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# ON COMPARING COHERENT SYSTEMS WITH HETEROGENEOUS COMPONENTS

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**Abstract**

In this paper we investigate different methods that may be used to compare coherent systems having heterogeneous components. We consider both the case of systems with independent components and the case of systems with dependent components. In the first case, the comparisons are based on the new concept of the survival signature due to Coolen and Coolen-Maturi (2012) which extends the well-known concept of system signatures to the case of components with lifetimes that need not be independent and identically distributed. In the second case, the comparisons are based on the concept of distortion functions. A graphical procedure (called an RR-plot) is proposed as an alternative to the analytical methods when there are two types of components.

*Keywords:* Coherent system; system signature; survival signature; distorted distribution; stochastic order

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## 1. Introduction

In this paper we deal with two specific analytical methods for comparing the reliability functions of two coherent systems (defined as in Barlow and Proschan [4]) whose components are assumed to have independent but not identically distributed lifetimes. This problem has been quite successfully treated using 'system signatures' as a tool under the more restrictive assumption that the components of both systems have lifetimes that are independent and identically distributed (i.i.d.) with a common reliability function $\overline{F}(t) = \mathbb{P}(X > t)$. Details on that work may be found in Samaniego [27]; it will be briefly reviewed in Section 2 of this paper. The development of useful expressions and representations of the reliability function of a coherent system when component lifetimes are independent but not identically distributed (i.n.i.d.) has been a challenging problem that has been approached heretofore by opportunistic calculations and through the use of elementary tools such as system structure functions. Our purpose here is to describe two new approaches to the problem which show substantial promise for advancing the state of the current methodology for system comparisons. Worthy of special mention is the fact that the distortion function method treated in Sections 4 and 5 of the paper is shown to extend to the case in which conponent lifetimes are dependent.

The first method we present involves an extension of the notion of system signatures to the i.n.i.d. case; it is built upon a recent breakthrough in this area by Coolen and Coolen-Maturi [7].

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The second method uses the notion of distortion functions (see [10]) to exploit the functional form of the reliability functions of coherent systems with i.n.i.d. components in developing the comparisons of interest. The two methods differ from each other quite significantly, both in the mathematical ideas involved and in the manner in which the reliability functions of the systems of interest are treated. The signature-based approach is constructive; its main purpose is to construct a representation of the reliability function using a method that is applicable to coherent systems having arbitrary size and structure and based on independent heterogeneous components. The second approach assumes that the reliability functions of such systems have been obtained (by whatever means possible; see, e.g. [18]) and it exploits the character of the associated distortion functions in comparing the reliability of two or more systems.

The complementary purposes of these two approaches suggest an interesting potential synergy between them. While both approaches can be used directly to compare two systems of interest, it is clear that the main strength of the first approach lies in obtaining an explicit representation of a system's reliability function, while the second approach focuses on comparing two reliability functions once both have been obtained in closed form. In the sections that follow, we will treat each of the two approaches in turn, giving examples of how each may be used to compare the reliability of two coherent systems in i.n.i.d. components. In Section 5 we provide some guidance on how the problem of comparing systems with independent heterogeneous components may be handled by numerical or graphical means when an analytical solution to the problem proves intractable. In Section 6 we discuss some possible directions for future research.

We close this section with a collection of ideas that will be employed in the sequel. Throughout this paper, we say that a function $h\colon \mathbb{R}^n \to \mathbb{R}$ is increasing (decreasing) if $h(x_1, \dots, x_n) \le h(y_1, \dots, y_n)$ $(\ge)$ for all $x_i \le y_i$, $i = 1, \dots, n$. Analogously, if $g, h$ are two functions $g, h\colon S \to \mathbb{R}$, then $g \le h$ means $g(z) \le h(z)$ for all $z \in S$.

We shall study the following stochastic orders. Their basic properties and applications to reliability studies can be seen in [4] and [29]. Let $F$ and $G$ be the distribution functions of two random variables $X$ and $Y$ with respective reliability functions $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$. We have the following:

(i) $X$ is said to be smaller than $Y$ in the usual stochastic order (denoted by $X \le_{\mathrm{ST}} Y$) if $\overline{F}(t) \le \overline{G}(t)$ for all $t$;

(ii) $X$ is said to be smaller than $Y$ in the hazard rate order (denoted by $X \le_{\mathrm{HR}} Y$) if $\overline{G}(t)/\overline{F}(t)$ is increasing in $t$;

(iii) $X$ is said to be smaller than $Y$ in the reversed hazard rate order (denoted by $X \le_{\mathrm{RH}} Y$) if $G(t)/F(t)$ is increasing in $t$;

(iv) $X$ is said to be smaller than $Y$ in the likelihood ratio order (denoted by $X \le_{\mathrm{LR}} Y$) if $g(t)/f(t)$ is increasing in $t$, where $g = G'$ and $f = F'$ are the respective probability density functions.

The following relationships are well known (see [29]):

$$\begin{array}{ccc} X \le_{\mathrm{LR}} Y & \Longrightarrow & X \le_{\mathrm{HR}} Y \\ \Downarrow & & \Downarrow \\ X \le_{\mathrm{RH}} Y & \Longrightarrow & X \le_{\mathrm{ST}} Y. \end{array}$$

## 2. Comparisons based on the survival signature

As described in Barlow and Proschan [4], the structure function $\psi$ of an $n$-component system maps the state vector $\boldsymbol{x} \in \{0, 1\}^n$ of the components of a given system (where '1' signifies 'working') to the state of the system $\psi(\boldsymbol{x}) \in \{0, 1\}$. A coherent system is one in which every component affects the working or failure of the system and for which the structure function is monotone in every component. While structure functions do characterize coherent systems, they have not proven very useful in comparative studies.

Samaniego [25] introduced an alternative index for coherent systems that, while narrower in scope than the structure function, is substantially more useful. Assuming that the lifetimes of the system's components are i.i.d., the signature $\boldsymbol{s}$ of a coherent system of order $n$ is the $n$-dimensional probability vector whose $i$th element is $s_i = \mathbb{P}(T = X_{i:n})$, where $T$ is the system lifetime and $X_{1:n}, \dots, X_{n:n}$ are the order statistics of the $n$ i.i.d. component lifetimes $X_1, \dots, X_n$. Under the i.i.d. assumption, the signature vector is a distribution-free function that constitutes a pure measure of the system's design. The i.i.d. assumption has the effect of 'leveling the playing field' among systems, eliminating anomalies such as the fact that a series system in good components can outperform a parallel system in poor components even though that latter system is clearly 'better' from a design point of view. The utility of signatures derives from the fact that combinatorial mathematics is applicable in their computation and, as is made clear below, the theory of order statistics for i.i.d. samples from a common continuous distribution $F$ is applicable for identifying the system's lifetime characteristics. Some examples of signatures: the signature $\boldsymbol{s} = (\frac{1}{3}, \frac{2}{3}, 0)$ of the 3-component coherent system with minimal cut sets $\{1\}$ and $\{2, 3\}$, the signature $\boldsymbol{s} = (0, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, 0)$ of the widely referenced 5-component bridge system, and the signature $\boldsymbol{s} = (0, \dots, 0, 1_k, 0, \dots, 0) \in [0, 1]^n$ of the $k$-out-of-$n$ system (which fails upon the $k$th component failure). Explicit expressions to obtain $\boldsymbol{s}$ from $\psi$ can be found in [5], [14], and [27]. Extensions of signatures to the case of two systems with shared components are treated in [11], [17], and [19].

