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ABSTRACT

The equilibrium and quasi-equilibrium properties of a system of interacting bosons are studied from a microscopic point of view. For equilibrium, the model of Bogolyubov is generalized to finite temperature by using the grand partition function. The thermodynamic properties and the pair-correlation function are calculated. The statistical mechanics for moving systems is then developed and applied to the problem of a rotating fluid. For quasi-equilibrium, general transport equations are derived from first principles, independent of statistics and model. For the Bogolyubov model, the familiar two-fluid hydrodynamics is then derived, leading to the phenomena of first and second sound.

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I. INTRODUCTION

Although useful phenomenological schemes have been developed to describe the properties of liquid helium at very low temperatures, no quantitative microscopic theory exists at present. Nevertheless, significant progress has been made towards solving, from first principles, the model problem of a low-density system of interacting bosons in equilibrium. The first step was made by Bogolyubov in 1947.[1] More recently, Lee and Yang and their collaborators have extended this work to the case of hard-sphere interactions.[2] The attractive feature of the low-density theory is that it unifies two ideas which are probably essential for the understanding of liquid helium: the condensation phenomenon associated with the ideal Bose-Einstein gas,[3] and the collective excitations which are the basis of the two-fluid model.[4]

The first objective of this paper is the generalization of Bogolyubov's theory to finite temperature. The method of the grand canonical ensemble is used for this purpose. In Section II, the ideas of Bogolyubov and of Lee and Yang are applied to the basic operator in this theory, $H - \mu N$, where H, μ, and N are the Hamiltonian, the chemical potential, and the operator for the total number of particles, respectively. Section III

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is devoted to a full discussion of the equilibrium statistical mechanics of such a system, with particular emphasis on the evaluation of the chemical potential, μ , and the mean occupation of the unperturbed ground state, N_0 . In Section IV, the pair correlation function is calculated for finite temperature.

The equilibrium statistical mechanics for moving systems is described in Section V, and natural definitions of the normal and superfluid velocities are made. The normal density may be defined either by the effective mass for drift or for rotation, as was originally asserted by Landau.[4] Our result differs from Landau's work in that it is based on a microscopic calculation using Bogolyubov's model.

A general transport theory is developed in Section VI, starting only from the equations of motion for the density matrix and the statistical average of any operator. These quantum hydrodynamic equations, derived from first principles, are of the same form as in classical physics. The only essential assumption is that of local equilibrium, i.e., equilibrium is established in regions whose linear dimensions are small on a macroscopic scale. Application to liquid helium is then made by using the Bogolyubov theory for the various equilibrium quantities that appear in the transport theory. In Section VII it is shown how the usual two-fluid hydrodynamic equations may be derived from the general transport theory of Section VI and the definitions of Section V. The paper is concluded with a discussion of first and second sound.

II. MANY-BOSON HAMILTONIAN

We consider $\bar{N}$ bosons interacting inside a volume $\mathcal{V}$, in the limit that the particle density $\bar{n} \equiv \bar{N}/\mathcal{V}$ remains finite as $\bar{N}$ and $\mathcal{V}$ become infinite. The pertinent operator in the density matrix for the grand canonical ensemble is

$$H - \mu N = \sum_{\underset{\sim}{k}} E_k \, a_{\underset{\sim}{k}}^\dagger \, a_{\underset{\sim}{k}} + \frac{1}{2} \mathcal{V}^{-1} \sum_{\underset{\sim}{k}\,\underset{\sim}{\ell}\,\underset{\sim}{m}\,\underset{\sim}{n}} v(\underset{\sim}{k} - \underset{\sim}{n}) a_{\underset{\sim}{k}}^\dagger \, a_{\underset{\sim}{\ell}}^\dagger \, a_{\underset{\sim}{m}} \, a_{\underset{\sim}{n}} \, \delta_{\underset{\sim}{k}+\underset{\sim}{\ell},\underset{\sim}{m}+\underset{\sim}{n}} \, . \tag{2.1}$$

The second-quantized operators $a_{\underset{\sim}{k}}$ and $a_{\underset{\sim}{k}}^\dagger$ destroy and create, respectively, free-particle states of momentum $\underset{\sim}{k}$, and satisfy the usual commutation rules for bosons. The quantities H, μ, and N are, respectively, the Hamiltonian, the chemical potential, and the operator for the total number of particles:

$$N = \sum_{\underset{\sim}{k}} N_{\underset{\sim}{k}} = \sum_{\underset{\sim}{k}} a_{\underset{\sim}{k}}^\dagger \, a_{\underset{\sim}{k}} \, . \tag{2.2}$$

Because of the inclusion of μN in Eq. (2.1), E_k is an "effective kinetic energy" for the state $\underset{\sim}{k}$ ($\hbar = 1$):[5]

$$E_k = \frac{k^2}{2m} - \mu \, . \tag{2.3}$$

The bosons are assumed to interact through a central two-body potential $V(r)$, whose Fourier transform is

$$v(\underset{\sim}{k}) = \int d^3r \; e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} V(r) \, . \tag{2.4}$$

An essential simplification, due to Bogolyubov,[1] is to regard the zero-momentum operators a_0, $a_0^\dagger$ as c numbers equal to $N_0^{1/2}$. The parameter N_0 is the occupation number of the free-particle (unperturbed) ground state. Its average value is supposed to be of the same order of

magnitude as $\overline{N}$. The Bogolyubov method consists of keeping in Eq. (2.1) only terms containing at least two zero-momentum operators, and of making the replacement $a_0,\ a_0^\dagger \to N_0^{1/2}$. Thus we have

$$H - \mu N = (H_B - \mu N) + V' \ , \tag{2.5}$$

where

$$H_B - \mu N = -\mu N_0 + \frac{1}{2} N_0^2 \mathcal{V}^{-1} v(0) + \sum_{\underset{\sim}{k}}{}' [f_k \, a_{\underset{\sim}{k}}^+ \, a_{\underset{\sim}{k}} + \frac{1}{2} h_k (a_{\underset{\sim}{k}}^+ \, a_{-\underset{\sim}{k}}^+ + a_{\underset{\sim}{k}} \, a_{-\underset{\sim}{k}})] . \tag{2.6}$$

In Eq. (2.6) we have

$$\begin{aligned} f_k &\equiv E_k + n_0[v(k) + v(0)] \ , \\ \text{and} \quad h_k &\equiv n_0 \, v(k) \ , \\ n_0 &\equiv N_0/\mathcal{V} \ . \end{aligned} \tag{2.7}$$

The "Bogolyubov operator," $H_B - \mu N$, is quadratic in operators referring to excited free-particle states, while V' contains products of three and four such operators; V' will be ignored in this paper.

As is well known,[1,6] the truncated Hamiltonian $H_B - \mu N$ can be diagonalized by a linear transformation with real coefficients that depend only on $k = |\underset{\sim}{k}|$,

$$a_{\underset{\sim}{k}} = u_k \, \alpha_{\underset{\sim}{k}} + v_k \, \alpha_{-\underset{\sim}{k}}^\dagger \tag{2.8}$$

(for $\underset{\sim}{k} \neq 0$). The new operators $\alpha_{\underset{\sim}{k}}$, $\alpha_{\underset{\sim}{k}}^\dagger$ destroy and create excitations or quasiparticles of momentum $\underset{\sim}{k}$.

In order that these operators conform to the boson commutation rules, the transformation coefficients must satisfy the equation

$$u_k^2 - v_k^2 = 1 \quad . \tag{2.9}$$

This type of transformation is possible if $f_k > |h_k|$. It is not difficult to show that the choice of coefficients that diagonalizes $H_B - \mu N$ is

$$\begin{cases} u_k^2 = \frac{1}{2}\left[1 + (f_k/\epsilon_k)\right] \\ v_k^2 = \frac{1}{2}\left[-1 + (f_k/\epsilon_k)\right] \\ 2u_k v_k = -h_k/\epsilon_k \ , \end{cases} \tag{2.10}$$

with

$$\epsilon_k \equiv (f_k^2 - h_k^2)^{1/2} \quad . \tag{2.11}$$

The diagonalized Bogolyubov operator then has the form

$$H_B - \mu N = \epsilon_0(\mu, N_0) + \sum_{\underset{\sim}{k}}{}' \epsilon_k \alpha_{\underset{\sim}{k}}^\dagger \alpha_{\underset{\sim}{k}} - \mu N_0 \ , \tag{2.12}$$

where ϵ_k is the excitation energy of a quasiparticle of momentum $\underset{\sim}{k}$. The "ground-state energy" is

$$\epsilon_0(\mu, N_0) = \frac{1}{2} N_0^2 \, v(0)/\mathcal{V} - \frac{1}{2} \sum_{\underset{\sim}{k}}{}'(f_k - \epsilon_k) \quad . \tag{2.13}$$

We note that the ϵ_k's are functions of the parameters μ and N_0 through f_k and h_k . The volume $\mathcal{V}$ is also a parameter, but it will be held fixed for the present considerations.

The above discussion, which follows Bogolyubov's original paper,[1] is restricted to weak interactions and low densities. Lee and Yang have obtained a similar result for the dilute hard-sphere gas at absolute zero.[2] To obtain their result in our formulation would require a study of the

effects of V', the part of the Hamiltonian that we neglect. In this paper we emphasize the extension of Bogolyubov's method to finite temperature, rather than the improvement of the model for strong interactions.

III. STATISTICAL MECHANICS

The statistical mechanics of the system will be studied with the grand partition function

$$\mathcal{Z} = \mathrm{Tr}[\, e^{-\beta(H - \mu N)}] \quad , \tag{3.1}$$

where Tr indicates the usual trace for the grand ensemble[7] and $\theta = \beta^{-1}$ is the absolute temperature times Boltzmann's constant. The statistical average of any operator F will be indicated by $\overline{F}$ and is given by the expression

$$\mathcal{Z}\overline{F} = \mathrm{Tr}[\, F\, e^{-\beta(H - \mu N)}] \, . \tag{3.2}$$

In this work, the Bogolyubov approximation is used, and the grand partition function is therefore

$$\mathcal{Z} = \mathrm{Tr}[\, e^{-\beta(H_B - \mu N)}] \, . \tag{3.3}$$

According to Section II, $H_B - \mu N$ is diagonal in the quasiparticle representation, i.e., the representation characterized by definite values for the quasiparticle occupation operators $\underset{\sim}{\nu}_k = \underset{\sim}{\alpha}_k^\dagger \underset{\sim}{\alpha}_k$. Therefore, the grand partition function may be written as the sum over quantum states of the entire system which are characterized by the parameter N_0 and the occupation numbers $\underset{\sim}{\nu}_k$:

$$\begin{aligned}\mathcal{Z}(\mu, \beta) &= \sum_{N_0} \exp\{ -\beta[\epsilon_0(\mu, N_0) - \mu N_0] \} \prod_k{}' \sum_{\underset{\sim}{\nu}_k} \exp(-\beta\, \epsilon_k\, \underset{\sim}{\nu}_k) \\ &= \sum_{N_0} \exp[\beta\mu N_0 - \beta\epsilon_0(\mu, N_0)] \prod_k{}' \{1 - \exp[-\beta\epsilon_k(\mu, N_0)] \}^{-1} \, .\end{aligned}$$

Rewriting this result as a single sum over N_0 , we have

$$\mathcal{Z}(\mu, \beta) = \sum_{N_0} e^{\beta\mu N_0} Z(\mu, N_0, \beta) , \tag{3.4}$$

where

$$-\beta^{-1} \ell n\, Z(\mu, N_0, \beta) \equiv \epsilon_0(\mu, N_0) + \beta^{-1} \sum_{\underset{\sim}{k}}{}' \ell n(1 - e^{-\beta\epsilon_k(\mu, N_0)}) . \tag{3.5}$$

The average number of particles may be found by taking the statistical average of Eq. (2.2). When this equation is rewritten in the quasiparticle representation, it becomes

$$N = N_0 + \sum_{\underset{\sim}{k}}{}' [v_k^2 + (u_k^2 + v_k^2)\nu_{\underset{\sim}{k}} + u_k v_k(\alpha_{\underset{\sim}{k}}^\dagger \alpha_{-\underset{\sim}{k}}^\dagger + \alpha_{\underset{\sim}{k}} \alpha_{-\underset{\sim}{k}})] .$$

Terms that are not diagonal in the quasiparticle representation do not contribute to the statistical average, and so $\overline{N}$ is simply

$$\overline{N} = \overline{N}_0 + \sum_{k}{}' [v_k^2 + (u_k^2 + v_k^2)\overline{\nu}_{\underset{\sim}{k}}] ,$$

where

$$\overline{\nu}_{\underset{\sim}{k}} = \frac{1}{\exp(\beta\epsilon_k) - 1} . \tag{3.6}$$

Using Eq. (2.10), we finally find that[8,9]

$$\overline{N} = \overline{N}_0 + \frac{1}{2} \sum_{\underset{\sim}{k}}{}' \left(\frac{f_k}{\epsilon_k} - 1 \right) + \sum_{\underset{\sim}{k}}{}' \frac{f_k}{\epsilon_k} \overline{\nu}_{\underset{\sim}{k}} . \tag{3.7}$$

At absolute zero there are no excitations ($\overline{\nu}_{\underset{\sim}{k}} = 0$) and the last term vanishes. This is not true of the second term. In other words, the interactions cause $\overline{N}$ and $\overline{N}_0$ to be different even at absolute zero.[9]

Because the summand of Eq. (3.4) is a rapidly varying function of N_0 with a sharp maximum at $\langle N_0 \rangle$, say, it will be a good approximation to retain only the largest term in the sum:

$$\mathcal{Z} = e^{\beta\mu\langle N_0\rangle} \, Z(\mu, \langle N_0\rangle, \beta) \quad . \tag{3.8}$$

The condition that this term is the largest is

$$\frac{\partial}{\partial N_0} \left[\beta\mu N_0 + \ell n \, Z(\mu, N_0, \beta)\right] \Big|_{N_0=\langle N_0\rangle} = 0 \quad ,$$

or, if we solve for μ ,

$$\mu = -\frac{\partial}{\partial N_0} \beta^{-1} \ell n \, Z(\mu, N_0, \beta) \Big|_{N_0=\langle N_0\rangle} \quad . \tag{3.9}$$

On the other hand, if we use Eq. (3.2) with $F = N_0$ and Eq. (3.4), the mean value $\overline{N}_0$ of the parameter N_0 is

$$\overline{N}_0 = \mathcal{Z}^{-1} \sum_{N_0} N_0 \, e^{\beta\mu N_0} \, Z(\mu, N_0, \beta) \, .$$

In the present (saddle-point) approximation, characterized by Eq. (3.8), this is

$$\overline{N}_0 = \mathcal{Z}^{-1} \langle N_0\rangle \, e^{\beta\mu\langle N_0\rangle} \, Z(\mu, \langle N_0\rangle, \beta) \, .$$

If we refer to Eq. (3.8) for $\mathcal{Z}$, it follows that the mean value $\overline{N}_0$ is the same as $\langle N_0\rangle$:

$$\overline{N}_0 = \langle N_0\rangle \quad .$$

Let us now evaluate μ , the chemical potential, by using Eqs. (3.9), (3.5), and (2.13):

$$\mu = \overline{n}_0 \, v(0) - \frac{1}{2} \sum_{\underset{\sim}{k}}{}' \frac{\partial}{\partial N_0} (f_k - \epsilon_k)_{N=\overline{N}_0} + \sum_{\underset{\sim}{k}}{}' \nu_k \frac{\partial \epsilon_k}{\partial N_0} \Big|_{N_0=\overline{N}_0} \quad . \tag{3.10}$$

From Eqs. (2.7) and (2.11) we then obtain

$$\mu = \bar{n}_0 v(0) - \frac{1}{2\bar{N}_0} \sum_{\underset{\sim}{k}}' (f_k - \epsilon_k)\left(1 + \frac{E_k}{\epsilon_k}\right) \Bigg|_{N=\bar{N}_0} + \frac{1}{\bar{N}_0} \sum_{\underset{\sim}{k}}' \bar{\nu}_k \left(\epsilon_k - \frac{f_k E_k}{\epsilon_k}\right) \Bigg|_{N=\bar{N}_0} . \tag{3.11}$$

Initially we assume that the first term of Eq. (3.11) is a good approximation for μ. Later we shall investigate the conditions for this to be so.[10] We therefore write

$$\mu \equiv \mu_0 + \Delta\mu , \tag{3.12}$$

where

$$\mu_0 \equiv \bar{n}_0 v(0) .$$

Then $\Delta\mu$ is given approximately by the last two terms of Eq. (3.11) with μ replaced by μ_0.

