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HOMOCLINIC SOLUTIONS OF NONLINEAR SECOND ORDER HAMILTONIAN SYSTEMS

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Abstract

We study the existence of homoclinic solutions for a second order non-autonomous dynamical system including both the kinetic and potential terms. We assume very little concerning the kinetic term, just enough to make the essential spectrum of the linear operator the same as that for free particles. Our theorems cover all cases and allow both sublinear and superlinear problems. We obtain ground state solutions.

1 Introduction

We consider the following problem. One wishes to solve

$$-\ddot{x}(t) = B(t)x(t) + \nabla_x V(t, x(t)), \tag{1}$$

where

$$x(t) = (x_1(t), \cdots, x_n(t)) \tag{2}$$

is a map from $\mathbb{R}$ to $\mathbb{R}^n$ such that each component $x_j(t)$ is a function in $H^1 = H^{1,2}(\mathbb{R})$, $B(t)$ is a symmetric matrix corresponding to the bilinear form

$$b(u, v) = -\sum_{j=1}^{n}\sum_{k=1}^{n}\int_{\mathbb{R}} b_{jk}(t)u_k(t)v_j(t)\,dt \tag{3}$$

and the function $V(t, x) = V(t, x_1, \cdots, x_n)$ is continuous from $\mathbb{R}^{n+1}$ to $\mathbb{R}$ with

$$\nabla_x V(t, x) = (\partial V/\partial x_1, \cdots, \partial V/\partial x_n) \in C(\mathbb{R}^{n+1}, \mathbb{R}^n). \tag{4}$$

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We shall study this problem under several sets of assumptions. They allow both sublinear and superlinear problems.

The system (1) has been studied by many researchers since 1977 (cf. the bibliography; a discussion is given at the end of this section). In many cases the matrix $B(t)$ is either missing are subsumed in the potential $V(t,x)$. When the full system is considered, most reseachers make such strong assumptions on the matrix $B(t)$ that the spectrum of the operator

$$\mathcal{D}_0 x = -\ddot{x}(t) - B(t)x(t)$$

is far different from the spectrum of the operator

$$\mathcal{A}_0 x = -\ddot{x}(t).$$

When this is done, there is little connection between the problem when $B(t) \neq 0$ and the problem when $B(t) = 0$. On the contrary, our assumptions on $B(t)$ are such that the essential spectrum is the same whether $B(t) = 0$ or $B(t) \neq 0$. Our assumption on $B(t)$ is

$(B1)$ Each component $b_{jk}(t)$ of $B(t)$ is a locally integrable function on $\mathbb{R}$ and satisfies

$$\int_t^{t+1} |b_{jk}(s)|\, ds \to 0 \; as \; |t| \to \infty.$$

This assumption implies that there is an extension $\mathcal{D}$ of the operator

$$\mathcal{D}_0 x = -\ddot{x}(t) - B(t)x(t)$$

having essential spectrum equal to $[0,\infty)$ and a (possibly empty) discrete, countable negative spectrum consisting of isolated eigenvalues of finite multiplicity with a finite lower bound $-L$

$$-\infty < -L \leq \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_l < \ldots < 0, \tag{5}$$

(see Section 2).

For the operator $\mathcal{D}$ there are three possibilities: (a) it has no negative eigenvalues, (b) it has only one negative eigenvalue, and (c) it has two or more negative eigenvalues. What is interesting is that each of these possibilities can be dealt with differently. We shall study all of them separately.

Concerning the potential $V(t,x)$ we assume

$$|V(t,x)| \leq C(S(t)^q|x|^q + S(t)|x|), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n,$$

$$|\nabla_x V(t,x)| \leq C(S(t)^q |x|^{q-1} + S(t)), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n,$$

for some $q \geq 2$, where $S(t)$ is a function in $L^q(t)$ such that

$$\|Sx\|_q \leq \|x\|_H, \quad x \in H.$$

The non-autonomous problem (1) has an extensive history in the case of singular systems (cf., e.g., Ambrosetti-Coti Zelati [1]). The first to consider it for potentials satisfying (4) were Berger and the author [5] in 1977. We proved the existence of solutions of (1) under the condition that

$$V(t,x) \to \infty \quad \text{as} \quad |x| \to \infty$$

uniformly for a.e. t. Subsequently, Willem [55], Mawhin [25], Mawhin-Willem [27], Tang [48, 49], Tang-Wu [52, 53], Wu-Tang [56] and others proved existence under various conditions (cf. the references given in these publications).

Most previous work considered the case when $B(t) = 0$. Ding and Girardi [11] considered the case of (1) when the potential oscillates in magnitude and sign,

$$-\ddot{x}(t) \ = \ B(t)x(t) \ + \ b(t)\nabla W(x(t)) \ , \tag{6}$$

and found conditions for solutions when the matrix $B(t)$ is symmetric and negative definite and the function $W(x)$ grows superquadratically and satisfies a homogeneity condition. Antonacci [3, 4] gave conditions for existence of solutions with stronger constraints on the potential but without the homogeneity condition, and without the negative definite condition on the matrix. Generalizations of the above results are given by Antonacci and Magrone [2], Barletta and Livrea [6], Guo and Xu [16], Li and Zou [24], Faraci and Livrea [15], Bonanno and Livrea [7, 8], Jiang [21, 22], Shilgba [40, 41], Faraci and Iannizzotto [14] and Tang and Xiao [54].

Some authors considered the second order system (1) where the potential function $V(t,x)$ is quadratically bounded as $|x| \to \infty$. Han [17] gave conditions for existence of solutions when $B(t)$ was a multiple of the identity matrix, the system satisfies the resonance condition, and the potential has upper and lower subquadratic bounds. Li and Zou [24] considered the case where $B(t)$ is continuous and nonconstant and the system satisfies the resonance condition, and showed existence of solutions when the potential is even and grows no faster than linearly. Tang and Wu [50] required the function that satisfies the resonance condition to pass through the zero vector, and gave upper and lower conditions for subquadratic growth of the magnitude of $V(t,x)$ without the requirement that the potential be even. Faraci [13] considered the case where for each t, the matrix $B(t)$ is negative definite with elements that are bounded but not necessarily continuous and the potential has an upper quadratic bound as $|x| \to \infty$, showing existence of a solution when the gradient of the potential is bounded near the origin and exceeds the matrix product in at least one direction.

