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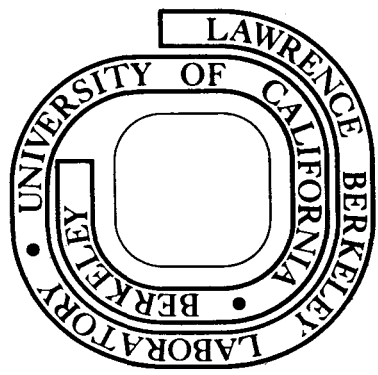
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Is the Taub-NUT Metric a Gravitational Instanton ?

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Abstract. Hawking has suggested that the Taub-NUT metric might give rise to the gravitational analog of the Yang-Mills pseudoparticle or instanton. We extend Hawking's treatment by using the Atiyah-Patodi-Singer index theorem for manifolds with boundaries and reach somewhat different conclusions regarding the nature of the bound states of the Dirac equation. In particular, the Taub-NUT metric does not seem to be a suitable candidate for a gravitational instanton.

The discovery of pseudoparticle solutions [1] of the Euclidean Yang-Mills SU(2) gauge theory has led to a new nonperturbative picture of the Yang-Mills vacuum [2]. There appear to be an infinite number of equivalent vacua |n> which are labeled by an integer n. The pseudoparticle solution with second Chern class k gives the lowest order quantum amplitude for tunneling between vacua which are labeled by n and by n+k. The existence of these tunneling amplitudes breaks the infinite vacuum degeneracy and implies that there is a single true Yang-Mills vacuum given by a superposition of all the vacua |n>.

Since Einstein's theory of gravitation is in many respects similiar to Yang-Mills theory in its geometric properties, one might naturally seek gravitational analogs of the Yang-Mills pseudoparticle solutions [3]. If such solutions of the Euclidean Einstein equations do exist, they would have profound implications for the structure of the gravitational vacuum.

Since all Yang-Mills pseudoparticle solutions are believed to produce self-dual field strengths which are geometrically interpretable as curvature tensors, it seems reasonable to begin by looking for gravitational metrics which yield self-dual curvature tensors. Hawking [4] has suggested that a possible candidate might be the Euclidean Taub-NUT metric on $R \times S^3$. While the parameters of the Taub-NUT metric can indeed be chosen to give self-dual (or anti-self-dual) curvature tensors, Taub-NUT space must be treated as a manifold with two boundaries, rather than as a closed manifold. Thus Hawking's arguments regarding the topological invariants of Taub-NUT space must be modified to take into account the results of Atiyah, Patodi, and Singer [5] concerning manifolds with boundaries.

In what follows, we show that when boundary corrections are included, the Dirac index of Taub-NUT space vanishes, rather than being fractional as indicated by Hawking. Furthermore, we find that the Euler-Poincaré characteristic of Taub-NUT space is also zero. These results can be understood intuitively by observing first that Taub-NUT space has the topological structure $R \times S^3$, and so is a generalization of a cylinder. A cylinder has no topological obstructions such as handles and has vanishing Euler-Poincaré characteristic. Secondly, since the Taub-NUT space is known to admit a spin structure, the index of the Dirac operator is the difference of two integers and therefore must be an integer. We shall give an argument which shows this integer must be zero for manifolds with the following properties: 1) $M_4 = [a,b] \times N_3$ where N_3 is a compact 3 dimensional manifold without boundary; 2) the scalar curvature of the restriction of the metric to N_3 at any point in the interval $[a,b]$ is everywhere positive. (By taking limits, we can apply these results to non-compact manifolds of the form $(a,b) \times N_3$.)

We begin our calculation by employing the elegant methods of Cartan [6] to examine the Taub-NUT metric. (For specific applications of these techniques to Taub-NUT space, see Miller [7] and especially Misner [8].) Following Misner and Taub [9], let the angles $\psi \theta \phi$ be the classical Euler angle coordinates on SO(3). Since we are considering the simply connected covering space S^3 instead of SO(3), we allow ψ to have fundamental domain $0 \leq \psi < 4\pi$ while θ and ϕ have their usual ranges $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. We let $(\sigma_x, \sigma_y, \sigma_z)$ be a basis for the space of left-invariant differential 1-forms on the three-sphere $S^3 = SU(2)$. In local parametric coordinates we have:

$$\sigma_x = \cos(\psi) d\theta + \sin(\psi)\sin(\theta)d\phi \ , \ \sigma_y = -\sin(\psi)d\theta + \cos(\psi)\sin(\theta)d\phi$$

$$\sigma_z = \cos(\theta)d\phi + d\psi. \tag{1}$$

The basis $(\sigma_x, \sigma_y, \sigma_z)$ obeys the Cartan structure equations in the exterior calculus,

$$d\sigma_x = -\sigma_y \wedge \sigma_z \ , \ d\sigma_y = -\sigma_z \wedge \sigma_x \ , \ d\sigma_z = -\sigma_x \wedge \sigma_y. \tag{2}$$