When μ_0 replaces μ, the expressions for f_k and ϵ_k [Eqs. (2.7) and (2.11)] become[11]

$$f_k = \frac{k^2}{2m} + n_0 v(k) \tag{3.13a}$$

and

$$\epsilon_k = \frac{k}{2m} \left[k^2 + 4m n_0 v(k)\right]^{1/2} . \tag{3.13b}$$

As Bogolyubov showed, the excitations for small k are phonons with energy

$$\epsilon_k = ck , \tag{3.14}$$

and sound velocity

$$c \equiv (\mu_0/m)^{1/2} . \tag{3.15}$$

For large k, the excitations have a single-particle character. The transition between these limits depends on the form of $v(k)$.

For purposes of calculation, it is useful to represent $v(k)$ by a simple step function:

$$v(k) = \begin{cases} v(0) & k < a^{-1} \\ 0 & k > a^{-1} , \end{cases} \tag{3.16}$$

where a is the range of the interaction. It is also convenient to introduce a coupling constant by the equation

$$v(0) = g\, a^3 . \tag{3.17}$$

In these terms, the condition for low density, which we assume in this paper, is

$$n\, a^3 \ll 1 , \tag{3.18}$$

since a is the same order of magnitude as the size of the atom. The condition for weak coupling, which we also assume, is

$$g\, m\, a^2 \ll 1 . \tag{3.19}$$

The first-order chemical potential of Eq. (3.12) may now be written as

$$\mu_0 = (n_0\, a^3) g . \tag{3.20}$$

Since we assume weak coupling and low density, we have

$$\mu \ll g \ll (ma^2)^{-1} .$$

If we wish to compare our results with calculations using hard spheres, we must set $g = 4\pi(ma^2)^{-1}$ and interpret a as the diameter of the hard

sphere. However, this transcription works only to lowest order.[6]

For the interaction (3.16), the expressions (3.13) become

$$f_k = \frac{k^2}{2m} + \mu_0 \tag{3.21a}$$

and

$$\epsilon_k = \frac{k}{2m}(k^2 + k_0^2)^{1/2} \tag{3.21b}$$

for $k < a^{-1}$, and $f_k = \epsilon_k = k^2/2m$ for $k > a^{-1}$. The parameter k_0^2 is

$$k_0^2 \equiv 4m\, n_0\, v(0) = 4m\, \mu_0 \ . \tag{3.22}$$

For weak coupling and low density, $k_0\, a \ll 1$.

With this model $\Delta\mu$ can be calculated from Eq. (3.11) with the result:

$$\frac{\Delta\mu}{\mu_0} = -\frac{gma^2}{2\pi^2}\left\{1 - \frac{1}{3}k_0\, a[5 + 6\, U_1(\gamma) - 8\,\gamma\, U_1'(\gamma) - 20\, U_3(\gamma)]\right\} , \tag{3.23}$$

where the functions $U_n(\gamma)$ are defined in Appendix A, and

$$\gamma \equiv 2\beta\mu_0 = \beta\, k_0^2/2m \ . \tag{3.24}$$

Since we are using μ_0 as an approximation to μ , we must require that

$$|\Delta\mu| \ll \mu_0 \ . \tag{3.25}$$

The first term of Eq. (3.23) evidently satisfies this from condition (3.19); the second term determines the temperature range for which (3.25) is valid. Using the "high-temperature"asymptotic form of $U_n(\gamma)$ given in Eq. (A.3) of Appendix A, we obtain the requirement for $\theta = \beta^{-1}$:

$$\theta \ll \theta_0 \ , \tag{3.26}$$

where

$$\theta_0 = \mu_0 (n_0 a^3)^{-1/5} (gma^2)^{-3/5} = (ma^2)^{-1} (gma^2)^{2/5} (n_0 a^3)^{4/5} . \qquad (3.27)$$

Using conditions (3.18) and (3.19) and Eq. (3.20), we conclude that θ_0 is restricted by the inequalities $\mu_0 \ll \theta_0 \ll (ma^2)^{-1}$. Note that, as the density n or the interaction strength g vanishes, the allowed temperature range vanishes too.

The thermodynamic properties of the system can now be studied with the model just described. For example, the fractional occupation of the unperturbed ground state, $\xi = \overline{N}_0/\overline{N}$, is readily calculated from Eq. (3.7) to be

$$\xi = 1 - 2\pi^{-2} (gma^2)^{3/2} (n_0 a^3)^{1/2} \left[\frac{1}{6} + U_1(\gamma) \right] . \qquad (3.28)$$

From conditions (3.18) and (3.19), it follows that

$$1 - \xi \ll 1 , \qquad (3.29)$$

so that we are able to discuss states where only a small (but macroscopic) fraction of the particles are out of the zero-momentum state. Using Eq. (A.3) for the asymptotic form of $U_1(\gamma)$ at low temperatures ($\gamma \gg 1$), it can now be shown that the depletion of the unperturbed ground state increases as the second power of the temperature. (The value at absolute zero agrees with the older work of Lee, Huang, and Yang,[12] but the temperature dependence is at variance with the recent paper by Lee and Yang.[2]) Because of the restriction (3.29), the possibility of a phase change corresponding to $\xi \to 0$ can not be discussed with this theory, which is restricted to temperatures much less than the transition temperature. In Appendix B we give a discussion which

approaches the transition from above, using linked diagram expansions for the grand partition function.[13] We show there that a slightly imperfect gas at low density does have a phase transition.

Next, we may evaluate $\epsilon_0(\mu, \bar{N}_0)$ from Eq. (2.13), obtaining

$$\epsilon_0 = \frac{1}{2} \mathcal{V} n_0 \mu_0 [1 - \frac{1}{2} \pi^{-2} (gma^2)(1 - \frac{16}{15} k_0 a)] . \tag{3.30}$$

After further algebra, expression (3.5) then becomes

$$-\beta^{-1} \ell n \, Z(\mu, \beta, \mathcal{V}) = \frac{1}{2} \mathcal{V} n_0 \mu_0 [1 - \frac{1}{2} \pi^{-2} (gma^2) \{ 1 - \frac{16}{15} k_0 a[1 - 5U_3(\gamma)]\}] . \tag{3.31}$$

The thermodynamic potential $\Omega(\mu, \beta, \mathcal{V})$ and the pressure $p(\mu, \beta)$ follow immediately, from Eq. (3.8):

$$\Omega(\mu, \beta, \mathcal{V}) \equiv -\beta^{-1} \ell n \, \mathcal{Z} \tag{3.32a}$$

$$= -\mu \mathcal{V} n_0(\mu, \beta) - \beta^{-1} \ell n \, Z(\mu, \beta, \mathcal{V})$$

$$p(\mu, \beta) = -\mathcal{V}^{-1} \Omega \tag{3.32b}$$

$$= \mu n_0(\mu, \beta) - \frac{1}{2} n_0 \mu_0 [1 - \frac{1}{2} \pi^{-2} (gma^2) \times \{ 1 - \frac{16}{15} k_0 a[1 - 5 U_3(\gamma)] \}] .$$

In the first two terms of p, let us express n_0 in terms of μ and $\Delta\mu$, by Eq. (3.12). We find that the first-order terms in $\Delta\mu$ cancel, and we obtain to first order in $\Delta\mu$

$$p(\mu, \beta) = \frac{1}{2} \mu^2 / v(0) + \frac{1}{4} \pi^{-2} \mu^2 ma^{-1} \{ 1 - \frac{32}{15} \mu^{1/2} m^{1/2} a[1 - 5 U_3(\gamma)] \} . \tag{3.33}$$

From the pressure we may obtain the particle density $n(\mu, \beta)$,

$$n = \mathcal{V}^{-1} N = -\mathcal{V}^{-1} \frac{\partial \Omega}{\partial \mu} = \frac{\partial p}{\partial \mu}\bigg|_{\theta} , \tag{3.34a}$$

and the entropy density $\mathcal{S}(\mu, \beta)$

$$\mathcal{S} = \mathcal{V}^{-1} S = -\mathcal{V}^{-1} \frac{\partial \Omega}{\partial \theta} = \frac{\partial p}{\partial \theta}\bigg|_{\mu} . \tag{3.34b}$$

When we calculate the particle density from Eqs. (3.34) and (3.33), and express the result in terms of n_0 by means of Eqs. (3.12) and (3.23), we are led to Eq. (3.28) for $\xi = n_0/n$. (This provides a check on the algebra and the consistency of our approximations.) The calculation of entropy density yields

$$\mathcal{S}(\mu, \theta) = \frac{16\, \mu^{7/2} m^{3/2}}{3 \pi^2 \theta} [- U'_3(\gamma)] . \tag{3.35}$$

IV. PAIR CORRELATION FUNCTION

We shall now calculate the pair correlation function, for finite temperature, in the Bogolyubov approximation. Lee, Huang, and Yang have determined this function at absolute zero for the dilute hard-sphere gas.[12] We define the pair correlation operator as

$$D(\underset{\sim}{r}_1, \underset{\sim}{r}_2) = n^{-2}\, \psi^\dagger(\underset{\sim}{r}_1)\, \psi^\dagger(\underset{\sim}{r}_2)\, \psi(\underset{\sim}{r}_2)\, \psi(\underset{\sim}{r}_1) \quad , \tag{4.1}$$

with

$$\psi(\underset{\sim}{r}) = \mathcal{V}^{-1/2} \sum_{\underset{\sim}{k}} a_{\underset{\sim}{k}}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} \quad . \tag{4.2}$$

Substitution leads to

$$D(\underset{\sim}{r}_1, \underset{\sim}{r}_2) = (\overline{N})^{-2} \sum_{\underset{\sim}{p}\underset{\sim}{p}'\underset{\sim}{q}\underset{\sim}{q}'} \exp\{ i[(\underset{\sim}{p} - \underset{\sim}{p}')\cdot\underset{\sim}{r}_1 + (\underset{\sim}{q} - \underset{\sim}{q}')\cdot\underset{\sim}{r}_2]\} a_{\underset{\sim}{p}'}^\dagger\, a_{\underset{\sim}{q}'}^\dagger\, a_{\underset{\sim}{q}}\, a_{\underset{\sim}{p}} \quad . \tag{4.3}$$

As in Section II, we replace the operators $a_0^\dagger$ and a_0 for the zero-momentum states by the c number $N_0^{1/2}$, obtaining

$$D(\underset{\sim}{r}_1, \underset{\sim}{r}_2) = (\overline{N})^{-2} \Bigg\{ N^2 + N_0 \sum_{\underset{\sim}{k}}{}' e^{i\underset{\sim}{k}\cdot(\underset{\sim}{r}_1 - \underset{\sim}{r}_2)} (2\, a_{\underset{\sim}{k}}^\dagger\, a_{\underset{\sim}{k}} + a_{\underset{\sim}{k}}^\dagger\, a_{-\underset{\sim}{k}}^\dagger + a_{\underset{\sim}{k}}\, a_{-\underset{\sim}{k}})$$

$$+ \sum_{\underset{\sim}{p}\underset{\sim}{p}'\underset{\sim}{q}\underset{\sim}{q}'}{}' \exp\{i[(\underset{\sim}{p} - \underset{\sim}{p}')\cdot\underset{\sim}{r}_1 + (\underset{\sim}{q} - \underset{\sim}{q}')\cdot\underset{\sim}{r}_2]\}\, a_{\underset{\sim}{p}'}^\dagger\, a_{\underset{\sim}{q}'}^\dagger\, a_{\underset{\sim}{q}}\, a_{\underset{\sim}{p}}$$

$$- \sum_{\underset{\sim}{p}\underset{\sim}{q}}{}' a_{\underset{\sim}{p}}^\dagger\, a_{\underset{\sim}{q}}^\dagger\, a_{\underset{\sim}{p}}\, a_{\underset{\sim}{q}} \Bigg\} \quad . \tag{4.4}$$

Here we have dropped terms of relative order $\overline{N}^{-1}$, and also terms containing an odd number of operators, since their expectation values vanish in the quasiparticle representation.

In expression (4.4), we make the Bogolyubov transformation (2.8) to the quasiparticle representation, and then take the statistical average. The algebra is straightforward; we present only the result:

$$\bar{D}(r) = 1 + 2\xi\,[F(r) + G(r)] + F^2(r) + G^2(r) \quad , \tag{4.5}$$

where

$$r \equiv |r_1 - r_2| \quad , \tag{4.6}$$

$$\begin{aligned} F(\underset{\sim}{r}) &\equiv (\bar{N})^{-1} \sum_{\underset{\sim}{k}}{}' e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} u_k v_k (1 + 2\bar{\nu}_k) \quad , \\ G(\underset{\sim}{r}) &\equiv (\bar{N})^{-1} \sum_{\underset{\sim}{k}}{}' e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} [v_k^2 + (u_k^2 + v_k^2)\bar{\nu}_k] \quad , \end{aligned} \tag{4.7}$$

and $\xi = N_0/N$, as before.