The homoclinic problem for system (1) is more difficult than the periodic problem for the same system. For the latter problem one must assume periodicity for $B(t)$ and $V(t,x)$. Moreover, the periodic problem has the advantage of the compact imbeddings allowed by the spaces. Study of the homoclinic problem is at a distinct disadvantage because of this.

We shall use the following framework. Let H be the set of vector functions $x(t)$ described above. It is a Hilbert space with norm satisfying

$$\|x\|_H^2 = \sum_{j=1}^n \|x_j\|_{H^1}^2.$$

We also write

$$\|x\|^2 = \sum_{j=1}^n \|x_j\|^2,$$

where $\|\cdot\|$ is the $L^2(\mathbb{R})$ norm. The energy functional corresponding to the system (1) is

$$G(x) = (\mathcal{D}x, x) - 2\int_{\mathbb{R}} V(t,x)\,dt, \quad x \in H. \tag{7}$$

It is easily checked that $x \in H$ is a (weak) solution of (1) iff it is a critical point of $G(x)$. Our methods will make use of this fact.

In the next section we give the construction of the operator $\mathcal{D}$. We obtain the largest self-adjoint extension of $\mathcal{D}_0$ which preserves the essential spectrum. We state our theorems in Sections 3 – 6 and prove them in Section 7. In particular, we obtain ground state solutions, i.e., solutions that minimize the energy functional.

In solving the problems, we use linking and sandwich methods of critical point theory. The theory of sandwich pairs began in [42] and [35, 36] and was developed in subsequent publications such as [37, 38].

2 The operator $\mathcal{D}$

In constructing the operator $\mathcal{D}$ we shall make use of the following considerations.

We define a bilinear form $a(\cdot,\cdot)$ on the set $L^2(\mathbb{R},\mathbb{R}^n) \times L^2(\mathbb{R},\mathbb{R}^n)$,

$$a(u,v) = (\dot{u},\dot{v}) + (u,v). \tag{8}$$

The domain of the bilinear form is the set $D(a) = H$, cosisting of those $x(t) = (x_1(t),\cdots,x_n(t)) \in L^2(\mathbb{R},\mathbb{R}^n)$ having weak derivatives in $L^2(\mathbb{R},\mathbb{R}^n)$. H is a dense subset of $L^2(\mathbb{R},\mathbb{R}^n)$. Note that H is a Hilbert space. Thus we can define an operator $\mathcal{A}$ such that $u \in D(\mathcal{A})$ if and only if $u \in D(a)$ and there exists $f \in L^2(\mathbb{R}, \mathbb{R}^n)$ such that

$$a(u, v) = (f, v), \quad v \in D(a). \tag{9}$$

If u and f satisfy this condition we say $\mathcal{A}u = f$.

Lemma 2.1. *The operator $\mathcal{A}$ is a self-adjoint Fredholm operator from $L^2(\mathbb{R}, \mathbb{R}^n)$ to $L^2(R, \mathbb{R}^n)$. It is one-to-one and onto.*

Proof. Let $f \in L^2(\mathbb{R}, \mathbb{R}^n)$. Then

$$(v, f) \leq \|v\| \cdot \|f\| \leq \|v\|_H \|f\|, \quad v \in H.$$

Thus (v, f) is a bounded linear functional on H. Since H is complete, there is a $u \in H$ such that

$$(u, v)_H = (f, v), \quad v \in H.$$

Consequently, $u \in D(\mathcal{A})$ and $\mathcal{A}u = f$. Moreover, if $\mathcal{A}u = 0$, then

$$(u, v)_H = 0, \quad v \in H.$$

Thus, $u = 0$. Hence, $\mathcal{A}$ is one-to-one and onto.

For any two functions $x, y \in D(\mathcal{A})$,

$$(\mathcal{A}x, y) = (\dot{x}, \dot{y}) + (x, y) = (x, \mathcal{A}y). \tag{10}$$

Thus, $\mathcal{A}$ is symmetric. It is now easy to show that $D(\mathcal{A}) \subset D(a)$ is also a dense subset of $L^2(\mathbb{R}, \mathbb{R}^n)$. In fact, if $f \in L^2(\mathbb{R}, \mathbb{R}^n)$ satisfies $(f, v) = 0 \ \forall v \in D(\mathcal{A})$, then $w = \mathcal{D}^{-1} f$ satisfies $(w, \mathcal{A}v) = (\mathcal{A}w, v) = 0 \ \forall \ v \in D(\mathcal{A})$. Since $\mathcal{A}$ is onto, $w = 0$. Hence , $f = \mathcal{D}w = 0$.

Next, we show that $\mathcal{A}$ is self-adjoint. Consider any $u, f \in L^2(\mathbb{R}, \mathbb{R}^n)$, and suppose for any $v \in D(\mathcal{A})$,

$$(u, \mathcal{A}v) = (f, v). \tag{11}$$

Since $\mathcal{A}$ is onto and $f \in L^2(\mathbb{R}, \mathbb{R}^n)$, there exists $w \in D(\mathcal{A})$ such that $\mathcal{A}w = f$. Then using (10),

$$(u - w, \mathcal{A}v) = (f, v) - (\mathcal{A}w, v) = 0.$$

Since $u - w \in L^2(\mathbb{R}, \mathbb{R}^n)$, we can find a $v \in D(\mathcal{A})$ such that $\mathcal{A}v = u - w$, and

$$\|u - w\|^2 = 0.$$

This implies $u = w$ in the space $L^2(\mathbb{R}, \mathbb{R}^n)$, and therefore $u \in D(\mathcal{A})$. Hence, $\mathcal{A}u = \mathcal{A}w = f$. □

The following theorems are proved in [31].

Theorem 2.2. *If*

$$\sup_t \int_t^{t+1} |V(s)|^2\, ds < \infty, \tag{12}$$

then for every $\varepsilon > 0$ *there is a constant* C *such that*

$$\int_{\mathbb{R}} |V(t)|^2 x(t)^2\, dt \le \varepsilon \int_{\mathbb{R}} \dot{x}(t)^2\, dt + C \int_{\mathbb{R}} x(t)^2\, dt, \quad x \in H. \tag{13}$$

Theorem 2.3. *If*

$$\int_t^{t+1} |V(s)|^2\, ds \to 0 \text{ as } |t| \to \infty, \tag{14}$$

then for each bounded sequence (x_k) *in* H *there is a renamed subsequence such that* (Vx_k) *converges in* $L^2(\mathbb{R}, \mathbb{R}^n)$.