We then choose our Euclidean Taub-NUT metric in the region $m<r<\infty$ of $R\times S^3$ to be that of ref. [7] with $\ell=im$,

$$ds^2= \frac{r+m}{r-m} dr^2+(r^2-m^2)\{\sigma_x^2+\sigma_y^2+(\frac{2m}{r+m})^2\sigma_z^2\}. \qquad (3)$$

One can also define a metric in the region $-m<r<m$ by changing the signs to obtain a positive definite metric.

We next take a dual orthonormal basis for the cotangent space of the form

$$e^0=(\frac{r+m}{r-m})^{\frac{1}{2}}dr \qquad , e^1=(r^2-m^2)^{\frac{1}{2}}\sigma_x$$

$$e^2=(r^2-m^2)^{\frac{1}{2}}\sigma_y \qquad , e^3= \frac{2m}{r+m}(r^2-m^2)^{\frac{1}{2}}\sigma_z, \qquad (4)$$

where the $e^a=e^a_\mu dx^\mu$ have been chosen so $ds^2=(e^0)^2+(e^1)^2+(e^2)^2+(e^3)^2$. Then we may compute the connection one-forms $\{\omega^a_{\ b}\}$ using the identities

$$de^a=-\omega^a_{\ b}\wedge e^b \qquad , \omega^a_{\ b}+\omega^b_{\ a}=0 \qquad (5)$$

We find

$$\omega^0_{\ 1}=-\frac{r}{r+m}\sigma_x \qquad , \omega^0_{\ 2}=-\frac{r}{r+m}\sigma_y \qquad , \omega^0_{\ 3}=-\frac{2m^2}{(r+m)^2}\sigma_z$$

$$\omega^2_{\ 3}=-\frac{m}{r+m}\sigma_x \qquad , \omega^3_{\ 1}=-\frac{m}{r+m}\sigma_y \qquad , \omega^1_{\ 2}=\{\frac{2m^2}{(r+m)^2}-1\}\sigma_z. \qquad (6)$$

Using the identity $R^a_{\ b}=d\omega^a_{\ b}+\omega^a_{\ c}\wedge\omega^c_{\ b}$, we find the curvature two-forms

$$R^0_{\ 1}=-R^2_{\ 3}=-m(r+m)^{-3}(e^0\wedge e^1-e^2\wedge e^3)$$

$$R^0_{\ 2}=-R^3_{\ 1}=-m(r+m)^{-3}(e^0\wedge e^2-e^3\wedge e^1) \qquad (7)$$

$$R^0_{\ 3}=-R^1_{\ 2}=2m(r+m)^{-3}(e^0\wedge e^3-e^1\wedge e^2).$$

One can easily show using Eq. (7) that the curvature tensor satisfies Einstein's source-free field equations and that the curvature is anti-self-dual, i.e. that $*R^a_{\ b}=-R^a_{\ b}$. (Reversing the orientation of Taub-NUT space yields a self-dual curvature.) It is remarkable that neither the connection forms (6) nor the curvature forms (7) show any pathology at $r=m$; in contrast, the metric (3) changes sign at $r=m$ and the spin structure coupling to the vierbeins (4) becomes undefined. Although this is reminiscent of the behavior exhibited at the Schwarzschild singularity, in our case we must change the overall sign of the metric in order to have an elliptic Dirac complex in the interior region as we pass from $\infty>r>m$ to $m>r>-m$.