For the evaluation of F and G, we use Eqs. (2.10) and (2.11) for u_k and v_k , and obtain, for $r \neq 0$,

$$F(\underset{\sim}{r}) = -(\bar{N})^{-1} \sum_{\underset{\sim}{k}}{}' e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} \frac{h_k}{\epsilon_k} \left(\frac{1}{2} + \bar{\nu}_k \right) \quad , \tag{4.8a}$$

and

$$G(\underset{\sim}{r}) = (\bar{N})^{-1} \sum_{\underset{\sim}{k}}{}' e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} \frac{f_k}{\epsilon_k} \left(\frac{1}{2} + \bar{\nu}_k \right) \quad . \tag{4.8b}$$

By Eq. (3.21), these expressions become

$$F(r) = -\pi^{-2}(g\,m\,a^2)\left(\frac{a}{r}\right) J(\alpha, \gamma) \quad , \tag{4.9a}$$

and

$$F(r) + G(r) = -2\pi^{-2}(g\,m\,a^2)\left(\frac{a}{r}\right) \frac{\partial^2}{\partial\alpha^2} J(\alpha, \gamma) \quad , \tag{4.9b}$$

where

$$J(\alpha, \gamma) = \int_0^\infty dx \frac{\sin \alpha x}{(1 + x^2)^{1/2}} \left(\frac{1}{2} + \{\exp[\gamma x(1 + x^2)^{1/2}] - 1\}^{-1} \right) \tag{4.10a}$$

$$= \frac{1}{2} \int_0^\infty dx (1 + x^2)^{-1/2} (\sin \alpha x) \operatorname{ctnh} \left[\frac{1}{2} \gamma x (1 + x^2)^{1/2} \right] , \tag{4.10b}$$

and

$$\alpha \equiv k_0 r \quad . \tag{4.11}$$

We shall evaluate $J(\alpha, \gamma)$ in some limiting cases:

(1a) $\gamma \gg \alpha \gg 1$. The condition $\gamma \gg 1$ means $\theta \ll \mu_0$, or a temperature so low that only phonons are excited. The condition $\gamma \gg \alpha$ allows us to consider the limiting case of $\gamma \to \infty$ for finite α, or absolute zero at finite r.

(1b) $\alpha \gg \gamma \gg 1$. This differs from (1a) in that we may consider $r \to \infty$ for finite temperature.

(2) $\alpha \gg 1 \gg \gamma$. The condition $\gamma \ll 1$ means $\theta \gg \mu_0$, or a temperature high enough to excite free-particle states. Of course, we still require $\theta \ll \theta_0$, so that $1 - \xi \ll 1$.

In these limiting cases we find for $J(\alpha, \gamma)$ the following forms, by integrating by parts for (1a) and (1b), and deforming a contour for (2):

$$\text{(1a)} \quad J(\alpha, \gamma) = \frac{1}{2\alpha} + \mathcal{O}(\alpha^{-3}) + \mathcal{O}(\alpha \gamma^{-2}) ,$$

$$\text{(1b)} \quad J(\alpha, \gamma) = \frac{\pi}{2\gamma} + \frac{1}{2\alpha^3} + \mathcal{O}(\alpha^{-5}) + \mathcal{O}(\alpha \gamma^{-2} e^{-2\alpha/\gamma}),$$

$$\text{(2)} \quad J(\alpha, \gamma) = \frac{\pi}{2\gamma} + \frac{1}{2} \exp[-\alpha(\pi/\gamma)^{1/2}] \sin[\alpha(\pi/\gamma)^{1/2}](1 + \mathcal{O}(\gamma^2)) \quad . \tag{4.12}$$

In these cases, we obtain the following expressions for $2(F + G)$, from Eq. (4.9), which are to be inserted in Eq. (4.5) for $D(r)$:

$$\text{(1a)} \quad 2(F + G) = -\frac{1}{2}\pi^{-2}(a/r)^4 (g\, m\, a^2)^{-1/2} (n\, a^3)^{-3/2} + \ldots$$

$$\text{(1b)} \quad 2(F + G) = -\frac{3}{4}\pi^{-2}(a/r)^6 (g\, m\, a^2)^{-3/2} (n\, a^3)^{-5/2} + \ldots$$

$$\text{(2)} \quad 2(F + G) = +\frac{2\, m\, \theta}{\pi\, n\, r}\, e^{-r/\lambda\!\!\!^{-}} + \ldots , \tag{4.13}$$

where

$$\lambda\!\!\!^{-} \equiv (2\pi\, m\, \theta)^{-1/2} \tag{4.14}$$

is the thermal wavelength. The leading terms in $F^2 + G^2$ are

$$\text{(1a)} \quad F^2 + G^2 = (8\pi^4)^{-1} (a/r)^2 (g\, m\, a^2)(n\, a^3)^{-1} + \ldots$$

$$\text{(1b)} \quad F^2 + G^2 = (8\pi^2)^{-1} \left(\frac{m\theta}{nr}\right)^2 + \ldots \tag{4.15}$$

$$\text{(2)} \quad F^2 + G^2 = (8\pi^2)^{-1} \left(\frac{m\theta}{nr}\right)^2 + \ldots .$$

The limiting cases are such that each of the expressions in (4.13) and (4.15) is much less than unity in magnitude, so that $|D - 1| \ll 1$. Let us now discuss each case separately.

Case (1a) leads to a correlation function independent of temperature, and includes absolute zero as a special case. In this case, $2(F + G) \gg F^2 + G^2$, so that only $2(F + G)$ need be kept in $D - 1$. The result, with an r^{-4} dependence, is in agreement with the calculation of Lee, Huang, and Yang[12] for absolute zero.

In case (1b), the term $2(F + G)$ predominates over $F^2 + G^2$ at finite r as $g \to 0$ or $n \to 0$, while $F^2 + G^2$ predominates as $r \to \infty$.

In the latter limit, the correlation is independent of the strength of the interaction for a given temperature, but the allowed temperature range depends on g [see Eq. (3.26)]. In any case the dependence on r is as an inverse power of r.

In case (2), the term $2(F + G)$ predominates as g approaches 0 (because the allowed temperature range depends on g), while $F^2 + G^2$ predominates as n approaches 0 or r approaches ∞. In the former case, the dependence on r is exponential.

There is some question as to the propriety of keeping the quadratic terms $(F^2 + G^2)$ in D, inasmuch as they arise from terms in Eq. (4.4) which are quadrilinear in the a operators. Such terms have been omitted in making the Bogolyubov approximation in the Hamiltonian. A final resolution of this point probably depends on a more accurate treatment of the Hamiltonian.

V. STATISTICAL MECHANICS FOR MOVING SYSTEMS

In the previous sections, we have considered the system under discussion to be at rest. We shall now study the fluid motion of this system. First, we shall generalize our treatment to include flow at a uniform, constant velocity. This will provide a means for defining "normal" and "superfluid" components and introducing their respective flow velocities. Next, a discussion of rotation with a uniform angular velocity will provide an evaluation of the moment of inertia. Finally, in the last two sections of this paper we shall derive the hydrodynamic equations for nonuniform flow.

In the statistical treatment given earlier, we have made a plausible assumption that it is the zero-momentum state that is macroscopically occupied. Galilean invariance, of course, implies that any momentum state can be so chosen. For the evaluation of the partition function $\mathcal{Z}$, we should, strictly speaking, have chosen a general state, say $\underset{\sim}{k}_s$, as macroscopically occupied. Then we should have summed $\mathcal{Z}$ over all $\underset{\sim}{k}_s$, subject to the constraint that the mean momentum of the entire system be specified. A saddle-point evaluation of the sum over $\underset{\sim}{k}_s$ would then lead us back to a specific choice for $\underset{\sim}{k}_s$, $\underset{\sim}{k}_s = 0$ being appropriate for a fluid at rest.

A state of uniform motion of the entire system can then be obtained in a trivial manner from the state of the system when at rest. A Galilean transformation, changing the macroscopically occupied state from $\underset{\sim}{k} = 0$ to $\underset{\sim}{k} = \underset{\sim}{k}_s$, will accomplish this. The total momentum of the system is just

$$\underset{\sim}{\overline{G}} = N \underset{\sim}{k}_s \,. \tag{5.1}$$

We can obtain a state of uniform motion with total momentum $\underset{\sim}{\overline{G}}$ in an alternative way, however. We can still maintain the constraint

that $\underset{\sim}{k} = 0$ labels the state of macroscopic occupation even though $\overline{\underset{\sim}{G}}$ is not zero. This constraint, which means that the system is not in true thermodynamic equilibrium but only in quasi-equilibrium, is possible because of the lack of coupling in the Bogolyubov model between the zero-momentum state and the excitations. This lack of coupling leads to the two-fluid model, as we shall see.[14]

To allow for a value of $\overline{\underset{\sim}{G}}$ differing from zero, we utilize the generalized partition function

$$\mathcal{Z}(\beta, \mu, \underset{\sim}{w}) = \mathrm{Tr}\ \{ \exp [-\beta(H_B - \mu N - \underset{\sim}{w}\cdot\underset{\sim}{G})] \} , \tag{5.2}$$

where again we drop the term $V^{1/2}$ of Eq. (2.5) and require that $\overline{N}_0$ is macroscopic. The total-momentum operator $\underset{\sim}{G}$ is

$$\underset{\sim}{G} = \sum_{\underset{\sim}{k}} \underset{\sim}{k}\ a_{\underset{\sim}{k}}^{\dagger}\ a_{\underset{\sim}{k}} . \tag{5.3}$$

Under the Bogolyubov transformation (2.8), this becomes

$$\underset{\sim}{G} = \sum_{\underset{\sim}{k}} \underset{\sim}{k}\ \alpha_{\underset{\sim}{k}}^{\dagger}\ \alpha_{\underset{\sim}{k}} , \tag{5.4}$$

so the elementary excitation corresponding to $\alpha_{\underset{\sim}{k}}^{\dagger}$ does indeed carry momentum $\underset{\sim}{k}$.

Because $\underset{\sim}{G}$ is diagonal in $\alpha_{\underset{\sim}{k}}^{\dagger}\ \alpha_{\underset{\sim}{k}}$, the evaluation of Eq. (5.2) proceeds just as before, the only change being that now ϵ_k is replaced by

$$(\epsilon_k - \underset{\sim}{w}\cdot\underset{\sim}{k}) . \tag{5.5}$$

Thus for $\mathcal{Z}$ we now have

$$\mathcal{Z}(\mu, \beta, \underset{\sim}{w}) = e^{\beta\mu\overline{N}_0}\, Z(\mu, \beta, \overline{N}_0, \underset{\sim}{w}) \tag{5.6}$$

where

$$-\beta^{-1} \ell n\, Z(\mu, \beta, N_0, \underset{\sim}{w}) = \epsilon_0(\mu, N_0) + \beta^{-1} \sum_{\underset{\sim}{k}}{}' \ell n[1 - e^{-\beta(\epsilon_k - \underset{\sim}{w}\cdot\underset{\sim}{k})}] , \tag{5.7}$$

and $\overline{N}_0$ is determined by

$$\mu = - \frac{\partial}{\partial N_0} \beta^{-1} \ell n\, Z(\mu, \beta, N_0, \underset{\sim}{w}) \Big|_{N_0=\overline{N}_0} . \tag{5.8}$$

The mean momentum is found from

$$\overline{\underset{\sim}{G}} = \beta^{-1} \frac{\partial\, \ell n\, \mathcal{Z}}{\partial\, \underset{\sim}{w}} \Big|_{\mu,\beta} . \tag{5.9}$$

We shall assume that $|\underset{\sim}{w}|$ is sufficiently small that we can neglect terms in $|\underset{\sim}{w}|^2$ or higher order. Then, since the thermodynamic functions such as μ and $N_0(\mu, \beta, \underset{\sim}{w})$ must be even in $\underset{\sim}{w}$ by symmetry, it is sufficient in Eq. (5.9) to differentiate only the terms in which $\underset{\sim}{w}$ appears explicitly, so that

$$\overline{\underset{\sim}{G}} = \sum_{\underset{\sim}{k}}{}' \underset{\sim}{k}\, \overline{\nu}_{\underset{\sim}{k}}(\underset{\sim}{w}) . \tag{5.10}$$

Here we have

$$\overline{\nu}_{\underset{\sim}{k}}(\underset{\sim}{w}) \equiv (e^{\beta(\epsilon_k - \underset{\sim}{w}\cdot\underset{\sim}{k})} - 1)^{-1} . \tag{5.11}$$

Upon expanding Eq. (5.10) to first order in $\underset{\sim}{w}$, we obtain

$$\overline{\underset{\sim}{G}} = - \underset{\sim}{w} \sum_{\underset{\sim}{k}}{}' \frac{1}{3} k^2 \frac{d\overline{\nu}_k}{d\epsilon_k} , \tag{5.12}$$

where the derivative is evaluated for $\underset{\sim}{w} = 0$.

Let us define the quantity ρ_n by

$$\overline{\underset{\sim}{G}} \equiv \rho_n \underset{\sim}{W} \mathcal{V} \quad . \tag{5.13}$$

We evaluate ρ_n from Eq. (5.12) and obtain

$$\rho_n = -\mathcal{V}^{-1} \sum_{\underset{\sim}{k}} \frac{1}{3} k^2 \frac{\partial \bar{\nu}_k}{\partial \epsilon_k} , \tag{5.14}$$

or[15]

$$\rho_n = \frac{1}{3} \beta \mathcal{V}^{-1} \sum_{\underset{\sim}{k}} k^2 \nabla_k (1 + \nu_k) \quad . \tag{5.15}$$

On evaluation of Eq. (5.15), we find

$$\frac{\rho_n}{\rho} = \frac{8}{3\pi^2} (n a^3)^{1/2} (g m a^2)^{3/2} \gamma[-W_3'(\gamma)] , \tag{5.16}$$

where $W_n(\gamma)$ is defined in Appendix A. Here ρ is the mass density $m n$; the quantity ρ_n is called the "density of the normal fluid." It is customary also to define the "density of the superfluid" ρ_s by

$$\rho_s \equiv \rho - \rho_n \quad . \tag{5.17}$$

Of course, ρ_s is <u>not</u> to be interpreted as the density of the particles in the zero-momentum state.

