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MATHEMATICAL FOUNDATIONS OF THE GENERAL ORDERED S-MATRIX AND ITS TOPOLOGICAL EXPANSION

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MATHEMATICAL FOUNDATIONS OF THE [illegible] S MATRIX

AND [illegible] EXPANSION

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ABSTRACT

The concept of order is analyzed for general hadronic amplitudes. It is shown that unitarity imposes restrictions on the graphs used to represent the ordered amplitudes and the most general graphs consistent with unitarity are derived and then analyzed in terms of their topological properties.

In the second part of the thesis, the mathematical ingredients for the classification of higher order corrections to the ordered amplitudes are introduced. Three "indices" characterizing the topological amplitudes and their products are defined and are shown to be consistent and complete for the classification.

A connection with the 1/N expansion, similar to the one introduced by Veneziano for mesons, is suggested.

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General Introduction to the Dissertation

The objective of this dissertation is to develop the mathematical aspects of an S-matrix theory of strong interactions based on the concept of Order. In the meson sector, this approach has registered sufficient success to make generalization to all hadronic amplitudes appear worthwhile.

The dissertation is presented in two separate and self-contained parts. In the first part we discuss the concept of "order"* in Hadron physics, a concept central to the theory. The basic point of the concept is to define a Hilbert space of hadronic states where channels are defined not only by particle momenta and helicities but also by an additional relation between any given particle A and a subset of particles $\{N\}_A$ called neighbors of A.

The most natural way to represent these relations is by means of graphs. The vertices of the graph represent the particles and the edges represent the relations between a vertex and its neighbors. Ordered amplitude graphs - representing reactions - are then formed by combining the channel graph according to well-defined rules. The properties of these amplitude graphs are examined.

* cf. Chew and Rosenzweig: "Dual Topological Expansion", LBL-6783 (1977), to be published in Physics Reports.

Throughout this dissertation, and in particular in the first part, the importance of the role of vertices of order three is exhibited. A potential explanation of this role had been discussed in a separate paper.* Another potential explanation, based on the operation of triangulation of a polygone - not developed in this thesis - is also worth considering. Briefly, the idea is as follows: Consider a vertex with n-edges. The dual of this graph is a polygone with n sides. There are in general several ways to triangulize the polygone. However, if the triangulation is considered as a graph, constructing its dual produces a tree graph with 3-vertices exclusively. Since the dual of the dual of a graph is the graph itself, we are essentially back to the original topology (i. e. connections). The triangulation performed on the polygone has, however, transformed the n-vertex into a tree graph composed of 3-vertices.

Thus, a n-vertex is eventually decomposed into a cubic tree graph. The special role of the 3-vertex is therefore related to the fact that every polygone can be trianguiarized, a fundamental result of Algebraic Topology.

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* J. P. Sursock. " On a Connection Between Charge Conjugation, Quark Structure and Ordered S-Matrix Theory", LBL-7515 (1978), submitted to Nucl. Phys. B.

Ordered amplitudes are only an approximation to physical amplitudes since, in the physical world, order is not observed.* Corrections to ordered amplitudes are needed. In fact, an infinite number of higher-order corrections must be introduced for each amplitude. These corrections constitute the "topological expansion", the subject of the second part of the thesis.

Here again, our emphasis is on the mathematical aspects of this expansion. The underlying physical framework is of course the unitarity properties of the S matrix and the resulting constraints on the singularity structure of the analytic connected parts (i. e., amplitudes).

Another avenue for the topological expansion was explored with some success <u>in the meson sector</u> but was later abandoned. The idea was to classify non-ordered products of amplitudes by their degree of "disorder". That is to say, given an intermediate state where particles do not have the same sequential order in both members of the product, the relevant index for the classification of the product would be the minimum number of permutations of intermediate particles in one member of the product to reproduce the order of

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* It is interesting to note that both S-matrix theory and Field theory introduce, as basic concepts, unobservable properties of matter.

the other member. It turns out that this number is equal to the minimum number of handles used, for instance, in Veneziano's approach and hence is perfectly relevant in the meson sector. The generalization was found to be complicated in the case of generalized order but the question remains open whether such an approach is fruitful or not. If it were useful, then it would provide a connection between the concept of order and disorder and the singularity structure of all topological amplitudes.

We realize that these comments may not be fully appreciated by a reader unfamiliar with the contents of the dissertation. We hope we have not eroded his/her interest before he/she has had a chance to read more about this fascinating subject.

PART I

MATHEMATICAL PROPERTIES OF ORDERED AMPLITUDE GRAPHS *

1. Introduction

The S-matrix approach to the topological expansion of Veneziano[1] is made possible by invoking the notion of "order" first introduced by Chew et. al.[2] for mesonic amplitudes and later generalized to all hadronic amplitudes.[3] The ordered S matrix is defined in an ordered Hilbert space and obeys unitarity as does the physical S matrix. This fact guarantees consistence between poles and asymptotic states. The attempt to get rid of the unobservable order by summing over all ordered amplitudes (planar amplitude) produces a breaking of the unitarity constraint. Unitarity is restored when all terms of the topological expansion are taken into account.

The objective of this paper is to discuss the mathematical framework of the material presented in Ref. 3.

2. The Concept of Order

The first step of the theory is to define a simpler S matrix (in terms of analytical structure) in an unobservable Hilbert space, called the Ordered Hilbert Space (OHS). In this space, a particle, in addition to its mass, momentum and spin has a well-defined collection of "neighbors".* For instance, each meson has two neighbors, each baryon has three neighbors and so on. A more general channel

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* The number of neighbors will turn out to be characterizable as the number of "quarks" and "antiquarks" that "build" a hadron.

defined by n particles is specified by the particles involved <u>and</u> by the specification of each particle neighbor (concept of order). For instance, in the simple case of a channel containing mesons only, the channel could be written as $|a\rangle = |M_1 \ldots M_i \ldots M_j \ldots M_n\rangle$. Here, the order matters as it specifies the neighbors of each mesons M_i. This channel is, in general, different from the channel $|b\rangle =$ $|M_1 \ldots M_j \ldots M_i \ldots M_n\rangle$.

In either channel $|a\rangle$ or $|b\rangle$, M_1 and M_n have each one neighbor. However, considered in an amplitude, they will eventually have two neighbors also. For example, in the amplitude $\langle M_k \ldots M_m | T | M_1 \ldots M_n \rangle$ M_1 will have M_2 and M_m as neighbors and M_n will have M_k and M_{n-1} as neighbors. Such a channel $|a\rangle$ can be graphically represented in Fig. 1a and the amplitude in Fig. 1b. Thus, although all the neighbors of a given particle are not necessarily completely specified in a channel they are unambiguously defined when the particle is considered in a process (transition amplitude).

In the more general case, a particle may have many neighbors (e. g., Fig. 2). Apriori, any graph can represent a channel of the OHS. We shall however impose the following restrictions:

(1) The graph must be connected.

(2) The graph must not contain loops or tadpoles (i. e., edges leaving and returning to the same vertex).

(3) The graph should not include vertices of degree higher than three.* This requirement is discussed in more detail in Appendix C and is independently justified in Ref. 4. It does not preclude the existence of particles with more than three neighbors because a particle does not necessarily correspond to a vertex but can correspond to a whole graph. So, for instance, particles with four and five neighbors are shown on Fig. 3**

(4) Each edge of the graph representing a channel of the OHS must be distinguishable. As will be shown in Section 4, this requirement is crucial to obey the constraints of the ordered S-matrix.

3. The Ordered Hilbert Space (OHS)

To construct the OHS we start from the cubic tree graphs (i.e., tree graphs such that all vertices are of degree three. Contrary to mathematician's practice, we drop the 1-vertices at the "tips" of the tree.). Examples are shown on Fig. 4.

All cubic tree graphs are allowed to represent ordered channels. In general, the topological structure of the tree graphs is sufficient to distinguish between the edges. However, some tree graphs exhibit a two- or a three-fold symmetry (e. g., Fig. 4b). It is possible to show (Appendix A) that in order to restore the distinguishability of every edge it is necessary and sufficient to orient and color the edges of the graph. The orientation is such that all edges are

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* The degree of a vertex is the number of edges attached to it.

** A more complete discussion of exotic states is found in Ref. 3.

incoming toward a vertex or outgoing away from it. Then every edge is given a color index 1, 2 or 3 such that, at any 3-vertex, the three edges all have different color indices. This operation insures that no graph is symmetric.* An example is shown on Fig. 5.

