Distributed Energy Consumption Control via Real-Time Pricing Feedback in Smart Grid

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Abstract—**This brief proposes a pricing-based energy control strategy to remove the peak load for smart grid. According to the price, energy consumers control their energy consumption to make a tradeoff between the electricity cost and the load curtailment cost. The consumers are interactive with each other because of pricing based on the total load. We formulate the interactions among the consumers into a noncooperative game and give a sufficient condition to ensure a unique equilibrium in the game. We develop a distributed energy control algorithm and provide a sufficient convergence condition of the algorithm. The energy control algorithm starts at the beginning of each time slot, e.g., 15 min. Finally, the energy control strategy is applied to control the energy consumption of the consumers with heating ventilation air conditioning systems. The numerical results show that the energy control strategy is effective in removing the peak load and matching supply with demand, and the energy control algorithm can converge to the equilibrium.**

Index Terms—**Demand response, energy control, Nash equilibrium, noncooperative game, real-time pricing (RTP), smart grid.**

I. INTRODUCTION

MATCHING supply with demand has been an active topic in operating electricity markets. Traditionally, we need enough generation capacity to meet the peak load, which requires substantial infrastructure to be idle for all but a few hours a year. Recently, demand response has been proposed to control the load instead of providing enough generation capacity. In practice, demand response can be implemented by direct load control or market-based pricing. For the direct load control, energy providers have the ability to remotely shut down consumer equipments on a short notice when needed [1]–[3]. For the market-based pricing, energy providers can adjust the load by flexible pricing, such as time of use, critical peak pricing, and real-time pricing (RTP) [4]. With the development of smart grid, which enables reliable and real-time communications between the energy providers and the consumers, the price can be provided to the consumers daily, hourly, or in even shorter intervals. The communications between the energy providers and the consumers are based on an advanced metering infrastructure, which supports the collection of meter readings and the announcement of electricity price [5].

Recently, game theory and convex optimization have been applied to model the pricing-based demand response. For example, the noncooperative game was used to study the cost minimization of interactive consumers [6], [7], the charging of large populations of plug-in electric vehicles (PEVs) [8], and the PEV charging with disturbances and delays [9]. The Stackelberg game was employed to formulate the energy exchange between the PEVs and the smart grid [10], and the interactions between the consumers and the energy providers [11], [12]. For demand response based on convex optimization [13]–[15], the energy control strategies and the RTP algorithms were obtained by dual decomposition. Nevertheless, a few papers are devoted to pricing conditions to ensure stable demand response. In this brief, we give a sufficient pricing condition to ensure a unique equilibrium in the pricing-based demand response. The interactions among the consumers are formulated into a noncooperative game, and the equilibrium in the demand response is aligned with the Nash equilibrium in the noncooperative game. We develop an energy control algorithm to search for the unique equilibrium in a distributed fashion and obtain a sufficient convergence condition of the algorithm. To the best of our knowledge, there is no work in the literature providing rigorous analysis of the pricing condition to ensure a unique equilibrium in demand response. The pricing condition can guide the energy provider to choose the pricing function to implement stable demand response.

The rest of this brief is organized as follows. Some preliminaries are given in Section II. An energy system with pricing is formulated into a noncooperative game in Section III. Section IV gives the pricing condition to ensure a unique equilibrium in the game and the convergence condition of the energy control algorithm. In Section V, the results are applied to control the energy consumption of the consumers with heating ventilation air conditioning (HVAC) systems. The numerical results are shown in Section VI, and the conclusion is drawn in Section VII.

II. PRELIMINARIES

A. Noncooperative Game

Definition 1 [16]: A noncooperative game is defined as a triple $G = \{\mathcal{N}, (S_i)_{i \in \mathcal{N}}, (U_i(\boldsymbol{l}))_{i \in \mathcal{N}}\}$, where $\mathcal{N} = \{1, 2, \ldots, N\}$ is the set of active players participating in the

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game

$$S_i = \{l_i | l_i \in [l_i^{\min}, l_i^{\max}]\} \tag{1}$$

is the set of possible strategies that player i can take, and $U_i(\boldsymbol{l})$ is the payoff function.

Definition 2 [16]: For a noncooperative game $G = \{\mathcal{N}, (S_i)_{i \in \mathcal{N}}, (U_i(\boldsymbol{l}))_{i \in \mathcal{N}}\}$, a vector of strategies $\boldsymbol{l}^* = (l_1^*, l_2^*, \ldots, l_N^*)$ is a Nash equilibrium if and only if $U_i(l_i^*, \boldsymbol{l}_{-i}^*) \geq U_i(l_i', \boldsymbol{l}_{-i}^*)$ for all $i \in \mathcal{N}$ and any other $l_i' \in S_i$, where $\boldsymbol{l}_{-i} = (l_1, l_2, \ldots, l_{i-1}, l_{i+1}, \ldots, l_N)$ denotes the set of strategies selected by all the consumers except for consumer i, $(l_i, \boldsymbol{l}_{-i}) = (l_1, l_2, \ldots, l_{i-1}, l_i, l_{i+1}, \ldots, l_N)$ denotes the strategy profile, and $U_i(l_i, \boldsymbol{l}_{-i})$ is the resulting payoff for the player i given the strategies of the other players.