The utility of signatures is evident from the following representation theorem drawn from [25]. It shows that the lifetime reliability of a mixed system (i.e. a stochastic mixture of coherent systems) based on $n$ components with i.i.d. lifetimes having common reliability function $\overline{F}$ can be written as a function of the system's signature and the underlying distribution of its components. This result can be stated as follows.

Consider a system of order $n$ based on components with i.i.d. lifetimes $X_1, \dots, X_n$ distributed according to a common lifetime distribution $F$. Let $T$ be the system's lifetime. Then

$$\overline{F}_T(t) \equiv \mathbb{P}(T > t) = \sum_{i=1}^{n} s_i \overline{F}_{i:n}(t) = \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} F^j(t) \overline{F}^{n-j}(t), \tag{2.1}$$

where $\overline{F}(t) = 1 - F(t)$ and $\boldsymbol{s} = (s_1, \dots, s_n)$ is the system's signature vector.

Signature vectors have proven especially useful when comparing the performance of competing systems. For example, Kochar *et al.* [12] established the following preservation results.

Let $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ be the signatures of two systems of order $n$, both based on components with i.i.d. lifetimes with common distribution $F$. Let $T_1$ and $T_2$ be their lifetimes.

(i) If $\boldsymbol{s}_1 \le_{\text{ST}} \boldsymbol{s}_2$ then $T_1 \le_{\text{ST}} T_2$.

(ii) If $\boldsymbol{s}_1 \le_{\text{HR}} \boldsymbol{s}_2$ then $T_1 \le_{\text{HR}} T_2$.

(iii) If $\boldsymbol{s}_1 \le_{\text{LR}} \boldsymbol{s}_2$ and $F$ is absolutely continuous, then $T_1 \le_{\text{LR}} T_2$.

Consider the possibility of comparing systems of arbitrary sizes. Two systems with i.i.d. component lifetimes having common reliability function $\overline{F}$ will be said to be equivalent if their reliability functions are identical. This will clearly occur if two systems have the same signature since their reliability functions admit the same representation displayed in (2.1). But the equivalence of systems goes well beyond this special circumstance. It is now known that, given an arbitrary mixed system in $n$ i.i.d. components, there exists an equivalent mixed system in $m$ i.i.d. components for any $m > n$. Samaniego [26] established the result below. Repeated use of this result accomplishes the stated goal for arbitrary $m > n$. It can be stated as follows.

Let $\boldsymbol{s} = (s_1, \ldots, s_n)$ be the signature of a mixed system in $n$ i.i.d. components with common reliability function $\overline{F}$. Then the mixed system with $(n+1)$-components with i.i.d. lifetimes having common reliability function $\overline{F}$ and corresponding to the signature vector $\boldsymbol{s}^*$ given by

$$\left(\frac{n}{n+1}s_1, \frac{1}{n+1}s_1 + \frac{n-1}{n+1}s_2, \frac{2}{n+1}s_2 + \frac{n-2}{n+1}s_3, \ldots, \frac{n-1}{n+1}s_{n-1} + \frac{1}{n+1}s_n, \frac{n}{n+1}s_n\right) \tag{2.2}$$

has the same reliability as the $n$-component mixed system with signature $\boldsymbol{s}$. Navarro *et al.* [23] provided an exact formula in the latter case showing that it is also valid in the case of exchangeable components.

The signature vector $\boldsymbol{s}$ of a given $n$-component system, as defined above, has the very desirable and useful property of being distribution-free under the assumption that the system's components have i.i.d. lifetimes. On the other hand, when the component lifetime distributions vary, the vector $\boldsymbol{s}$ with $i$th element $s_i = \mathbb{P}(T = X_{i:n})$ for $i = 1, \ldots, n$ will, in general, depend on the underlying component distributions. This fact renders this particular metric inappropriate in the development of representation and preservation results such as those given above for the i.i.d. case; see [23] for a more detailed discussion. Indeed, it has not been clear, until very recently, that the notion of system signatures could be generalized to apply to the case of heterogeneous components. An important advance was presented by Coolen and Coolen-Maturi [7]. The authors defined a new metric, which they called the *survival signature*, which is distribution-free and depends only on the system design. This metric is defined as follows.

**Definition 2.1.** Consider an $n$-component system with components of $r$ different types. Suppose that the system has $m_k$ components of type $k$, where $k = 1, \ldots, r$. Assume that the lifetimes of components of the same type are exchangeable and that the lifetimes of components of different types are independent. Then the *survival signature* of the system is a nonnegative function $\phi$ of $r$ variables, where $\phi(i_1, \ldots, i_r)$ for $i_k = 0, \ldots, m_k$ and $k = 1, \ldots, r$, represents the probability that the system works when precisely $i_k$ components of type $k$ are working for $k = 1, \ldots, r$.

Coolen and Coolen-Maturi [7] discussed the calculation of the survival signature under the conditions stated above, and showed that, under these assumptions, $\phi$ does not depend on the component distributions. Note that, in the preceding definition, the assumptions (about exchangeability of the components of the same type and independence of components of different types) can be removed but, in this case, $\phi$ may depend on the joint distribution of the component lifetimes.

Coolen and Coolen-Maturi [7] also showed that, under the stronger assumption that, for $k = 1, \ldots, r$, components of type $k$ have i.i.d. lifetimes with common distribution $F_k$, the system's reliability function can be obtained via the representation

$$\overline{F}_T(t) = \sum_{i_1=0}^{m_1} \cdots \sum_{i_r=0}^{m_r} \phi(i_1, \dots, i_r) \prod_{k=1}^{r} \binom{m_k}{i_k} F_k^{m_k - i_k}(t) \overline{F}_k^{i_k}(t). \tag{2.3}$$

Under the stated assumptions, the representation in (2.3) follows easily from the law of total probability. In particular, if the components are i.i.d., that is, $r = 1$ and $m_1 = n$, then we have

$$\phi(k) = \mathbb{P}(\text{the system works} \mid \text{exactly } k \text{ components work}) = \sum_{i=n-k+1}^{n} s_i$$

for $k = 1, \dots, n$ and $\phi(0) = 0$; see [7, Equation (18)]. Hence, (2.3) reduces to

$$\overline{F}_T(t) = \sum_{k=1}^{n} \phi(k) \binom{n}{k} F^{n-k}(t) \overline{F}^k(t) = \sum_{k=1}^{n} \biggl( \sum_{i=n-k+1}^{n} s_i \biggr) \binom{n}{k} F^{n-k}(t) \overline{F}^k(t)$$

(see [7, Equation (30)]) which, upon letting $j = n - k$, yields

$$\overline{F}_T(t) = \sum_{j=0}^{n-1} \biggl( \sum_{i=j+1}^{n} s_i \biggr) \binom{n}{j} F^j(t) \overline{F}^{n-j}(t),$$

which is equivalent to the representation given in (2.1); see also [5].

Another particular case is $r = n$ and $m_k = 1$ for $k = 1, \dots, n$ (independent nonidentically distributed components), in which (2.3) reduces to

$$\overline{F}_T(t) = \sum_{i_1, \dots, i_n \in \{0,1\}} \psi(i_1, \dots, i_n) \prod_{k=1}^{n} F_k^{1-i_k}(t) \overline{F}_k^{i_k}(t),$$

where $\psi$ is the structure function (which in this case coincides with the survival signature $\phi$). This expression was given in [4, Equation (1.8)].