A colored cubic tree graph is called a skeleton. All possible ordered states are built from skeletons by the following operation (shown on Fig. 6):

(1) Select an edge a (Fig. 6a) with a color index i and cut it into two parts a_1 and a_2 (Fig. 6b).

(2) Connect the two ends with two edges b and c with color indices j and k such that i, j and k are all different and with opposite orientation relative to a (Fig. 6c).

It is easy to see that the new edges (b and c) are still distinguishable. First they can be distinguished from one another because they have different color and second they can be distinguished from any other edge in the graph because they are "inserted" in the old edge a which was distinguishable by construction.

The two operations described above can be iterated at will on any edge or edges of the new graph thus constructed. An example is shown on Fig. 7 (where the arrows and color indices are omitted for clarity).

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* This operation is called induced symmetry breaking.

Finally, any number of vertices of degree two (mesons) can be inserted in any edge of such graphs without altering its topology. Sometimes, however, a meson will be represented as the cubic graph of Fig. 6e. For the ordered S matrix, the two notations are equivalent. It can be shown that the graphs thus constructed are the most general graphs consistent with unitarity. However, the proof will be postponed until the end of Section 4.

Thus, every graph of the OHS (allowed graph) can be related to a unique skeleton. The OHS is divided in disjoint subspaces called <u>sectors</u>. Each sector is defined by a skeleton and all graphs derived therefrom by the operations described earlier (Fig. 6).

4. The Ordered Amplitude Graphs (OAG)

We shall postulate that the sectors defined above are orthogonal. That is, the ordered S matrix does not allow communication between channels belonging to different sectors. Thus, for a transition amplitude to be non-vanishing, the following requirements must be met:

(1) Poincaré invariance

(2) Conservation of internal quantum numbers

(3) Conservation of order (i. e. sectors)

The transition amplitude between two states belonging to the same sector is represented by a graph such that all the neighbors of all particles are defined. An example is shown on Fig. 8.

The OAG is constructed by matching all* the corresponding outer** edges of each state. One can see now the importance of these edges being all distinguishable. If this were not the case, several matching would have been equally possible. This would be contrary to the principle that given the initial and final states, the transition amplitude is unambiguously determined.

The fact that every edge should be distinguishable, not only the outer edges, will result from the requirements of crossing symmetry (every edge can be an outer edge).

An important property of a unitary S matrix is factorization. In this context it means that each cut separating the amplitude graph into two connected parts (bisection) defines a collection of poles belonging to the same sector (the cut edges) and two channels that should belong to the OHS. Several bisections of the amplitude graph of Fig. 8 are shown on Fig. 9. The cuts can all be thought of as planes cutting a sphere on which the amplitude graph is imbedded.***

It is now possible to prove the statement made earlier that the most general channels allowed by unitarity are those defined in Section 3. We show in Appendix C that vertices of degree higher than 3 should be discarded and in the remainder we concentrate on

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* This point will be discussed in Section 5.

** "Outer" or "dangling" edge refers to the peripheral edges of a tree graph (the "tips" of the tree).

*** Although, for the considerations discussed in this paper, we do not require such an imbedding.

graphs constructed with 3-vertices only (2-vertices can be added anywhere).

We first remark that the graph of Fig. 10 cannot be allowed in the OHS because the edges are not all distinguishable.* Then we show that any state with the same topology as Fig. 10 is inadmissible regardless of the coloring. Let's assume that we can use as many colors or indices as we wish to distinguish the edges and we start from Fig. 11a which we combine with itself to form the amplitude graph of Fig. 11b. This should always be possible. Next we cut (11b) as shown on Fig. 11c to obtain the state of Fig. 11d on the right hand side. This new state is made to combine with itself in a new amplitude shown on Fig. 11e which we cut again as shown on Fig. 11f. The resulting states are the unwanted states of Fig. 10. Thus, regardless of coloring and orientation, the graph of Fig. 10 should not appear in the OHS as a graph or as a subgraph.

The immediate consequence of this result is that any graph containing a subgraph of the type of Fig. 12a should not be allowed, because by combining this state with itself in an amplitude (Fig. 12b) we generate the unwanted graph of Fig. 10 as shown by the bisection in Fig. 12c. Finally, it is easy to see that a graph without subgraphs of the type (12a) and with only one outer edge

* The undesirable aspect of this graph was first noted by G. Weissmann.

is not colorable (Fig. 13) and therefore cannot be made consistent with unitarity.

This completes the proof that the only allowed states of the OHS are those constructed by the operations of Section 3 (Fig. 6). In summary, an allowed channel graph has the following characteristics:

(1) It has more than one outer edge.

(2) Every cycle is connected to the set of outer edges by two and only two edges (Fig. 6c in contrast to Fig. 12a). These two edges will be called links. The vertices at which the links connect to the cycle will be called gates of the cycle and the disjoint paths of the cycle connecting the two gates will be called legs of the cycle.

In the remainder of this paper we shall use the notation $\langle g_f | g_i \rangle$ to denote the OAG obtained from the graph g_f and g_i corresponding to the states $|f\rangle$ and $|i\rangle$ respectively. γ_f and γ_i are the corresponding skeletons and we must have $\gamma_f = \gamma_i$ for $\langle g_f | g_i \rangle$ to be non-vanishing.

5. Topological Properties of the Ordered Amplitude Graphs

(1) Up to a homeomorphism, an Ordered Amplitude Graph (OAG) must have the topology of a regular cubic graph (i. e., all vertices are of degree three). In other words, all outer edges of the channel graphs must be connected as mentioned earlier. Failure to do so would result in a possibility of cutting the graph in

three parts when attempting to isolate certain ordered states.[3] This is shown on Fig. 14. Inasmuch as a single quark would be associated with a one-vertex, one can see that the failure to reconcile graphs such as Fig. 14 with unitarity guarentees that quarks cannot be poles of the S-matrix and thus cannot be considered as asymptotic states (quark confinement).[3*]

(2) The OAG are connected and planar (evident).

(3) Every cycle of the OAG has an even number of vertices. This results from the fact that each vertex of the left hand skeleton has <u>one</u> counterpart on the right hand skeleton (cf. Fig. 8) and all cycles within each skeleton contains by construction of Section 3 an even number of vertices on either leg of the cycle. As discussed in Ref. 3, this property enables one to specify Fermi statistics for baryons.

(4) Therefore the OAG is bipartite (theorem of graph theory).[5] That is, it is possible to divide the set of vertices of the graph into two disjoint parts A and B such that no vertex of A is adjacent to another vertex of A and similarly for B.

(5) If we assign a coefficient +1 to the vertices of A and -1 to the vertices of B, then, because of properties 3 and 4, the algebraic sum of these coefficients on any cycle is zero. This

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* This argument was originally put forth by G. Weissman. A somewhat different argument for quark confinement is developed in Ref. 4.

is also true for the overall amplitude. If we associate these coefficients with the baryon number we automatically obtain baryon number conservation.

(6) Property 5 is equivalent to the possibility of coloring the edges of the OAG with three colors.*[5] Furthermore, it is possible to assign the coefficients ±1 to a vertex such that when the graph is imbedded on an orientable surface (e. g., sphere), the coefficient +1 is associated with the clockwise cyclic order of the color indices 1, 2 and 3 around the vertex and the coefficient -1 to the opposite orientation.[5] Thus, if the OAG is imbedded on an orientable surface the edge orientation becomes redundant. The orientation of the color indices is associated with the charge conjugation operation.[4] That is, an odd permutation of color indices transforms an amplitude into its charge conjugate.

(7) Existence and uniqueness of <u>mates</u>: In an OAG we define the <u>mate</u> of a vertex A as a vertex A' which could be reached from A by three disjoint paths (i. e., paths that have no edge in common). The fact that to each vertex of an OAG corresponds one and only one mate is a crucial result that will enable a

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* This is also true for the faces of the OAG: namely, the faces of the OAG can be colored using three colors only such that no two adjacent faces are colored with the same color.

consistent topological expansion.[8] We establish this result in Appendix B.