Lemma 2.4. *The essential spectrum of* $\mathcal{A}$ *is* $[1, \infty)$.

Theorem 2.5. *Let* $a(\cdot,\, \cdot)$ *be a closed Hermitian bilinear form with dense domain in* $L^2(\mathbb{R}, \mathbb{R}^n)$. *If for some real number* N,

$$a(\, u,\, u\,) \;+\; N\|u\|^2 \;\ge\; 0\;, \tag{15}$$

then the operator $\mathcal{A}$ *associated with* $a(\cdot,\, \cdot)$ *is self-adjoint and* $\sigma(\mathcal{A}) \subset [-N, \infty)$.

Theorem 2.6. *Suppose* $a(\cdot,\, \cdot)$ *is a bilinear form satisfying the hypotheses of Theorem 2.5. Let* $b(\cdot,\, \cdot)$ *be a Hermitian bilinear form such that* $D(a) \subset D(b)$ *and for some positive real number* K, *for any* $u \in D(a)$,

$$|b(\, u,\, u\,)| \;\le\; K a(\, u,\, u\,)\;. \tag{16}$$

Assume that every sequence $(u_k) \subset D(a)$ *which satisfies*

$$\|u_k\|^2 \;+\; a(\, u_k,\, u_k\,) \;\le\; C \tag{17}$$

has a subsequence (v_j) *such that*

$$b(\, v_j - v_k,\, v_j - v_k\,) \;\to\; 0\;. \tag{18}$$

Assume also that if (17),(18) *hold and* $v_j \to 0$ *in the* $L^2(\mathbb{R}, \mathbb{R}^n)$ *norm, then* $b(\, v_j,\, v_j\,) \to 0$. *Set*

$$c(\, u,\, v\,) \;=\; a(\, u,\, v\,) \;+\; b(\, u,\, v\,)\;. \tag{19}$$

and let $\mathcal{A}, \mathcal{C}$ *be the operators associated with* a, c, *respectively. Then*

$$\sigma_e(\mathcal{A}) \;=\; \sigma_e(\mathcal{C}).$$

Let

$$b(\,u,\,v\,)\;=\;-\sum_{j=1}^{n}\sum_{k=1}^{n}\int_{\mathbb{R}}(b_{jk}(t))\,u_k(t)v_j(t)\,dt \tag{20}$$

and

$$c(u,v)=a(u,v)+b(u,v). \tag{21}$$

We define

$$d(x)=c(x)-\|x\|^2,\quad x\in H,$$

and let $\mathcal{D}$ be the corresponding operator. We shall prove

Lemma 2.7. *The operator $\mathcal{D}$ associated with the bilinear form $d(\cdot,\,\cdot)$ under assumption (B1) is self-adjoint. Its essential spectrum is $[0,\infty)$ and there exists a finite real value L such that $\sigma(\mathcal{D})\subset[-L,\infty)$. $\mathcal{D}$ has a discrete, countable spectrum below 0 consisting of isolated eigenvalues of finite multiplicity with a finite lower bound -L*

$$-\infty\;<\;-L\;\le\;\lambda_0\;<\;\lambda_1\;<\;\lambda_2\;<\;\ldots\;<\;\lambda_l\;<\;\ldots\;<0. \tag{22}$$

To show the bilinear form $b(\cdot,\,\cdot)$ is Hermitian, we can use the symmetry of the matrix $B(t)$ to rearrange the order of the finite summation,

$$\begin{aligned}
b(\,u,\,v\,)\;&=\;-\sum_{j=1}^{n}\sum_{k=1}^{n}\int_{\mathbb{R}}(b_{jk}(t))\,u_k(t)v_j(t)\,dt\\
&=\;-\sum_{k=1}^{n}\sum_{j=1}^{n}\int_{\mathbb{R}}(b_{jk}(t))\,v_j(t)u_k(t)\,dt\\
&=\;-\sum_{k=1}^{n}\sum_{j=1}^{n}\int_{\mathbb{R}}(b_{kj}(t))\,v_j(t)u_k(t)\,dt\\
&=\;b(\,v,\,u\,)\,,
\end{aligned}$$

since $b_{kj}(t)=b_{jk}(t)$. By Theorem 2.2 the magnitude of $b(u)=b(u,u)$ is bounded by a multiple of the bilinear form $a(\cdot,\,\cdot)$ and satisfies (16),

$$\begin{aligned}
|b(\,u\,)|\;&\le\;(M\|u\|_H)^2\\
&=\;Ka(\,u\,).
\end{aligned}$$

Consider a sequence $(x_k)\subset D(\mathcal{A})$ which is bounded by a constant C in the H norm. By Theorem 2.3 we can find a subsequence $(x_{\bar{k}})$ for which

$$|b(\,x_{\bar{\jmath}}-x_{\bar{k}}\,)|\;\to\;0. \tag{23}$$

If in addition the subsequence $(x_{\bar{k}})$ converges to zero in $L^2(\mathbb{R},\mathbb{R}^n)$, then

$$b(\,x_{\bar{k}}\,)\to 0.$$

Then the bilinear forms $a(\cdot,\,\cdot)$ and $b(\cdot,\,\cdot)$ satisfy the conditions of Theorem 2.6. The bilinear form $c(\cdot,\,\cdot)$ is the sum of these two bilinear forms as in (19). By this theorem, the operator $\mathcal{C}$ associated with this bilinear form has the same essential spectrum as the operator $\mathcal{A}$ associated with the bilinear form $a(\cdot,\,\cdot)$. By Theorem 2.3, for any constant $\epsilon > 0$ there exists a positive constant K_ϵ such that

$$(24) \qquad |\, b(\,x\,)\,| \;\leq\; \epsilon\|\dot{x}\|^2 \;+\; K_\epsilon\|x\|^2 \quad x \in D(\mathcal{A}).$$

To show $c(\cdot,\,\cdot)$ is closed, first apply (24) with $\epsilon = 1/2$. Thus there is a constant C_0 such that

$$(25) \qquad |\, b(\,u\,)\,| \;\leq\; \frac{1}{2}\,a(\,u\,) \;+\; C_0\|u\|^2 \,.$$