Lemma 1 [17]: A Nash equilibrium exists in the game $G = \{\mathcal{N}, (S_i)_{i \in \mathcal{N}}, (U_i(\boldsymbol{l}))_{i \in \mathcal{N}}\}$, if for all $i \in \mathcal{N}$.

1) S_i is a nonempty, convex, and compact subset of some Euclidean space $\Re^N$.
2) $U_i(\boldsymbol{l})$ is continuous in $\boldsymbol{l}$ and quasi-concave in l_i.

B. Taguchi Loss Function

The Taguchi loss function is a statistical method that captures the cost to society due to the manufacture of imperfect products [18]. The loss function is given as

$$L = \theta(y - \hat{y})^2 \tag{2}$$

where y is the value of quality characteristic, $\hat{y}$ is the target value of y, L is the loss in dollars, and θ is a constant coefficient. The quadratic representation of the loss function is minimum at $y = \hat{y}$, increases as y deviates from $\hat{y}$. The Taguchi loss function defines the relationship between the economic loss and the deviation of the quality characteristic from the target value. For a product with target value $\hat{y}$, $\hat{y} \pm \Delta_0$ represents the deviation at which functional failure of the product occurs. When a product is manufactured with the quality characteristic at the extremes, $\hat{y} + \Delta_0$ or $\hat{y} - \Delta_0$, some countermeasure must be undertaken by the customers. Assuming the cost of countermeasure is A_0 at $\hat{y} + \Delta_0$ or $\hat{y} - \Delta_0$, we define the constant θ as

$$\theta = \frac{A_0}{\Delta_0^2}. \tag{3}$$

III. PROBLEM FORMULATION

A. System Model

As shown in Fig. 1, we consider an energy system consisting of an energy provider and several consumers. The energy provider purchases electricity from the wholesale markets and sells it to the consumers. We assume that the consumers can communicate with the energy provider in real time and schedule the energy usage of appliances with energy management controllers (EMCs). The set of consumers is denoted by $\mathcal{N} = \{1, 2, \ldots, N\}$, and the intended operation cycle is divided into K time slots, indexed by k ($k \in \{1, 2, \ldots, K\}$). We denote the energy consumption of consumers as $\boldsymbol{l}^k = (l_1^k, \ldots, l_i^k, \ldots, l_N^k)$, where l_i^k is the energy consumption of consumer i ($i \in \mathcal{N}$) in time slot k. At the beginning of each
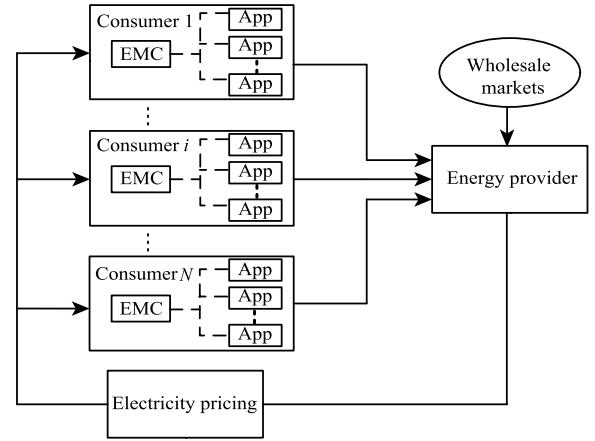

Fig. 1.   Energy system with pricing.

time slot, the energy provider publishes the electricity price $p(\boldsymbol{l}^k)$, and the consumers determine the energy consumption to make a tradeoff between the electricity cost and the load curtailment cost. Then, the total cost to consumer i can be defined as

$$C_i(\boldsymbol{l}^k) = \begin{cases} V_0 + p(\boldsymbol{l}^k)l_i^k, & \text{if } l_i^k < l_i^{\min} \\ V_i(l_i^k) + p(\boldsymbol{l}^k)l_i^k, & \text{if } l_i^{\min} \leq l_i^k \leq l_i^{\max} \\ V_1 + p(\boldsymbol{l}^k)l_i^k, & \text{if } l_i^k > l_i^{\max} \end{cases} \tag{4}$$

where $l_i^{\min}$ and $l_i^{\max}$ are the minimum and maximum energy consumption of consumer i, $V_i(l_i^k)$ denotes the load curtailment cost, and $p(\boldsymbol{l}^k)l_i^k$ denotes the electricity cost. To make $C_i(\boldsymbol{l}^k)$ continuous, we assume $V_i(l_i^{\min}) = V_0$ and $V_i(l_i^{\max}) = V_1$.