Let us now change to a new frame of reference moving with a velocity $-\underset{\sim}{v}_s$ relative to the old frame. In the new frame, the macroscopically occupied state has momentum $\underset{\sim}{k}_s = m \underset{\sim}{v}_s$, and the system has the total momentum

$$\begin{aligned} \overline{\underset{\sim}{G}} &= M \underset{\sim}{v}_s + \rho_n \underset{\sim}{w} \mathcal{V} \\ &= (\rho \underset{\sim}{v}_s + \rho_n \underset{\sim}{w}) \mathcal{V} \\ &= [\rho_s \underset{\sim}{v}_s + \rho_n(\underset{\sim}{v}_s + \underset{\sim}{w})] \mathcal{V} , \end{aligned} \tag{5.18}$$

by (5.13) and (5.15). We may now introduce the notation

$$\underset{\sim}{v}_n \equiv \underset{\sim}{v}_s + \underset{\sim}{w} , \tag{5.19}$$

so that

$$\overline{\underset{\sim}{G}} = (\rho_s \underset{\sim}{v}_s + \rho_n \underset{\sim}{v}_n) \mathcal{V} . \tag{5.20}$$

This form suggests calling $\underset{\sim}{v}_s$ and $\underset{\sim}{v}_n$ the "velocities of the superfluid and normal fluid," respectively. We note that the parameter $\underset{\sim}{w}$, introduced in (5.2), is just the "relative velocity of the two fluids."

Using Eq. (5.20) we may write the mean momentum density as

$$\overline{\underset{\sim}{\mathcal{G}}} = \rho_s \underset{\sim}{v}_s + \rho_n \underset{\sim}{v}_n . \tag{5.21}$$

The definitions of the normal and superfluid components, introduced above, will be supplemented in Sections VI and VII with the derivation of appropriate equations of motion.

We next consider the fluid to be confined in a cylindrical bucket which rotates at angular velocity ω about its symmetry axis $\underset{\sim}{n}$. The partition function for thermal equilibrium is now[16]

$$\mathcal{Z} = \mathrm{Tr} \{ \exp[-\beta(H - \mu N - \omega L)] \} , \tag{5.22}$$

where L is the projection of the total angular momentum operator $\underset{\sim}{L}$ along $\underset{\sim}{n}$. The mean value of L is

$$\bar{L} = \mathcal{Z}^{-1} \mathrm{Tr} \{ L \exp [-\beta(H - \mu N - \omega L)] \} . \tag{5.23}$$

The moment of inertia is defined in the usual way from

$$\bar{L}(\omega) = I(\omega)\omega . \tag{5.24}$$

Following the method of Blatt, Butler, and Schafroth,[17] we consider the limit

$$I(0) = \lim_{\omega \to 0} \frac{\bar{L}(\omega)}{\omega} = \frac{\partial \bar{L}(\omega)}{\partial \omega} \Bigg|_{\omega=0} . \tag{5.25}$$

In differentiating $\bar{L}$ in Eq. (5.23), the dependence of $\mathcal{Z}$ on ω may be neglected in the limit $\omega \to 0$, since $\mathcal{Z}$ is an even function of ω. For a cylindrical bucket, H and L commute, so we have

$$\begin{aligned} I(0) &= \beta \mathcal{Z}^{-1} \mathrm{Tr} \{ L^2 \exp[-\beta(H - \omega L - \mu N)] \} \Big|_{\omega=0} \\ &= \beta \mathcal{Z}^{-1} \mathrm{Tr} \{ L^2 \exp[-\beta(H - \mu N)] \} . \end{aligned} \tag{5.26}$$

Thus, $I(0)$ is proportional to the fluctuations of the angular momentum operator for a <u>stationary</u> system:[17]

$$I(0) = \beta \langle L^2 \rangle . \tag{5.27}$$

The calculation of the moment of inertia in the Bogolyubov approximation is now straightforward. We first write the projection of the total angular momentum along the symmetry axis in second quantized form as

$$L = -i \int d^3r \; \psi^\dagger(\underset{\sim}{r})(\underset{\sim}{n} \cdot \underset{\sim}{r} \times \nabla)\psi(\underset{\sim}{r}) , \tag{5.28}$$

where

$$\psi(\underset{\sim}{r}) = \mathcal{V}^{-1/2} \sum_{\underset{\sim}{k}} a_{\underset{\sim}{k}} e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} .$$

Thus we have

$$L = \sum_{\underset{\sim}{k}\,\underset{\sim}{k}'} L_{\underset{\sim}{k}\underset{\sim}{k}'} a_{\underset{\sim}{k}}^{\dagger} a_{\underset{\sim}{k}'} \tag{5.29}$$

and

$$L_{\underset{\sim}{k}\underset{\sim}{k}'} = -i \mathcal{V}^{-1} \int d^3r \; e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}} (\underset{\sim}{n}\cdot\underset{\sim}{r} \times \nabla) e^{i\underset{\sim}{k}'\cdot\underset{\sim}{r}} . \tag{5.30}$$

The matrix element $L_{\underset{\sim}{k}\underset{\sim}{k}'}$ is real and symmetric; some other of its essential properties are that $L_{\underset{\sim}{k}0} = 0$, $L_{\underset{\sim}{k}\underset{\sim}{k}'} = 0$ if $|\underset{\sim}{k}| = |\underset{\sim}{k}'|$, and $L_{-\underset{\sim}{k},-\underset{\sim}{k}'} = -L_{\underset{\sim}{k}'\underset{\sim}{k}}$.

From Eq. (5.29), we can write

$$L^2 = \sum_{\underset{\sim}{k}\underset{\sim}{k}'\,\underset{\sim}{\ell}\underset{\sim}{\ell}'} L_{\underset{\sim}{k}\underset{\sim}{k}'} L_{\underset{\sim}{\ell}\underset{\sim}{\ell}'} a_{\underset{\sim}{k}}^{\dagger} a_{\underset{\sim}{k}'} a_{\underset{\sim}{\ell}}^{\dagger} a_{\underset{\sim}{\ell}'} . \tag{5.31}$$

We now introduce the quasiparticle transformation of Eq. (2.8) for each operator in this expression. Only the terms diagonal in the quasiparticle representation need be kept, since the off-diagonal elements vanish when the statistical average is performed. Using the above properties of the matrix elements $L_{\underset{\sim}{k}\underset{\sim}{k}'}$ and the definitions of u_k and v_k, we find that

$$\langle L^2 \rangle = \sum_{\underset{\sim}{k}} (L^2)_{\underset{\sim}{k}\underset{\sim}{k}} \bar{\nu}_k(1 + \bar{\nu}_k) . \tag{5.32}$$

Introducing a rectangular coordinate system for which z is parallel to the symmetry axis, we may show from Eq. (5.30) that $L^2_{\underset{\sim}{k}\underset{\sim}{k}}$, averaged over the directions of $\underset{\sim}{k}$, is[17]

$$(4\pi)^{-1} \int d\Omega_{\underset{\sim}{k}} (L^2)_{\underset{\approx}{kk}} = \frac{1}{3} \langle x^2 + y^2 \rangle k^2 , \tag{5.33}$$

where

$$\langle x^2 + y^2 \rangle = \mathcal{V}^{-1} \int_{\mathcal{V}} d^3r \, (x^2 + y^2) . \tag{5.34}$$

The final expression for the moment of inertia $I(0)$ is therefore[18]

$$I(0) = \frac{1}{3} \beta \langle x^2 + y^2 \rangle \sum_{\underset{\sim}{k}} k^2 \bar{\nu}_k (1 - \bar{\nu}_k) . \tag{5.35}$$

The corresponding moment of inertia for a rigid body is

$$I_0 = \rho \mathcal{V} \langle x^2 + y^2 \rangle . \tag{5.36}$$

Using Eq. (5.15), we find the ratio

$$\frac{I(0)}{I_0} = \frac{\rho_n}{\rho} . \tag{5.37}$$

Landau stated this result in his original paper on superfluidity.[4] It is reassuring to derive this formula from a microscopic point of view.

Equation (5.37) is convenient for calculation, and from it one easily obtains the well-known result for the ideal Bose gas:[17]

$$\frac{I(0)}{I_0} = \bar{N}^{-1} \sum'_{\underset{\sim}{k}} \nu_k = 1 - \frac{\bar{N}_0}{\bar{N}} . \tag{5.38}$$

In this case the ratio approaches zero as the 3/2 power of the temperature. At very low temperatures, one finds for the Bogolyubov Hamiltonian that[4]

$$\frac{I}{I_0} = \frac{(2\pi)^2}{90} \frac{\theta^4}{\rho c^5} . \tag{5.39}$$

VI. GENERAL TRANSPORT THEORY

In this section we shall derive hydrodynamic and thermodynamic equations for the fluid motion of a single-component, quantum-mechanical system, applicable for either Fermi-Dirac or Bose-Einstein statistics. Conventional treatments in kinetic theory employ the Boltzmann equation (for the single-particle distribution function), which is based on the assumption of molecular chaos. Their applicability is thus limited to ideal gases. Because our discussion is to be applied to liquids and nonideal gases, we shall use the many-particle density matrix $\underset{\sim}{\rho}(t)$, whose time development is given by the quantum Liouville equation[19]

$$\frac{\partial \underset{\sim}{f}}{\partial t} = -\frac{i}{\hbar} [H, \underset{\sim}{\rho}] , \tag{6.1}$$

where H is the many-particle Hamiltonian.

As in the Chapman-Enskog method of solution of the Boltzmann equation, we shall assume that the fluid is in <u>local</u> thermodynamic equilibrium, an assumption that requires that macroscopic physical quantities (such as pressure and density) vary appreciably only over <u>macroscopic</u> distances. (In the next section, we shall modify this assumption to that of local quasi-equilibrium to treat the Bogolyubov Hamiltonian.) In accordance with this assumption, we write

$$\underset{\sim}{\rho} = \underset{\sim}{\rho}_{eq} + \delta\underset{\sim}{\rho} , \tag{6.2}$$

where $\underset{\sim}{\rho}_{eq}$ is the density matrix appropriate to local thermodynamic equilibrium, and $\delta\underset{\sim}{\rho}$ is a small correction.

The present discussion will be restricted to a consideration of reversible flow, for which only $\underset{\sim}{\rho}_{eq}$ is to be used. The corrections due to $\delta\underset{\sim}{\rho}$

give rise to irreversible phenomena; they will be discussed in a future publication.[20] Thus, in the remainder of this section, the density matrix $\underset{\sim}{\rho}$ is to be understood to mean $\underset{\sim}{\rho}_{eq}$. We shall make explicit use of the property of $\underset{\sim}{\rho}_{eq}$ that, in the neighborhood of any point $\underset{\sim}{r}$ and in a coordinate system moving with the local fluid velocity $\underset{\sim}{v}(\underset{\sim}{r}, t)$, the momentum distribution of the particles is isotropic. That this is so follows from the definition of $\underset{\sim}{\rho}_{eq}$ as representing local equilibrium.

For any operator A, the mean value $\bar{A}$ is

$$\bar{A}(t) = \mathrm{Tr}(\underset{\sim}{\rho}(t)A) \quad . \tag{6.3}$$

The time rate of change of $\bar{A}$ is, if A is time-independent,

$$\begin{aligned} \frac{\partial \bar{A}}{\partial t} &= \mathrm{Tr}\left\{ -\frac{i}{\hbar} [\, H, \underset{\sim}{\rho} \,] A \right\} \\ &= \frac{i}{\hbar} \mathrm{Tr}\left\{ \underset{\sim}{\rho} [H, A] \right\} \\ &= \frac{i}{\hbar} \langle [H, A] \rangle \quad . \end{aligned} \tag{6.4}$$

This equation, applied to various operators, yields the hydrodynamic equations of change. The Hamiltonian H is that of Eq. (2.1):

$$H = K + V \quad ,$$

where

$$K = \sum_{\underset{\sim}{k}} \frac{k^2}{2m} a_{\underset{\sim}{k}}^{\dagger} a_{\underset{\sim}{k}} \tag{6.5}$$

and

$$V = \frac{1}{2} \mathcal{V}^{-1} \sum_{\underset{\sim}{k}\,\underset{\sim}{\ell}\,\underset{\sim}{m}\,\underset{\sim}{n}} v(\underset{\sim}{k} - \underset{\sim}{n})\, a_{\underset{\sim}{k}}^{\dagger}\, a_{\underset{\sim}{\ell}}^{\dagger}\, a_{\underset{\sim}{m}}\, a_{\underset{\sim}{n}}\, \delta_{\underset{\sim}{k}+\underset{\sim}{\ell},\underset{\sim}{m}+\underset{\sim}{n}} \quad . \tag{6.6}$$

We shall assume that the interaction v depends upon $(\underset{\sim}{k} - \underset{\sim}{n})^2$ only.