(8) Algebraic notation: Since each edge of the OAG is distinguishable let us relabel the edges with labels α_i. Then a vertex A is associated with three labels α_i, α_j, α_k. We adopt the following conventions:

(a) We write A with the α-indices as superscript or subscript according to whether the arrows are outgoing or incoming (or, if the graph is imbedded on an orientable surface, according to whether the color cyclic order around A is clockwise or anticlockwise).

(b) The order of the indices is such that the first one applies to color index 1, the second to color index 2 and the third to color index 3.

(c) Finally, we associate to each mate pair a number which helps identify the members of the pair. An example is shown on Fig. 15. One can see that each α-index should appear twice, once as a superscript and once as a subscript. The algebraic notation stresses the fact that only the connections between vertices are important.

If the subgraph (AB) of Fig. 15 represents one particle (a meson) then we could have used the more compact notation

$M^{..\alpha_3}_{..\alpha_6} = A^{\alpha_1\alpha_2\alpha_3} B_{\alpha_1\alpha_2\alpha_6}$. Similarly, if (AD) is a baryonium, we can write $E^{\alpha_1\alpha_2 .}_{\alpha_4\alpha_5 .} = A^{\alpha_1\alpha_2\alpha_3} D_{\alpha_4\alpha_5\alpha_3}$.

One can readily see the connection between the edges of the OAG and the quark structure usually associated to hadronic states. The concept of order thus implies the quark structure, including zero triality and quark confinement.[2,3,4,7]

6. Unitarity and Cluster Decomposition

In this section we make no distinction between graphs representing a connected part T_{fi} and its complex conjugate T^*_{fi}.

As discussed in the introduction, the unitarity condition is expressed through the discontinuity relations in a given process. The process is specified by an initial state $|i\rangle$ and a final state $|f\rangle$. Let us denote the corresponding graphs g_i and g_f and the corresponding skeletons by γ_i and γ_f. Furthermore, for any intermediate state $|n\rangle$ to which we associate a graph g_n and a skeleton γ_n we can define two OAG which we denote $\langle g_f|g_n\rangle$ and $\langle g_n|g_i\rangle$. The product of the two ordered amplitudes, $\langle g_f|g_n\rangle \langle g_n|g_i\rangle$ will be non-zero only if $\gamma_i = \gamma_n = \gamma_f$. The result $\gamma_i = \gamma_f$ indicates that the product of two ordered amplitudes can always be interpreted as part of the discontinuity of an ordered amplitude $\langle g_f | g_i \rangle$. This fact is essential for the consistency of the theory. It is a consequence of the requirement of conservation of order discussed earlier.

We now turn to Cluster Decomposition. Consider the two states $|i\rangle$ and $|f\rangle$ introduced above. Each graph g_i and g_f can be decomposed into n connected subgraphs g_{ij} and g_{fj}; $j = 1,\ldots,n$. and correspondingly, each skeleton into n connected subskeletons γ_{ij} and γ_{fj}. The following rules however must be obeyed by the decomposition:

(1) To each skeleton γ_{ij} there corresponds an identical skeleton γ_{fj} at the same relative location in the original tree. In other words, the two skeletons γ_i and γ_f must be decomposed identically.

(2) A subgraph representing a single particle must lie entirely in the same g_{ij} or g_{fj}.

We then define n ordered amplitudes $\langle g_{fj} | g_{ij} \rangle$ for each couple of graphs (g_{fj}, g_{ij}) separately. If one or more of these amplitudes vanish, the particular cluster decomposition is set equal to zero. An example of cluster decomposition is shown on Fig. 16.

Finally, we address the question of crossing. Consider an amplitude in the phase space region where $|a\rangle$ and $|b\rangle$ are the initial and final states respectively. This ordered amplitude is associated to an OAG $\langle g_b|g_a\rangle$. It is possible to analytically continue this amplitude to another region of phase space where $|c\rangle$ and $|d\rangle$ are the initial and final states respectively, provided $\langle g_d|g_c\rangle = \langle g_b|g_a\rangle$. In other words, given an OAG, the regions of phase space related by analytical continuation are those corresponding to all possible bisections of the OAG such that one connected graph thus obtained corresponds to an initial state and the other to a final state.

Conclusion

The objective of this paper was to expand on the mathematical aspects of the ordered hadronic amplitudes presented in Ref. 3. In particular we have presented a systematic way to construct the ordered Hilbert space of Ref. 3 and we have shown that it is the most general Hilbert space consistent with unitarity.

We have also discussed a number of topological properties of the ordered amplitude and, in particular, the concept of mates was exhibited and will turn out to be crucial in the topological expansion.[8] Finally the ordered S matrix turns out to be cluster decomposable and therefore crossing symmetric.

Acknowledgments

I would like to thank Professor G. F. Chew for numerous discussions and helpful suggestions during the preparation of this paper. I also would like to thank G. Weissmann for valuable conversations pertaining to this paper.

APPENDIX A: INDUCED SYMMETRY BREAKING

To establish that cubic tree graphs have only three possible symmetries we recall Jordan and Sylvester's theorem:[6] "Every tree has a center* consisting of either one vertex or two adjacent vertices." Thus there are two cases to consider.

(a) There is only one central point. Since it is a 3-vertex, the symmetry requires either that the three branches of the point be identical yielding a 3-fold symmetry (e. g., Fig. A1) or two branches only are identical yielding a 2-fold symmetry (e. g., Fig. A2).

(b) There are two central points. In that case, the theorem states that the two points are connected by one edge: e. The only possible two-fold symmetry of a cubic tree graph is then obtained where the two subgraphs on either side of e are identical (e. g., Fig. A3).

In the case (a), the symmetry around the central point is manifestly broken by coloring the three edges of the central vertex with different colors and in the case (b), the symmetry is broken by giving to the edge e an orientation.

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* To each vertex of a tree graph one can associate an integer corresponding to the (unique) path from that vertex to the most remote "tip" of the tree (relative to the vertex). A central vertex is a vertex with the smallest integer and the center of the tree graph is the set of all central vertices.

As we take a path on one (now well-defined) branch, away from the center of the tree graph and toward an outer vertex we encounter at each vertex a bifurcation. The choice at the bifurcation can be made well defined by the same coloring procedure. Thus, any edge of the graph can be sepcified by the colored path one must take to reach it with the center of the graph as starting point.

As an example, in Fig. A4, edge a is defined by the path that starts at the center of the graph c and follows the colored edge, 1, 2 and 3 successively.

APPENDIX B: EXISTENCE AND UNIQUENESS OF MATES

Existence: Given a vertex A in an ordered amplitude graph G we want to show that there exists another vertex A' such that A and A' are connected by three disjoint paths.

Since in a regular cubic graph each vertex is at least on two circuits (in fact on three circuits), A is on some circuit K. We bisect G such that K is entirely on one channel graph. Again, this is always possible because it suffices to isolate part of an edge of G as the bisection. Then by the construction of Section 3, A is necessarily the gate of some circuit C. Then let A' be the other gate of the same circuit C. It is easy to see that two of the three disjoint paths connecting A to A' are the two legs of the circuit C. The third path is obtained by following a path containing the links of C -- which always exists since the amplitude graph is regular.

Uniqueness: Consider $A'' \neq A'$. If A'' is on one leg of C then the path from A to A'' containing the links and the path containing the other leg are no longer disjoint. If A'' is not on C, then the paths containing the two legs have at least one link in common. Thus, no other vertex than A' can fulfill the definition of a mate.

Alternatively, it can be shown that two mates A and A' share the same three Edmond orbits (i. e., faces of G).

APPENDIX C: CONSTRUCTION OF HILBERT SPACE WITH n-VERTICES IN GENERAL

First we note that the sector displayed in Fig. C1 (two edges with different colors) would conflict with unitarity unless it does not communicate with any other state. This could be seen in the amplitude of Fig. C2 where a state with indistinguishable edges can be extracted.*

Now we discuss the possibility of admitting n-vertices ($n > 3$) in the graphs. This automatically requires at least n colors to distinguish the edges of a single vertex. There are two possibilities:

(a) The n-vertices are admitted concurrently with the 3-vertices. In this case, one can readily construct the unwanted state of Fig. C1 (e. g., Fig. C3).* Incidently, Fig. C3 also shows that we cannot have more than three colors even if we restrict ourselves to 3-vertices.