In the general case (i.e. $1 \le r \le n$), without loss of generality (by choosing an appropriate structure function), we can assume that the $m_1$ components of type 1 are placed in the system in the first $m_1$ positions, the $m_2$ components of type 2 are placed in the positions $m_1 + 1$ to $m_1 + m_2$, and so on. Under this assumption, the survival signature $\phi$ can be obtained from the structure function $\psi$ as

$$\phi(i_1, \dots, i_r) = \frac{1}{\binom{m_1}{i_1} \cdots \binom{m_r}{i_r}} \sum_{\sum_{j=1}^{m_1} x_j = i_1} \; \sum_{\sum_{j=m_1+1}^{m_1+m_2} x_j = i_2} \cdots \sum_{\sum_{j=n-m_r+1}^{n} x_j = i_r} \psi(x_1, \dots, x_n),$$

where $x_1, \dots, x_n \in \{0, 1\}$ and $i_k \in \{0, \dots, m_k\}$ for $k = 1, \dots, r$. If the components are i.i.d., this expression reduces to the expression obtained by Boland [5, p. 599]; see also [14, Equation (3)].

**Remark 2.1.** The notation used in Coolen and Coolen-Maturi [7] differs from the standard approach to signature computation found in the existing literature on the subject. System signatures are defined in terms of a system's 'failure' rather than its 'survival'. However, it is clear that there is a deterministic relationship between the survival signature defined above and what these authors might have called a system's 'failure signature' $\phi^*$. Assuming, as the authors do, that there are $m_k$ components of type $k$, it follows that

$$\begin{aligned}\phi(i_1, \ldots, i_r) &= \mathbb{P}(\text{the system works} \mid \text{exactly } i_k \text{ comp. of type } k \text{ work}, k = 1, \ldots, r)\\ &= \mathbb{P}(\text{the system works} \mid \text{exactly } m_k - i_k \text{ comp. of type } k \text{ fail}, k = 1, \ldots, r)\\ &= 1 - \mathbb{P}(\text{the system fails} \mid \text{exactly } m_k - i_k \text{ comp. of type } k \text{ fail}, k = 1, \ldots, r)\\ &= 1 - \phi^*(m_1 - i_1, \ldots, m_r - i_r). \end{aligned} \tag{2.4}$$

The results discussed in the sequel are stated in terms of Coolen and Coolen-Maturi's survival signatures but can be easily rewritten in terms of the failure signatures using (2.4) above.

While Coolen and Coolen-Maturi's treatment of survival signatures opens the door to studying the behavior of systems with heterogeneous components, it does not provide guidance on whether, and how, the survival signature can be utilized in comparisons between competing systems. The determination of whether or not one system has better performance than another remains to be investigated. Furthermore, questions regarding possible 'preservation' results such as those mentioned above for the i.i.d. case remain unanswered. Here, we propose to investigate the questions involving the comparative performance of systems with components of different types. As the following examples show, the availability of this new tool will prove useful in this investigation.

A quick glance at the representation in (2.3) above does in fact suggest a very strong condition which guarantees that the lifetimes of two particular heterogeneous competing systems will be stochastically ordered, that is, that one system will be uniformly superior to the other. Suppose that two $n$-component systems have precisely the same number of components of each of $r$ types. Assuming that components of the same type have i.i.d. lifetimes and components of different types have independent lifetimes, it follows from (2.3) that a system with a uniformly larger survival signature than another system will provide better performance than the second system. We record this fact as follows.

**Theorem 2.1.** *Consider two systems in independent components with $m_k$ components of type $k$ having the distribution function $F_k$ for $k = 1, \ldots, r$. Let system 1 have lifetime $T_1$ and survival signature $\phi_1$ and the system 2 have lifetime $T_2$ and survival signature $\phi_2$. If, for all vectors $(i_1, \ldots, i_r)$, with $i_k = 0, \ldots, m_k$ and $k = 1, \ldots, r$, the inequality*

$$\phi_1(i_1, \ldots, i_r) \leq \phi_2(i_1, \ldots, i_r)$$

*holds, then it follows that $T_1 \leq_{\mathrm{ST}} T_2$ for all distribution functions $F_1, \ldots, F_r$.*

Even though the domination of one system's survival signature over another's is quite a strong assumption, the result mentioned here is not vacuous. A comparison in which the ST ordering of two system results from such domination is shown below.

**Example 2.1.** Assume that the two systems shown in Figures 1 and 2 each have two components of type A and one component of type B, with all components independent.

System 2 is uniformly ST-better than system 1 when all three components have i.i.d. lifetimes since the respective signatures are $\boldsymbol{s}_1 = (\frac{1}{3}, \frac{2}{3}, 0)$ and $\boldsymbol{s}_2 = (0, \frac{2}{3}, \frac{1}{3})$. However, it may not be obvious that this is also the case when components of type A and type B have different lifetime distributions. The latter domination does follow, however, from the uniform domination of the two survival signatures shown in Table 1, where the number of working components of type A and of type B are recorded in the first and second arguments, respectively, of each $\phi$.

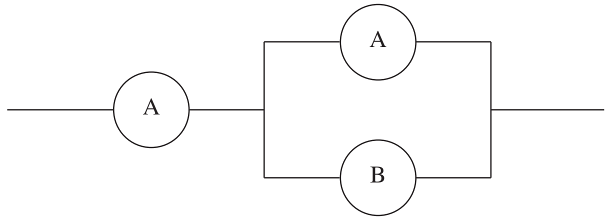


FIGURE 1: System 1 in Example 2.1.

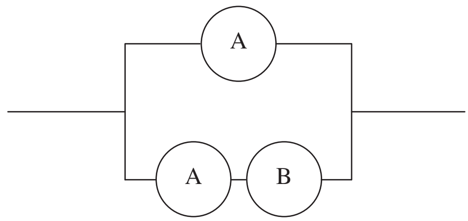


FIGURE 2: System 2 in Example 2.1.

TABLE 1: The survival signatures of the systems shown in Figures 1 and 2.

| $(i_1, i_2)$ | (0, 0) | (0, 1) | (1, 0) | (1, 1) | (2, 0) | (2, 1) |
|---|---|---|---|---|---|---|
| $\phi_1(i_1, i_2)$ | 0 | 0 | 0 | $\frac{1}{2}$ | 1 | 1 |
| $\phi_2(i_1, i_2)$ | 0 | 0 | $\frac{1}{2}$ | 1 | 1 | 1 |

From Theorem 2.1, it follows that $T_1 \leq_{\mathrm{ST}} T_2$.

The uniform domination of survival signatures as in Example 2.1 is not a common occurrence. To see how systems might be compared when such a domination fails to occur, we include the following example.

**Example 2.2.** Consider the comparison of the two systems in Figures 3 and 4, where we will assume that the lifetimes of components of type A are i.i.d. with distribution $F_{\mathrm{A}}$, the lifetimes of components of type B are i.i.d. with distribution $F_{\mathrm{B}}$ and components of different types have independent lifetimes.

System 1 has the structure of a 5-component bridge system connected in series with a single component, while system 2 consists of two 3-component parallel systems connected

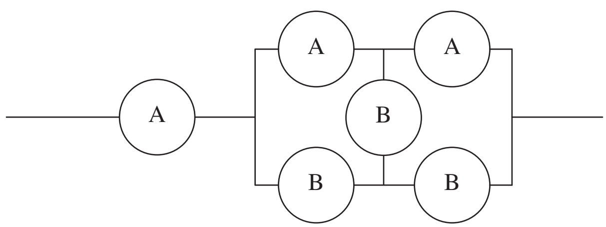


FIGURE 3: System 1 in Example 2.2.