B. Energy Consumption Game

The consumers determine their energy consumption to minimize the total cost. This can be described as the following individual optimization problems:[1]

$$l_i^{k*} = \operatorname*{arg\,max}_{l_i^{\min} \leq l_i^k \leq l_i^{\max}} -V_i(l_i^k) - p(\boldsymbol{l}^k)l_i^k, \quad i \in \mathcal{N}. \tag{5}$$

The individual optimization problems are coupled with each other by assuming the pricing function is known to the consumers, i.e., the energy consumption strategy of each consumer is affected by the energy consumption strategies of the other consumers. This coupled optimization problem can be formulated into a noncooperative energy consumption game, where the consumers act as the players and select the energy consumption strategies. The strategy space is defined by (1) and the payoff function is denoted as $U_i(\boldsymbol{l}^k) = -C(\boldsymbol{l}^k)$. Before proceeding further, we need to analyze the Nash equilibrium of the noncooperative energy consumption game. According to Definition 2, the Nash equilibrium is a set of strategies where no consumer has an incentive to change its strategy unilaterally given the strategies of the other consumers. Then, the

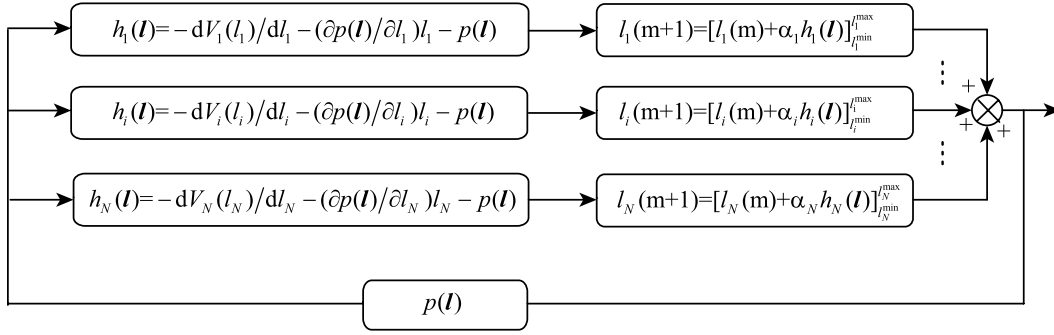[1]The maximization is equivalent to minimizing the total cost.

Fig. 2. Energy consumption control with RTP feedback.

Nash equilibrium can be obtained from the following equations:

$$\frac{\partial U_i(\boldsymbol{l}^k)}{\partial l_i^k} = -\frac{dV_i(l_i^k)}{dl_i^k} - \frac{\partial p(\boldsymbol{l}^k)}{\partial l_i^k}l_i^k - p(\boldsymbol{l}^k) = 0, \ i \in \mathcal{N}. \quad (6)$$

In the subsequent sections, we omit k for convenience.

IV. MAIN RESULTS

In this section, we first give a pricing condition to ensure the uniqueness of Nash equilibrium in the noncooperative energy consumption game.

Theorem 1: The noncooperative energy consumption game has a unique Nash equilibrium if $p(\boldsymbol{l})$ is a linear rotational symmetric function and satisfies

$$(N-1)\left|\frac{\partial p(\boldsymbol{l})}{\partial l_i}\right| - 2\frac{\partial p(\boldsymbol{l})}{\partial l_i} \leq \frac{d^2 V_i(l_i)}{dl_i^2}, \ i \in \mathcal{N}. \quad (7)$$

Proof: The proof is moved to Appendix A. □

The condition (7) requires the cost function to be convex when the number of consumers is larger than three, which is reasonable for consumers in smart grid. For a noncooperative game, we cannot ensure that the players always find the Nash equilibrium even if it exists. Therefore, we will turn to another question: how can the consumers pursue a distributed search for the unique Nash equilibrium? We develop an energy control algorithm based on the gradient of the payoff function

$$l_i(m+1) = [l_i(m) + \alpha_i h_i(\boldsymbol{l})]_{l_i^{\min}}^{l_i^{\max}}, \quad i \in \mathcal{N} \quad (8)$$

where

$$h_i(\boldsymbol{l}) = -\frac{dV_i(l_i)}{dl_i} - \frac{\partial p(\boldsymbol{l})}{\partial l_i}l_i - p(\boldsymbol{l}), \quad i \in \mathcal{N} \quad (9)$$

where m is the iterative step and α_i is the step size. As shown in Fig. 2, the energy control algorithm (8) is embedded inside the EMC in the energy system with RTP feedback. The distributed implementation of the control algorithm is dependent on the derivative of the price, which should be obtained by each consumer with local information. The distributed implementation will be discussed in detail in Section V. Next, we give the condition to ensure the convergence of the energy control algorithm in the following theorem.

Theorem 2: Suppose the energy consumption game has a unique Nash equilibrium, the energy control algorithm (8) converges to the equilibrium if the step size satisfies

$$\alpha_i < \frac{2}{d^2 V_i(l_i)/dl_i^2 + (N+1)\partial p(\boldsymbol{l})/\partial l_i}, \quad i \in \mathcal{N}. \quad (10)$$

Proof: The proof is given in Appendix B. □

The condition (10) gives an upper bound on the step size, within which the algorithm can converge to the Nash equilibrium. We see that the upper bound is dependent on both the second derivative of the cost function and the first derivative of the pricing function. Particularly, the upper bound is changing during the iterations of the algorithm.

Remark 1: In practice, the cost function may be a combination of multiple step functions [19]. In that case, we can employ the quadratic convex function to approximate it and obtain the optimal energy consumption of the consumers. Then, the suboptimal energy consumption is obtained by approximating the optimal energy consumption to the step values.

Remark 2: The convergence speed of the algorithm is dependent on the choice of step size [20]. From the proof of Theorem 2, we see that the energy control algorithm can converge in a single step when the step size is set to be half of the upper bound, which requires the consumers to know the pricing function and the number of consumers. This means that the convergence speed can be optimized when the whole pricing curve is known to the consumers. However, this knowledge is subjected to errors stemming from infrequent communications of that curve, which may be changing rapidly as the gap between supply and demand is changing. Too large errors will result in a large deviation in the step size and further incur instability to the energy control algorithm.