Next we recall that the index $I_D(M)$ of the Dirac operator on a manifold M is defined as the difference between the number of normalizable positive and negative chirality solutions of the Dirac equation, that is

$$I_D(M)=n_+-n_-. \qquad (8)$$

Since a space with anti-self-dual curvature like that shown in Eq. (7) could treat the positive and negative chirality spinors differently, one might intuitively expect $I_D\neq 0$. For compact manifolds M without boundary (dM=0), $I_D(M)$ is related to the intrinsic topological structure of the manifold by the Atiyah-Singer index theorem [10]. For the Dirac complex, this yields the formula

$$I_D(M)= \frac{-1}{24} P_1[M] ; \qquad (9)$$

Where $P_n[M]$ is the nth Pontrjagin number. $P_1[M]$ is given in terms of the curvature two-form $R^a_{\ b}$ by

$$P_1[M]= \frac{-1}{8\pi^2} \int_M Tr(R\wedge R). \qquad (10)$$

The quantity $\frac{-1}{24} P_1[M]$ is known as the *A-roof genus* of M and is usually denoted by $\hat{A}[M]$. This is a topological invariant of M and is independent of the choice of the metric. We note that while n_+ and n_- may depend individually on the choice of the metric, their *difference* is always a topological invariant.

The Taub-NUT space, however, has the structure of $R\times S^3$ and must be treated as a manifold with *boundaries*, one at either end of the chosen interval $[r_1,r_2]$ on the real line R. We must thus use the generalized index theorem of Atiyah, Patodi, and Singer [5] for manifolds with boundary. If the metric is product near the boundary dM of M, then the index theorem becomes

$$I_D(M)= \frac{1}{8\cdot 24\pi^2}\int_M Tr(R\wedge R) - \frac{1}{2}\{h(dM)+n_D(dM)\}. \qquad (11)$$

h(dM), the dimension of the space of harmonic spinors on dM, will vanish if the scalar curvature of the restriction of the metric to dM is everywhere positive

[11]. $\eta_D(dM)$ is the Atiyah-Patodi-Singer *eta-invariant* of the Dirac complex. If $\{\lambda_k\}$ is the set of eigenvalues of the tangential part of the Dirac operator restricted to the boundary dM, we define

$$\eta_D(s,dM) = \sum_{\lambda_k \neq 0} \text{sign}(\lambda_k) |\lambda_k|^{-s} \quad \text{for } s>2. \tag{12}$$

It can be shown that $\eta_D(s,dM)$ admits a meromorphic extension to the complex plane which is regular at s=0, $\eta_D(dM) \equiv \eta_D(0,dM)$. It is worth noting that $\eta_D(dM)$ is independent of change of scale in the metric since we are evaluating Eq.(12) at s=0. Furthermore, $\eta_D(dM)=0$ if there is an orientation reversing isometry of dM with the given metric [5]. If dM has more than one component, the quantity $\frac{1}{2}\{h(dM)+\eta_D(dM)\}$ becomes an oriented sum over these pieces. (We use the orientation $e^0 \wedge e^1 \wedge e^2 \wedge e^3$ on M).

However, the Taub-NUT metric is *not* product on dM, so that Eq. (11) must be modified by a surface term [12] which makes use of the Chern-Simons formula [13]. Let ω_0 be the connection 1-form of the product metric coinciding with Eq. (3) on a given piece of the boundary. The second fundamental form is $\theta=\omega-\omega_0$. We now define

$$Q = \frac{1}{8\cdot24\pi^2} \text{Tr}(2\theta \wedge R + \frac{2}{3}\theta \wedge \theta \wedge \theta - \theta \wedge \omega \wedge \theta - \theta \wedge \theta \wedge \omega - \theta \wedge d\theta). \tag{13}$$

The exterior derivative of Q is the integrand of Eq. (10), $dQ=\text{Tr}(R \wedge R)/(8\cdot24\pi^2)$. The new term in the index formula then can be written (after some algebra) as

$$-\int_{dM} Q = \frac{-1}{8\cdot24\pi^2} \int_{dM} \text{Tr}(\theta \wedge R). \tag{14}$$

In order to perform the calculations, we choose the constant r level sets of the Taub-NUT space. dM then consists of two pieces, one at r=m+ε, the other at $r=\frac{1}{\varepsilon}$. By reversing the sign of the metric (3), we can also examine the region 0<r<m by choosing the boundary level sets at r=ε and at r=m-ε. The resulting structures are represented in Fig. 1.