(b) The 3-vertices are not admitted. There we have two subcases:

(b-1) n is even: In this case, one cannot construct skeletons** with an odd number of outer edges. This results from the fact that any skeleton is constructed with k

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* This part of the argument is due to G. Weissmann.[7]

** For $n > 3$, skeleton graphs may include cycles as long as no more than two vertices reside on any cycle.

vertices. We thus have nk edges at our disposal but nk is even and every connection reduces that number by two. Hence the final number of outer edges is always even. Therefore the resulting Hilbert space is much more restrictive than the OHS (constructed with 3-vertices).

(b-2): n is odd (and $n > 3$). In this case it is easy to see that one cannot construct a skeleton graph with three outer edges. In fact, more generally, it is impossible to construct a skeleton graph with m outer edges and with $m < n$ if m and n are odd. This can be seen as follows: if k is the number of vertices and e the number of internal edges we have $kn - 2e = m$. On the other hand, if k is even, $e_{max} = kn/2 - 1$ and if k is odd, $e_{max} = (k - 1)n/2$. The latter case is of interest here $(m,n$ odd) and we have $m_{min,odd} = kn - (k - 1)n = n$. So the resulting Hilbert space is again more restrictive than the OHS.

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* This work was supported by the U. S. Department of Energy.

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FIGURE CAPTIONS

Fig. 1: (a) Mesonic ordered state. (b) mesonic amplitude

Fig. 2: Generalization of the sequential order

Fig. 3: Particle graph representing (a) a particle with four neighbors and (b) with five neighbors

Fig. 4: Cubic tree graphs: basis of the ordered Hilbert space

Fig. 5: Oriented and colored tree graph (skeleton)

Fig. 6: Elementary operation generating all graphs of the ordered Hilbert space

Fig. 7: Successive iterations of the operations of Fig. 6.

Fig. 8: Combining states to obtain an ordered amplitude graph (OAG)

Fig. 9: Bisections of ordered amplitude graphs

Fig. 10: This graph is illegal because the diagonal edges are not distinguishables.

Fig. 11: Constructing the graph of Fig. 10 starting with an arbitrary number of indices to distinguish between edges

Fig. 12: (a) Prototype of inadmissible subgraph. (b) The result of its combination with itself and (c) showing the emergence of the unwanted graph of Fig. 10

Fig. 13: Uncolorable graph with only one external vertex.

Fig. 14: Possibility of cutting the OAG in three parts if 1-vertices are allowed

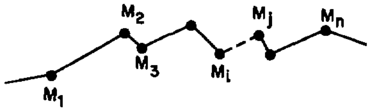


Fig. 1a

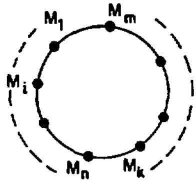


Fig. 1b

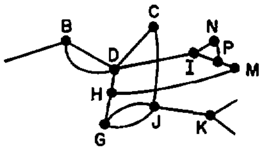


Fig. 2

Fig. 3a

Fig. 3b

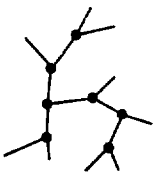


Fig. 4a

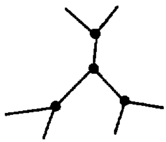


Fig. 4b

XBL 785-715

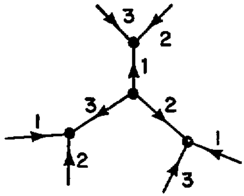


Fig. 5

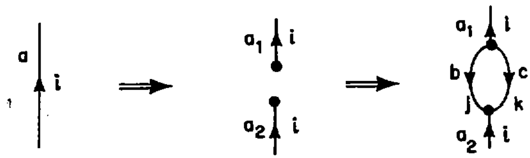


Fig. 6a Fig. 6b Fig. 6c

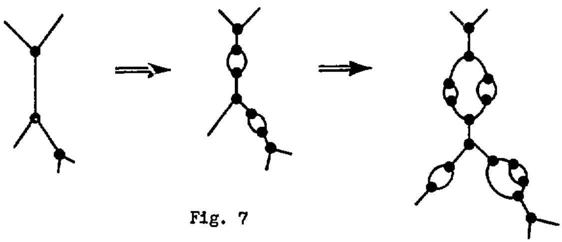


Fig. 7

XBL 784-714

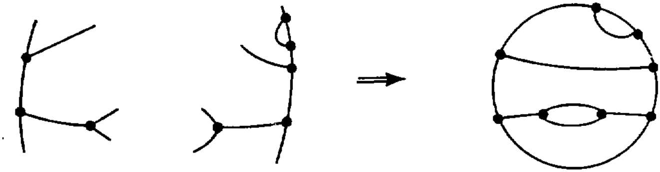


Fig. 8

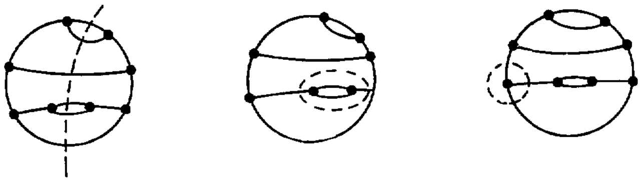


Fig. 9

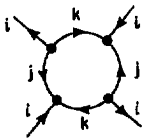


Fig. 10

XBL 784-716

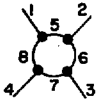


Fig. 11a

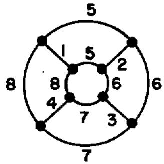


Fig. 11b

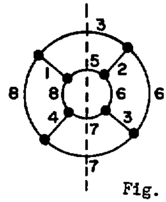


Fig. 11c

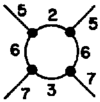
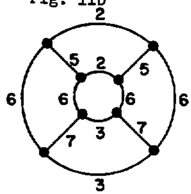


Fig. 11d



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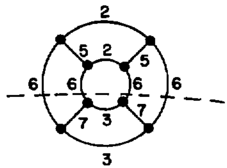

Fig. 11f

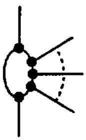
Fig. 12a

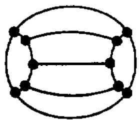
Fig. 12b

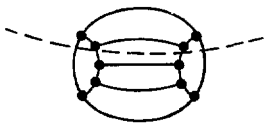
Fig. 12c

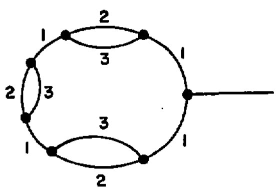

Fig. 13

XBL 784-713

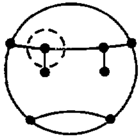

Fig. 14

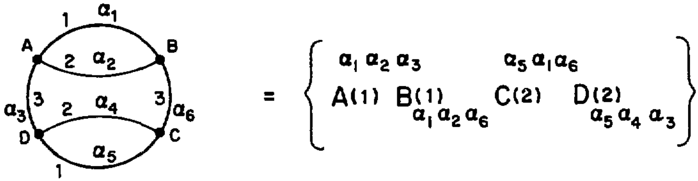

Fig 15

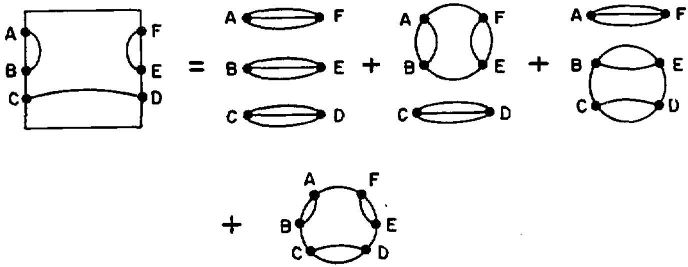

Fig. 16

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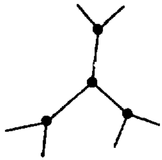

Fig. A1

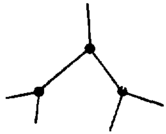

Fig. A2

Fig. A3

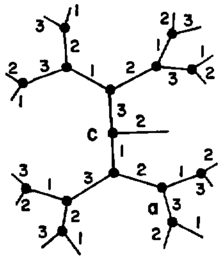

Fig. A4

Fig. C1

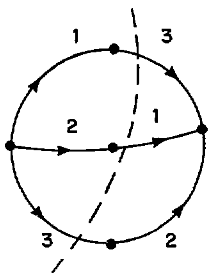

Fig. C2

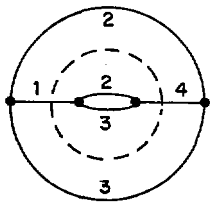

Fig. C3

XBL 784-710

PART II

GENERAL TOPOLOGICAL EXPANSION OF HADRONIC AMPLITUDES*

1. Introduction*

The objective of the Dual Topological Unitarization (DTU) program is to systematically approach complete satisfaction of unitarity for physical transition amplitude through an infinite sum of simpler structures called topological amplitudes. Formally, one may write the transition amplitude between states $|i\rangle$ and $|f\rangle$ as

$$T_{fi} = \sum_{\alpha} (T_\alpha)_{fi} \tag{1}$$

where α stands for a number of "indices" completely characterizing the topological amplitudes. The word "indices" is used rather loosely here (for historical reasons). It refers not only to the integers g and r to be discussed in Sec. 5 but also to the boundary structure of the amplitude (cf. Sec. 5) which is a graph, not an integer. (In the early days of the topological expansion only the number of boundaries was considered important for the 1/N expansion). Thus, the symbol α represents a couple of integers (g and r) as well as the specification of interconnections between vertices representing particles (the boundary structure). Henceforth, this set will be referred to as the "indices" of a topological amplitude.