Consider first the operators for particle density $n(\underset{\sim}{r})$ and momentum density $\underset{\sim}{\mathcal{J}}(\underset{\sim}{r})$:

$$n(\underset{\sim}{r}) = \mathcal{V}^{-1} \sum_{\underset{\sim}{k}\underset{\sim}{k}'} a_{\underset{\sim}{k}'}^{\dagger} a_{\underset{\sim}{k}} e^{i(\underset{\sim}{k} - \underset{\sim}{k}')\cdot\underset{\sim}{r}} \tag{6.7}$$

$$\underset{\sim}{\mathcal{J}}(\underset{\sim}{r}) = \mathcal{V}^{-1} \sum_{\underset{\sim}{k}\underset{\sim}{k}'} \tfrac{1}{2}(\underset{\sim}{k} + \underset{\sim}{k}') a_{\underset{\sim}{k}'}^{\dagger} a_{\underset{\sim}{k}} e^{i(\underset{\sim}{k} - \underset{\sim}{k}')\cdot\underset{\sim}{r}} \quad . \tag{6.8}$$

The equation of change for the mean density,

$$\bar{n}(\underset{\sim}{r}, t) = \mathrm{Tr}\,[\underset{\sim}{\rho}(t)\, n(\underset{\sim}{r})] \quad ,$$

is obtained from Eq. (6.4). The evaluation of $[K, n(\underset{\sim}{r})]$ is accomplished by Eq. (C.2) of Appendix C:

$$[K, n(\underset{\sim}{r})] = (2m\mathcal{V})^{-1} \sum_{\underset{\sim}{k}\underset{\sim}{k}'} (\underset{\sim}{k}' - \underset{\sim}{k})\cdot(\underset{\sim}{k}' + \underset{\sim}{k}) a_{\underset{\sim}{k}'}^{\dagger} a_{\underset{\sim}{k}} e^{i(\underset{\sim}{k} - \underset{\sim}{k}')\cdot\underset{\sim}{r}} \tag{6.9}$$

$$= i\hbar m^{-1} \nabla\cdot\underset{\sim}{\mathcal{J}}(\underset{\sim}{r}) \quad ,$$

while, from Eq. (C.4), we have

$$[V, n(\underset{\sim}{r})] = 0 \quad . \tag{6.10}$$

Equation (6.4) then yields

$$\frac{\partial \bar{n}(\underset{\sim}{r}, t)}{\partial t} = -m^{-1} \nabla\cdot\underset{\sim}{\bar{\mathcal{J}}}(\underset{\sim}{r}, t) \quad . \tag{6.11}$$

We may define the fluid mass density and velocity by

$$\rho(\underset{\sim}{r},\ t) \equiv m\,\bar{n}(\underset{\sim}{r},\ t) \tag{6.12}$$

and

$$\rho\,\underset{\sim}{v}(\underset{\sim}{r},\ t) \equiv \overline{\underset{\sim}{\mathcal{J}}}(\underset{\sim}{r},\ t)\ . \tag{6.13}$$

We thus obtain the conventional continuity equation

$$\frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho\,\underset{\sim}{v})\ . \tag{6.14}$$

We next apply the same procedure to $\underset{\sim}{\mathcal{J}}(\underset{\sim}{r})$, to obtain the equation of motion. The relevant commutators are, from Eqs. (C.2) and (C.4),

$$[K,\ \underset{\sim}{\mathcal{J}}(\underset{\sim}{r})] = i\hbar(4m\mathcal{V})^{-1}\ \nabla\cdot\left[\sum_{\underset{\sim}{k}\underset{\sim}{k}'}(\underset{\sim}{k}+\underset{\sim}{k}')(\underset{\sim}{k}+\underset{\sim}{k}')a_{\underset{\sim}{k}'}^{\dagger}\,a_{\underset{\sim}{k}}\,e^{i(\underset{\sim}{k}-\underset{\sim}{k}')\cdot\underset{\sim}{r}}\right] \tag{6.15}$$

and

$$[V,\ \underset{\sim}{\mathcal{J}}(\underset{\sim}{r})] = \frac{1}{2}\mathcal{V}^{-2}\sum_{\underset{\sim}{p}\underset{\sim}{p}'\underset{\sim}{q}\underset{\sim}{q}'} a_{\underset{\sim}{p}}^{\dagger}\,a_{\underset{\sim}{p}'}^{\dagger}\,a_{\underset{\sim}{q}'}\,a_{\underset{\sim}{q}}\,e^{i(\underset{\sim}{q}+\underset{\sim}{q}'-\underset{\sim}{p}-\underset{\sim}{p}')\cdot\underset{\sim}{r}} \tag{6.16}$$

$$\times\{\,v(\underset{\sim}{p}-\underset{\sim}{q})[\underset{\sim}{p}-\underset{\sim}{q}] + v(\underset{\sim}{p}'-\underset{\sim}{q}')[\underset{\sim}{p}'-\underset{\sim}{q}']\,\}\ .$$

To simplify Eq. (6.16), we recall our assumption that macroscopic quantities vary appreciably only over macroscopic distances. That is, when $\mathrm{Tr}\,\{\,\underset{\sim}{\rho}[V,\ \underset{\sim}{\mathcal{J}}(\underset{\sim}{r})]\}$ is taken, the dependence on $\underset{\sim}{r}$ is slow. Therefore we may replace this quantity by its average over a small macroscopic volume. But the integration over volume will, because of the exponential in Eq. (6.16), have contributions only for

$$\underset{\sim}{p} + \underset{\sim}{p}' \approx \underset{\sim}{q} + \underset{\sim}{q}'\ . \tag{6.17}$$

It is thus convenient to expand $v(\underset{\sim}{p}' - \underset{\sim}{q}')$ in the small quantity $(\underset{\sim}{p} + \underset{\sim}{p}' - \underset{\sim}{q} - \underset{\sim}{q}')$:

$$v(\underset{\sim}{p}' - \underset{\sim}{q}') = v(\underset{\sim}{p} - \underset{\sim}{q}) + 2(\underset{\sim}{p} + \underset{\sim}{p}' - \underset{\sim}{q} - \underset{\sim}{q}')\cdot(\underset{\sim}{p}' - \underset{\sim}{q}') \frac{dv(\underset{\sim}{p} - \underset{\sim}{q})}{d(\underset{\sim}{p} - \underset{\sim}{q})^2} + \cdots \quad . \tag{6.18}$$

The expression in braces in Eq. (6.16) is then

$$(\underset{\sim}{p} + \underset{\sim}{p}' - \underset{\sim}{q} - \underset{\sim}{q}')\cdot \left\{ \underset{\sim}{I}\, v(\underset{\sim}{p} - \underset{\sim}{q}) + 2(\underset{\sim}{p} - \underset{\sim}{q})(\underset{\sim}{p} - \underset{\sim}{q}) \frac{dv(\underset{\sim}{p} - \underset{\sim}{q})}{d(\underset{\sim}{p} - \underset{\sim}{q})^2} \right\} + \cdots \quad , \tag{6.19}$$

where $\underset{\sim}{I}$ is the unit dyadic. We insert expression (6.19) in Eq. (6.16) and remove the factor $(\underset{\sim}{p} + \underset{\sim}{p}' - \underset{\sim}{q} - \underset{\sim}{q}')$ as a spatial gradient, because of the exponential. Also, because of the assumed isotropy of the momentum distribution (in the local rest frame) implied by $\underset{\sim}{\rho}$, we may average $(\underset{\sim}{p} - \underset{\sim}{q})(\underset{\sim}{p} - \underset{\sim}{q})$ over solid angle and replace it by $\frac{1}{3}(\underset{\sim}{p} - \underset{\sim}{q})^2 \underset{\sim}{I}$. We then obtain

$$\langle [V, \underset{\sim}{\mathcal{J}}(\underset{\sim}{r})] \rangle = i\hbar \nabla \bar{V}_{eff}(\underset{\sim}{r}, t) + \cdots \tag{6.20}$$

where

$$V_{eff}(\underset{\sim}{r}) \equiv \frac{1}{2} \mathcal{V}^{-2} \sum_{\underset{\sim}{p}\underset{\sim}{p}'\underset{\sim}{q}\underset{\sim}{q}'} a_{\underset{\sim}{p}}^{\dagger} a_{\underset{\sim}{p}'}^{\dagger} a_{\underset{\sim}{q}'} a_{\underset{\sim}{q}} e^{i(\underset{\sim}{q} + \underset{\sim}{q}' - \underset{\sim}{p} - \underset{\sim}{p}')\cdot\underset{\sim}{r}}$$

$$\times \left[v(\underset{\sim}{p} - \underset{\sim}{q}) + \frac{2}{3}(\underset{\sim}{p} - \underset{\sim}{q})^2 \frac{dv}{d(\underset{\sim}{p} - \underset{\sim}{q})^2} \right] . \tag{6.21}$$

Higher-order terms in the expansion, represented by ..., lead to higher-order gradients and are dropped in accordance with our assumption that the gradients are small.

Let us now return to (6.15). Here it is convenient to introduce the momentum

$$\underset{\sim}{\kappa} \equiv \underset{\sim}{k} - m \underset{\sim}{v}(\underset{\sim}{r}, t) \quad , \tag{6.22}$$

i.e., the momentum measured in the local rest frame. Upon using Eq. (6.22) in Eq. (6.15), we obtain

$$[K, \underset{\sim}{\mathcal{J}}(\underset{\sim}{r})] = i\hbar \nabla\cdot \Big[m\, n(\underset{\sim}{r}) \underset{\sim}{v}\, \underset{\sim}{v} - 2 \underset{\sim}{\mathcal{J}}'(\underset{\sim}{r}) \underset{\sim}{v} + (4m\mathcal{V})^{-1} \sum_{\underset{\sim}{\kappa}\underset{\sim}{\kappa}'} (\underset{\sim}{\kappa} + \underset{\sim}{\kappa}')(\underset{\sim}{\kappa} + \underset{\sim}{\kappa}') a_{\underset{\sim}{k}'}^{\dagger} a_{\underset{\sim}{k}} e^{i(\underset{\sim}{\kappa} - \underset{\sim}{\kappa}')\cdot\underset{\sim}{r}} \Big],$$

where $\underset{\sim}{\mathcal{J}}'(\underset{\sim}{r})$ is the momentum density operator in the local rest frame, i.e., with $\underset{\sim}{k}$ replaced by $\underset{\sim}{\kappa}$. When we take the statistical average, $\underset{\sim}{\mathcal{J}}'$ vanishes, since $\underset{\sim}{\mathcal{J}}' = \rho \underset{\sim}{v}' = 0$, and $(\underset{\sim}{\kappa} + \underset{\sim}{\kappa}')(\underset{\sim}{\kappa} + \underset{\sim}{\kappa}')$ can be replaced by $\frac{1}{3}(\underset{\sim}{\kappa} + \underset{\sim}{\kappa}')^2 \underset{\sim}{I}$. We then have

$$\langle [K, \underset{\sim}{\mathcal{J}}(\underset{\sim}{r})] \rangle = i\hbar [\nabla\cdot(\rho \underset{\sim}{v}\, \underset{\sim}{v}) + \tfrac{2}{3} \nabla \overline{\mathcal{K}}(r, t)] \quad , \tag{6.23}$$

where

$$\overline{\mathcal{K}}(\underset{\sim}{r}, t) \equiv (8m\mathcal{V})^{-1} \sum_{\underset{\sim}{\kappa}\underset{\sim}{\kappa}'} (\underset{\sim}{\kappa} + \underset{\sim}{\kappa}')^2 \langle a_{\underset{\sim}{k}'}^{\dagger} a_{\underset{\sim}{k}} \rangle e^{i(\underset{\sim}{\kappa} - \underset{\sim}{\kappa}')\cdot\underset{\sim}{r}} \quad .$$

Because we are allowed to average over a small microscopic volume, $\overline{\mathcal{K}}$ may be written as

$$\overline{\mathcal{K}}(\underset{\sim}{r}, t) = \mathcal{V}^{-1} \sum_{\underset{\sim}{k}} \frac{\kappa^2}{2m} \langle a_{\underset{\sim}{k}}^{\dagger} a_{\underset{\sim}{k}} \rangle \quad , \tag{6.24}$$

showing that $\overline{\mathcal{K}}$ is the local kinetic-energy density. Equations (6.23) and (6.20) may now be combined to yield the equation of change (6.4) for $\underset{\sim}{\mathcal{J}}$:

$$\frac{\partial \underset{\sim}{\mathcal{J}}(\underset{\sim}{r},\, t)}{\partial\, t} = -\nabla\cdot(\rho\, \underset{\sim}{v}\, \underset{\sim}{v}) - \nabla\left(\frac{2}{3}\,\overline{\mathcal{K}} + \overline{V}_{eff}\right) \quad . \tag{6.25}$$

We shall now show that the combination $\left(\frac{2}{3}\,\overline{\mathcal{K}} + \overline{V}_{eff}\right)$ is just the thermodynamic pressure p :

$$p = \frac{2}{3}\,\overline{\mathcal{K}} + \overline{V}_{eff} \quad . \tag{6.26}$$

Our proof will be restricted to our present approximation $\underset{\sim}{\rho} = \underset{\sim}{\rho}_{eq}$, for which the expressions $\overline{\mathcal{K}}$ and $\overline{V}_{eff}$ were defined. The generalization to include $\delta\, \underset{\sim}{\rho}$ will be given in another paper.[20]

For a system at rest and in thermodynamic equilibrium, the pressure is defined as

$$p = \theta\, \partial\, \ell n\, \mathcal{Z}/\partial\, \mathcal{V} \quad , \tag{6.27}$$

where

$$\mathcal{Z} = \mathrm{Tr}\, e^{-\beta(H - \mu N)} \quad , \tag{6.28}$$

and the volume (of quantization) is to be varied infinitesimally. Thus we have

$$p = \mathcal{Z}^{-1}\, \mathrm{Tr}\left[-\frac{\partial H}{\partial \mathcal{V}}\, e^{-\beta(H - \mu N)} \right]$$

or (6.29)

$$p = -\left\langle \frac{\partial H}{\partial \mathcal{V}} \right\rangle ,$$

since for thermodynamic equilibrium

$$\underset{\sim}{\rho} = \mathcal{Z}^{-1}\, e^{-\beta(H - \mu N)} \quad .$$

In calculating

$$\frac{\partial K}{\partial \mathcal{V}} = \frac{1}{2m} \sum_{\underset{\sim}{k}} \frac{\partial k^2}{\partial \mathcal{V}} a_{\underset{\sim}{k}}^{\dagger} a_{\underset{\sim}{k}} ,$$

we must realize that the values of $\underset{\sim}{k}$ are determined by the quantization volume $\mathcal{V}$, so that, for a given set of quantum numbers, we have

$$k^2 \propto \mathcal{V}^{-2/3} .$$

It follows immediately that

$$\frac{\partial K}{\partial \mathcal{V}} = -\frac{2}{3} \frac{K}{\mathcal{V}} . \tag{6.30}$$

Let us consider $\mathcal{V}$ as a small macroscopic volume of the fluid. In accord with the assumption of thermodynamic equilibrium in that volume, the fluid is at rest in the local rest frame. Hence, from Eqs. (6.30) and (6.24), we have

$$\left\langle \frac{\partial K}{\partial \mathcal{V}} \right\rangle = -\frac{2}{3} \overline{\mathcal{K}} . \tag{6.31}$$

To calculate $\partial V/\partial \mathcal{V}$ from Eq. (6.6), we note not only the explicit dependence on $\mathcal{V}$, but also the implicit dependence from $v(\underset{\sim}{k} - \underset{\sim}{n})$, since, as just noted, the values of $\underset{\sim}{k}$ depend upon $\mathcal{V}$. A short calculation yields

$$\left\langle \frac{\partial K}{\partial \mathcal{V}} \right\rangle = -\overline{V}_{eff} . \tag{6.32}$$

Upon summing Eqs. (6.32) and (6.31), we see that Eq. (6.29) leads to Eq. (6.26).

Thus the equation of motion (6.25) can be written

$$\frac{\partial \overline{\underset{\sim}{J}}}{\partial t} = -\nabla\cdot(\rho \underset{\sim}{v}\,\underset{\sim}{v}) - \nabla p \quad , \tag{6.33}$$

or, with the use of Eqs. (6.13) and (6.14),

$$\rho \frac{d\underset{\sim}{v}}{dt} = -\nabla p \quad , \tag{6.34}$$

where $d/dt \equiv \partial/\partial t + \underset{\sim}{v}\cdot\nabla$. Equation (6.34) is the Eulerian equation of motion, derived from the Liouville equation.

