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Publication Date

1963-03-07

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Berkeley, California

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UCRL-10716

UNIVERSITY OF CALIFORNIA

Lawrence Radiation Laboratory
Berkeley, California

Contract No. W-7405-eng-48

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ABSTRACT

The uniqueness condition introduced by Tadjbakhsh & Keller in their method for finding the higher order effects for two-dimensional finite-amplitude standing gravity waves is considered. The condition, which excludes certain fluid depths, is shown to be unacceptable because it is impossible to satisfy in practice. A revision of the condition is suggested which retains the validity of the other aspects of the method. The revision also pertains to the forms of the uniqueness condition presented by other authors in extending the method to the solution of more complex standing wave problems.

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I. INTRODUCTION

In a recent paper, Tadjbakhsh & Keller (1960) introduced a useful mathematical method for finding the higher order effects for two-dimensional standing gravity waves of finite amplitude in an inviscid incompressible fluid of arbitrary depth. Experimental results of Fultz (1962) are essentially in agreement with the theoretical results obtained by Tadjbakhsh & Keller. Their method has been extended by Verma & Keller (1962) to solve the same problem for a three-dimensional wave and by Concus (1962) to solve the problem for a two-dimensional wave including the effects of surface tension. The method utilizes a uniqueness condition which requires that the fluid depth satisfy a certain relationship in order that the solution obtained be unique and that certain resonances be avoided. A similar uniqueness condition also appears in the general discussion of finite-amplitude waves presented by Moiseyev (1958).

The uniqueness condition is investigated here, and it is shown that the condition is an inappropriate one because it is impossible to satisfy in practice. A revision of the condition is suggested along the lines of the one used by Penney & Price (1952) and later by Mack (1962) in their studies of finite-amplitude standing waves. The revision does not limit the essential validity of the method nor does it alter the form of the solutions previously obtained by the method.

2. UNIQUENESS CONDITION

The simplest form of the uniqueness condition is the one for two-dimensional waves without surface tension, as given by Tadjbakhsh & Keller as (10) in their paper. It is

$$\frac{n \tanh nh}{\tanh h} \neq (\text{integer})^2 \quad (n = 2, 3, \dots), \quad (1)$$

where h is the mean depth of the fluid. The role that the condition plays in the development of the solution can be studied by considering the velocity potential ϕ .

The general solution for the velocity potential subject to the fixed-wall boundary condition, which is the solution to their (1) and (4), can be written as

$$\phi(x, y, t) = \sum_{n=0}^{\infty} A_n(t) \cos nx \cosh n(y+h),$$

where the $A_n(t)$ are function of time to be determined from the other boundary conditions and from the periodicity requirement that

$$A_n(t + 2\pi) = A_n(t). \quad (2)$$

An expansion for ϕ in terms of a small parameter ϵ about $\epsilon = 0$ is sought, where ϵ is a measure of the amplitude of the linearized wave motion. The expansion is given by Tadjbakhsh and Keller as (41), which is

$$\epsilon\phi = \epsilon\phi^0 + \epsilon^2\phi^1 + 1/2\epsilon^3\phi^2 + O(\epsilon^4),$$

where the superscripts on ϕ denote the orders of the expansion terms. In essence, such an expansion corresponds to an expansion of the $A_n(t)$ as

$$\epsilon A_n(t) = \epsilon A_n^0(t) + \epsilon^2 A_n^1(t) + 1/2\epsilon^3 A_n^2(t) + O(\epsilon^4), \quad (3)$$

where, again, the superscripts on $A_n(t)$ denote the orders of the expansion terms and not exponents.

Each stage in the perturbation solution consists of finding and solving the linear differential equations satisfied by the appropriate order of $A_n(t)$.

For the j th order these equations are of the form

$$\frac{d^2}{dt^2} A_n^j(t) + \frac{n \tanh nh}{\omega_0^2} A_n^j(t) = f_n^j(t), \quad (n = 0, 1, 2, \dots), \quad (4)$$

where ω_0 is the frequency of the linearized or zero order wave and the $f_n^j(t)$ are functions of time calculated from the lower order solutions. The relationship

$$f_n^j(t) \equiv 0 \quad \text{for } n \geq j + 2$$

always holds. (See Tadjbakhsh & Keller's (21) through (24) for the form which the equations take for $j = 1$.)

For $j = 0$ the equations are homogeneous of the form

$$\frac{d^2}{dt^2} A_n^0(t) + \frac{n \tanh nh}{\omega_0^2} A_n^0(t) = 0, \quad (n = 0, 1, 2, \dots). \quad (5)$$

Of the zero-order waves which satisfy (5) and the three other required conditions-- which are the periodicity condition (2) and the phase and amplitude conditions given by Tadjbakhsh & Keller as (7) and (8) in their paper -- the one consisting purely of the fundamental mode is the one of interest. For this wave

$$A_1^0(t) = A \cos t, \quad A_n^0(t) \equiv 0 \quad \text{for } n \neq 1, \quad (6)$$

and $\omega_0 = (\tanh h)^{1/2}$,

where A is determined by the amplitude condition.