Remark 3: The energy control algorithm (8) is dependent on the gradient of the payoff function, which may not be known to the consumers. In this case, we need advanced optimization methods to search for the equilibrium. Several iterative algorithms without gradient are studied in [21]–[23]. These methods can be used to design the energy control algorithm without gradient information.

V. ENERGY CONSUMPTION CONTROL FOR CONSUMERS WITH HVAC SYSTEMS

In this section, the energy control strategy is applied to the consumers with HVAC systems. Before giving the control

algorithm, we first formulate the cost and the price based on the conditions in Theorem 1.

A. Load Curtailment Cost

For consumers with HVAC systems, the cost of changing temperature settings is defined by the Taguchi loss function

$$V_i\big(T_i^{\mathrm{in}}(k)\big) = \theta\big(T_i^{\mathrm{in}}(k) - \hat{T}_i^{\mathrm{in}}(k)\big)^2, \quad i \in \mathcal{N} \qquad (11)$$

where θ is the cost coefficient, and $\hat{T}_i^{\mathrm{in}}(k)$ and $T_i^{\mathrm{in}}(k)$ denote the target indoor temperature and the actual indoor temperature in time slot k, respectively. The indoor temperature of consumer i evolves according to the following linear dynamics [24], [25]:

$$T_i^{\mathrm{in}}(k) = T_i^{\mathrm{in}}(k-1) + \beta\big(T_i^{\mathrm{out}}(k) - T_i^{\mathrm{in}}(k-1)\big) + \gamma l_i^k \quad (12)$$

where β and γ specify the thermal characteristics of the HVAC system and the operating environment, T_i^{out} denotes the outdoor temperature, $\beta(T_i^{\mathrm{out}}(k) - T_i^{\mathrm{in}}(k-1))$ models the heat transfer, and γl_i^k models the energy–heat transformation of HVAC: $\beta > 0$ if HVAC is a heater and $\beta < 0$ if HVAC is a cooler. Assuming consumer i requires $\hat{l}_i^k$ kWh energy to maintain the target indoor temperature, we have

$$\hat{T}_i^{\mathrm{in}}(k) = T_i^{\mathrm{in}}(k-1) + \beta\big(T_i^{\mathrm{out}}(k) - T_i^{\mathrm{in}}(k-1)\big) + \gamma \hat{l}_i^k. \quad (13)$$

Substituting (12) and (13) into (11), we omit k and obtain the load curtailment cost

$$V_i(l_i) = \theta\gamma^2(l_i - \hat{l}_i)^2, \quad i \in \mathcal{N}. \qquad (14)$$

B. Real-Time Pricing

Recalling the pricing condition obtained in Theorem 1, the pricing function should be linear and rotational symmetric. Then, we formulate the following pricing function:

$$p(\boldsymbol{l}) = \lambda \sum_{i \in \mathcal{N}} l_i + p_0 \qquad (15)$$

where λ is a positive parameter to implement elastic pricing and p_0 is a basic price for unit energy consumption. Following (7), we have:

$$\lambda \le \frac{2\theta\gamma^2}{N-3}, \quad \text{for} \quad N > 3. \qquad (16)$$

It is easy to see that the pricing condition is satisfied for $N \le 3$. The role of the electricity price is similar to the lever principle in economics. Specifically, the energy provider will increase the price to remove the peak load and decrease the price to fill the valley load, which can be implemented by regulating the parameter λ. Substituting (14) and (15) into the payoff function $U_i(\boldsymbol{l})$, we have

$$U_i(\boldsymbol{l}) = -\theta\gamma^2(l_i - \hat{l}_i)^2 - \left(\lambda \sum_{i \in \mathcal{N}} l_i + p_0\right)l_i, \quad i \in \mathcal{N}. \quad (17)$$

Next, we will give the method for setting the pricing parameter λ to match supply with demand at the Nash equilibrium.

Theorem 3: The matching between supply and demand is achieved at the Nash equilibrium if the pricing parameter λ is set to

$$\lambda^* = \frac{2\theta\gamma^2\big(\sum_{i \in \mathcal{N}} \hat{l}_i - L\big) - Np_0}{(N+1)L} \qquad (18)$$

where L is the energy supply.

Proof: The proof can be found in Appendix C.   □

Substituting (18) into (15), we obtain the pricing function

$$p(\boldsymbol{l}) = \left(\frac{2\theta\gamma^2\big(\sum_{i \in \mathcal{N}} \hat{l}_i - L\big) - Np_0}{(1+N)L}\right) \sum_{i \in \mathcal{N}} l_i + p_0 \quad (19)$$

with which the balance of supply and demand is achieved.