* This paper is a continuation of Ref. 1.

The adjective "topological" describes the fact that the analytical structure of the T_α's is related to the topological properties of associated dual-like diagrams. This adjective will be used for all amplitudes T_α except those corresponding to the lower order of the expansion (the ordered amplitudes).

2. General ordered amplitude

The leading terms in the expansion of Eq. (1) are called ordered amplitudes.[2,3] The ordered amplitudes are defined in an ordered Hilbert space and obey unitarity.[1,3] In this section we only briefly recall the basic properties of ordered hadronic amplitudes. For more details, the reader is referred to Refs. 1, 3, and 9.

1. The order between particles of a given state is represented by a colored and oriented graph relating each particle to its neighbors. In the ordered Hilbert space, two states with the same particles (same spin and momenta) but differing by the order i. e., by the graph representing the connections between particles are considered different.

2. Not all graphs are allowed to represent ordered states. It can be shown[1] that the most general graphs are those constructed starting with cubic trees - i. e., such that all internal

vertices are of degree three - and expanding according to the following operation (cf. Fig. 1):

(a) Select an edge and cut it in two parts (color i).

(b) Connect the two vertices newly created with two edges colored j and k $(j \neq k \neq i)$ and with opposite orientation.

(c) Iterate as many times as necessary with any edge.

It can be shown that in such graphs all edges are distinguishable.

3. Particles (poles) are represented by subgraphs of these graphs. In general particles are represented by subtrees. Mesons could be represented either by 2-vertices or by the graph of Fig. 1c. In this paper we use the latter for convenience of presentation (all vertices are 3-vertices) but the 2-vertex representation can be shown to be equivalent.

4. A colored oriented tree graph and all graphs derived from it by the operation above form a <u>sector</u> of the ordered Hilbert space. The tree graph is called the <u>skeleton</u> of the sector.

5. Ordered amplitudes are non-vanishing only between states belonging to the same sector. The ordered amplitude graph (OAG) is obtained by matching the "corresponding outer edges" of each state. In this context, two "corresponding" outer edges refer to the same outer edge of the common skeleton graph.

6. Among many topological properties of the OAG one of the most important is the existence of <u>mates</u>. Given a vertex A, there exists a unique vertex A' such that A and A' are connected by three disjoint paths (equivalently, A and A' are on the same three faces). A and A' are called mates. An example is shown on Fig. 2. A_1 and A_2 are mates as well as B_1 and B_2; and C_1 and C_2.

7. OAG's are planar, 3-colorable and bipartite. As a result, without loss of generality, the graphs can be imbedded on a sphere such that the orientation of the vertices alternates on any cycle. (By convention, a vertex is positively oriented if the cyclic order of the colored edges is clockwise and negatively oriented if the cyclic order is anticlockwise). For instance, in the graph of Fig. 2, A_1, C_1 and B_2 are positively oriented whereas A_2, B_1, and C_2 are negatively oriented. Once imbedded on an orientable surface (e. g., sphere), the orientation of the edges (i. e., arrows) becomes redundant and henceforth will be dropped.

3. Topological amplitudes

In order to get rid of the order degree of freedom, one is led to define a new amplitude as the sum of all possible ordered amplitudes containing the relevant particles.[3] For historical reasons, this new amplitude is called "planar amplitude". An example is shown on Fig. 3.

One quickly realizes that planar amplitudes do not obey unitarity since the product of two planar amplitudes includes unordered products.* The unordered products can be classified in terms of their topological structure. More specifically, it will be shown that they can be classified in terms of three "indices". The topological structure of a product being related to the analytic structure of the associated complex function. All products belonging to a given classification (i. e. with a specific set of "indices") are then attributable to the discontinuity formula of a topological amplitude characterized by the same "indices."

Since the planar amplitude is not unitary it must be corrected. The corrections include these topological amplitudes generated through unordered products by the mechanism described above. One expects a better approximation (i. e., closer to unitarity) to be achieved by adding to the planar amplitude

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* The expression "ordered product" refers to products of amplitudes that could appear in discontinuity formulae of ordered amplitudes. The "unordered products" never appear in such formulae.

all the topological amplitudes found in products of planar amplitudes. This new amplitude is still not unitary because a product of two of these new amplitudes will generate still higher order topological amplitudes. The process needs to be repeated indefinitely. Of course, to be meaningful this topological expansion must comverge at a reasonable rate. There are numerous indications - although no rigorous proof - that such is the case.

4. Mesonic sector revisited

Mesonic amplitudes, with simple sequential order have been extensively studied. The original concept, due to Veneziano[4], made use of dual diagrams in the Feynman spirit and interpreted the expansion in terms of the parameter $1/N$ where N is the number of flavors. Two topological properties of the graph were identified as affecting the power of $1/N$.

(a) The genus of the graph (h_v)

(b) The number of boundaries (B_v). This latter number is essentially the number of orbits (see Appendix A) to which external particles are attached. Later, Chew and Rosenzweig[2] reinterpreted the diagrams in the S -matrix spirit and have shown that

(1) the genus of the graph is related to the strength of the discontinuity

(2) the number of boundaries is related to the pole structure.

Thus, h_v and B_v characterize the analytic structure of the function associated to the graph.

Three types of graphs have been used to represent the topological amplitudes of mesons. In the mesonic sector they are all equivalent. Examples are shown in Fig. 4. Figure 4a is a representative of the quark (or dual) diagrams(lines represent quarks); whereas Fig. 4c is the equivalent particle diagram (lines represent particles - they are not oriented). Figure 4b is the equivalent hybrid diagram (oriented lines represent quarks, non-oriented lines represent particles). It turns out that the most useful type of diagram for generalization purposes is a modified version of the hybrid diagrams. Figure 5 is equivalent to the diagrams of Fig. 4 ($h_v = 1$; $B_v = 1$) except that each particle line is replaced by two particle lines.* At each vertex, the cyclic order is alternatively quark line and particle line. A particle line either starts and ends "outside"

* It is tempting to associate the pair of particle lines with quarks as for dual diagrams. However, such association will not be made in this paper; our particle lines are not oriented.

the quark rings or starts and ends "inside" the quark rings. ("Outside the ring" means that at any vertex, a clockwise rotation goes from the incoming quark edge to the outgoing quark edge. The situation is opposite if we are "inside the ring" as in Fig. 5 for instance).

The genus, h, of the graph in Fig. 5 and the number of orbits, f, are higher than in the corresponding graphs of Fig. 4. However, it is easy to see that we have a general relation relating the new and old quantities:

$$\begin{aligned} h &= 2h_v + 1 \\ f &= 2f_v . \end{aligned} \qquad (2)$$

Now, the orbits of the new graphs can be coupled in pairs such that both members of a given pair have the same particle structure (number and order). The boundary is now redefined as being a couple of orbits with the same particle structure. Further, if the number of quark rings (i. e., the number of connected graphs when all particle lines are removed) is d instead of 2 as in the previous example, then we have the more general formula:

$$\begin{aligned} h &= 2h_v + d - 1 \\ B &= B_v \end{aligned} \qquad (3)$$

As will be discussed later, the amplitude is still characterized by h_v(and B_v) and not h. (More specifically, the index g to be discussed in Sect. 5 is equal to $2h_v$ for mesonic amplitudes).