The 3-metric on each piece of dM is given by Eq. (3) with r held constant,

$$ds_3^2 = |r^2-m^2| (\sigma_x^2 + \sigma_y^2 + (\frac{2m}{r+m})^2 \sigma_z^2). \tag{15a}$$

Since η_D is invariant under change of scale, the eta invariant of this metric is the same as the eta invariant of the metric

$$d\bar{s}_3^2 = \frac{1}{4}(\sigma_x^2 + \sigma_y^2 + (\frac{2m}{r+m})^2 \sigma_z^2). \tag{15b}$$

The scalar three-curvature of the metric of Eq. (15a) is given by:

$$K^{(3)} = \frac{1}{|r^2-m^2|} \{1 - \frac{m^2}{(r+m)^2}\}. \tag{16}$$

Since $K^{(3)}$ is positive in the regions considered, 0<r<m and m<r<∞ , there are no harmonic spinors [11] on dM. A separate argument based on [11] shows there are also no harmonic spinors for $-\frac{1}{2}$m<r≤0. Therefore

$$h(dM(r))=0 , \quad -\frac{1}{2}m<r<m ,$$
$$h(dM(r))=0 , \quad m<r<\infty . \tag{17}$$

Since there are no harmonic spinors, $\eta_D(dM)$ is continuous; thus $\lim_{\varepsilon \to 0} \eta_D(dM(r=m\pm\varepsilon))$ is the eta invariant for the standard metric $d\bar{s}_3^2 = \frac{1}{4}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2)$. But this metric admits an orientation reversing isometry and therefore the eta invariant vanishes:

$$\lim_{\varepsilon \to 0} \eta_D(dM(r=m\pm\varepsilon))=0. \tag{18}$$

Finally, we choose as our product metric near each boundary $dM(r=r_i)$

$$ds_i^2 = \pm\{ \frac{r_i+m}{r_i-m} dr^2 + (r_i^2-m^2)(\sigma_x^2 + \sigma_y^2 + (\frac{2m}{r_i+m})^2 \sigma_z^2)\} \tag{19}$$

This defines the connection forms ω_i. The formula for the index between two boundaries r_1 and r_2 with $r_1 < r_2$ becomes

$$I_D(r_1,r_2)(M) = \frac{1}{24\cdot8\pi^2}\{\int_M \text{Tr}(R \wedge R) - \int_{dM_2}\text{Tr}(R \wedge (\omega-\omega_2)) + \int_{dM_1}\text{Tr}(R \wedge (\omega-\omega_1))\}$$
$$-\frac{1}{2}\{\eta_D(dM_2)-\eta_D(dM_1)\}. \tag{20}$$

The interior and boundary integrals are continuous functions of r_1, r_2. Since h≡0 by Eq. (17), η_D is a continuous function as well [5]. This implies that $I_D(r_1,r_2)$ is a continuous integer valued function and is therefore *constant*. As $r_1 \to r_2$, $I(r_1,r_2) \to 0$ so

$$I(r_1,r_2) \equiv 0. \tag{21}$$

We note that if $h(dM) \neq 0$ anywhere in the interval $[r_1, r_2]$, this argument fails. We are using the product structure of $R \times S^3$ in an essential fashion.

Direct evaluation of the right hand side of Eq. (20) confirms the validity of our general argument:

$$I_D(r_1, r_2) = \{\frac{4m^3(m-2r)}{3(r+m)^4}\}|_{r_1}^{r_2} - \{\frac{2m^2(r-m)^2}{3(r+m)^4}\}|_{r_1}^{r_2} - \frac{1}{12}(1 - 2(\frac{2m}{r+m})^2 + (\frac{2m}{r+m})^4)|_{r_1}^{r_2}$$
$$\cong 0. \tag{22}$$

The first term gives the volume integral, the second term gives the surface integrals, and the last term gives the eta invariants calculated by Hitchin [11]. For example, we can *solve* Eqs. (20) and (21) for the potentially unknown eta invariants:

$$0 = I_D(m, \infty) = \frac{1}{12} - \frac{1}{2}\eta_D(\infty), \tag{23a}$$

$$0 = I_D(o, m) = \frac{-3}{4} + \frac{1}{2}\eta_D(0), \tag{23b}$$

since $\eta_D(dM(r=m)) = 0$ by Eq. (16). We therefore conclude that

$$n_+ = n_- \quad , \quad \eta_D(\infty) = \frac{1}{6} \quad , \quad \eta_D(0) = \frac{3}{2}. \tag{24}$$

Thus the index theorem applied to the Taub-NUT metric gives no information about the existence of harmonic spinors on the full Euclidean manifold besides the statement $n_+ = n_-$. In particular, the index is not fractional. Since the Taub-NUT space admits a spin structure, the index *must* be an integer. The difference between Hawking's result and our result arises from the inclusion of the eta-invariant and the surface terms which combine to give a vanishing index.