Once ω_0 is determined by (6), the periodicity condition can then be satisfied only by those solutions of (5) for $n \geq 2$ that oscillate with frequencies that are integral multiples of the fundamental frequency. Arbitrary amounts of these higher harmonics may be added to $A \cos t$, and the solution for the resulting wave will still satisfy the phase and amplitude conditions. To make the pure fundamental the only solution of the zero order problem, the uniqueness condition (1) is imposed, thus ensuring that no nonzero solution of (5) for $n \geq 2$ can also satisfy (2).

The uniqueness condition is used in a similar manner for the higher order equations to ensure that the only solutions to the homogeneous equations of (4) for $n \geq 2$ are

$$A_n^j(t) = 0,$$

thus making the perturbation solution unique at each stage. For the three-dimensional problem and the two-dimensional problem including surface tension, the uniqueness conditions are slightly more complex, but they operate in essentially the same manner to make the solution unique.

The uniqueness condition also performs another function which is brought out more clearly in the low-order stage of the solution for the two-dimensional problem including surface tension. The situation corresponds to $f_n^j(t)$ in (4) being of the form

$$f_n^j(t) = B_1 \sin mt + B_2 \cos mt, \quad (7)$$

where m is an integer that is even if n is even and odd if n is odd, and

$$\frac{n \tanh nh}{\tanh h} = m^2. \quad (8)$$

In this case a resonance condition would be present and the solution obtained for $A_n^j(t)$ would contain secular terms unbounded in time and would not satisfy

the periodicity condition. In general, $f_{22}^j(t)$ consists of terms of the form (7); however, the uniqueness condition, in prohibiting (8) from ever being satisfied, ensures that the corresponding resonant conditions are excluded.

3. UNSUITABILITY OF UNIQUENESS CONDITION

It is now shown that the uniqueness condition is not mathematically acceptable. Consider the uniqueness condition for the two-dimensional problem without surface tension given by (1). For any $n > 1$ the ratio $\tanh nh / \tanh h$ is continuous and decreases monotonically from n to 1 as h increases from 0 to ∞ . Thus, for each n there are approximately $n - \sqrt{n}$ distinct values of the fluid depth excluded by (1). The totality of all such excluded depths obtained by letting n run through its range of positive integers will form a denumerably infinite set of points ranging over the positive real line $(0, \infty)$.

It is next shown that this set is densely distributed over $(0, \infty)$. Let h_1 and h_2 be any two values of h such that $0 < h_1 < h_2$, and consider the interval (h_1, h_2) . As h increases through all the values in (h_1, h_2) the ratio $n \tanh nh / \tanh h$ decreases through all the values in the interval

$$\left(\frac{n \tanh nh_2}{\tanh h_2}, \frac{n \tanh nh_1}{\tanh h_1} \right). \quad (9)$$

Consider the interval

$$\left(\frac{\tanh nh_2}{\tanh h_2}, \frac{\tanh nh_1}{\tanh h_1} \right). \quad (10)$$

Since $\tanh h_1 < \tanh h_2$ and $\tanh nh_2 < 1$, and since an N can be found such that†

$$\tanh nh_1 > \frac{1}{2} + \frac{1}{2} \frac{\tanh h_1}{\tanh h_2} \quad \text{for all } n \geq N,$$

†The author is indebted to Dr. G. Nooney for suggesting this and other features of the proof.

the interval (10) contains the smaller interval

$$\left(\frac{1}{\tanh h_2}, \frac{1}{2} \left[\frac{1}{\tanh h_1} + \frac{1}{\tanh h_2} \right] \right)$$

for all $n \geq N$. Choose a rational number p/q in this interval. Then (9) contains the rational number np/q . Choose $n = N^2 pq$. Then $n \tanh nh / \tanh h$ takes on the value of the perfect square $N^2 p^2$ for some h in the interval (h_1, h_2) . Hence, any such interval (h_1, h_2) contains at least one value of h excluded by (1).

Because the set of excluded values of h is dense, it is impossible to satisfy (1) in practice. When one measures a fluid depth, he does not do so exactly, but he specifies some small interval in which the depth is known to lie. However, any such interval contains an infinity of points excluded by (1), making it impossible to determine that the measured fluid depth is an allowed one.

In the same way it can be shown that the forms of the uniqueness condition given for three-dimensional waves and two-dimensional waves including surface tension are also impossible to satisfy in practice. One is therefore left no choice but to abandon the uniqueness criterion as it is given and to devise a replacement or revision.