Now suppose a sequence $(u_k) \subset D(c)$ satisfies

$$(26) \qquad c(\,u_j \;-\; u_k\,) \;\to\; 0\,,$$

and $(u_k) \to u$ in $L^2(\mathbb{R}, \mathbb{R}^n)$. The sequence is Cauchy in $L^2(\mathbb{R}, \mathbb{R}^n)$ and as j, k increase

$$\|\, u_j \;-\; u_k\,\|^2 \;\to\; 0\,.$$

Suppose that $u \notin D(c)$. Because the domains of $c(\cdot,\,\cdot)$ and $a(\cdot,\,\cdot)$ are the same, $u \notin D(a)$. We have shown above that $a(\cdot,\,\cdot)$ is closed, so the sequence cannot be Cauchy and as j, k increase $a(\,u_j - u_k\,)$ does not approach zero. But by (26),

$$a(\,u_j - u_k\,) \;-\; b(\,u_j - u_k\,) \;\to\; 0\,.$$

Applying the inequality in (25) bounds the magnitude of each $b(\cdot,\,\cdot)$ term, and since $a(u, u) \geq 0$, the following inequality is satisfied,

$$a(\,u_j - u_k\,) \;-\; b(\,u_j - u_k\,) \;\geq\; \tfrac{1}{2}a(\,u_j - u_k\,) \;-\; C_0\|u_j - u_k\|^2\,.$$

Adding the last term to both sides leaves only the positive bilinear form on the right side,

$$\begin{aligned} a(\,u_j - u_k\,) - b(\,u_j - u_k\,) \;+\; C_0\|u_j - u_k\|^2 & \\ \geq\; \tfrac{1}{2}a(\,u_j - u_k\,) & \\ \geq\; 0\,. & \end{aligned}$$

As j, k increase the left side of this equation approaches zero so the center term must also approach zero, a contradiction to the statement above. Therefore, $u \in D(a) = D(c)$, and $c(\cdot,\,\cdot)$ is also a closed bilinear form.

Next we show that there exists a positive constant N such that for any $x \in D(a)$,

$$(27) \qquad d(\,x\,) \;+\; N\|x\|^2 \;\geq\; 0\,.$$

For any positive constant $\epsilon > 0$ we can find K_ϵ which satisfies (24) and thereby find a lower bound for $b(\,x\,,x\,)$,

$$a(\,x\,) \,+\, b(\,x\,) \,+\, N\|x\|^2 \,\geq\, a(\,x\,) \,-\, \epsilon\|\dot{x}\|^2 \,-\, K_\epsilon\|x\|^2.$$

We have shown that $d(\cdot,\,\cdot)$ is closed, and as the sum of two Hermitian bilinear forms, $d(\cdot,\,\cdot)$ is clearly Hermitian. Its domain is dense in $L^2(\mathbb{R},\mathbb{R}^n)$ and the N in (27) satisfies the conditions of Theorem 2.5, so the operator $\mathcal{D}$ associated with this bilinear form is self-adjoint and has its spectrum bounded below by $-N$. We have shown that the essential spectrum of this operator is $[0,\infty)$, so the negative spectrum is discrete and we can number the eigenvalues in increasing order, and each eigenvalue is of finite multiplicity.

3 No negative eigenvalues

In this case,

$$d(x) = (\mathcal{D}x, x) \geq 0, \quad x \in H.$$

To solve, we can look for a minimum of $G(x)$. To get $G(x)$ bounded from below, we need

$$2\int V(t,x)\,dt \leq C, \quad x \in H.$$

This can be accomplished by assuming

1. There is a positive constant θ such that

$$2V(t,x) \leq -\theta|x|^2 + W(t), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n,$$

where $W(t) \in L^1(\mathbb{R})$.

We have

Theorem 3.1. *Under the above hypotheses, the system (1) has a solution.*

Remark 3.2. *It will be clear from the proof that the solution obtained will be nontrivial if*

$$\nabla_x V(t,0) \neq 0.$$

4 One negative eigenvalue

Let $\lambda_0 < 0$ be the negative eigenvalue of $\mathcal{D}$. In this case we have

$$d(x) \geq \lambda_0\|x\|^2, \quad x \in H.$$

We can still look for a minimum of $G(x)$ provided

$$2V(t,x) \le \lambda_0|x|^2 + W(t), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n.$$

Thus, we have

Theorem 4.1. *Assume*

1. $$2V(t,x) \le -\theta|x|^2 + W(t), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n,$$
   where $\theta > -\lambda_0$.

Then the system (1) has a solution.

Hypothesis 1 of Theorem 4.1 is stronger than Hypothesis 1 of Theorem 3.1. However, we can weaken it by making use of the fact that there is a gap in the spectrum of $\mathcal{D}$. We can then look for critical points which are not extrema.

Let $N = E(\lambda_0)$, the eigenspace of λ_0. Take $M = N^\perp$. Then

$$d(v_0) = \lambda_0\|v_0\|^2, \quad d(x) \ge 0,\ x \in M,$$

where $v_0 \in E(\lambda_0)$. Since M, N are sandwich pairs (cf. Theorem 3.17, p.26 of [38]), we can obtain a solution provided

$$\sup_N G < \infty, \quad \inf_M G > -\infty.$$

For $v = sv_0 \in N$, we have

$$G(v) = s^2\lambda_0\|v_0\|^2 - 2\int V(t, sv_0)\,dt \le \int W(t)\,dt < \infty$$

provided

1. $$2V(t,x) \ge \lambda_0|x|^2 - W(t), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n,$$
   where $W(t) \in L^1(\mathbb{R})$.

   For $w \in M$, we have

   $$G(w) = d(w) - 2\int V(t,w)\,dt \ge -2\int V(t,w)\,dt \ge -\int W(t)\,dt > -\infty,$$

   provided

2. There is a constant $\theta > 0$ such that

$$2V(t,x) \leq -\theta|x|^2 + W(t), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n,$$

where $W(t) \in L^1(\mathbb{R})$.

We have

Theorem 4.2. *Under the above hypotheses, the system (1) has a solution.*

5 At least two negative eigenvalues

Theorem 4.1 applies to this case as well, but again we can weaken the hypotheses by searching for critical points which are not extrema.