C. Control Algorithm Implementation

For consumer i with the HVAC system, the energy control algorithm is denoted as

$$l_i(m+1) = \left[l_i(m) + \alpha_i\big(2\theta\gamma^2(\hat{l}_i - l_i) - \lambda l_i - p(\boldsymbol{l})\big)\right]_{l_i^{\min}}^{l_i^{\max}} \quad (20)$$

where $p(\boldsymbol{l})$ is assumed to be known to the consumers. Then, Algorithm (20) can be implemented in a distributed fashion because each consumer does not need the information of the others. In practice, the number of consumers is very large. It is desirable to analyze the case that the number of consumers approaches to infinity. We first give the limit of the pricing parameter as

$$
\begin{aligned}
\lambda^\infty &= \lim_{N \to \infty} \frac{2\theta\gamma^2\big(\sum_{i \in \mathcal{N}} \hat{l}_i - L\big) - Np_0}{(1+N)L} \\
&= \lim_{N \to \infty} \frac{2\theta\gamma^2 \sum_{i \in \mathcal{N}} \hat{l}}{(1+N)L} - \frac{2\theta\gamma^2}{1+N} - \frac{Np_0}{(1+N)L} \\
&\approx \frac{2\theta\gamma^2 \mu - p_0}{L}
\end{aligned}
\qquad (21)
$$

where $\mu = \sum_{i \in \mathcal{N}} \hat{l}/N$ is the average demand of consumers. The energy provider can estimate the average demand of consumers from the historical data. Substituting (21) into (15), we obtain the limit of the electricity price

$$p^\infty = \lim_{N \to \infty} p(\boldsymbol{l}) \approx \frac{2\theta\gamma^2 \mu - p_0}{L} \sum_{i \in \mathcal{N}} l_i + p_0. \qquad (22)$$

The results in (21) and (22) show that the energy provider can set the price approximately when the number of consumers is sufficiently large. In this case, the energy control algorithm can be implemented with low communication overhead, because the energy provider does not need to acquire the individual parameters of the consumers.

D. Error Analysis

The electricity price will deviate from the optimum when the pricing parameter is set to (21). This will further result

TABLE I

TABLE I

RESPONSE PERFORMANCE

	Load curtailment(kWh)	Daily load (kWh)	Daily payments($)	Average price($/kWh)	PAR	Ψ
RTP	1.98×10^4	2.89×10^5	5.41×10^4	0.19	1.48	0.10
NP	1.78×10^4	2.99×10^5	2.99×10^4	0.10	1.59	0

in some errors in the matching between supply and demand. Combining with (18), we obtain the matching errors

$$e_l = \left| \sum_{i \in \mathcal{N}} l_i - L \right|$$
$$= \left| \frac{(2\theta\gamma^2 \sum_{i \in \mathcal{N}} \hat{l}_i - Np_0)(N+1)e_\lambda}{\Gamma^2 + e_\lambda(N+1)\Gamma} \right| \quad (23)$$

where

$$\Gamma = 2\theta\gamma^2 + \lambda^*(N+1) \quad (24)$$

and e_λ is the estimation error of the pricing parameter λ

$$e_\lambda = |\lambda^\infty - \lambda^*|$$
$$= \left| \frac{2\theta\gamma^2\mu - p_0}{L} - \frac{2\theta\gamma^2(\sum_{i \in \mathcal{N}} \hat{l}_i - L) - Np_0}{(1+N)L} \right|$$
$$= \left| \frac{2\theta\gamma^2((1+N)\mu - \sum_{i \in \mathcal{N}} \hat{l}_i + L) - p_0}{(1+N)L} \right|. \quad (25)$$

Remark 4: In practice, the energy provider first estimates the average demand of consumers based on the historical data and then publishes the electricity price to the consumers. Each smart meter calculates the energy consumption plan of the corresponding consumer by the energy control Algorithm (20) according to the published price and then sends it back to the energy provider. Then, the energy provider will publish a new price to the consumers and the smart meter will send back a new energy consumption plan to the provider. The iterations end until all of the consumers converge to the Nash equilibrium.

VI. NUMERICAL RESULTS

In the simulations, the entire time cycle is divided into 24 time slots representing the 24 h in a day. We evaluate the proposed energy control strategy in an energy system with 10^4 consumers. The cost coefficient θ and the heat transfer parameter γ are normalized to one. The basic price p_0 is set to 0.1$/kWh. We assume that the actual load of consumers obey normal distribution, i.e., $l_i \sim N(\mu, \sigma_i^2)$, where the average demand μ is obtained from [26] and the standard deviation σ_i is assumed to be 0.05 kWh.

The total load with RTP and normal pricing (NP) strategies are shown in Fig. 3. We see that the peak load is reduced to the limited supply with the RTP-based energy control strategy at the peak time. In Table I, we compare the amount of load curtailment, the daily load, the daily payments, the average price, the peak-to-average ratio (PAR), and the response as percent of normal day loads (Ψ) for the two pricing strategies.
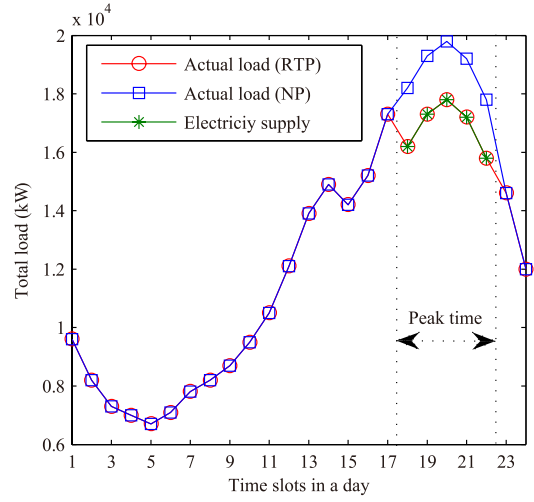