4. DISCUSSION

The fact that (1) cannot be satisfied implies that the fundamental solution is, in actuality, never the unique solution of the posed mathematical problem. There must always be some other independent solutions of (4) and (5) satisfying the periodicity conditions no matter what fluid depth is chosen. One can never exclude them all. However, it is possible to define a procedure for choosing a particular solution and to discard all other solutions which are independent of the one of interest. The possible existence of solutions other than the chosen one is not denied, but they are ignored because they do not affect the particular solution under study.

It is essentially such a procedure that Penney & Price and Mack use in their studies of finite-amplitude waves. In terms of the present problem the appropriate selection procedure for replacing (1) is to require that the solution of the problem, when linearized, reduce to the fundamental linear solution (6). That is, (6) is chosen to be the solution of (5), and all $A_n^0(t)$ are chosen to be zero for $n \neq 1$, even though some of them would not have to be zero to also satisfy all the other conditions. For the higher order equations, i. e. $j \geq 1$ in (4), only those solutions which represent higher order corrections to the fundamental linear solution are selected, and those that are independent of the fundamental linear solution are ignored. This corresponds to taking the solutions of (4) to be only the particular integrals arising from nonzero $f_n^j(t)$ and taking zero for the solutions of the homogeneous equations.

Such a procedure selects the desired solution of the problem in a unique way and the solution is, in fact, the same one that would be obtained were the uniqueness condition (1) used. The above procedure has the advantage, moreover of being mathematically more acceptable in that it does not attempt to unjustifiably exclude other solutions from also being possible.

The procedure satisfactorily handles the problem of uniqueness, but the one of resonance, as represented by (7) and (8), still remains to be considered. In a condition of resonance, a higher order term in (3), for example $\epsilon^{j+1} A_n^j(t)/j!$, becomes so large as to be of comparable magnitude with the fundamental $\epsilon A_1^0(t)$. In such a case, the derivation of the perturbation equations would have to be modified since the tacitly assumed condition that

$$\epsilon^{-k} [A_n(t) - \sum_{j=0}^k \epsilon^j A_n^j(t)] = O(\epsilon) \text{ for all } k$$

would be violated. (For an example of the modifications required for one such resonant term see Mack.) The previous analysis has shown, moreover, that

there are infinitely many such resonant terms with which one must contend. Thus it seems that there are always too many high order terms which cannot be neglected in the development, because they are of comparable magnitude with the low-order terms. Such considerations make it appear that the expansion procedure is not a suitable one for finding the desired periodic motion.

A solution to this difficulty lies in the realization that the problem is formulated in terms of the laminar inviscid theory. At the lower frequencies one is justified in using this theory for fluids of low viscosity. For higher frequencies (larger n), however, the theory becomes invalid, and the appearance of the high frequency waves will, in actuality, be inhibited or at least their amplitudes diminished in magnitude. One need therefore consider the possibility of a higher order term being of comparable magnitude with the fundamental term only for the smaller values of n . For larger n , the physical situation is no longer described by the mathematical formulation, and the high frequency waves will not be present in the amounts predicted by the theory.

5. CONCLUSION

It is possible to discard the unsatisfactory uniqueness condition given by (1) for the two-dimensional problem without surface tension, or by the generalization of (1) for the more complex cases, and replace it by a usable one based partially on physical reasoning. The solution given by the method will, as a result, be unchanged. Equation (1) or its generalisation should be utilized, however, for the smaller values of n in order to predict the depths for which there may exist a coupling for which a higher order mode is of comparable magnitude with the linearized fundamental mode.

This work was carried out under the auspices of the U. S. Atomic Energy Commission.

REFERENCES

- Concus, Paul 1962, Standing capillary-gravity waves of finite amplitude, J. Fluid Mech. 14, 568-576.
- Fultz, Dave 1962, An experimental note on finite-amplitude standing gravity waves, J. Fluid Mech. 13, 193-212.
- Mack, Lawrence R. 1962, Periodic finite-amplitude, axisymmetric gravity waves, J. Geophys. Res. 67, 829-843.
- Moiseyev, M. N. 1959, On the theory of nonlinear vibrations of a liquid of finite volume, J. Appl. Math. Mech. 22, 860-872. (Translated from Prikl. Mat. Meh. 22, 612-21).
- Penney, W. G. & Price, A. T. 1952, Some gravity wave problems in the motion of perfect liquids. Part II. Finite periodic stationary gravity waves in a perfect liquid, Phil. Trans. Roy. Soc. London, Ser. A 244, 254-84.
- Tadjbakhsh, I. & Keller, J. B. 1960, Standing surface waves of finite amplitude, J. Fluid Mech. 8, 442-451.
- Verma, G. R. & Keller, J. B. 1962, Three-dimensional standing surface waves of finite amplitude, Phys. Fluids 5, 52-56.

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