Let $\lambda_{l-1},\ \lambda_l$ be two consecutive negative eigenvalues of $\mathcal{D}$. Define the subspaces M and N of H as,

$$N \ = \ \bigoplus_{k\,<\,l} E(\lambda_k)\ , \quad M \ = \ N^{\perp}\ , \quad H \ = \ M \oplus N\ ,$$

where $E(\lambda_k)$ is the eigenspace of λ_k. Let

$$G(x) = d(x) - 2\int_{\mathbb{R}} V(t,x)\,dt. \tag{28}$$

We have

Theorem 5.1. *Assume*

1.

$$2V(t,x) \geq \lambda_{l-1}|x|^2, \quad t \in \mathbb{R},\ x \in \mathbb{R}^n.$$

2. There are constants $\mu \leq \lambda_l$ and $m > 0$ such that

$$2V(t,x) \leq \mu|x|^2, \quad |x| \leq m,\ x \in \mathbb{R}^n.$$

3. There is a constant $\theta > 0$ such that

$$2V(t,x) \leq -\theta|x|^2 + W(t), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n,$$

where $W(t) \in L^1(\mathbb{R})$.

4. The function given by

$$H(t,x) = \nabla_x V(t,x) \cdot x - 2V(t,x) \tag{29}$$

satisfies

$$H(t,x) \le W(t) \in L^1(\mathbb{R}), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n, \tag{30}$$

and

$$H(t,x) \to -\infty, \quad |x| \to \infty,\ t \in \mathbb{R},\ x \in \mathbb{R}^n, \tag{31}$$

where $W(t) \in L^1(\mathbb{R})$.

Then the system

$$\mathcal{D}x(t) = \nabla_x V(t, x(t)) \tag{32}$$

has a solution.

Theorem 5.2. *The system (32) has a solution if we assume*

1. $$\lambda_{l-1}|x|^2 - W_1(t) \le 2V(t,x) \le \lambda_l |x|^2 + W_2(t), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n,$$
   where $W_j(t) \in L^1(\mathbb{R})$.
2. *There is a constant* $\theta > 0$ *such that*
   $$2V(t,x) \le -\theta|x|^2 + W(t), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n,$$
   where $W(t) \in L^1(\mathbb{R})$.
3. *The function* $H(t,x)$ *given by (29) satisfies (30) and (31).*

We can improve Theorem 5.2 in the following way.

Theorem 5.3. *Assume that there are numbers* a_1, a_2 *such that*

1. $\alpha_l < a_1 \le a_2$ *and*
   $$a_1(x^-)^2 + \gamma_l(a_1)|x^+|^2 - W_1(t) \le 2V(t,x) \tag{33}$$
   $$\le a_2|x^-|^2 + \Gamma_l(a_2)|x^+|^2 + W_2(t), \quad x \in \mathbb{R}^n,\ t \in \mathbb{R},$$
   where
   $$\alpha_l := \max\{d(v) : v \in N, v \ge 0, \|v\| = 1\},$$
   the W_j *are in* $L^1(\mathbb{R})$ *and the functions* $\gamma_l(a)$, $\Gamma_l(a)$ *are defined by*
   $$\gamma_l(a) := \max\{d(v) - a\|v^-\|^2 \ :\ v \in N,\ \|v^+\| = 1\} \tag{34}$$

and

$$\Gamma_l(a) := \inf\{d(w) - a\|w^-\|^2 :\ w \in M,\ \|w^+\| = 1\}, \tag{35}$$

where $x^{\pm} = \max\{\pm u, 0\}$.

2. *There is a constant* $\theta > 0$ *such that*

$$2V(t,x) \leq -\theta|x|^2 + W(t), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n,$$

   where $W(t) \in L^1(\mathbb{R})$.

3. *The function* $H(t,x)$ *given by (29) satisfies (30) and (31).*

Then the system (32) has a solution.

Theorem 5.4. *The conclusions of Theorems 5.1 - 5.3 hold if we replace (30) and (31) with*

$$H(t,x) \geq -W_1(t) \in L^1(\mathbb{R}), \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}, \tag{36}$$

and

$$H(t,x) \to \infty, \quad |x| \to \infty,\ t \in \mathbb{R},\ x \in \mathbb{R}^n, \tag{37}$$

where $W_1(t) \in L^1(\mathbb{R})$.

Theorem 5.5. *The conclusions of Theorems 5.1 - 5.3 hold if in place of (30), (31) we assume*

$$H(t,x) \geq -W_1(t)(|x|^\alpha + 1), \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}, \tag{38}$$

and

$$\nu(t) := \liminf_{|x|\to\infty} H(t,x)/|x|^\alpha > 0 \quad a.e. \tag{39}$$

for some $\alpha > 0$, *where* $W_1(t) \in L^1(\mathbb{R})$.

Theorem 5.6. *The conclusions of Theorems 5.1 - 5.3 hold if in place of (30), (31) we assume that there is an* $\alpha > 0$ *such that*

$$H(t,x) \leq W_1(t)(|x|^\alpha + 1), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n, \tag{40}$$

and

$$\nu(t) := \limsup_{|x|\to\infty} H(t,x)/|x|^\alpha < 0 \quad a.e., \tag{41}$$

where $W_1(t) \in L^1(\mathbb{R})$.

Theorem 5.7. *If, in Theorem 5.1, we assume that* $\mu < \lambda_l$ *and there is a constant* $\gamma > \lambda_l$*, and a function* $W(t) \in L^1(\mathbb{R})$ *such that*

$$2V(t,x) \geq \gamma|x|^2 - W(t), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n,$$

then the system (32) has a nontrivial solution.

6 Ground state solutions

Let $\mathcal{M}$ be the set of all solutions of

$$\mathcal{D}x(t) = \nabla_x V(t, x(t)). \tag{42}$$

A solution $\tilde{x}$ is called a "ground state solution" if it minimizes the functional

$$G(x) = d(x) - 2\int_{\mathbb{R}} V(t,x)\,dt \tag{43}$$

over the set $\mathcal{M}$.

We have

Theorem 6.1. *Under the hypotheses of Theorem 5.4, system (42) has a ground state solution.*

7 Proofs of the theorems

We now give the proof of Theorem 3.1.