To complete the set of hydrodynamic equations, we need an equation for the entropy flow. For this purpose we introduce the energy-density operators:

$$\mathcal{U}(\underset{\sim}{r}) = \mathcal{U}_K(\underset{\sim}{r}) + \mathcal{U}_V(\underset{\sim}{r}) \quad , \tag{6.35}$$

where

$$\mathcal{U}_K(\underset{\sim}{r}) \equiv \mathcal{V}^{-1} \sum_{\underset{\sim}{k}\underset{\sim}{k}'} \frac{1}{2}\left(\frac{k'^2}{2m} + \frac{k^2}{2m}\right) a_{\underset{\sim}{k}'}^{\dagger}\, a_{\underset{\sim}{k}}\, e^{i(\underset{\sim}{k} - \underset{\sim}{k}')\cdot\underset{\sim}{r}} \tag{6.36}$$

and

$$\mathcal{U}_V(\underset{\sim}{r}) \equiv \frac{1}{2}\mathcal{V}^{-2} \sum_{\underset{\sim}{p}\underset{\sim}{p}'\underset{\sim}{q}\underset{\sim}{q}'} v(\underset{\sim}{p} - \underset{\sim}{q})\, a_{\underset{\sim}{p}}^{\dagger}\, a_{\underset{\sim}{p}'}^{\dagger}\, a_{\underset{\sim}{q}'}\, a_{\underset{\sim}{q}}\, e^{i(\underset{\sim}{q} + \underset{\sim}{q}' - \underset{\sim}{p} - \underset{\sim}{p}')\cdot\underset{\sim}{r}} \tag{6.37}$$

are the <u>total</u> kinetic-energy-density and potential-energy-density operators, respectively.

To find $\partial\overline{\mathcal{U}}/\partial t$ from Eq. (6.4), we need the commutators of $\mathcal{U}_K$ and $\mathcal{U}_V$ with K and V. Evaluating these by the relations of Appendix C and methods used above, we find

$$[K, \mathcal{U}_V] + [V, \mathcal{U}_K] = i\hbar(2m\mathcal{V}^2)^{-1} \nabla\cdot \left\{ \sum_{\underset{\sim}{pp'}\underset{\sim}{qq'}} a_{\underset{\sim}{p}}^\dagger a_{\underset{\sim}{p'}}^\dagger a_{\underset{\sim}{q'}} a_{\underset{\sim}{q}} \right.$$

$$\left. \times e^{i(\underset{\sim}{q} + \underset{\sim}{q'} - \underset{\sim}{p} - \underset{\sim}{p'})\cdot\underset{\sim}{r}} [v(\underset{\sim}{p} - \underset{\sim}{q})\underset{\sim}{I} + \frac{dv}{d(\underset{\sim}{p} - \underset{\sim}{q})^2}(\underset{\sim}{p} - \underset{\sim}{q})(\underset{\sim}{p} - \underset{\sim}{q})]\cdot(\underset{\sim}{p} + \underset{\sim}{q}) \right\} , \tag{6.38}$$

$$[K, \mathcal{U}_K] = i\hbar\mathcal{V}^{-1} \nabla\cdot \left\{ \sum_{\underset{\sim}{kk'}} \left(\frac{\underset{\sim}{k'} + \underset{\sim}{k}}{4m} \right)\left(\frac{k'^2}{2m} + \frac{k^2}{2m} \right) a_{\underset{\sim}{k'}}^\dagger a_{\underset{\sim}{k}} e^{i(\underset{\sim}{k} - \underset{\sim}{k'})\cdot\underset{\sim}{r}} \right\} , \tag{6.39}$$

and

$$[V, \mathcal{U}_V] = 0 .$$

In these expressions we make the rest-frame transformation (6.22), and take the statistical average. Our assumptions of slow spatial variation and local isotropy in the rest frame then yield, after some algebra,

$$\frac{\partial \bar{\mathcal{U}}}{\partial t} = -\nabla\cdot[(\bar{\mathcal{U}} + p)\underset{\sim}{v}] , \tag{6.40}$$

the equation of change for energy density, i.e., the equation of state. To convert Eq. (6.40) into more conventional form, we subtract from $\mathcal{U}$ the hydrodynamic energy density:

$$\mathcal{U} - \frac{1}{2}\rho v^2 \equiv \mathcal{U}_{th} , \tag{6.41}$$

so that $\mathcal{U}_{th}$ is the thermal energy density. Introducing the specific thermal energy u by $\mathcal{U}_{th} \equiv \rho u$, we may write Eq. (6.41) as

$$\rho \frac{du}{dt} = -p\nabla\cdot\underset{\sim}{v} , \tag{6.42}$$

with the help of the continuity equation (6.14) and the equation of

motion (6.34). This equation is equivalent to

$$\frac{du}{dt} + p \frac{d\rho^{-1}}{dt} = 0 ,$$

or

$$\frac{ds}{dt} = 0 , \tag{6.43}$$

where s is the specific entropy. Finally, the last equation, which expresses conservation of entropy, can be written in terms of the entropy density $\mathcal{S} = \rho s$ as

$$\frac{\partial \mathcal{S}}{\partial t} + \nabla\cdot(\underset{\sim}{v}\, \mathcal{S}) = 0 , \tag{6.44}$$

a continuity equation.

VII. APPLICATION OF TRANSPORT THEORY TO THE DEGENERATE BOSE-EINSTEIN SYSTEM

The discussion of the previous section is quite general. It applies to the reversible flow of any single-component quantum fluid in local thermodynamic equilibrium. In studying the Bogolyubov model of liquid helium, however, we have to consider instead quasi-quilibrium situations, as discussed in Section V. We recall that in the Bogolyubov model the existence of an unperturbed state with macroscopic occupation leads naturally to a two-fluid model. The velocity of the superfluid component is just $\underset{\sim}{v}_s = \underset{\sim}{k}_s/m$, where $\underset{\sim}{k}_s$ is the momentum of the macroscopically occupied state. The normal velocity $\underset{\sim}{v}_n$ represents the drift velocity of the excitations with respect to this state. To derive the equations of change for these velocities, we shall have to modify some of the discussion in the previous section.

First of all, the equation of continuity (6.11) is still valid,

$$\frac{\partial \rho}{\partial t} = - \nabla \cdot \underset{\sim}{\overline{\mathcal{J}}} \ , \tag{7.1}$$

but for ρ and $\underset{\sim}{\overline{\mathcal{J}}}$ we shall now use expressions (5.17) and (5.21):

$$\rho = \rho_n + \rho_s \ , \tag{7.2}$$

and

$$\underset{\sim}{\overline{\mathcal{J}}} = \rho_n \underset{\sim}{v}_n + \rho_s \underset{\sim}{v}_s \ . \tag{7.3}$$

Next we note that the equation of motion (6.33), which was based on the assumption of local isotropy, is no longer valid in quasi-equilibrium. The term ∇p should be generalized to the divergence $\nabla \cdot \underset{\sim}{p}$ of a stress tensor $\underset{\sim}{p}$, which will have a nonscalar part proportional to $\underset{\sim}{w}\,\underset{\sim}{w}$, where $\underset{\sim}{w} = \underset{\sim}{v}_n - \underset{\sim}{v}_s$. We shall, however, limit our discussion to <u>small-amplitude</u>

disturbances, with velocities so small that terms quadratic in them can be neglected. The equation of motion then reduces to

$$\frac{\partial \underset{\sim}{\mathcal{G}}}{\partial t} = - \nabla p \quad , \tag{7.4}$$

where the pressure is to be evaluated for $\underset{\sim}{w} = 0$, i.e., true local equilibrium.

Finally, we will modify Eq. (6.44) for the conservation of entropy and use in its place,

$$\frac{\partial \mathcal{S}}{\partial t} + \nabla \cdot [\underset{\sim}{v}_n \mathcal{S}] = 0 \quad . \tag{7.5}$$

The appearance of $\underset{\sim}{v}_n$ in this equation may be qualitatively described with the statement, "the entropy is associated only with the normal fluid." In Appendix D, a proof will be given of Eq. (7.5) in the Bogolyubov approximation, subject only to the condition that the normal and super velocities, $\underset{\sim}{v}_n$ and $\underset{\sim}{v}_s$, are small.

One more equation is needed, to determine the change with time of the relative velocity $\underset{\sim}{w} \equiv \underset{\sim}{v}_n - \underset{\sim}{v}_s$. This may be obtained by considering the momentum-density operator $\underset{\sim}{\mathcal{G}}(\underset{\sim}{r})$ (6.8) in a frame moving locally with the superfluid velocity. In that frame the momentum of the macroscopically occupied state $\underset{\sim}{k}_s$ is zero; we denote $\underset{\sim}{\mathcal{G}}(\underset{\sim}{r})$ for that frame as $\underset{\sim}{\mathcal{G}}_n(\underset{\sim}{r})$, and have, from Eq. (5.21),

$$\underset{\sim}{\mathcal{G}}_n(\underset{\sim}{r}) = \rho_n(\underset{\sim}{v}_n - \underset{\sim}{v}_s) \quad . \tag{7.6}$$

Interpreting the momentum label $\underset{\sim}{k}$ in Eq. (6.8) as referring to the moving frame, we may apply the Bogolyubov transformation (2.8), and obtain an expression for $\underset{\sim}{\mathcal{G}}_n$ in terms of $\alpha_{\underset{\sim}{k}}$, $\alpha_{\underset{\sim}{k}}^{+}$, the quasi-particle operators.

Terms independent of $\alpha_{\underset{\sim}{k}}$, $\alpha_{\underset{\sim}{k}}^{+}$ vanish by symmetry, while those in $\alpha_{\underset{\sim}{k}'}$ $\alpha_{\underset{\sim}{k}}$ and $\alpha_{\underset{\sim}{k}'}^{+}$ $\alpha_{\underset{\sim}{k}}^{+}$ do not contribute to the equation of change, as may be verified directly. Keeping only the terms which do contribute, we have

$$\underset{\sim}{\mathcal{J}}_n(\underset{\sim}{r}) = \mathcal{V}^{-1} \sum_{\underset{\sim}{k}\underset{\sim}{k}'} \frac{1}{2}(\underset{\sim}{k}' + \underset{\sim}{k})\alpha_{\underset{\sim}{k}'}^{+} \alpha_{\underset{\sim}{k}} e^{i(\underset{\sim}{k} - \underset{\sim}{k}')\cdot\underset{\sim}{r}} . \qquad (7.7)$$

(Alternatively, this expression could have been deduced from Eq. (5.4).)

Now in the laboratory frame the equation of change for $\overline{\underset{\sim}{\mathcal{J}}}$ is [from Eq. (6.4)]

$$\frac{\partial \overline{\underset{\sim}{\mathcal{J}}}}{\partial t} = i \langle [H, \underset{\sim}{\mathcal{J}}] \rangle .$$

Since N commutes with $\underset{\sim}{\mathcal{J}}$ (before the Bogolyubov transformation), we may replace H by H - μN in the commutator above. Then we go to the superfluid rest frame, replacing $\underset{\sim}{\mathcal{J}}$ by $\underset{\sim}{\mathcal{J}}_n$, and make the Bogolyubov approximation, replacing H - μN by H_B - μN :

$$\frac{\partial \overline{\underset{\sim}{\mathcal{J}}}_n}{\partial t} = i \langle [H_B - \mu N, \underset{\sim}{\mathcal{J}}_n] \rangle . \qquad (7.8)$$

(The transformation of $\partial/\partial t$ involves a correction of order $\underset{\sim}{v}_s \cdot \nabla$, which we neglect.) Because H_B - μN is just

$$\sum_{\underset{\sim}{k}}{}' \epsilon_k \alpha_{\underset{\sim}{k}}^{+} \alpha_{\underset{\sim}{k}}$$

plus terms independent of $\alpha_{\underset{\sim}{k}}$, the commutator of Eq. (7.8) is easy to evaluate, by Eq. (C.2):

$$[H_B - \mu N, \underset{\sim}{\mathcal{G}}_n] = \mathcal{V}^{-1} \sum_{\underset{\sim}{k}\underset{\sim}{k}'} \frac{1}{2}(\epsilon_{k'} - \epsilon_k)(\underset{\sim}{k}' + \underset{\sim}{k})\alpha_{\underset{\sim}{k}'}^{+}\alpha_{\underset{\sim}{k}} e^{i(\underset{\sim}{k} - \underset{\sim}{k}')\cdot\underset{\sim}{r}}$$

$$= i\nabla\cdot\left\{ \mathcal{V}^{-1} \sum_{\underset{\sim}{k}\underset{\sim}{k}'} \frac{\partial \epsilon_k}{\partial k^2} \underset{\sim}{k}\underset{\sim}{k} \alpha_{\underset{\sim}{k}'}^{+}\alpha_{\underset{\sim}{k}} e^{i(\underset{\sim}{k} - \underset{\sim}{k}')\cdot\underset{\sim}{r}} \right\} ,$$

under the assumption of slow spatial variation. We then use the assumption of isotropy (in the small-amplitude limit), and obtain for Eq. (7.8)

$$\frac{\partial \underset{\sim}{\mathcal{G}}_n}{\partial t} = -\nabla p_n \tag{7.9}$$

where

$$p_n \equiv \frac{2}{3} \mathcal{V}^{-1} \sum_{\underset{\sim}{k}} k^2 \frac{d\epsilon_k}{dk^2} \overline{\nu}_k \quad . \tag{7.10}$$

Upon evaluation, this is

$$p_n = 8(3\pi^2)^{-1} \mu^{5/2} m^{3/2} U_3(\gamma) \tag{7.11}$$

in the notation of Appendix A. We note that p_n is the last term in expression (3.33) for p.

We now have a complete set of equations of change--Eqs. (7.1), (7.4), (7.5), and (7.9)--to describe the small-amplitude motions of the fluid. They are supplemented by the thermodynamic expressions, Eq. (3.33) for pressure,

$$p = \frac{1}{2}\mu^2/v(0) + \frac{1}{4}\pi^{-2}\mu^2 m a^{-1}\{1 - \frac{32}{15}\mu^{1/2} m^{1/2} a[1 - 5U_3(\gamma)]\} , \tag{7.12}$$

Eq. (3.35) for entropy density,

$$\mathcal{S} = 16(3\pi^2\theta)^{-1} \mu^{7/2} m^{3/2} [-U_3'(\gamma)] , \tag{7.13}$$

Eq. (5.16) for ρ_n ,

$$\rho_n = 8(3\pi^2)^{-1} \rho(\, n\, a^3)^{1/2} (g\, m\, a^2)^{3/2} \gamma[\, - W_3'(\gamma)] \,, \tag{7.14}$$

Eq. (7.11) for p_n , and the algebraic relations (7.2), (7.3), and (7.6).