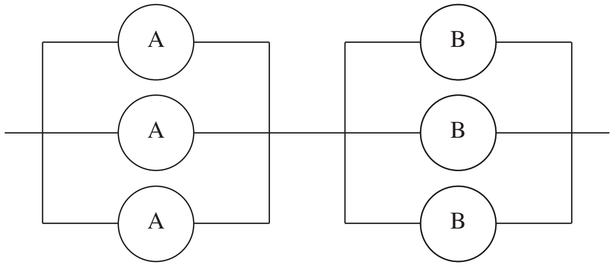


FIGURE 4: System 2 in Example 2.2.

TABLE 2: The survival signature of the system shown in Figure 3.

| $\phi_1(i_1, i_2)$ | $i_2 = 0$ | $i_2 = 1$ | $i_2 = 2$ | $i_2 = 3$ |
|---|---|---|---|---|
| $i_1 = 0$ | 0 | 0 | 0 | 0 |
| $i_1 = 1$ | 0 | 0 | $\frac{1}{9}$ | $\frac{1}{3}$ |
| $i_1 = 2$ | 0 | 0 | $\frac{4}{9}$ | $\frac{2}{3}$ |
| $i_1 = 3$ | 1 | 1 | 1 | 1 |

TABLE 3: The survival signature of the system shown in Figure 4.

| $\phi_2(i_1, i_2)$ | $i_2 = 0$ | $i_2 = 1$ | $i_2 = 2$ | $i_2 = 3$ |
|---|---|---|---|---|
| $i_1 = 0$ | 0 | 0 | 0 | 0 |
| $i_1 = 1$ | 0 | 1 | 1 | 1 |
| $i_1 = 2$ | 0 | 1 | 1 | 1 |
| $i_1 = 3$ | 0 | 1 | 1 | 1 |

to each other in series. Since the respective signatures are $\boldsymbol{s}_1 = (\frac{5}{30}, \frac{9}{30}, \frac{13}{30}, \frac{3}{30}, 0, 0)$ and $\boldsymbol{s}_2 = (0, 0, \frac{1}{10}, \frac{3}{10}, \frac{6}{10}, 0)$, then $\boldsymbol{s}_1 \leq_{\mathrm{ST}} \boldsymbol{s}_2$ and hence it is clear that system 2 is uniformly ST-superior to system 1 when all six components have i.i.d. lifetimes. However, neither of the survival signatures of these two systems dominates the other, as can be seen from the Tables 2 and 3.

The values of $\phi_1$ and $\phi_2$ from Tables 2 and 3 show that $\phi_1(1, 2) < \phi_2(1, 2)$ while $\phi_1(3, 0) > \phi_2(3, 0)$. Thus, the domination of one survival signature exploited in Example 2.1 does not occur in the systems considered here. A comparison of these two systems will require a more refined analysis. Let $T_1$ and $T_2$ be the lifetimes of the two 6-component systems above. Using the representation of a system's reliability function in (2.3), the difference $\overline{F}_{T_2}(t) - \overline{F}_{T_1}(t) = \mathbb{P}(T_2 > t) - \mathbb{P}(T_1 > t)$ can be written as

$$\overline{F}_{T_2}(t) - \overline{F}_{T_1}(t) = \sum_{i_1=0}^{m_1} \cdots \sum_{i_r=0}^{m_r} [\phi_2(i_1, \dots, i_r) - \phi_1(i_1, \dots, i_r)] \prod_{k=1}^{r} \binom{m_k}{i_k} F_k^{m_k - i_k}(t) \overline{F}_k^{i_k}(t).$$

To simplify the notation and facilitate the desired comparison between these systems, we will replace the variable $\overline{F}_{\mathrm{A}}(t) = 1 - F_{\mathrm{A}}(t)$ by the variable $x$ and replace the variable $\overline{F}_{\mathrm{B}}(t) = 1 - F_{\mathrm{B}}(t)$ by the variable $y$. The pair $(x, y)$ varies in the unit square as $t$ varies from 0 to $\infty$.

For $t \in [0, \infty)$, the difference $\overline{F}_{T_2}(t) - \overline{F}_{T_1}(t)$ can be written as the multinomial expression

$$\begin{aligned} D(x, y) &= 9x(1-x)^2y(1-y)^2 + 8x(1-x)^2y^2(1-y) + 2x(1-x)^2y^3 \\ &\quad + 9x^2(1-x)y(1-y)^2 + 5x^2(1-x)y^2(1-y) + x^2(1-x)y^3 - x^3(1-y)^3. \end{aligned} \tag{2.5}$$

The calculation of $D(x, y)$ in (2.5) at a grid of points in the unit square quickly reveals that the function can be either positive or negative for specific $(x, y) \in [0, 1]^2$. While uniform domination of the survival signatures is absent in this example, it is nonetheless possible to identify a sufficient condition on the reliability of components of types A and B that ensures that one system is superior to the other. The result can be stated as follows.

**Theorem 2.2.** *Consider the two systems shown in Figures 3 and 4. For $i = 1, 2$, let $T_i$ be the lifetime of system $i$. Assume that the lifetimes of components of type A are i.i.d. with reliability $\overline{F}_{\mathrm{A}}$, the lifetimes of components of type B are i.i.d. with reliability $\overline{F}_{\mathrm{B}}$ and components of different types have independent lifetimes. If $\overline{F}_{\mathrm{A}}(t) \le \overline{F}_{\mathrm{B}}(t)$ for all $t$, then $T_1 \le_{\mathrm{ST}} T_2$.*

*Proof.* To prove this result consider the function $D(x, y)$ in (2.5), where $x = \overline{F}_{\mathrm{A}}(t)$ and $y = \overline{F}_{\mathrm{B}}(t)$. Then the stated result follows from the following facts:

$$\begin{aligned} 0 < x \le y < 1 \quad &\Longrightarrow \quad \frac{1-x}{x} \ge \frac{1-y}{y} \\ &\Longrightarrow \quad 9\frac{1-x}{x} \ge \frac{1-y}{y} \\ &\Longrightarrow \quad 9x^2(1-x)y(1-y)^2 \ge x^3(1-y)^3 \\ &\Longrightarrow \quad D(x, y) \ge 0. \end{aligned}$$

In addition, $D(x, y) = 0$ if $x = 0$ and $D(x, y) \ge 0$ if $y = 1$. □

The result above illustrates that systems with heterogeneous components can indeed be compared, and that conditions under which one system provides better performance than another can indeed be identified. A new tool, the survival signature of a system with heterogeneous components, provides entry into the study of the comparative behavior of competing systems of this type. While the two results above are restricted to the comparison of systems which have the same number of components of type $k$, for $k = 1, \dots, r$, we show, in the next section, that developments involving equivalent systems of different sizes allow us to expand the scope of such comparisons to systems of different sizes. Theorem 2.2 above has been used with profit in comparing systems of the same size having components with i.i.d. lifetimes. We begin Section 3 with a result that provides the recursive relationship between the survival signatures of equivalent systems of sizes $n$ and $(n + 1)$, a result that can be used repeatedly to obtain two systems which satisfy the conditions of Theorem 2.1 and thus facilitate the type of analysis illustrated in this section.