5. Hadronic Amplitudes

5.1. Construction of Topological Amplitudes

The construction of higher order topological amplitudes is carried through products of amplitudes of lower order. In as much as these products define the higher amplitudes, a topological amplitude is defined by its discontinuities.

A necessary and sufficient set of conditions to define a consistent topological expansion can be stated as follows:

(a) The first condition is to define a multiplication rule for the OAG consistent with the unitarity requirements at the ordered level. In other words, the multiplication rule should be such that the ordered product of two OAG can be interpreted as representing the discontinuity of an ordered amplitude in an ordered discontinuity formula.

(b) The next condition is to characterize the graphs representing the product of two OAG by a set of "indices" such that substitution of an ordered subgraph of the intermediate state* of the product by its skeleton (single particle) - or conversely - does not alter the indices and substitution of unordered subgraphs by their skeletons alters at least one index. In other words, these "indices" completely determine the product relative to an ordered product.

(c) The particular product - or discontinuity - can then be attributed to a topological amplitude carrying the same "indices." Therefore, by definition, the topological amplitude can be represented by any (product) graph carrying the relevant "indices". In practice however one chooses the simplest graph carrying these "indices". The algebraic notation, to be introduced later, automatically incorporates this feature.

(d) The product of two general topological amplitudes must be defined by the same multiplication rule as in condition (a) and carry the same set of "indices" as found in (b). Furthermore, knowledge of the "indices" of both members of the product and

---

* A subgraph of the intermediate state is a connected subgraph such that all vertices belong to the intermediate state. This subgraph is "ordered" if the corresponding vertices on either side of the product are connected in exactly the same v (ordered product) and is "unordered" otherwise.

specification of the particular intermediate state should be sufficient to determine the set of "indices" characterizing the product.

In summary, the topological amplitudes are defined through products of lower order amplitudes. They are characterized by a set of "indices" related to the topological structure of the associated graphs. These "indices" are selected by the multiplication rule of the lower order graphs and by the requirements of consistency and completeness (i. e., products of topological amplitudes are defined by the same set of "indices" as either member of the product and this set is completely determined by the indices of the members of the product and the intermediate state under consideration).

5.2 Product of two amplitudes

Generalizing the rule for products of mesonic amplitude graphs discussed in Sec. 4 to general ordered amplitude graphs (regular cubic graphs), we connect corresponding vertices belonging to the intermediate states (henceforth called intermediate vertices) with <u>three</u> "particle lines"; each starting between a different couple of "quark lines" (i. e., colored edges) (i,j), (j,k) or (k,i) and ending between the corresponding couple of quark lines on the other graph. Simple examples are shown in Figs. 6 and 7. When the detailed structure of the intermediate lines (particle lines) is not needed, a wavy line will be drawn instead of three particle lines (e. g., Fig. 8).

We now discuss the topological indices in details. These indices are first obtained in products of amplitude graphs as discussed in Sec. 5.1. They are then - and only then - associated with amplitude graphs.

5.3. The boundary structure

As in the mesonic case, the boundary structure is defined in terms of Edmonds' orbits (Appendix A). We shall say that two orbits are <u>connected</u> if they have at least one vertex in common.

Then, given a graph representing either an amplitude or a discontinuity we group the orbits in sets of connected orbits. Each set will be called a <u>boundary</u>. The number of boundaries will be designated by B and the external vertices belonging to a given boundary are unambiguously interconnected. Vertices (and particles) belonging to different boundaries are completely disconnected. As an example, the graph of Fig. 6 has a single boundary whereas the graph of Fig. 7 has two boundaries. In general, every boundary can be represented by a regular cubic graph. Note that the boundary structure need not be an ordered graph or even a planar graph but it must be bipartite. It is, however, very important to recognize that once the connections between vertices in a boundary are specified, the number of orbits of the boundary is completely determined.*

As was true for the mesonic amplitudes, the boundary structure is related to the pole structure of the amplitude.[2] The amplitude has a pole in a particular channel - defined by a specific partition of the external vertices into initial and final states - if and only if there exist a graph representing the amplitude (i. e., with the relevant "indices") that could be cut into two connected subgraphs such that one connected subgraph contains the vertices of the initial (or final) state only and the other subgraph contains all other vertices (i. e., all intermediate vertices and all vertices of the final (initial) state). For instance, the amplitude of Fig. 8 has poles in the channel $AD \to BC$ but not in the channel $AB \to CD$.

---

* In contrast with a discontinuity graph, a graph representing an amplitude contains by convention the <u>minimum</u> number of orbits compatible with the boundary structure.

5.4. The mate-remoteness index (r-index)

We first define the concept of mate walk. All graphs we are concerned with are composed of a certain number of ordered amplitude graphs (OAG) connected to one another by particle lines (wavy lines in Fig. 9). In each OAG, the mating is well-defined. However, some vertices may be mated to intermediate vertices; (i. e., to which wavy lines are attached). For instance, in Fig. 9a, A is mated to B. A thereby "loses" its mate but eventually retrieves one in F. The path ABCDEF is called a mate walk. Similarly, the path MNPQRSTU is a mate walk. A and F, M and U are called remote mates.

Formally, the operation of finding the remote mate of A is to go to its mate in the OAG (B); then jump to the counterpart of that mate in the other OAG (C); go to its mate (D), and so on, until one reaches an external vertex (no particle - or wavy line attached).

Since no such operation is required for X and Y in Fig. 9a, they are called close mates.

In Fig. 9b similar situations are displayed except that no external vertex appears in the walk. Such a closed walk will be called circuit. Thus ABCD is a circuit and EFGHIJKL is another. X and Y as well as Z and T are close mates.

Finally a circuit with only four vertices, as ABCD in Fig. 9b, is called a trivial circuit or <u>ordered circuit</u> because it involves an ordered product.

We can now define the remoteness index of the product of two OAG as

$$r \equiv \sum_{w} \left[\frac{b}{2}\right]_w - C_o \tag{4}$$

where the sum is over all walks (open and closed), b is the number of wavy lines belonging to a given walk and C_o is the number of ordered circuits. The notation $[x]$ is defined by

$$[x] \equiv \sup \{n \in \mathcal{N} / n \leqslant x\} \tag{5}$$

for any real number x. That is, $[x]$ is the largest integer smaller than or equal to x. (e. g., $[3/2] = 1$)

Note that r is completely determined by the boundary structure of the members of the product and by the specification of the intermediate state.

A discontinuity with $r \neq 0$ belongs to an unordered topological amplitude with corresponding r-index. We can now define the remoteness index for a general product. Consider a product of two amplitudes M_1 and M_2 with respective indices r_1 and r_2. Then the resulting remoteness index is

$$r = r_1 + r_2 + r_i \tag{6a}$$

where

$$r_i \equiv \sum_{w_i} \left[\frac{b}{2}\right]_{w_i} - C_{oi} \tag{6b}$$

the sum on the right hand side of Eq. (6b) and the term C_{oi} refer, respectively, to the walks and ordered circuits created by the product of M_1 and M_2. Note again that r_i is completely determined by the boundary structure of M_1 and M_2 and by the specification of the intermediate state. We also define for future needs the index r':

$$r' = r_1 + r_2 + \sum_{w_i} \left[\frac{b}{2}\right]_{w_i} - C_i \tag{7}$$

where C_i is the total number of intermediate circuits created by the product.

In general, $2r$ represents the minimum number of additional 3-vertices (uncorrelated baryons*) in the simplest discontinuity of the amplitude relative to the corresponding ordered amplitude. Thus, a topological amplitude with index r will not have a discontinuity with less than $2r$ 3-vertices in the channel graph representing the intermediate state.

5.5 The g-index

The g-index can be thought of as a renormalized handle or genus index of the mesonic amplitudes. The g-index of a product of two graphs representing amplitudes is defined by

$$g = h - b + c - d + 1 \tag{8}$$

where h is the topological number of handles (genus) computed with Euler's formula and Edmonds' orbits; b is the number of wavy lines (intermediate 3-vertices on either side of the product); c is the number of circuits and d is the number of OAG included in the product (or number of connected parts if all particle lines are eliminated).

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* Uncorrelated baryons are pairs of baryons antibaryon that cannot be reduced to a single meson. Prof. G. F. Chew has suggested a potential suppression mechanism related to this fact.

For ordered products, g is always equal to zero.