Let us simplify these equations, making further use of the small-amplitude assumption. Equations (7.1) and (7.4) combine to yield

$$\frac{\partial^2 \rho}{\partial t^2} = \nabla^2 p \quad . \tag{7.15}$$

Equation (7.5) becomes

$$\frac{\partial \mathcal{S}}{\partial t} = - \mathcal{S} \nabla \cdot \underset{\sim}{v}_n \quad , \tag{7.16}$$

and Eqs. (7.9) and (7.6) yield

$$\rho_n \frac{\partial}{\partial t} (\underset{\sim}{v}_n - \underset{\sim}{v}_s) = - \nabla p_n \quad . \tag{7.17}$$

It is convenient to consider temperature and density as the dependent variables. We thus write

$$\nabla p_n = \left.\frac{\partial p_n}{\partial \theta}\right|_\rho \nabla \theta + \left.\frac{\partial p_n}{\partial \rho}\right|_\theta \nabla \rho \quad . \tag{7.18}$$

From Eqs. (7.11) and (7.13) we note that

$$\left.\frac{\partial p_n}{\partial \theta}\right|_\rho = \mathcal{S} \,, \tag{7.19}$$

while the second term of Eq. (7.18), when evaluated, can be shown to give a negligible contribution to the results to be obtained below in the temperature range of Eq. (3.26) where our theory is valid. Dropping that term, then, we

may write Eq. (7.17) as

$$\rho_n \frac{\partial}{\partial t} (\underset{\sim}{v}_n - \underset{\sim}{v}_s) = - \mathcal{S} \nabla \theta \quad . \tag{7.20}$$

The three equations (7.15), (7.16), and (7.20) are now identical with those of the phenomenological two-fluid theory.[21]

In analogy to Eq. (7.18), we may express $\nabla^2 p$ in Eq. (7.15) as

$$\nabla^2 p = \left.\frac{\partial p}{\partial \theta}\right|_\rho \nabla^2 \theta + \left.\frac{\partial p}{\partial \rho}\right|_\theta \nabla^2 \rho$$

in the small-amplitude limit, and can now show that the first term makes a negligible contribution (to lowest order) because the temperature dependence of p in Eq. (7.12) is weak at low temperatures. Equation (7.15) then becomes

$$\frac{\partial^2 \rho}{\partial t^2} = c_I^2 \nabla^2 \rho \, , \tag{7.21}$$

where

$$c_I^2 = \left.\frac{\partial p}{\partial \rho}\right|_\theta \approx m^{-1} \mu_0 = c^2 \quad . \tag{7.22}$$

Thus density fluctuations propagate with a wave velocity equal to the phonon velocity of the small-momentum excitations. For these waves ("first sound") the temperature fluctuations are negligible, and, by Eq. (7.20), the two "fluids" are in phase:

$$\underset{\sim}{v}_n \approx \underset{\sim}{v}_s \quad . \tag{7.23}$$

Temperature waves, or "second sound," involve negligible density fluctuations, but appreciable temperature fluctuations. For these modes, by Eq. (7.1) $\underset{\sim}{\mathcal{J}}$ is negligible, so that by Eq. (7.3) we have

$$\underset{\sim}{v}_n \approx \frac{\rho_s}{\rho_n} \underset{\sim}{v}_s \ .$$

But in our theory, we have $\rho_n \ll \rho_s$ [see Eq. (7.14)], so that for second sound we have

$$v_n \gg v_s \ , \tag{7.24}$$

in contrast to (7.23) for first sound. Thus Eq. (7.20) becomes

$$\rho_n \frac{\partial \underset{\sim}{v}_n}{\partial t} = - \mathcal{S} \nabla \theta \ , \tag{7.25}$$

and in Eq. (7.16), we replace $\partial \mathcal{S} / \partial t$ by

$$\frac{\partial \mathcal{S}}{\partial \theta} \bigg|_\rho \frac{\partial \theta}{\partial t} \ ,$$

to obtain

$$\frac{\partial^2 \theta}{\partial t^2} = c_{II}^2 \nabla^2 \theta \ , \tag{7.26}$$

where

$$c_{II}^2 \equiv \frac{\mathcal{S}^2}{\rho_n \frac{\partial \mathcal{S}}{\partial \theta} \Big|_\rho} \ . \tag{7.27}$$

(It should be pointed out that the various approximations made after Eq. (7.17) are not necessary. One could solve Eqs. (7.15) to (7.17) directly for the normal modes, and then note from the final results for the two propagation velocities that these approximations are indeed valid.)

We may evaluate c_{II}, from Eqs (7.13) and (7.14), as a function of temperature. For the two limiting cases, (1) $\theta \ll \mu_0$ and (2) $\mu_0 \ll \theta \ll \theta_0$,

we find

$$(1) \qquad c_{II}^{\,2} = \frac{1}{3}\, c_{I}^{\,2} \,, \tag{7.28}$$

a familiar result, and

$$(2) \qquad c_{II}^{\,2} = \frac{16}{15}\, \frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})}\, \frac{\mu_0}{\theta}\, c_{I}^{\,2} \,, \tag{7.29}$$

where ζ is the Riemann zeta function.

APPENDIXES

A. Integrals

A discussion of the statistical mechanical properties of the Bogolyubov Hamiltonian involves the functions

$$U_n(\gamma) = \int_0^\infty dy \, \frac{x^n}{e^{\gamma y} - 1} , \tag{A.1}$$

where $x = k/k_0$ and $y = x(x^2 + 1)^{1/2}$, k_0 and γ having been defined by Eqs. (3.22) and (3.24), respectively. In the "low temperature" limit ($\gamma \gg 1$), we have $x \approx y$, and the Debye functions are obtained. In the "high temperature" limit ($\gamma \ll 1$), we have $x \approx y^{1/2}$, and these functions go over to the Bose-Einstein integrals. The functions $U_n(\gamma)$ can therefore be obtained in these limits in terms of the Riemann ζ functions:

$$U_n(\gamma) = \frac{\Gamma(\frac{n}{2} + 1)\, \zeta(\frac{n}{2} + 1)}{\gamma^{n/2 + 1}} \tag{A.2}$$

for $\gamma \ll 1$ and

$$U_n(\gamma) = \frac{\Gamma(n + 1)\, \zeta(n + 1)}{\gamma^{n+1}} \tag{A.3}$$

for $\gamma \gg 1$.

A similar function that is needed is

$$W_n(\gamma) = \int_0^\infty dy \, \frac{x^n}{e^{\gamma y} - 1} \, \frac{1}{2x^2 + 1} . \tag{A.4}$$

For large γ, the extra factor $(2x^2 + 1)^{-1}$ may be ignored, and the asymptotic form is the same as $U_n(\gamma)$:

$$W_n(\gamma) \longrightarrow U_n(\gamma) . \tag{A.5}$$

For small γ , the extra factor is $(2x^2)^{-1}$; so we have

$$W_n(\gamma) \longrightarrow \frac{1}{2} U_{n-2}(\gamma) \quad . \tag{A.6}$$

B. Statistical Properties above the Transition Temperature

Using the recently developed linked-diagram expansions,[13] we shall briefly discuss the properties of the system above the transition temperature. In this method, the grand partition function is written

$$\mathcal{Z} = \mathcal{Z}_0 \exp(Q_1 + Q_2 + \dots) \quad , \tag{B.1}$$

where $\mathcal{Z}_0$ is the function for an ideal gas,

$$\mathcal{Z}_0 = \prod_{\underset{\sim}{k}} (1 + \overline{\nu}_k) \quad , \tag{B.2}$$

and Q_1, Q_2, ... give the effect of interactions in a linked-diagram perturbation series.

In our case of a dilute gas with weak interactions, we need keep only the leading term Q_1, which is[13]

$$Q_1 = -\frac{1}{2}\beta \mathcal{V}^{-1} \sum_{\underset{\sim}{k},\underset{\sim}{k}'} [v(\underset{\sim}{k} - \underset{\sim}{k}') + v(0)] \overline{\nu}_k \overline{\nu}_{k'} \quad . \tag{B.3}$$

Upon evaluating the mean number of particles $\overline{N} = -\partial\Omega/\partial\mu$, we obtain

$$\overline{N} = \sum_{\underset{\sim}{k}} \overline{\nu}_k \{ 1 - \beta \mathcal{V}^{-1} \sum_{\underset{\sim}{k}'} [v(\underset{\sim}{k} - \underset{\sim}{k}') + v(0)] \overline{\nu}_{k'}(1 + \overline{\nu}_{k'}) \} \quad . \tag{B.4}$$

At low temperatures we may approximate

$$v(\underset{\sim}{k} - \underset{\sim}{k}') \approx v(0) \quad ,$$

and then can express (B.5) in terms of the Bose-Einstein integral[22]

$$F_{3/2}(\alpha) = \frac{1}{\Gamma(\frac{3}{2})} \int_0^\infty \frac{dz\, z^{1/2}}{e^{z+\alpha} - 1} , \qquad (B.5)$$

and the thermal wave length $\lambda = 2\pi \lambda\!\!\!^{-}$ [Eq. (4.14)]:

$$n \lambda^3 = F_{3/2}(\alpha) [1 + A F_{3/2}'(\alpha)] . \qquad (B.6)$$

The quantities α and A are

$$\alpha = -\beta \mu \qquad (B.7)$$

and

$$A = 2n \beta v(0) \lambda^{-3} . \qquad (B.8)$$

Equation (B.6) has a solution for all temperatures Θ greater than some "transition" temperature Θ_c, determined by

$$n \lambda_c^3 = \max_\alpha \{ F_{3/2}(\alpha) [1 + A F_{3/2}'(\alpha)] \} . \qquad (B.9)$$

For the ideal gas, A vanishes, and one finds[22] for the transition temperature Θ_{ci}

$$n \lambda_{ci}^3 = F_{3/2}(0) = 2.612 . \qquad (B.10)$$

For weak repulsive interactions, the maximum occurs at non-zero α. From the expansion[22] of α:

$$F_{3/2}(\alpha) = -3.545 \alpha^{1/2} + 2.612 + 1.460 \alpha - 0.104 \alpha^2 + \ldots \qquad (B.11)$$

we find that the maximum is at

$$\alpha = \frac{1}{2} F_{3/2}(0) \quad A \quad . \tag{B.12}$$

From Eqs. (B.7), (B.8), and (B.10), this implies that, just above θ_c,

$$\mu \approx - n v(0) \tag{B.13}$$

and, from (B.9) and (B.11), that the shift in θ_c is

$$\frac{\theta_c - \theta_{ci}}{\theta_{ci}} = 4.45 (\beta_{ci} n v(0))^{1/2} \quad . \tag{B.14}$$

We have thus shown that a Bose gas with weak repulsive interactions has a phase transition at a temperature slightly higher than that for an ideal gas.

C. Commutation Relations

We list here several commutation relations which are needed in Sec. VI. Let $s_{\underset{\sim}{k}}$ and $e_{\underset{\sim}{k}}$ be two functions of momentum $\underset{\sim}{k}$. We define

$$\Gamma_1(\underset{\sim}{r}) \equiv \mathcal{V}^{-1} \sum_{\underset{\sim}{k}\underset{\sim}{k}'} \frac{1}{2}(s_{\underset{\sim}{k}} + s_{\underset{\sim}{k}'})\, a_{\underset{\sim}{k}'}^{\dagger}\, a_{\underset{\sim}{k}}\, e^{i(\underset{\sim}{k} - \underset{\sim}{k}')\cdot\underset{\sim}{r}} \tag{C.1}$$

and

$$D_1 \equiv \sum_{\underset{\sim}{k}} e_{\underset{\sim}{k}}\, a_{\underset{\sim}{k}}^{\dagger}\, a_{\underset{\sim}{k}} .$$

Using the relation

$$[a_{\underset{\sim}{\ell}}^{\dagger} a_{\underset{\sim}{\ell}},\ a_{\underset{\sim}{k}'}^{\dagger} a_{\underset{\sim}{k}}] = (\delta_{\underset{\sim}{\ell}\underset{\sim}{k}'} - \delta_{\underset{\sim}{\ell}\underset{\sim}{k}}) a_{\underset{\sim}{k}'}^{\dagger} a_{\underset{\sim}{k}} ,$$

which is valid for either Bose-Einstein or Fermi-Dirac statistics, we find that

$$[D_1, \Gamma_1(\underset{\sim}{r})] = \frac{1}{2}\, \mathcal{V}^{-1} \sum_{\underset{\sim}{k}\underset{\sim}{k}'} (e_{\underset{\sim}{k}'} - e_{\underset{\sim}{k}})(s_{\underset{\sim}{k}'} + s_{\underset{\sim}{k}}) a_{\underset{\sim}{k}'}^{\dagger}\, a_{\underset{\sim}{k}}\, e^{i(\underset{\sim}{k} - \underset{\sim}{k}')\cdot\underset{\sim}{r}} . \tag{C.2}$$

Next we define

$$D_2 \equiv \frac{1}{2}\, \mathcal{V}^{-1} \sum_{\underset{\sim}{p}\underset{\sim}{p}'\underset{\sim}{q}\underset{\sim}{q}'} v(\underset{\sim}{p} - \underset{\sim}{q})\, a_{\underset{\sim}{p}}^{\dagger}\, a_{\underset{\sim}{p}'}^{\dagger}\, a_{\underset{\sim}{q}'}\, a_{\underset{\sim}{q}}\, \delta_{\underset{\sim}{p}+\underset{\sim}{p}',\underset{\sim}{q}+\underset{\sim}{q}'} , \tag{C.3}$$

and obtain

$$[D_2, \Gamma_1(\underset{\sim}{r})] = \frac{1}{4}\, \mathcal{V}^{-2} \sum_{\underset{\sim}{p}\underset{\sim}{p}'\underset{\sim}{q}\underset{\sim}{q}'} a_{\underset{\sim}{p}}^{\dagger}\, a_{\underset{\sim}{p}'}^{\dagger}\, a_{\underset{\sim}{q}'}\, a_{\underset{\sim}{q}}\, e^{i(\underset{\sim}{q} + \underset{\sim}{q}' - \underset{\sim}{p} - \underset{\sim}{p}')\cdot\underset{\sim}{r}}$$

$$\times \left[v(\underset{\sim}{p} - \underset{\sim}{q})(s_{\underset{\sim}{q}'} + s_{\underset{\sim}{p}+\underset{\sim}{p}'-\underset{\sim}{q}} - s_{\underset{\sim}{p}'} - s_{\underset{\sim}{q}+\underset{\sim}{q}'-\underset{\sim}{p}}) \right.$$

$$\left. + v(\underset{\sim}{p}' - \underset{\sim}{q}')(s_{\underset{\sim}{q}} + s_{\underset{\sim}{p}+\underset{\sim}{p}'-\underset{\sim}{q}'} - s_{\underset{\sim}{p}} - s_{\underset{\sim}{q}+\underset{\sim}{q}'-\underset{\sim}{p}'}) \right] . \tag{C.4}$$