## 3. Comparisons of heterogeneous systems with different numbers of one or more component types

Let $\phi$ be the survival signature of a system with independent components of $r$ different types. Assume that there are $m_k$ components of type $k$ for $k = 1, \dots, r$. Recall that $\phi$ is defined as follows. For $k = 1, \dots, r$ and $1 \le i_k \le m_k$, $\phi(i_1, \dots, i_r)$ is the probability that the system will work when exactly $i_k$ components of type $k$ are working. The size of the system will be denoted by $n = m_1 + \cdots + m_r$.

In developing the recursive formula below connecting the survival signatures of two equivalent heterogeneous systems in independent components, we will utilize the approach taken by Lindqvist and Samaniego [13] for constructing equivalent systems. We will first briefly review that work, as the alternative approach taken there to the proof of (2.2) and its generalization to two equivalent systems of arbitrary sizes is the approach we will utilize in comparing the heterogeneous systems discussed in this section. The 'one-step' version of the result may be stated as follows. If an independent irrelevant component is added to a coherent or mixed system based on $n$ components with i.i.d. lifetimes, the resulting $(n+1)$-component system is a monotone system whose signature is given in (2.2) and is equivalent (i.e. has the same lifetime distribution) as the original $n$-component system. Lindqvist and Samaniego [13] also provide a constructive argument showing that, for any positive integer $r$, a system of size $(n+r)$ that is equivalent to a given $n$-component system may be obtained by adding $r$ irrelevant components to the original system.

Now, suppose that an irrelevant component whose type is among the $r$-types of components present in the original system is added to this system, resulting in a monotone system of size $n+1$. Since the irrelevant component has no effect on the performance of the larger system, the larger and smaller systems have the same reliability functions in the arguments $u_k = \overline{F}_k(t), k = 1, \ldots, r$, where $t \in [0, \infty)$ is a fixed but arbitrary positive time point. Let $\phi^*$ be the survival signature of the system of size $n+1$ obtained by adding a component of type $k$. The following result identifies the relationship between $\phi^*$ and $\phi$. As we shall see, this recursion will prove to be a useful tool in the comparison of two systems of arbitrary sizes whose components are drawn from a collection of components of $r$ possible types, where $r \geq 2$.

**Theorem 3.1.** *Consider a system with n independent components, where $m_j$ components are of type j for $j = 1, \ldots, r$. Let the system have survival signature $\phi$. Suppose that an irrelevant component of type k is added to the system, and let $\phi^*$ be the survival signature of the resulting $(n+1)$-component system. The relationship between the survival signatures $\phi^*$ and $\phi$ of these two systems is shown below.*

(i) *For* $0 \leq i_j \leq m_j$, $j = 1, \ldots, k-1, k, \ldots, r$,

$$\phi^*(i_1, \ldots, i_{k-1}, 0, i_{k+1}, \ldots, i_r) = \phi(i_1, \ldots, i_{k-1}, 0, i_{k+1}, \ldots, i_r).$$

(ii) *For* $0 \leq i_j \leq m_j$, $j = 1, \ldots, k-1, k, \ldots, r$, *and for* $1 \leq i_k \leq m_k$,

$$\begin{aligned}\phi^*(i_1, \ldots, i_{k-1}, i_k, i_{k+1}, \ldots, i_r) = {} & \frac{i_k}{m_k+1}\phi(i_1, \ldots, i_{k-1}, i_k - 1, i_{k+1}, \ldots, i_r) \\ & + \frac{m_k - i_k + 1}{m_k + 1}\phi(i_1, \ldots, i_{k-1}, i_k, i_{k+1}, \ldots, i_r).\end{aligned}$$

(iii) *For* $0 \leq i_j \leq m_j$, $j = 1, \ldots, k-1, k, \ldots, r$,

$$\phi^*(i_1, \ldots, i_{k-1}, m_k + 1, i_{k+1}, \ldots, i_r) = \phi(i_1, \ldots, i_{k-1}, m_k, i_{k+1}, \ldots, i_r).$$

*Proof of Theorem 3.1(i).* This follows from the fact that, if $i_j$ components of type $j$ are working, for $j \neq k$, and 0 components of type $k$ are working in the system of size $n+1$, then 0 components of type $k$ are working in the system of size $n$. Thus, the survival signatures $\phi$ and $\phi^*$ are identical. □

*Proof of Theorem 3.1(ii).* Consider the computation of $\phi^*(i_1, \ldots, i_{k-1}, i_k, i_{k+1}, \ldots, i_r)$ for $1 \leq i_k \leq m_k$. Since it is assumed that the $(n+1)$-component system has $i_k$ working components of type $k$, the equivalent $n$-component system must have either $i_k$ or $i_k - 1$ components of type $k$ working. Since components of type $k$ have i.i.d. lifetimes with common distribution $F_k$, the probability that the $n$-component system has exactly $i_k$ working components (that is, the probability that the irrelevant component does not belong to the group of $i_k$ working components) is given by

$$\frac{\binom{m_k}{i_k}\binom{1}{0}}{\binom{m_k+1}{i_k}} = \frac{m_k - i_k + 1}{m_k + 1}.$$

The complementary probability that the $n$-component system has exactly $i_k - 1$ working components is thus given by

$$\frac{\binom{m_k}{i_k-1}\binom{1}{1}}{\binom{m_k+1}{i_k}} = \frac{i_k}{m_k + 1}$$

(that is, the probability that the irrelevant component belongs to the group of $i_k$ working components). Then it follows that $\phi^*(i_1, \ldots, i_{k-1}, i_k, i_{k+1}, \ldots, i_r)$ is given by the mixture of $\phi(i_1, \ldots, i_{k-1}, i_k, i_{k+1}, \ldots, i_r)$ and $\phi(i_1, \ldots, i_{k-1}, i_k - 1, i_{k+1}, \ldots, i_r)$ with mixing coefficients as stated in Theorem 3.1(ii). □

*Proof of Theorem 3.1(iii).* This follows from the fact that, if $i_j$ components of type $j$ are working, for $j \neq k$, and all the components of type $k$ are working in the system of size $n+1$, then all components of type $k$ are working in the system of size $n$. Thus, the two survival signatures with the respective arguments are identical as stated in Theorem 3.1(iii). □

A similar result for the $k = 1$ case was obtained in [7, Equation (26)]. The expressions for this case were also obtained in [23, Equation (2.1)]. The inclusion of the other component types is straightforward as can be seen in the preceding proof.

The following example illustrates the use of the survival signatures, together with the recursive relationship between the survival signatures of two equivalent systems to execute a comparison of the reliability of two systems in heterogeneous components. While the example only compares two systems of moderate size, it serves to make the point that the tools we have discussed in this and the preceding section have quite wide applicability.

**Example 3.1.** Consider the system in five components given in Figure 5 which has two components of type A and three components of type B. Assume that the A components have i.i.d. lifetimes with distribution $F_{\mathrm{A}}$ and the B components have i.i.d. lifetimes with distribution $F_{\mathrm{B}}$, and the lifetimes of A and B components are independent.

We begin with the computation of the survival signature of this system. In Table 4 we show the number of working A and B components (that is, $i_1$ and $i_2$) and the survival signature $\phi_1(i_1, i_2)$ for each $i_1, i_2$ pair.

In order to compare this system with the system given in Figure 4 (system 2 in Example 2.2) which has three components of type A and three components of type B, we consider the equivalent system of order 6 obtained by adding an irrelevant component of type A to the system given in Figure 5. Thus, by using Theorem 3.1, we obtain the survival signature of order 6 given in Table 5.