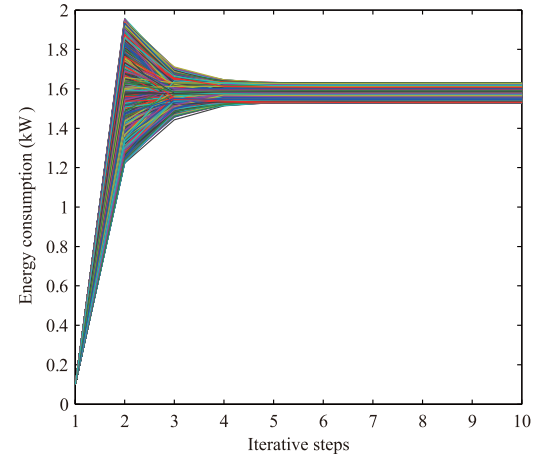Fig. 3. Load curtailment and matching with RTP.

Fig. 4. Convergence of the energy control algorithm (convergence is defined within 10^{-3} of the equilibrium).

We see that the peak load, the daily load, and the PAR are reduced with RTP feedback, whereas the daily payments and the average price increase a lot. Next, we study the convergence of the energy control algorithm in one time slot during the peak time. Assuming the step size errors are 10%, we show the convergence of the energy control algorithm in Fig. 4 and the changing of the electricity price during the iterations in Fig. 5. The energy consumption converges within five steps, and the price converges with two steps because the total load is almost same after two iterations. To see the impact of step size errors on the convergence speed, we give the simulation results of the iterative steps to reach convergence versus the step size errors in Fig. 6. We find that the iterative steps to reach convergence increase with the step size errors when the errors are bounded in [−90%, 90%], whereas the energy control algorithm cannot converge to the equilibrium when the errors are larger than 90%. We also give the electricity price versus the number of consumers in Fig. 7. It is concluded that the electricity price is almost constant with the number of consumers. This shows that the price will not change a lot when some of the consumers enter or quit the grid.
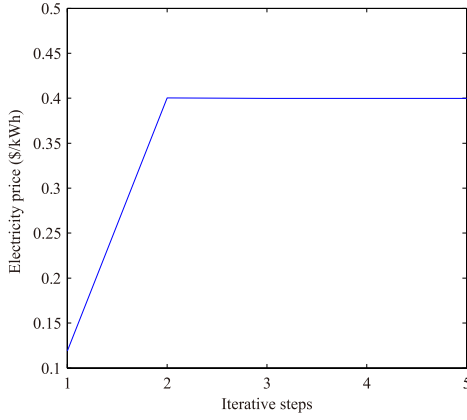
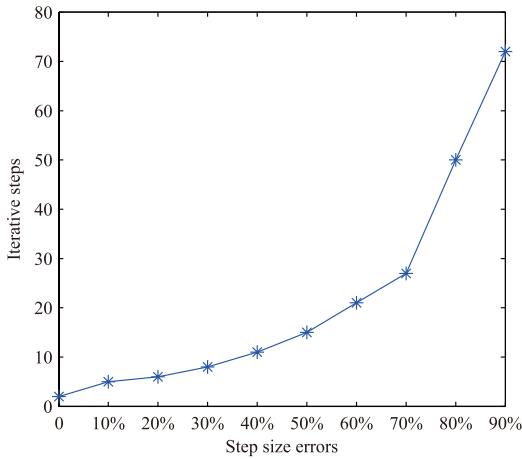Fig. 5.   Changing of the electricity price with iterations.

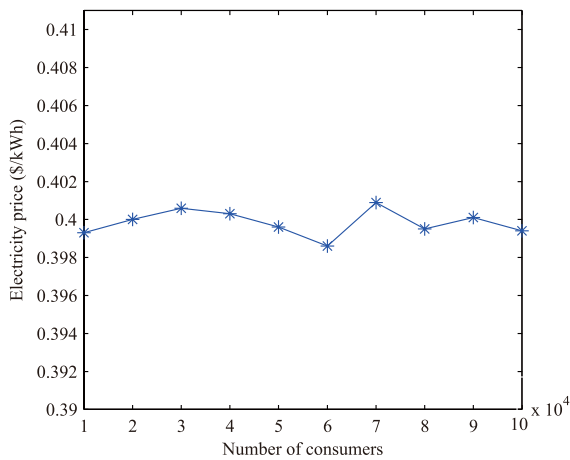Fig. 6.   Iterative steps to reach convergence versus the step size errors.

Fig. 7.   Electricity price versus the number of consumers.

VII. CONCLUSION

In this brief, we propose a distributed energy control strategy with RTP feedback. We find the pricing condition to ensure a unique equilibrium in demand response and the convergence condition of an energy control algorithm. The results are further applied to the consumers with HVAC systems. We find that the energy control strategy with RTP feedback

can remove the peak load and match supply with demand, and the convergence speed of the energy control algorithm decreases with the step size errors. This brief only gives a sufficient pricing condition to ensure a unique equilibrium. It is more meaningful to find the necessary pricing condition or more relaxed conditions to ensure the unique equilibrium. In addition, the energy control strategy is focused on the peak load curtailment within the time slot. However, it is more challenging to consider the peak load shifting across different time slots.