We note that while no interesting structure is evident in the regions $0 < r < m$ and $m < r < \infty$ of Euclidean Taub-NUT space, the region $-m < r < 0$ has some remarkable properties. From Eq. (16), the scalar three-curvature becomes negative for $r < 0$ so that $h(dM)$ is no longer necessarily zero. In fact, using Hitchin's method [11] for metrics on S^3, one may show that harmonic spinors first appear at $r = -\frac{1}{2}m$. As r passes $-\frac{1}{2}m$ in the negative direction, the eta invariant jumps by two units and the index $I_D(M)$ for the interval $[r, m]$ no longer vanishes. As $r \to -m$, the curvature becomes singular and the value of the index diverges.

Finally, we can also carry out a similiar calculation to find the surface correction to Hawking's result for the Euler-Poincaré characteristic of Taub-NUT space. Chern's formula [14] gives

$$\chi(M) = \frac{1}{32\pi^2}[\int_M \epsilon_{abcd} R^a_{\ b} R^c_{\ d} - \{\int_{dM_r} \epsilon_{abcd}(2\theta^a_{\ b} \hat{} R^c_{\ d} - \frac{4}{3}\theta^a_{\ b} \hat{} \theta^c_{\ e} \hat{} \theta^e_{\ d})\}|_{r_1}^{r_2}]. \tag{25}$$

Direct evaluation of each term for $r_1 = m$ and $r_2 = \infty$ gives

$$\chi(M) = 1 - 0 + (-1) = 0. \tag{26}$$

The only surface contribution arises from the $\theta \hat{} \theta \hat{} \theta$ integral at $r_1 = m$. For arbitrary $[r_1, r_2]$ in each domain, we also find that $\chi(M) = 0$.

We would like to thank B.Lawson, I.M.Singer, and A.Taub for many helpful discussions. After completion of this work, we learned that Römer and Schroer [15] have independently reached similiar conclusions concerning the vanishing of the index for the $m < r < \infty$ portion of Taub-NUT space.
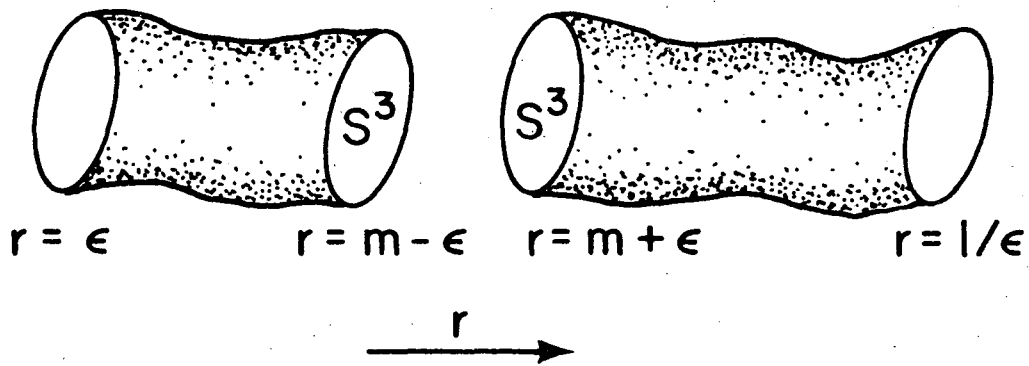
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Figure Caption

Fig.1. Two regions of Euclidean Taub-NUT space. For m<r<∞ , the boundaries are at
r=m+ε,r= $\frac{1}{\varepsilon}$. For o<r<m, the opposite sign of the metric must be chosen and
the boundaries are at r=ε, r=m-ε. All boundaries have the topology of the
three-sphere S^3, but only at r=m is the metric proportional to the canonical
metric on S^3.

$$r = \epsilon \qquad r = m - \epsilon \qquad r = m + \epsilon \qquad r = 1/\epsilon$$

$$r \longrightarrow$$

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Fig. 1