Comparing this survival signature with that of the system in Figure 4 given in Example 2.2, we see that $\phi_1^*(i_1, i_2) \leq \phi_2(i_1, i_2)$ for all $i_1, i_2$, and therefore, from Theorem 2.1, the system in Figure 4 is ST-better than that in Figure 5 for all distributions function $F_{\mathrm{A}}, F_{\mathrm{B}}$.

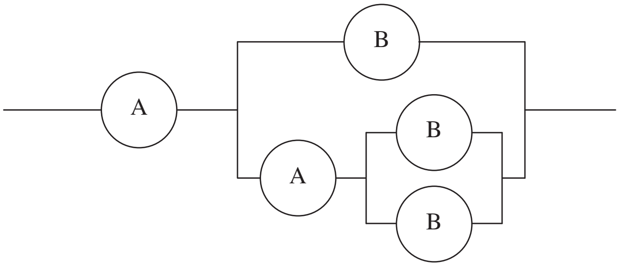


FIGURE 5: System 1 in Example 3.1.

TABLE 4: The survival signature of the system shown in Figure 5.

| $\phi_1(i_1, i_2)$ | $i_2 = 0$ | $i_2 = 1$ | $i_2 = 2$ | $i_2 = 3$ |
|---|---|---|---|---|
| $i_1 = 0$ | 0 | 0 | 0 | 0 |
| $i_1 = 1$ | 0 | $\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{1}{2}$ |
| $i_1 = 2$ | 0 | 1 | 1 | 1 |

TABLE 5: The survival signature of the system shown in Figure 5 with an additional independent irrelevant component of type A.

| $\phi_1^*(i_1, i_2)$ | $i_2 = 0$ | $i_2 = 1$ | $i_2 = 2$ | $i_2 = 3$ |
|---|---|---|---|---|
| $i_1 = 0$ | 0 | 0 | 0 | 0 |
| $i_1 = 1$ | 0 | $\frac{1}{9}$ | $\frac{2}{9}$ | $\frac{1}{3}$ |
| $i_1 = 2$ | 0 | $\frac{4}{9}$ | $\frac{5}{9}$ | $\frac{2}{3}$ |
| $i_1 = 3$ | 0 | 1 | 1 | 1 |

**Remark 3.1.** While Example 3.1 compares two systems which each contains components of two particular types, it is worth noting that the results of this section are applicable to any two heterogeneous systems with independent components. For example, if system 1 has two components of type A and three components of type B while system 2 has three components of type B and three components of type C, then the two systems can be compared using their survival signatures by three irrelevant components of type C to the first system and adding two irrelevant components of type A to the second system. The original systems need not have the same types of components nor the same number of components of a given type to proceed with the approach considered here.

In order to calculate the survival signature of a system that is equivalent to a given heterogeneous system, it may appear that one must use Theorem 3.1 repeatedly, and this can be somewhat cumbersome when comparing two systems for which the number of components of a given type differ substantially and such differences occur for many or even all of the types from 1 to $r$. The following result can simplify the calculation substantially, guaranteeing its completion in at most $r$ steps, one step for each component type whose frequency differs in the two systems of interest.

**Theorem 3.2.** *Consider a system with n independent components, where $m_j$ components are of type j for $j = 1, \ldots, r$. Let the system have survival signature $\phi$. For a given component type, say type k, suppose that $d_k$ irrelevant components of type k are added to the system, and let $\phi^*$ be the survival signature of the resulting $(n + d_k)$-component system. The relationship between the survival signatures $\phi^*$ and $\phi$ of these two systems is given by*

$$\phi^*(i_1, \ldots, i_r) = \sum_{j=\max(0,i_k-d_k)}^{\min(i_k,m_k)} \frac{\binom{m_k}{j}\binom{d_k}{i_k-j}}{\binom{m_k+d_k}{i_k}} \phi(i_1, \ldots, i_{k-1}, j, i_{k+1}, \ldots, i_r) \tag{3.1}$$

*for all vectors $(i_1, \ldots, i_r)$ with $0 \leq i_j \leq m_j$ for $j \neq k$ and with $0 \leq i_k \leq m_k + d_k$.*

*Proof.* Consider, first, the computation of $\phi^*(i_1, \ldots, i_r)$ for a vector $(i_1, \ldots, i_r)$ with $0 \leq i_j \leq m_j$ for $j \neq k$ and with $0 < i_k < m_k + d_k$. We assume that the $(n + d_k)$-component system has $i_k$ working components of type $k$. Suppose that the equivalent $n$-component system has $j$ working components of type $k$. It is evident that the integer $j$ must obey the following constraints:

$$0 \leq j \leq m_k, \qquad 0 \leq i_k - j \leq d_k.$$

Now assuming that these constrains are satisfied, and since components of type $k$ have i.i.d. lifetimes with common distribution $F_k$, the probability that the $n$-component system has exactly $j$ working components is given by

$$\frac{\binom{m_k}{j}\binom{d_k}{i_k-j}}{\binom{m_k+d_k}{i_k}}.$$

From the law of total probability, it follows that the survival probability $\phi^*$ of the $(n + d_k)$-component system is correctly specified in (3.1). We may extend (3.1) to the cases where $i_k = 0$ or $i_k = m_k + d_k$ by noting the following. In the first instance, there are no components of type $k$ working in the $(n + d_k)$-component system, so there can be no components of type $k$ working in the $n$-component system. Thus, both the upper and lower bound on the index $j$ in (3.1) are equal to 0 and, in this case, it reduces to

$$\phi^*(i_1, \ldots, i_{k-1}, 0, i_{k+1}, \ldots, i_r) = \phi(i_1, \ldots, i_{k-1}, 0, i_{k+1}, \ldots, i_r)$$

which is, of course, the value of $\phi^*$ in (3.1) when $i_k = 0$. On the other hand, when $i_k = m_k + d_k$, all $m_k$ components in the $n$-component system must be working, so that both the upper and lower bound on the index $j$ in (3.1) are equal to $m_k$ and, in this case, it reduces to

$$\phi^*(i_1, \ldots, i_{k-1}, m_k + d_k, i_{k+1}, \ldots, i_r) = \phi(i_1, \ldots, i_{k-1}, m_k, i_{k+1}, \ldots, i_r)$$

which is, of course, the value of $\phi^*$ in (3.1) when $i_k = m_k + d_k$. Thus, (3.1) holds for any fixed values of $i_1, \ldots, i_r$ with $0 \leq i_j \leq m_j$ for $j \neq k$ and with $0 \leq i_k \leq m_k + d_k$. □

The following example illustrates the use of Theorem 3.2.

**Example 3.2.** Consider the coherent system displayed in Figure 6 having seven independent components (one component of type A and six components of type B).

Suppose that one wished to compare the performance of this system to a competing system that happens to have six components of type A and six components of type B. A first step in the comparison process would be to use Theorem 3.2 to construct the survival signature of a system that is equivalent to the system in Figure 6 yet has six components of type A. Denoting

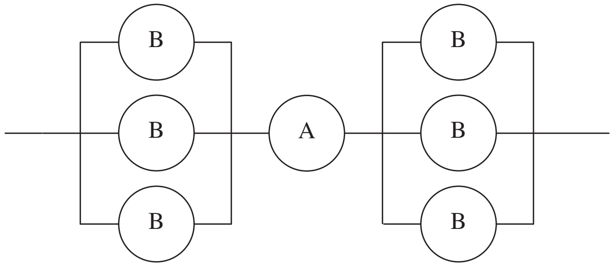


FIGURE 6: System in Example 3.2.