As was true for the h-index, to each topological amplitude we associate an index g such that all its discontinuities possess the same value of g.

When performing the product of two amplitudes M_1 and M_2 with respective indices g_1 and g_2 we generally get a graph with a new index g. We now shown how to compute g in terms of g_1, g_2 and the boundary structure of M_1 and M_2. In this section, the subscripts 1, 2 and i refer to the members of the product M_1 and M_2 and to the intermediate state respectively. The symbols without subscripts refer to the graph product. The letters h, v, e, f symbolize the number of handles, vertices, edges and orbits (faces) while g is defined by Eq. 8. We shall extensively use Euler's formula:

$$2h = 2 - v + e - f . \tag{9}$$

We first determine h as a function of h_1 and h_2. Starting from Eq. 9 applied to M_1 and M_2 respectively and adding, we get,

$$2(h_1 + h_2) = 4 - (v_1 + v_2) + (e_1 + e_2) - (f_1 + f_2).$$

Now, subtracting from Eq. (9) and using $v = v_1 + v_2$; $e = e_1 + e_2 + e_i$; we get:

$$h = h_1 + h_2 + \frac{1}{2}(e_i - f_i - 2) \qquad (10)$$

with the formal definition:

$$f_i \equiv f - (f_1 + f_2). \qquad (11)$$

We can also obviously write:

$$b = b_1 + b_2 + b_i \qquad (12)$$

$$c = c_1 + c_2 + c_i \qquad (13)$$

$$d = d_1 + d_2 \quad , \qquad (14)$$

and therefore, after substitution of Eq. (10) through (14) in Eq. (8) and some straightforward algebra we get:*

* Note: using the formal definition $v_i \equiv 4$; $d_i \equiv 2$; $2h_i \equiv 2 - v_i + e_i - f_i$ and $g_i \equiv h_i - b_i + c_i - d_i + 1$ we get the formulae $h = h_1 + h_2 + h_i$ and $g = g_1 + g_2 + g_i$ (comparable to Eq. (6a)).

$$g = g_1 + g_2 + \frac{1}{2}(e_i - f_i - 4) - b_i + c_i. \qquad (15)$$

Now, e_i, b_i and c_i are completely determined by the specification of the intermediate state (note the identity $e_i = 3b_i$), f_1 and f_2 are determined by the boundary structure of M_1 and M_2 and f is determined by the boundary structure and the specification of the intermediate state since, to determine the Edmond orbits we only need to know the connections between the vertices. Thus, f_i is completely determined by the boundary structure of M_1 and M_2 and by the specification of the intermediate state. Therefore, given g_1 and g_2, the boundary structures of M_1 and M_2 and the intermediate state of the product, g is completely specified by Eq. (15).

5.6. Algebraic notation

We now discuss how to write algebraically (as opposed to graphically) the boundary structure of an amplitude and its r and g indices. The major advantage of this notation is to exhibit the fact that topological amplitudes are independent of their graphical representation. Only the essential features common to all graphs (i. e., the connections between external vertices) are preserved in this notation.

(a) Representation of boundary structure: We recognize that, within a boundary, only the connections between vertices are significant. Therefore, we associate to each vertex V a set of three indices: α_i, α_j, α_k (representing the three edges attached to V) and we write $V(\overrightarrow{\alpha_i\alpha_j\alpha_k})$. The order or the α-indices is important as it corresponds to the color index of the associated edge. The left-to-right arrow corresponds to a positively oriented vertex and the right-to-left arrow $[V(\overleftarrow{\alpha_i\alpha_j\alpha_k})]$ corresponds to a negatively oriented vertex.

The boundary structure is defined by listing all vertices of the amplitude with the relevant α-indices. Two vertices connected by k edges in the graphical representation will have k α-indices in common in the algebraic notation (k = 1, 2 or 3). Since different boundaries have no α-index (edge) in common they are separated by a semi-column in the list.

Finally, it is useful to specify pairs of mates by an arbitrary integer. The two members of the pair would then carry the same number and would be readily identified. For example, $\{V(n,\overrightarrow{\alpha_1\alpha_2\alpha_3})\ W(n,\overleftarrow{\alpha_1\alpha_2\alpha_5})\}$.

(b) r and g indices. The remoteness index r and the g index of a topological amplitude M will be specified by a superscript and a subscript respectively (e. g., M_g^r { boundary structure}). Furthermore, the remote mates must be specified since two adjacent remote mates correspond to a different pole structure than two close mates. The specification is done by connecting the remote mates by a bar over the amplitude.

As an example, the algebraic notation corresponding to the graph of Fig. 10 is:

$$M_0^1 \ \{\overline{A(1,\overrightarrow{\alpha_1\alpha_2\alpha_3})\ B}(1,\overleftarrow{\alpha_1\alpha_2\alpha_3});\ C(2,\overrightarrow{\alpha_4\alpha_5\alpha_6})D(2,\overleftarrow{\alpha_4\alpha_5\alpha_6})\}$$

Products of two amplitudes can be computed with this notation. The procedure is detailed in Appendix B. This fact clearly shows that all the necessary ingredients to determine the product of two topological amplitudes are included in r, g and the boundary structure.

Another advantage of the algebraic notation is that it enables a sharp distinction between amplitudes and products of amplitudes or discontinuities. Although they are both represented by the same type of graphs it is crucial - in order to avoid confusion - to have a clear idea whether a given graph is supposed to represent an amplitude or a discontinuity. In the latter case, the intermediate state is well defined and cannot be altered while in the former case, there is no intermediate state - or particles and some particle lines may be removed - so long as the indices are not altered. Conversely, analytic continuation from one reaction to a crossed reaction is a property of amplitudes, not discontinuities. Failure to recognize these simple facts could lead to a paradoxical situation. The consistent use of the algebraic notation guarantees a clear distinction between amplitudes and discontinuities.

5.7 The $1/N$ expansion

We conjecture* that the suppression of non-ordered products relative to the corresponding ordered products (i. e., the ordered product with the same particles in the initial, final and intermediate states) can be understood as a $1/N$ suppression as defined by Veneziano.[4] If the product is characterized by g and r' defined by Eq. (8) and (7) respectively and by B boundaries then in all the examples we have studied there are $g + (B - 1) + r'$ fewer free quark lines than in the corresponding ordered amplitude. Thus the suppression factor is $(1/N)^{g+(B-1)+r'}$.

---

* This conjecture was originally suggested by Prof. G. F. Chew.

6. Conclusion

A classification of the product of general amplitude graphs in terms of three topological indices was presented. The classification was shown to be self-consistent; i. e., the indices corresponding to the product of two amplitudes are completely determined by the indices of the members of the product and by the specification of the intermediate state. The indices completely characterize the graph product and are related to the singularity structure of the associated complex function.

The new classification reduces to the "standard" one in the mesonic sector.

The next phase of the theory should be to analyze the spectrum of particles generated by this approach, to determine more accurately the connection with the analytical structure of the indices, to investigate the mechanisms of convergence and to apply it to practical considerations.

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* This work was supported by the U. S. Department of Energy.

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Appendix A: Edmonds' orbits

The following technique developed by Edmonds[7] and by Young[8] enables one to determine the faces (orbits) of a graph imbedded on any orientable two-dimensional surface.

The principle is to start from some point and to move alongside an edge such that the edge is always on the right (say), until a vertex is reached. At the vertex, we make a clockwise rotation until we hit the next edge* and we "travel" again with the edge to the right until the next vertex where we make a clockwise rotation, and so on. In the end, we come back to the starting point. This round trip is called an orbit. When all the orbits of the graph are determined we should have traversed each edge twice; once in each direction.

The number of orbits, being the number of faces of the graph enables us to compute the genus of the graph through Euler's formula (Eq. (9)) and also to specify the connections between vertices by recording the sequence in which they appear on each orbit.

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* This is why the position of particle edges with respect to quark edges is crucial. In fact, when we are on a quark edge of a given color in one OAG, the prescription will keep us on the same color after "landing" on the other OAG.