APPENDIX A
PROOF OF THEOREM 1

Proof: Given the strategy space defined by (1), S_i is a nonempty, convex, and compact subset of the Euclidean space $\Re^N$. It is straightforward to see that the payoff function $U_i(l)$ is continuous in l. Taking the second derivative of $U_i(l)$ with respect to l_i, we have

$$\frac{\partial^2 U_i(l)}{\partial l_i^2} = -\frac{d^2 V_i(l_i)}{dl_i^2} - 2\frac{\partial p(l)}{\partial l_i}, \quad i \in \mathcal{N} \qquad (26)$$

where the terms containing the second derivative of the pricing function are omitted because of $\partial^2 p(l)/\partial l_i^2 = 0$ for the linear pricing function. Combining with (7), it is sufficient to obtain $\partial^2 U_i(l)/\partial l_i^2 \leq 0$, i.e., the utility function is a quasi-concave function [20]. Following Lemma 1, the noncooperative game has Nash equilibrium. In general, the noncooperative game may have more than one Nash equilibrium, some of which are not efficient solutions for the game. Next, we will prove the uniqueness of the Nash equilibrium. First, we denote the Jacobian matrix of $dU_i(l)/dl_i$ as

$$J = \begin{bmatrix} \frac{\partial^2 U_1(l)}{\partial l_1^2} & \frac{\partial^2 U_1(l)}{\partial l_1 \partial l_2} & \cdots & \frac{\partial^2 U_1(l)}{\partial l_1 \partial l_N} \\ \frac{\partial^2 U_2(l)}{\partial l_2 l_1} & \frac{\partial^2 U_2(l)}{\partial l_2^2} & \cdots & \frac{\partial^2 U_2(l)}{\partial l_2 \partial l_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 U_N(l)}{\partial l_N \partial l_1} & \frac{\partial^2 U_N(l)}{\partial l_N \partial l_2} & \cdots & \frac{\partial^2 U_N(l)}{\partial l_N^2} \end{bmatrix} \qquad (27)$$

where the diagonal elements are denoted by (26) and the other elements are denoted as

$$\frac{\partial^2 U_i(l)}{\partial l_i \partial l_j} = -\frac{\partial p(l)}{\partial l_j}, \quad i, j \in \mathcal{N}, \; i \neq j. \qquad (28)$$

We construct a symmetric matrix $H = J + J^T$, where

$$H_{i,i} = -2\frac{d^2 V_i(l_i)}{dl_i^2} - 4\frac{\partial p(l)}{\partial l_i}, \quad i \in \mathcal{N} \qquad (29)$$

and

$$H_{i,j} = -\frac{\partial p(l)}{\partial l_j} - \frac{\partial p(l)}{\partial l_i}, \quad i, j \in \mathcal{N}, \; i \neq j. \qquad (30)$$

Combining with the rotational symmetry of the pricing function, we have

$$\frac{\partial p(l)}{\partial l_j} = \frac{\partial p(l)}{\partial l_i}, \quad i, j \in \mathcal{N}, \; i \neq j. \qquad (31)$$

Then, $H_{i,j}$ can be reduced to

$$H_{i,j} = -2\frac{\partial p(\boldsymbol{l})}{\partial l_i}, \quad i, j \in \mathcal{N}, \; i \neq j. \tag{32}$$

From (7), we obtain $H_{i,i} < 0$. Combining with (7), it is sufficient to show that H is strictly diagonally dominant

$$|H_{i,i}| \geq \sum_{j \neq i, j \in \mathcal{N}} |H_{i,j}|, \; |H_{i,i}| \geq \sum_{j \neq i, j \in \mathcal{N}} |H_{j,i}| \; \forall i \in \mathcal{N}. \tag{33}$$

Following Gershgorin's theorem [27], all the eigenvalues are negative, and H is a negative definite matrix. According to the Rosen's result in [28], the Nash equilibrium is unique. $\square$

APPENDIX B
PROOF OF THEOREM 2

Proof: The proof follows the similar analysis in [29]. First, we define a mapping $\Phi_i(\tau) : [0, 1] \rightarrow \Re$ as

$$\Phi_i(\tau) = \tau l_i + (1 - \tau)l_i^* + \alpha_i h_i(\tau \boldsymbol{l} + (1 - \tau)\boldsymbol{l}^*), \; i \in \mathcal{N} \tag{34}$$

where $h_i(\cdot)$ is defined by (9). We denote

$$G_i(\boldsymbol{l}) = l_i + \alpha_i \left(-\frac{dV_i(l_i)}{dl_i} - \frac{\partial p(\boldsymbol{l})}{\partial l_i} l_i - p(\boldsymbol{l}) \right), \; i \in \mathcal{N}. \tag{35}$$