TABLE 6: The survival signature of the system shown in Figure 6.

| $\phi_1(i_1, i_2)$ | $i_2 = 0$ | $i_2 = 1$ | $i_2 = 2$ | $i_2 = 3$ | $i_2 = 4$ | $i_2 = 5$ | $i_2 = 6$ |
|---|---|---|---|---|---|---|---|
| $i_1 = 0$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $i_1 = 1$ | 0 | 0 | $\frac{3}{5}$ | $\frac{9}{10}$ | 1 | 1 | 1 |

TABLE 7: The survival signature of the system shown in Figure 6 with five additional independent irrelevant components of type A.

| $\phi_1^*(i_1, i_2)$ | $i_2 = 0$ | $i_2 = 1$ | $i_2 = 2$ | $i_2 = 3$ | $i_2 = 4$ | $i_2 = 5$ | $i_2 = 6$ |
|---|---|---|---|---|---|---|---|
| $i_1 = 0$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $i_1 = 1$ | 0 | 0 | $\frac{1}{10}$ | $\frac{3}{20}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |
| $i_1 = 2$ | 0 | 0 | $\frac{1}{5}$ | $\frac{3}{10}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| $i_1 = 3$ | 0 | 0 | $\frac{3}{10}$ | $\frac{9}{20}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $i_1 = 4$ | 0 | 0 | $\frac{2}{5}$ | $\frac{3}{5}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ |
| $i_1 = 5$ | 0 | 0 | $\frac{1}{2}$ | $\frac{3}{4}$ | $\frac{5}{6}$ | $\frac{5}{6}$ | $\frac{5}{6}$ |
| $i_1 = 6$ | 0 | 0 | $\frac{3}{5}$ | $\frac{9}{10}$ | 1 | 1 | 1 |

the number of working A and B components by $i_1$ and $i_2$, respectively, the survival signature $\phi_1(i_1, i_2)$ of the system in Figure 6 is shown in Table 6.

When five irrelevant A components are added to the system in Figure 6, we obtain, using (3.1), the following survival signature $\phi_1^*$ of the 12-component monotone system satisfying the specification above. Denoting the number of working A and B components by $i_1$ and $i_2$, respectively, we obtain $\phi_1^*$ as shown in Table 7.

It is apparent from the example above that it is, in theory, possible to compare any two systems in heterogeneous components using the Coolen and Coolen-Maturi representations of the respective systems' reliability functions. In order to obtain survival signatures of the same dimension, one may employ the artifact of adding irrelevant components of various types to each system, as needed. Assuming that there are a total of $r$ component types involved in both systems taken together, the two reliability functions that result can be fairly complex multinomial expressions in the variables $\overline{F}_1(t), \ldots, \overline{F}_r(t)$. It will often be found that neither system will uniformly dominate the other for all possible values of these variables. As noted in Example 2.2 above, one may, in some cases, identify sufficient conditions of these $r$ variables to ensure that one system dominates the other; see also Section 5. In the worst case scenario where an analytical comparison appears to be completely intractable, the developments of this section can still be useful. Specifically, since the survival signature approach leads to closed-form expressions of the reliability functions of both systems, it is possible to explore the question of selected domination of one system over the other (and vice versa) using numerical searches, computing, and comparing reliability functions in a lattice within the unit hypercube $[0, 1]^r$; such a procedure is proposed in Section 5 in the special case in which $r = 2$.

We now turn our attention to a different but potentially quite useful approach to the same problem given in the following section.

## 4. Comparisons based on generalized distorted distributions

Let $T$ be the lifetime of a coherent (or mixed) system with $r$ different types of (possibly dependent) components having reliability functions $\overline{F}_1, \dots, \overline{F}_r$. Then, from the results given in [4], [21], and [18], its reliability function can be written as

$$\overline{F}_T(t) = \overline{Q}(\overline{F}_1(t), \dots, \overline{F}_r(t)), \tag{4.1}$$

where $\overline{Q}$ is a continuous increasing function which does not depend on $\overline{F}_1, \dots, \overline{F}_r$ such that $\overline{Q}(0, \dots, 0) = 0$ and $\overline{Q}(1, \dots, 1) = 1$. The function $\overline{Q}$ was called a *dual distortion* function by Hürlimann [10]. A similar expression holds for the distribution functions, that is,

$$F_T(t) = Q(F_1(t), \dots, F_r(t)), \tag{4.2}$$

where $Q(u_1, \dots, u_r) = 1 - \overline{Q}(1 - u_1, \dots, 1 - u_r)$ and $Q$ satisfies the same properties as $\overline{Q}$. The function $Q$ is called a *distortion* function. These kinds of distributions were called generalized distorted distributions (GDD) in [21] and they are the natural extensions of the distorted distribution (DD) defined in [10], [24], and [30]–[32] which are obtained when $r = 1$. To see how the functions $Q$ and $\overline{Q}$ can be computed, we refer the reader to [18] and [21].

Under the assumption that the components are independent, (4.1) can be obtained from (2.3), and it follows that the reliability function of the system can be written as a multinomial function $\overline{Q}$ of the components' reliability functions (as shown in Example 2.2). When $r = n$, where $n$ is the number of components, $\overline{Q}$ coincides with the *reliability function h* defined in [4, Section 2.1]. However, note that $\overline{Q}$ is not a reliability function in the usual sense. Moreover, if the components are independent and $r = 1$ (i.e. they are i.i.d.), then $\overline{Q}$ is a polynomial called the domination (or reliability) polynomial in [28] which can be computed from (2.1). As in the preceding section, these representations can be extended to the equivalent systems and the primary systems do not have the same number of components of each type (by adding irrelevant components to the system if necessary).

Comparison results for DD and GDD were obtained in [16], and [20]–[22]. Following a similar approach to that used in [22], we can obtain the following distribution-free comparison results for the systems considered in this paper.

**Theorem 4.1.** *Suppose that two coherent (or mixed) systems with lifetimes $T_1$ and $T_2$ have components taken from r different types with distributions $F_1, \dots, F_r$. Let $Q_1$, $\overline{Q}_1$ and $Q_2$, $\overline{Q}_2$ be the respective distortion functions obtained from (4.2) and (4.1), respectively. We have the following:*

(i) *$T_1 \le_{\mathrm{ST}} T_2$ holds for all $F_1, \dots, F_r$ if and only if $\overline{Q}_1 \le \overline{Q}_2$ in $(0, 1)^r$;*

(ii) *$T_1 \le_{\mathrm{HR}} T_2$ holds for all $F_1, \dots, F_r$ if and only if $\overline{Q}_2/\overline{Q}_1$ is decreasing in $(0, 1)^r$;*

(iii) *$T_1 \leq_{\mathrm{RH}} T_2$ holds for all $F_1, \dots, F_r$ if and only if $Q_2/Q_1$ is increasing in $(0,1)^r$.*

(iv) *If the distributions of $T_1$ and $T_2$ are absolutely continuous, $T_1 \leq_{\mathrm{LR}} T_2$ holds for all $F_1, \dots, F_r$ if $\gamma(u_1, \dots, u_r, v_2, \dots, v_r)$ is decreasing in $u_1, \dots, u_r$ and increasing (decreasing) in $v_i$ in the set $(0,1)^r \times (0,\infty)^{r-1}$ and $F_1 \leq_{\mathrm{LR}} F_i$ ($\geq_{\mathrm{LR}}$) for $i = 2, \dots, r$, where*

$$\gamma(u_1, \dots, u_r, v_2, \dots, v_r) = \frac{D_1\overline{Q}_2(u_1, \dots, u_r) + \sum_{i=2}^{r} v_i D_i \overline{Q}_2(u_1, \dots, u_r)}{D_1\overline{Q}_1(u_1, \dots, u_r) + \sum_{i=2}^{r} v_i D_i \overline{Q}_1(u_1, \dots, u_r)}$$