Proof. Let H be the set of vector functions $x(t)$ described above. It is a Hilbert space with norm satisfying

$$\|x\|_H^2 = \sum_{j=1}^n \|x_j\|_{H^1}^2.$$

We also write

$$\|x\|^2 = \sum_{j=1}^n \|x_j\|^2,$$

where $\|\cdot\|$ is the $L^2(\mathbb{R})$ norm.

We define

$$G(x) = d(x) - 2\int V(t, x(t))\,dt, \quad x \in H. \tag{44}$$

This is bounded below by Hypothesis 1. Let

$$c = \inf_H G.$$

By Corollary 3.22, p. 29, of [38] there is a sequence $\{x^{(k)}\} \subset H$ such that

$$G(x^{(k)}) = d(x^{(k)}) - 2 \int V(t, x^{(k)}(t))\, dt \to c, \tag{45}$$

$$(G'(x^{(k)}), z)/2 = d(x^{(k)}, z) - \int \nabla_x V(t, x^{(k)}) \cdot z(t)\, dt \to 0, \quad z \in H \tag{46}$$

and

$$(G'(x^{(k)}), x^{(k)})/2 = d(x^{(k)}) - \int \nabla_x V(t, x^{(k)}) \cdot x^{(k)}\, dt \to 0. \tag{47}$$

If

$$\rho_k = \|x^{(k)}\|_H \leq C,$$

then there is a renamed subsequence such that $x^{(k)}$ converges to a limit $x \in H$ weakly in H and uniformly on any bounded interval I. Taking the limit of this subsequence in (46), we see that

$$(G'(x), z)/2 = d(x, z) - \int \nabla_x V(t, x(t)) \cdot z(t)\, dt = 0, \quad z \in C_0^\infty,$$

from which we conclude easily that x is a solution of (1).

If

$$\rho_k = \|x^{(k)}\|_H \to \infty,$$

let $\tilde{x}^{(k)} = x^{(k)}/\rho_k$. Then, $\|\tilde{x}^{(k)}\|_H = 1$. There is a renamed subsequence such that $\tilde{x}^{(k)}$ converges to a function $\tilde{x}(t) \in H$ weakly in H, strongly in $L^2_{loc}(\mathbb{R})$, a.e. in $\mathbb{R}^n$, and such that $\|\nabla \tilde{x}^{(k)}\| \to r$ and $\|\tilde{x}^{(k)}\| \to \tau$, where $r^2 + \tau^2 = 1$. Since $\tilde{x}^{(k)}$ converges to $\tilde{x}$ weakly in H, we see that $b(\tilde{x}^{(k)}) \to b(\tilde{x})$. Note that by Hypothesis 1,

$$\liminf G(x^{(k)})/\rho_k^2 \geq a(\tilde{x}) + \theta\tau^2 = r^2 + b(\tilde{x}) + \theta\tau^2.$$

Since $r^2 + b(\tilde{x}) \geq 0$, in order for $G(x^{(k)})$ to be bounded, we must have $r^2 + b(\tilde{x}) = 0$ and $\tau = 0$. This means that $\tilde{x} = 0$ and $r = 1$, so that

$$1 + b(\tilde{x}) = 0.$$

But this implies that $b(\tilde{x}) \neq 0$, and consequently $\tilde{x}(t) \not\equiv 0$. Hence, the ρ_k are bounded, and the proof is complete. □

Proof of Theorem 4.1. We follow the proof of Theorem 3.1. We know that $r^2 + b(\tilde{x}) \geq \lambda_0 \|\tilde{x}\|^2$. We must show that $r = 1$ and $r^2 + b(\tilde{x}) = 0$. By Hypothesis

$$\liminf G(x^{(k)})/\rho_k^2 \geq d(\tilde{x}) + \theta\tau^2 = r^2 + b(\tilde{x}) + \theta\tau^2.$$

In order for $G(x^{(k)})$ to be bounded (on a minimizing sequence), this expression must be ≤ 0. Since $\theta + \lambda_0 > 0$, we have

$$(\theta + \lambda_0)\tau^2 \leq r^2 + b(\tilde{x}) + \theta\tau^2 \leq 0.$$

Consequently, we must have $r^2 + b(\tilde{x}) = 0$ and $\tau = 0$. Thus $r = 1$, and the proof proceeds as before.

In proving Theorem 4.2, we make use of the fact that M and N are a sandwich pair (cf. Theorem 3.17, p.26 of [38]). Hypotheses 1 and 2 allow us to conclude that there is a sequence in H satisfying (45) - (47). We can now follow the proof of Theorem 5.1 given below to obtain the desired conclusion.

Proof of Theorem 5.1.

Define the subspaces M and N of H as

$$N = \bigoplus_{k<l} E(\lambda_k), \quad M = N^\perp, \quad H = M \oplus N.$$

Let

$$G(x) = d(x) - 2\int_{\mathbb{R}} V(t,x)\,dt. \tag{48}$$

We note that Hypothesis 1 implies

$$G(v) \leq 0, \quad v \in N. \tag{49}$$

In fact, we have

$$G(x) = d(x) - 2\int_{\mathbb{R}} V(t,x)\,dt \leq \int_{\mathbb{R}} [\lambda_{l-1}|x|^2 - 2V(t,x)]\,dt \leq 0, \quad x \in N.$$

Note that there is a positive $\rho > 0$ such that

$$|x(t)| < m$$

when $\|x\|_H = \rho$. In fact, we have $|x(t)| \leq c_0\|x\|_H$. If $x \in M$ and $\|x\|_H = \rho$, then

$$G(x) = d(x) - 2\int_{|x|<m} V(t,x)\,dt \geq d(x) - \mu\|x\|^2 \geq 0.$$

Take

$$\begin{aligned} A &= \partial \mathbf{B}_\rho \cap M, \\ B &= N, \end{aligned}$$

where

$$\mathbf{B}_\rho = \{x \in H : \|x\|_H < \rho\}.$$

By Example 8, p. 22 of [38], A links B. Moreover,

$$\sup_A [-G] \le 0 \le \inf_B [-G]. \tag{50}$$

Hence, we may apply Corollary 2.8.2 of [32] to conclude that there is a sequence $\{x^{(k)}\} \subset H$ such that

$$G(x^{(k)}) = d(x^{(k)}) - 2\int_{\mathbb{R}} V(t, x^{(k)}(t))\, dt \to c, \tag{51}$$

$$(G'(x^{(k)}), z)/2 = d(x^{(k)}, z) - \int_{\mathbb{R}} \nabla_x V(t, x^{(k)}) \cdot z(t)\, dt \to 0, \quad z \in H \tag{52}$$

and

$$(G'(x^{(k)}), x^{(k)})/2 = d(x^{(k)}) - \int_{\mathbb{R}} \nabla_x V(t, x^{(k)}) \cdot x^{(k)}\, dt \to 0. \tag{53}$$