One more operator which has to be considered is

$$\Gamma_2(\underset{\sim}{r}) \equiv \frac{1}{2}\,\mathcal{V}^{-1} \sum_{\underset{\sim}{p}\underset{\sim}{p}'\underset{\sim}{q}\underset{\sim}{q}'} v(\underset{\sim}{p}-\underset{\sim}{q})\, a_{\underset{\sim}{p}}^{\dagger}\, a_{\underset{\sim}{p}'}^{\dagger}\, a_{\underset{\sim}{q}'}\, a_{\underset{\sim}{q}}\, e^{i(\underset{\sim}{q}+\underset{\sim}{q}'-\underset{\sim}{p}-\underset{\sim}{p}')\cdot\underset{\sim}{r}} \,.$$

Its commutators with D_1 and D_2 are

$$[D_1, \Gamma_2(\underset{\sim}{r})] = \frac{1}{2}\,\mathcal{V}^{-1} \sum_{\underset{\sim}{p}\underset{\sim}{p}'\underset{\sim}{q}\underset{\sim}{q}'} v(\underset{\sim}{p}-\underset{\sim}{q})\, a_{\underset{\sim}{p}}^{\dagger}\, a_{\underset{\sim}{p}'}^{\dagger}\, a_{\underset{\sim}{q}'}\, a_{\underset{\sim}{q}}(e_{\underset{\sim}{p}}+e_{\underset{\sim}{p}'}-e_{\underset{\sim}{q}}-e_{\underset{\sim}{q}'})$$

$$\times\, e^{i(\underset{\sim}{q}+\underset{\sim}{q}'-\underset{\sim}{p}-\underset{\sim}{p}')\cdot\underset{\sim}{r}} \qquad \text{(C.6)}$$

and

$$[D_2,\ \Gamma_2(\underset{\sim}{r})\ ] = 0 \,. \qquad \text{(C.7)}$$

D. The Equation for Entropy Change

We give here a direct proof of Eq. (7.5), using the Bogolyubov model. In obtaining this, we shall assume that the velocities $\underset{\sim}{v}_s$ and $\underset{\sim}{v}_n$ are small, so second-order terms in them may be dropped.

We need first a general form for $H_B - \mu N$. Let a momentum state $\underset{\sim}{k}$ be written as

$$\underset{\sim}{k} = \underset{\sim}{\kappa} + m \underset{\sim}{v}_s \quad , \tag{D.1}$$

where $\underset{\sim}{\kappa} = 0$ refers to the macroscopically occupied state. The kinetic energy of a particle is therefore

$$\frac{k^2}{2m} \approx \frac{\kappa^2}{2m} + \underset{\sim}{v}_s \cdot \underset{\sim}{\kappa} \tag{D.2}$$

to first order. This expression leads to the Bogolyubov "Hamiltonian"

$$H_B - \mu N = \sum_{\underset{\sim}{\kappa}}{}' (\epsilon_\kappa + \underset{\sim}{v}_s \cdot \underset{\sim}{\kappa}) \alpha_{\underset{\sim}{\kappa}}^+ \alpha_{\underset{\sim}{\kappa}} \quad , \tag{D.3}$$

where the ground-state "energy" has been omitted.

The density matrix $\underset{\sim}{\rho}$ for the case $\underset{\sim}{v}_n \neq \underset{\sim}{v}_s \neq 0$ can be obtained from Eq. (5.2), which pertains to the case $\underset{\sim}{v}_s = 0$, $\underset{\sim}{v}_n \neq 0$. The former case can be obtained from the latter by means of a Galilean transformation, which leads to the replacement[17]

$$H_B - \mu N - \underset{\sim}{w} \cdot \underset{\sim}{G} \rightarrow H_B - \mu N - \underset{\sim}{w} \cdot \underset{\sim}{G} - \underset{\sim}{v}_s \cdot \underset{\sim}{G} = H_B - \mu N - \underset{\sim}{v}_n \cdot \underset{\sim}{G} \quad .$$

The density matrix is thus

$$\begin{aligned} \underset{\sim}{\rho} &= \mathcal{Z}^{-1} \exp[-\beta (H_B - \mu N - \underset{\sim}{v}_n \cdot \underset{\sim}{G})] \\ &= \mathcal{Z}^{-1} \exp[-\beta \sum_{\underset{\sim}{\kappa}}{}' (\epsilon_\kappa - \underset{\sim}{w} \cdot \underset{\sim}{\kappa}) \alpha_{\underset{\sim}{\kappa}}^+ \alpha_{\underset{\sim}{\kappa}}] \quad , \end{aligned} \tag{D.4}$$

and $\mathcal{Z}$ is determined by

$$\mathrm{Tr}\ \underset{\sim}{\rho} = 1 \ .$$

The entropy of the system is

$$S(\underset{\sim}{w}) = -\mathrm{Tr}(\underset{\sim}{\rho}\ \ell n\ \underset{\sim}{\rho})$$

$$= \beta \sum_{\underset{\sim}{\kappa}}{}' (\epsilon_\kappa - \underset{\sim}{w}\cdot\underset{\sim}{\kappa})\, \overline{\nu}_{\underset{\sim}{\kappa}}(\underset{\sim}{w}) - \sum_{\underset{\sim}{\kappa}}{}' \ell n[1 - e^{-\beta(\epsilon_\kappa - \underset{\sim}{w}\cdot\underset{\sim}{\kappa})}] \ , \tag{D.5}$$

where

$$\overline{\nu}_{\underset{\sim}{\kappa}}(\underset{\sim}{w}) = [e^{\beta(\epsilon_\kappa - \underset{\sim}{w}\cdot\underset{\sim}{\kappa})} - 1]^{-1} \ . \tag{D.6}$$

For small $|\underset{\sim}{w}|$, we expand $S(\underset{\sim}{w})$ to linear terms, perform some integrations by parts, and write

$$S(\underset{\sim}{w}) = \sum_{\underset{\sim}{\kappa}}{}' I(\underset{\sim}{\kappa}, \underset{\sim}{w})\, \overline{\nu}_{\underset{\sim}{\kappa}}(\underset{\sim}{w}) \ , \tag{D.7}$$

where

$$I(\underset{\sim}{\kappa}, \underset{\sim}{w}) = \beta^2 \epsilon_\kappa (\tfrac{1}{3}\kappa \frac{d\epsilon_\kappa}{d\kappa} + \underset{\sim}{w}\cdot\underset{\sim}{\kappa})(1 - e^{-\beta\epsilon_\kappa})^{-1} \ . \tag{D.8}$$

Using Eq. (D.7) as a guide, we define the operator

$$\mathcal{S} \equiv \mathcal{V}^{-1} \sum_{\underset{\sim}{\kappa}\underset{\sim}{\kappa}'} \tfrac{1}{2}[I(\underset{\sim}{\kappa}', \underset{\sim}{w}) + I(\underset{\sim}{\kappa}, \underset{\sim}{w})]\, \alpha_{\underset{\sim}{\kappa}'}^{+}\, \alpha_{\underset{\sim}{\kappa}}\, e^{i(\underset{\sim}{\kappa} - \underset{\sim}{\kappa}')\cdot\underset{\sim}{r}} \ , \tag{D.9}$$

with the property that its average,

$$\mathcal{S} \equiv \mathrm{Tr}(\underset{\sim}{\rho}\ \mathcal{S}) \ ,$$

is the entropy density for equilibrium $\underset{\sim}{\rho}$.

The equation of change for $\overline{\mathscr{S}}$ is [see Eq. (7.8)]

$$\frac{\partial \overline{\mathscr{S}}}{\partial t} = i \langle (H_B - \mu N, \mathscr{S}) \rangle \ , \tag{D.10}$$

where Eqs. (D.3) and(D.9) are to be used in the commutator. The evaluation proceeds as in Sections VI and VII, and we obtain

$$\frac{\partial \overline{\mathscr{S}}}{\partial t} = - \nabla \cdot \{ \mathcal{V}^{-1} \sum_{\underset{\sim}{\kappa}} \left(\frac{\underset{\sim}{\kappa}}{\kappa} \frac{d\epsilon_\kappa}{d\kappa} + \underset{\sim}{v}_s \right) I(\underset{\sim}{\kappa}, \underset{\sim}{w}) \overline{\nu}_{\underset{\sim}{\kappa}}(\underset{\sim}{w}) \} \ . \tag{D.11}$$

(The parameter $\underset{\sim}{v}_s$ that appears here is now a function of position and time, because of the statistical averaging with $\underset{\sim}{\rho}$.) The second term in the curly bracket of (D.11) is just $\underset{\sim}{v}_s \overline{\mathscr{S}}$ by Eq. (D.9). From Eq. (D.8), the first term is, to lowest nonvanishing order,

$$\mathcal{V}^{-1} \sum_{\underset{\sim}{\kappa}} \frac{\underset{\sim}{\kappa}}{\kappa} \frac{d\epsilon_\kappa}{d\kappa} \beta^2 \epsilon_\kappa \underset{\sim}{w} \cdot \underset{\sim}{\kappa} (1 - e^{-\beta\epsilon_\kappa})^{-1} \overline{\nu}_{\underset{\sim}{\kappa}} \ .$$

Upon averaging over the directions of $\underset{\sim}{\kappa}$ in the sum, we obtain

$$\mathcal{V}^{-1} \frac{1}{3} \beta^2 \underset{\sim}{w} \sum_{\underset{\sim}{\kappa}} \kappa \frac{d\epsilon_\kappa}{d\kappa} \epsilon_\kappa (1 - e^{-\beta\epsilon_\kappa})^{-1} \overline{\nu}_{\underset{\sim}{\kappa}} = \underset{\sim}{w} \overline{\mathscr{S}} \ ,$$

so that Eq. (D.11) is

$$\begin{aligned} \frac{\partial \overline{\mathscr{S}}}{\partial t} &= - \nabla \cdot [(\underset{\sim}{w} + \underset{\sim}{v}_s) \overline{\mathscr{S}}] \\ &= - \nabla \cdot (\underset{\sim}{v}_n \overline{\mathscr{S}}) \ , \end{aligned} \tag{D.12}$$

as stated in Eq. (7.5).

FOOTNOTES

1. N. N. Bogolyubov, J. Phys. (U. S. S. R.) 11, 23 (1947).

2. T. D. Lee and C. N. Yang, Phys. Rev. 112, 1419 (1959), and previous papers cited there.

3. F. London, Phys. Rev. 54, 947 (1938).

4. L. D. Landau, J. Phys. (U. S. S. R.) 5, 71 (1941).

5. In most of this paper the following conventions will be adopted. First, $\hbar$ will be set equal to unity. A bar or angular brackets will mean statistical average, as defined in Section 3. A prime after a summation symbol, such as occurs in Eq. (2.6), indicates that the term for $\underset{\sim}{k} = 0$ is to be omitted.

6. See also L. D. Landau and E. M. Lifshitz, Statistical Physics (Pergamon Press, London, 1958), Chap. VII, p. 240.

7. See for example p. 105 of reference 6.

8. Equation (3.7) may also be obtained from the familiar formula $N = \theta \mathcal{Z}^{-1} \frac{\partial \mathcal{Z}}{\partial \mu}$, which may be obtained from Eq. (3.2) for $F = N$.

9. In the limit of zero temperature, Eq. (3.7) agrees with the result of Lee, Huang, and Yang [Phys. Rev. 106, 1135 (1957)]. However, the third term of Eq. (3.7), which is important at finite temperatures, differs significantly from the recent result of Lee and Yang [reference (2)]. These authors use the equation

$$\overline{N} = \overline{N}_0 + \sum_{\underset{\sim}{k}}{}' \nu_{\underset{\sim}{k}} ,$$

which implies that the number of excitations, $\Sigma' \alpha_{\underset{\sim}{k}}^{+} \alpha_{\underset{\sim}{k}}$, equals the number of particles in excited unperturbed states, $\Sigma' a_{\underset{\sim}{k}}^{+} a_{\underset{\sim}{k}}$. In a consistent theory this is strictly true only if the interactions are completely ignored.

10. This is the lowest-order chemical potential obtained by N. M. Hugenholtz and D. Pines [Phys. Rev. 116, 489 (1959)]. Although these authors did not study the thermal behavior of the system, they used essentially the same statistical procedure used here to determine μ and $\bar{N}$.

11. In order to simplify the notation, the bar will henceforth be omitted from $\bar{n}_0$ and $\bar{N}_0$, as in Eq. (3.13).

12. T. D. Lee, K. Huang, and C. N. Yang, Phys. Rev. 106, 1135 (1957).

13. C. Bloch and C. de Dominicis, Nuclear Phys. 7, 459 (1958); A. E. Glassgold, W. Heckrotte, and K. M. Watson, Phys. Rev. 115, 1374 (1959).

14. This is essentially the point of view of Landau [reference (4)] and more recently, T. D. Lee and C. N. Yang [Phys. Rev. 113, 1406 (1959)].

15. The latter form, Eq. (15.5), is due to R. B. Dingle, Advances in Physics, 1, 112 (1952).

16. See, for example, paragraph 34, p. 103, of reference 6.

17. J. B. Blatt, S. T. Butler, and M. S. Schafroth, Phys. Rev. 100, 481 (1955).

18. A similar result holds for Fermi-Dirac particles interacting through pairing forces, as in the theory of superconductivity. The only difference is the replacement of minus signs in Eq. (5.35) by plus signs.

19. In this section only, we do not set $\hbar = 1$.

20. A. E. Glassgold, A. N. Kaufman, and K. M. Watson, in preparation.

21. L. D. Landau and E. M. Lifshitz, Fluid Mechanics, Chapter XVI (Pergamon Press, London, 1959).

22. F. London, Superfluids (John Wiley and Sons, Inc., New York, N. Y., 1955).