Combining with (8), (9), and (34), we obtain

$$\begin{aligned}|l_i(m + 1) - l_i^*| &\leq |G_i(\boldsymbol{l}) - l_i^*| \\ &= |\Phi_i(1) - \Phi_i(0)| \\ &= \left| \int_0^1 \frac{d\Phi_i(\tau)}{d\tau} d\tau \right| \\ &\leq \int_0^1 \left| \frac{d\Phi_i(\tau)}{d\tau} \right| d\tau \\ &\leq \max_{0 \leq \tau \leq 1} \left| \frac{d\Phi_i(\tau)}{d\tau} \right|, \; i \in \mathcal{N}\end{aligned} \tag{36}$$

where the first inequality is because $|[G_i(\boldsymbol{l})]_{l_i^{\min}}^{l_i^{\max}} - l_i^*| \leq |G_i(\boldsymbol{l}) - l_i^*|$ for all $i \in \mathcal{N}$ when $l_i^* \in [l_i^{\min}, l_i^{\max}]$. Let $\tilde{\boldsymbol{l}} = \tau \boldsymbol{l} + (1 - \tau)\boldsymbol{l}^*$, and then $|d\Phi_i(\tau)/d\tau|$ can be further bounded by

$$\begin{aligned}&\left| \frac{d\Phi_i(\tau)}{d\tau} \right| \\ &= \left| (l_i - l_i^*)(1 + \alpha_i \frac{\partial U_i^2(\tilde{\boldsymbol{l}})}{\partial \tilde{l}_i^2}) + \alpha_i \sum_{j \neq i} \frac{\partial U_i^2(\tilde{\boldsymbol{l}})}{\partial \tilde{l}_i \partial \tilde{l}_j}(l_j - l_j^*) \right| \\ &\leq \left| 1 + \alpha_i(\frac{\partial U_i^2(\tilde{\boldsymbol{l}})}{\partial \tilde{l}_i^2} + \sum_{j \neq i} \frac{\partial U_i^2(\tilde{\boldsymbol{l}})}{\partial \tilde{l}_i \partial \tilde{l}_j}) \right| \cdot \|\boldsymbol{l} - \boldsymbol{l}^*\|_\infty, \; i \in \mathcal{N}\end{aligned} \tag{37}$$

where $\|\boldsymbol{l}\|_\infty = \max_i |l_i|$. To guarantee

$$0 < \left| 1 + \alpha_i(\frac{\partial U_i^2(\boldsymbol{l})}{\partial l_i^2} + \sum_{j \neq i} \frac{\partial U_i^2(\boldsymbol{l})}{\partial l_i \partial l_j}) \right| < 1, \; i \in \mathcal{N} \tag{38}$$

we need

$$-1 < 1 + \alpha_i \left(\frac{\partial U_i^2(\boldsymbol{l})}{\partial l_i^2} + \sum_{j \neq i} \frac{\partial U_i^2(\boldsymbol{l})}{\partial l_i \partial l_j} \right) < 1, \; i \in \mathcal{N}. \tag{39}$$

Then, we obtain

$$\alpha_i < \frac{-2}{\frac{\partial U_i^2(\boldsymbol{l})}{\partial l_i^2} + \sum_{j \neq i} \frac{\partial U_i^2(\boldsymbol{l})}{\partial l_i \partial l_j}}, \; i \in \mathcal{N}. \tag{40}$$

Substitute (26) and (28) into (40), we obtain (10) with the condition that the pricing function is a linear function, i.e., the second derivative is zero. Then, we have

$$\max_{\tau \in [0,1]} \left| \frac{d\Phi_i(\tau)}{d\tau} \right| \leq \Psi_i \|\boldsymbol{l} - \boldsymbol{l}^*\|_\infty, \; i \in \mathcal{N} \tag{41}$$

where

$$0 < \Psi_i = \left| 1 + \alpha_i \left(\frac{\partial U_i^2(\boldsymbol{l})}{\partial l_i^2} + \sum_{j \neq i} \frac{\partial U_i^2(\boldsymbol{l})}{\partial l_i \partial l_j} \right) \right| < 1, \; i \in \mathcal{N}. \tag{42}$$

Combining (36) and (41), it is proved that the energy control algorithm converges to the Nash equilibrium as $m \rightarrow \infty$. $\square$

APPENDIX C
PROOF OF THEOREM 3

Proof: Let the first derivative of $U_i(\boldsymbol{l})$ with respect to l_i equal to zero, we obtain

$$-2\theta\gamma^2(l_i - \hat{l}_i) - \lambda l_i - \lambda \sum_{i \in \mathcal{N}} l_i - p_0 = 0, \; i \in \mathcal{N}. \tag{43}$$

Adding (43) from one to N, we have

$$\begin{aligned}-2\theta\gamma^2 \sum_{i \in \mathcal{N}} l_i + 2\theta\gamma^2 \sum_{i \in \mathcal{N}} \hat{l}_i - \lambda \\ \sum_{i \in \mathcal{N}} l_i - \lambda N \sum_{i \in \mathcal{N}} l_i - Np_0 = 0\end{aligned} \tag{44}$$

from which, we obtain the total load

$$\sum_{i \in \mathcal{N}} l_i = \frac{2\theta\gamma^2 \sum_{i \in \mathcal{N}} \hat{l}_i - Np_0}{2\theta\gamma^2 + \lambda(N + 1)}. \tag{45}$$

To match supply with demand at the equilibrium, we need

$$\sum_{i \in \mathcal{N}} l_i = L. \tag{46}$$

Substituting (46) into (45), we obtain λ^* denoted by (18). $\square$

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