Moreover, (51) and (53) imply that

$$\int_{\mathbb{R}} H(t, x^{(k)}(t))\, dt \to c. \tag{54}$$

If

$$\rho_k = \|x^{(k)}\|_H \le C,$$

there is a renamed subsequence such that $x^{(k)}$ converges to a limit $x \in H$ weakly in H and uniformly on any bounded interval I. From (52) we see that

$$(G'(x), z)/2 = d(x, z) - \int_{\mathbb{R}} \nabla_x V(t, x(t)) \cdot z(t)\, dt = 0, \quad z \in C_0^\infty(\mathbb{R}),$$

from which we conclude easily that x is a solution of (1). On the other hand, if

$$\rho_k = \|x^{(k)}\|_H \to \infty,$$

let $\tilde{x}^{(k)} = x^{(k)}/\rho_k$. Then $\|\tilde{x}^{(k)}\|_H = 1$, and there is a renamed subsequence such that $\tilde{x}^{(k)}$ converges to a limit $\tilde{x} \in H$ weakly in H and uniformly on any bounded interval I. Since

$$d(\tilde{x}^{(k)}) - 2\int_{\mathbb{R}} V(t, x^{(k)}(t))\, dt/\rho_k^2 \to 0,$$

we have

$$\|\tilde{x}^{(k)}\|_H^2 + b(\tilde{x}^{(k)}) - \|\tilde{x}^{(k)}\|^2 - 2\int_{\mathbb{R}} V(t, x^{(k)}(t))\, dt/\rho_k^2 \to 0.$$

Since $b(x)$ is compact in H, this implies by Hypothesis 3,

$$1 + b(\tilde{x}) \leq \liminf (1-\theta)\|\tilde{x}^{(k)}\|^2 \leq 1-\theta.$$

Hence, $\tilde{x}(t) \not\equiv 0$. Let $\Omega_0 \subset \mathbb{R}$ be the set on which $\tilde{x}(t) \neq 0$. The measure of Ω_0 is positive. Moreover, $|x^{(k)}(t)| \to \infty$ as $k \to \infty$ for $t \in \Omega_0$. Thus,

$$\int_{\mathbb{R}} H(t, x^{(k)}(t))\, dt \leq \int_{\Omega_0} H(t, x^{(k)}(t))\, dt + \int_{\mathbb{R}\setminus\Omega_0} W(t)\, dt \to -\infty$$

by hypothesis. But this contradicts (54). Hence, the ρ_k are bounded, and the proof is complete.

Proof of Theorem 5.2. We note that Hypothesis 1 implies

$$G(v) \leq Q, \quad v \in N, \tag{55}$$

where

$$Q = \int_{\mathbb{R}} W(t)\, dt.$$

In fact, we have

$$G(x) = d(x) - 2\int_{\mathbb{R}} V(t,x)\, dt \leq \int_{\mathbb{R}} [\lambda_{l-1}|x|^2 - 2V(t,x)]\, dt \leq Q, \quad x \in N.$$

If $x \in M$, then

$$\begin{aligned} G(x) &\geq d(x) - \int \lambda_l |x(t)|^2\, dt - Q \\ &\geq (\lambda_l - \lambda_l)\|x\|^2 - Q \geq -Q. \end{aligned} \tag{56}$$

By Theorem 3.17, p.26 of [38], M and N form a sandwich pair. We can now follow the proof of Theorem 5.1 to come to the same conclusion.

In proving Theorem 5.3 we note that

$$d(v) \leq a_1\|v^-\|^2 + \gamma_l(a_1)\|v^+\|^2, \quad v \in N \tag{57}$$

and

$$a_2\|w^-\|^2 + \Gamma_l(a_2)\|w^+\|^2 \leq d(w), \quad w \in M. \tag{58}$$

Hence

$$G(v) \leq Q_1, \quad v \in N$$

and

$$G(w) \geq -Q_2, \quad w \in M$$

by (33). We now follow the proof of Theorems 5.2 and 5.1.

In proving Theorem 5.4 we merely replace $H(t,x)$ by $-H(t,x)$ in Theorems 5.1 – 5.3.

Proof of Theorem 5.5.
In this case we have

$$\underline{\lim} \int_{\Omega_0} \frac{H(t,x^{(k)})}{\rho_k^\alpha} \geq \int_{\Omega_0} \underline{\lim}\, \frac{H(t,x^{(k)})}{|x^{(k)}|^\alpha} \, |\tilde{x}^{(k)}|^\alpha \geq \int_{\Omega_0} \nu(t)\, |\tilde{x}|^\alpha,$$

where Ω_0 is the set where $\tilde{x}(t) \neq 0$. Since $\nu(t) > 0$ a.e., it follows that

$$\underline{\lim} \int_{\Omega_0} \frac{H(t,x^{(k)})}{\rho_k^\alpha} > 0.$$

The rest of the proof proceeds as before.

Proof of Theorem 5.6.
In this case we have

$$\overline{\lim} \int_{\Omega_0} \frac{H(t,x^{(k)})}{\rho_k^\alpha} \leq \int_{\Omega_0} \overline{\lim}\, \frac{H(t,x^{(k)})}{|x^{(k)}|^\alpha} \, |\tilde{x}^{(k)}|^\alpha \leq \int_{\Omega_0} \nu(t)\, |\tilde{x}|^\alpha,$$

where Ω_0 is the set where $\tilde{x}(t) \neq 0$. Since $\nu(t) < 0$ a.e., it follows that

$$\underline{\lim} \int_{\Omega_0} \frac{H(t,x^{(k)})}{\rho_k^\alpha} < 0.$$

The rest of the proof proceeds as before.