Appendix B: Edmonds' orbits and algebraic notation

It is possible, although tedious, to determine Edmonds' orbits from the algebraic notation alone. This method is analogous to the one introduced in Ref. 5. We shall describe the general rules and illustrate them with the simple product of Fig. B1. In the algebraic notation, this product is

$$\begin{aligned} P = M_0^0 \; &\{A(1,\overleftarrow{\alpha_1\alpha_2\alpha_3})B(1,\overrightarrow{\alpha_4\alpha_2\alpha_3})F(2,\overleftarrow{\alpha_4\alpha_5\alpha_6}) \\ &E(2,\overrightarrow{\alpha_1\alpha_5\alpha_6})\} \otimes M_0^0 \; \{E(1,\overleftarrow{\beta_1\beta_2\beta_3})F(1,\overrightarrow{\beta_1\beta_2\beta_4}) \\ &C(2,\overleftarrow{\beta_5\beta_6\beta_4})D(2,\overrightarrow{\beta_5\beta_6\beta_3})\} \; . \end{aligned} \tag{B1}$$

We first introduce some definitions. At each step described below two α-indices are crossed. The first one is called <u>in-index</u> and the second <u>out-index</u>. There are also two types of vertices: those without particle lines (e. g., A, B, C, D) called external vertices and those with particle lines (e. g., E, F) called <u>intermediate vertices</u>. The <u>correspondent</u> of an upper (lower) α-index is the same α-index but appearing as a subscript (superscript) in the <u>same</u> amplitude. The <u>counterpart</u> of an α-index is the α-index appearing in the same position (i. e., corresponding intermediate vertex and same order in the <u>other</u>

amplitude, e. g., the counterpart of α_5 is β_2) and the follower of an index is the next index (cyclic order specified below) on the same vertex. We now specify the general rules to determine the orbits:

(1). The clockwise order is followed on vertices with negative orientation as the counterclockwise order is followed on vertices with positive orientation (hence, the directions of the arrows in Eq. (B1)).

(2) To start, select any index in one amplitude as the first in-index of the sequence.

(3) Now cross the indices two by two in the following sequence:

(a) If you are on an external vertex

- after the in-index go to the follower.
- after the out-index go to the correspondent.

(b) If you are on an intermediate vertex

- after the in-index go to the counterpart.
- after the out-index go to the correspondent.

Thus the "correspondent" is always an in-index; the "follower" or the "counterpart" are always out-indices.

(4) An orbit is completed when the starting index is reached again as an in-index. The sequence should contain an even number of indices.

(5) At the end of the operation, every index should be crossed twice; once as an in-index and once as an out-index.

As an example, the sequences for the product introduced above are given below. The notation $\alpha(V)$ means the index appearing on vertex V. We thus have:

1st orbit $= \alpha_2(A)\alpha_1(A)\alpha_1(E)\beta_1(E)\beta_1(F)\alpha_4(F)\alpha_4(B)\alpha_2(B)$

2nd orbit $= \alpha_3(A)\alpha_2(A)\alpha_2(B)\alpha_3(B)$

3rd orbit $= \alpha_1(A)\alpha_3(A)\alpha_3(B)\alpha_4(B)\alpha_4(F)\beta_1(F)\beta_1(E)\alpha_1(E)$

4th orbit $= \beta_5(C)\beta_4(C)\beta_4(F)\alpha_6(F)\alpha_6(E)\beta_3(E)\beta_3(D)\beta_5(D)$

5th orbit $= \beta_6(C)\beta_5(C)\beta_5(D)\beta_6(D)$

6th orbit $= \beta_4(C)\beta_6(C)\beta_6(D)\beta_3(D)\beta_3(E)\alpha_6(E)\alpha_6(F)\beta_4(F)$

7th orbit $= \alpha_5(E)\beta_2(E)\beta_2(F)\alpha_5(F)$

8th orbit $= \alpha_5(F)\beta_2(F)\beta_2(E)\alpha_5(E)$.

Since there are eight orbits*, Euler's formula yields $h = 2$ and Eq. (8) yields $g = 0$. But we see that $B = 2$ because A and B do not have any orbit in common with C and D. As a result, the amplitude to which the discontinuity belongs can be written as:

$$M_0^0 \ \{A(1,\overleftarrow{\alpha_1\alpha_2\alpha_3})B(1,\overrightarrow{\alpha_1\alpha_2\alpha_3});C(2,\overleftarrow{\alpha_4\alpha_5\alpha_6})D(2,\overrightarrow{\alpha_4\alpha_5\alpha_6})\}.$$

The remote mates can be obtained simply by following a walk of mates as discussed earlier. In our example, all mates are close and the only circuit is ordered.

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* Note that they are compos   an <u>even</u> number of α-indices and an <u>even</u> number of β-indices.

FIGURE CAPTIONS

Fig. 1: Elementary operation generating all ordered state graphs

Fig. 2: An ordered amplitude. A_1, B_2, C_1 are positively oriented and A_2, B_1, C_2 are negatively oriented (color cycles). (A_1, A_2); (B_1, B_2); (C_1, C_2) are mates.

Fig. 3: Planar amplitude of four baryons.

Fig. 4: Three equivalent representations of mesonic amplitudes: (a) dual diagram; (b) hybrid diagram; (c) particle diagram.

Fig. 5: New version of hybrid diagram with two particle lines at each intermediate vertex

Fig. 6: Product of ordered amplitudes. The particle lines start and end between the same color couples (i,j).

Fig. 7: Same graph as Fig. 6, but with two boundaries.

Fig. 8: In this product, the two mates A and B (and C and D) are not adjacent. There is no pole in the channel $AB \to CD$.

Fig. 9: Walks of mates: (a) open walks. (b) closed walks (circuit).

Fig. 10: In this product A and B are remote mates. C and D are close mates.

Fig. B1: Product of two ordered amplitudes yielding $B = 2$.

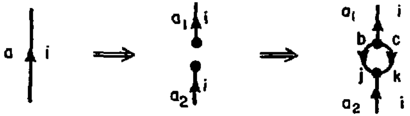

Fig. 1

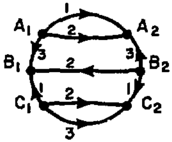

Fig. 2

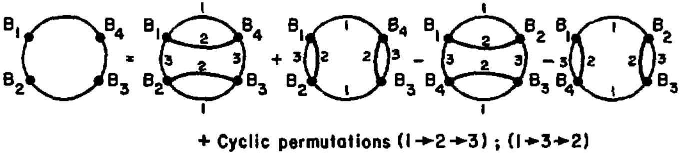

Fig. 3

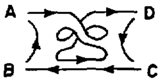

Fig. 4a

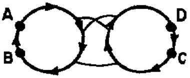

Fig. 4b

Fig. 4c

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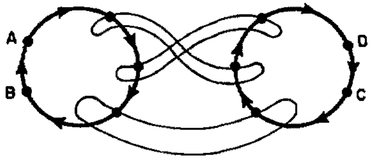

Fig. 5

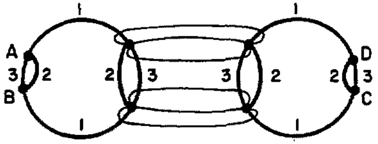

Fig. 6

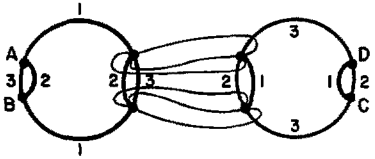

Fig. 7

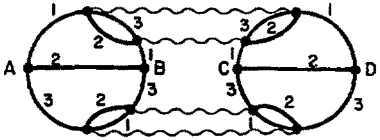

Fig. 8

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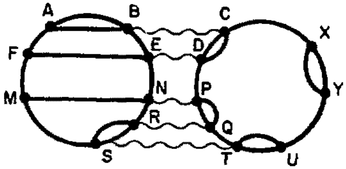

Fig. 9a

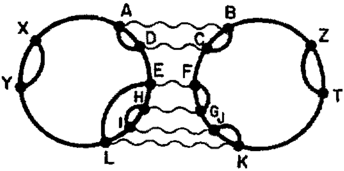

Fig. 9b

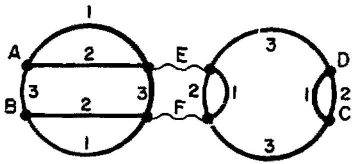

Fig. 10

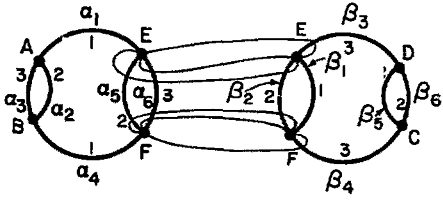

Fig. B1

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