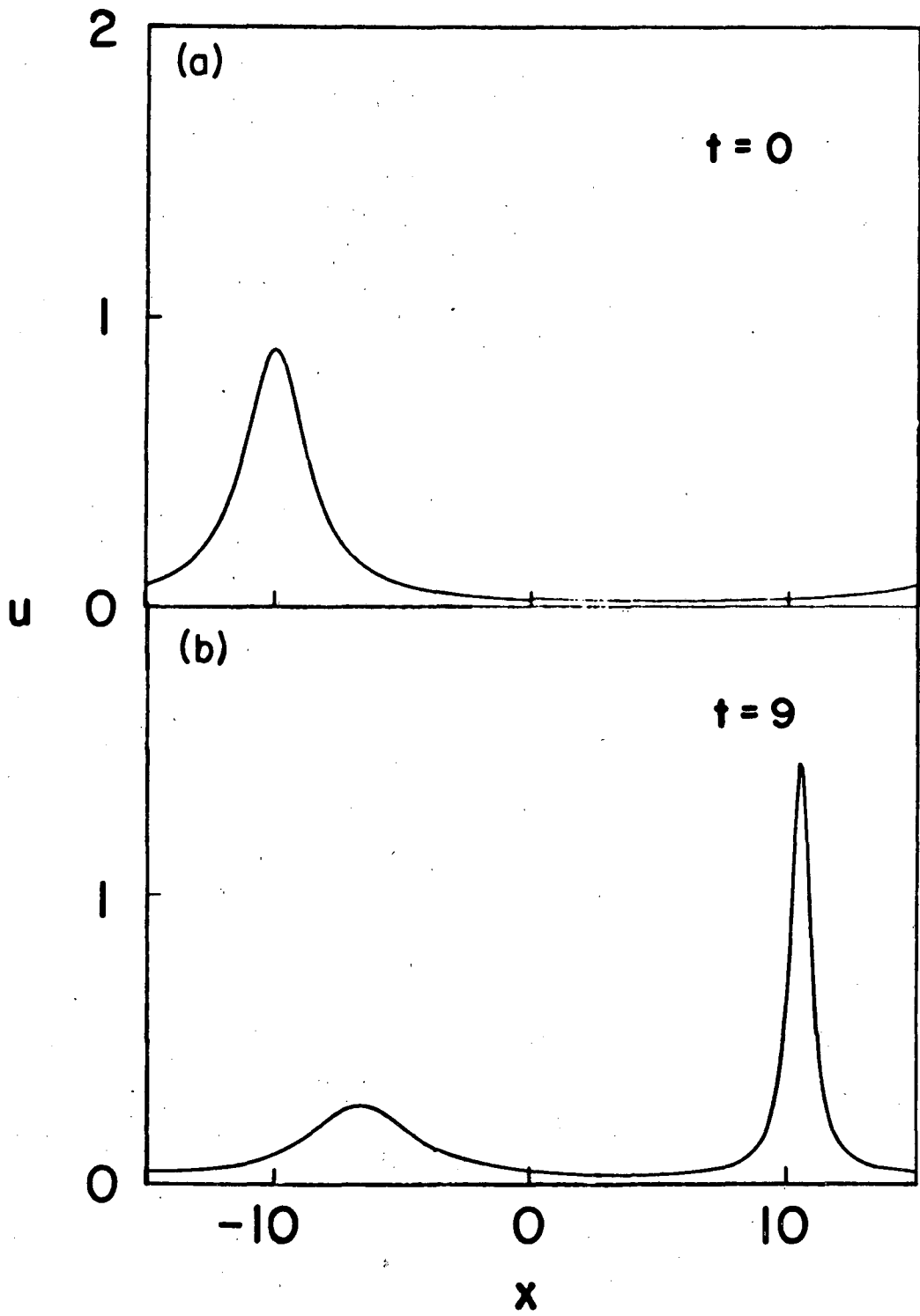
Figure 1 Three-soliton collision in the Benjamin-Ono equation. The solitons in the final state have the same velocities as initially. The positions of non-interacting solitons with these velocities are indicated by the arrows.

Figure 2 Breakup of a large Lorentzian pulse with initial velocity $V = 2/3$ and amplitude factor $A = 2$. The final state contains only two solitons with velocities 0.4 and 2.3.



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Fig. 1



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Fig. 2

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