Proof of Theorem 5.7. Let

$$y(t) = v + sw_0,$$

where $v \in N$, $s \geq 0$, and $w_0 \in M$ is an eigenfunction of $\mathcal{D}$ corresponding to λ_l. Consequently,

$$\begin{aligned} G(y) &= s^2 d(w_0) + d(v) - 2\int_{\mathbb{R}} V(t,y(t))\,dt \\ &\leq \lambda_l s^2 \|w_0\|^2 + \lambda_{l-1}\|v\|^2 - \gamma \int_{\mathbb{R}} |y(t)|^2\,dt + Q \\ &\leq (\lambda_{l-1} - \gamma)\|v\|^2 + (\lambda_l - \gamma)s^2\|w_0\|^2 + Q \\ &\to -\infty \ \text{ as } \ s^2 + |v|^2 \to \infty, \end{aligned}$$

where

$$Q = \int_{\mathbb{R}} W(t)\,dt.$$

Take

$$\begin{aligned} A &= \{v \in N : \|v\|_H \le R\} \cup \{sw_0 + v : v \in N, s \ge 0, \|sw_0 + v\|_H = R\}, \\ B &= \partial \mathbf{B}_\rho \cap M,\ 0 < \rho < R. \end{aligned}$$

By Example 3, p.38, of [32], A links B. Moreover, if R is sufficiently large,

$$\sup_A G \le 0 < \varepsilon \le \inf_B G. \tag{59}$$

To see this, we note that $G(v) \le 0$ for $v \in N$ as in the proof of Theorem 5.1. This, together with the argument given above, shows that $\sup_A G \le 0$. Also, note that there is a positive $\rho > 0$ such that

$$|x(t)| < m$$

when $\|x\|_H = \rho$. In fact, we have $|x(t)| \le c_0 \|x\|_H$. If $x \in M$ and $\|x\|_H = \rho$, then

$$G(x) = d(x) - 2\int_{|x|<m} V(t,x)\,dt \ge d(x) - \mu\|x\|^2 \ge \varepsilon \|x\|_H^2$$

since $\mu < \lambda_l$. We may now apply Corollary 2.8.2 of [32] to conclude that that there is a sequence $\{x^{(k)}\} \subset H$ such that

$$G(x^{(k)}) = d(x^{(k)}) - 2\int_{\mathbb{R}} V(t, x^{(k)}(t))\,dt \to c \ge \varepsilon > 0, \tag{60}$$

$$(G'(x^{(k)}), z)/2 = d(x^{(k)}, z) - \int_{\mathbb{R}} \nabla_x V(t, x^{(k)}) \cdot z(t)\,dt \to 0, \quad z \in H \tag{61}$$

and

$$(G'(x^{(k)}), x^{(k)})/2 = d(x^{(k)}) - \int_{\mathbb{R}} \nabla_x V(t, x^{(k)}) \cdot x^{(k)}\,dt \to 0. \tag{62}$$

As in the proof of Theorem 5.1, the hypotheses on $H(t,x)$ imply

$$\rho_k = \|x^{(k)}\|_H \le C.$$

Thus, there is a renamed subsequence such that $x^{(k)}$ converges to a limit $x \in H$ weakly in H and uniformly on any bounded interval I. From (61) we see that

$$(G'(x), z)/2 = d(x,z) - \int_{\mathbb{R}} \nabla_x V(t, x(t)) \cdot z(t)\,dt = 0, \quad z \in C_0^\infty(\mathbb{R}),$$

from which we conclude easily that x is a solution of (1). Now

$$\int_{\mathbb{R}} H(t, x^{(k)}(t))\,dt = G(x^{(k)}) - (G'(x^{(k)}), x^{(k)})/2 \to c \ge \varepsilon > 0.$$

Since

$$H(t,x^{(k)}(t)) \le W(t) \in L^1(\mathbb{R})$$

and

$$H(t,x^{(k)}(t)) \to H(t,x(t)) \ a.e.,$$

we have

$$\int_{\mathbb{R}} H(t,x(t))\,dt \ge \limsup \int_{\mathbb{R}} H(t,x^{(k)}(t))\,dt = c \ge \varepsilon > 0.$$

Consequently,

$$(G(x) - (G'(x),x)/2 = \int_{\mathbb{R}} H(t,x(t))\,dt \ge c \ge \varepsilon > 0.$$

But

$$G(0) = -2\int_{\mathbb{R}} V(t,0)\,dt = 0.$$

Hence, $x(t) \neq 0$. This completes the proof.

Proof of Theorem 6.1.

Let

$$\alpha = \inf_{\mathcal{M}} G(x).$$

There is a sequence $\{x^{(k)}\} \in \mathcal{M}$ such that

$$G(x^{(k)}) = d(x^{(k)}) - 2\int_{\mathbb{R}} V(t,x^{(k)}(t))\,dt \to \alpha, \tag{63}$$

$$(G'(x^{(k)}),z)/2 = d(x^{(k)},z) - \int_{\mathbb{R}} \nabla_x V(t,x^{(k)}) \cdot z(t)\,dt = 0, \quad z \in H \tag{64}$$

and

$$(G'(x^{(k)}),x^{(k)})/2 = d(x^{(k)}) - \int_{\mathbb{R}} \nabla_x V(t,x^{(k)}) \cdot x^{(k)}\,dt = 0. \tag{65}$$

Thus,

$$\int_{\mathbb{R}} H(t,x^{(k)}(t))\,dt = G(x^{(k)}) \to \alpha.$$

As in the proof of Theorem 5.1, the hypotheses on $H(t,x)$ imply that

$$\rho_k = \|x^{(k)}\|_H \le C.$$

Hence, there is a renamed subsequence such that $x^{(k)}$ converges to a limit $x \in H$ weakly in H and uniformly on any bounded interval I. From (64) we see that

$$(G'(x), z)/2 = d(x, z) - \int_{\mathbb{R}} \nabla_x V(t, x(t)) \cdot z(t)\, dt = 0, \quad z \in C_0^\infty(\mathbb{R}),$$

from which we conclude easily that x is a solution of (1). Hence, $x \in \mathcal{M}$. Moreover,

$$G(x) = \int_{\mathbb{R}} H(t, x(t))\, dt \leq \liminf \int_{\mathbb{R}} H(t, x^{(k)}(t))\, dt = \liminf G(x^{(k)}) = \alpha.$$

Thus, $G(x) = \alpha$. This completes the proof.

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