*and where $D_i\overline{Q}_j$ represents the partial derivative of $\overline{Q}_j$ with respect to the $i$th component for $i = 1, \dots, r$ and $j = 1, 2$.*

(v) *If the distributions of $T_1$ and $T_2$ are absolutely continuous, $T_1 \leq_{\mathrm{LR}} T_2$ holds for all $F_1, \dots, F_r$ if $\delta(u_1, \dots, u_r, v_1, \dots, v_r)$ is decreasing in $u_1, \dots, u_r$ and increasing (decreasing) in $v_i$ in the set $(0,1)^r \times (0,\infty)^r$ and $F_i$ is an increasing (decreasing) hazard rate—denoted IHR (DHR)—for $i = 1, \dots, r$, where*

$$\delta(u_1, \dots, u_r, v_1, \dots, v_r) = \frac{\sum_{i=1}^{r} v_i u_i D_i \overline{Q}_2(u_1, \dots, u_r)}{\sum_{i=1}^{r} v_i u_i D_i \overline{Q}_1(u_1, \dots, u_r)}.$$

*Proof.* The proof of Theorem 4.1(i) is immediate from (4.1).

To prove Theorem 4.1(ii), note that from (4.1),

$$\frac{\overline{F}_{T_2}(t)}{\overline{F}_{T_1}(t)} = \frac{\overline{Q}_2(\overline{F}_1(t), \dots, \overline{F}_r(t))}{\overline{Q}_1(\overline{F}_1(t), \dots, \overline{F}_r(t))}.$$

Therefore, $T_1 \leq_{\mathrm{HR}} T_2$ holds if and only if

$$\frac{\overline{Q}_2(\overline{F}_1(t), \dots, \overline{F}_r(t))}{\overline{Q}_1(\overline{F}_1(t), \dots, \overline{F}_r(t))}$$

is increasing in $t$ for all $F_1, \dots, F_r$. This obviously holds if $\overline{Q}_2/\overline{Q}_1$ is decreasing in $(0,1)^r$ since $\overline{F}_k$, $k = 1, \dots, r$ are decreasing functions. Conversely, if we assume that $T_1 \leq_{\mathrm{HR}} T_2$ holds for all $F_1, \dots, F_r$ and we want to prove that $\overline{Q}_2/\overline{Q}_1$ is decreasing, that is,

$$\frac{\overline{Q}_2(u_1, \dots, u_r)}{\overline{Q}_1(u_1, \dots, u_r)} \geq \frac{\overline{Q}_2(v_1, \dots, v_r)}{\overline{Q}_1(v_1, \dots, v_r)} \quad \text{for } u_i \leq v_i,$$

we need only to consider reliability functions $\overline{F}_1, \dots, \overline{F}_r$ for which $u_i = \overline{F}_i(t_2)$ and $v_i = \overline{F}_i(t_1)$ for $i = 1, \dots, r$ with $t_1 \leq t_2$ and to use the fact that $\overline{F}_{T_2}(t)/\overline{F}_{T_1}(t)$ is increasing in $t$ for these distributions.

The proof of Theorem 4.1(iii) is similar to that of Theorem 4.1(ii) taking into account (4.2).

To prove Theorem 4.1(iv), note that $T_1 \leq_{\mathrm{LR}} T_2$ holds if and only if $f_{T_2}(t)/f_{T_1}(t)$ is increasing. From (4.1), we have

$$f_{T_j}(t) = \sum_{i=1}^{r} f_i(t) D_i \overline{Q}_j(\overline{F}_1(t), \dots, \overline{F}_r(t)) \quad \text{for } j = 1, 2,$$

where $f_i$ is the probability density function of $F_i$. Therefore,

$$\frac{f_{T_2}(t)}{f_{T_1}(t)} = \gamma\left(\overline{F}_1(t), \dots, \overline{F}_r(t), \frac{f_2(t)}{f_1(t)}, \dots, \frac{f_r(t)}{f_1(t)}\right).$$

Hence, it is increasing in $t$ if the assumptions in Theorem 4.1(iv) hold.

Analogously, the proof of Theorem 4.1(v) is obtained from

$$f_{T_j}(t) = \sum_{i=1}^{r} h_i(t)\overline{F}_i(t) D_i \overline{Q}_j(\overline{F}_1(t), \dots, \overline{F}_r(t)) \quad \text{for } j = 1, 2,$$

where the hazard rate function $h_i = f_i/\overline{F}_i$ of the $i$th component is increasing (decreasing) when $F_i$ is IHR (DHR). □

Note that the preceding comparison results can be used to compare systems which have a different number of components from each group. Also note that in the first three cases we have necessary and sufficient conditions for the distribution-free comparison results. However, in the case of the LR order, we only have sufficient conditions and we need some conditions for the distributions of the components. If we add some conditions about the components in the groups, then the necessary and sufficient conditions in the preceding theorem can be changed accordingly. For example, if we assume that $\overline{F}_1(t) \le \cdots \le \overline{F}_r(t)$, then, for example, in Theorem 4.1(i), we need the condition

$$\overline{Q}_1(u_1, \dots, u_r) \le \overline{Q}_2(u_1, \dots, u_r)$$

only for $u_1 \le \cdots \le u_r$. The conditions in Theorem 4.1(ii)–(v) can be changed in a similar way. Let us examine some examples to illustrate these theoretical results.

**Example 4.1.** Let us consider the systems given in Figures 1 and 7 and assume that they have independent components.

A straightforward calculation shows that these systems cannot be ordered by using the survival signatures. From (2.3), we obtain that the dual distortion functions of these systems are

$$\overline{Q}_1(x, y) = x^2 + xy - x^2 y, \quad \overline{Q}_2(x, y) = 2xy - x^2 y.$$

Hence, the difference $D = \overline{Q}_2 - \overline{Q}_1$ is

$$D(x, y) = xy - x^2 = x(y - x).$$

Therefore, these systems are not ST-ordered for all distribution functions. However, note that $D \ge 0$ in $(0, 1)^2$ if and only if $x \le y$. Therefore, $T_1 \le_{\mathrm{ST}} T_2$ holds for all $\overline{F}_{\mathrm{A}} \le \overline{F}_{\mathrm{B}}$, that is, when the components of type A are ST-worse than those of type B. The ordering is reversed if $\overline{F}_{\mathrm{A}} \ge \overline{F}_{\mathrm{B}}$.

To study if they are HR-ordered, we compute the ratio $R = \overline{Q}_2/\overline{Q}_1$, obtaining

$$R(x, y) = \frac{2y - xy}{x + y - xy},$$

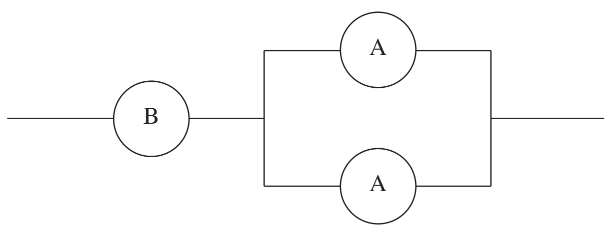


FIGURE 7: Systems 2 in Example 4.1.

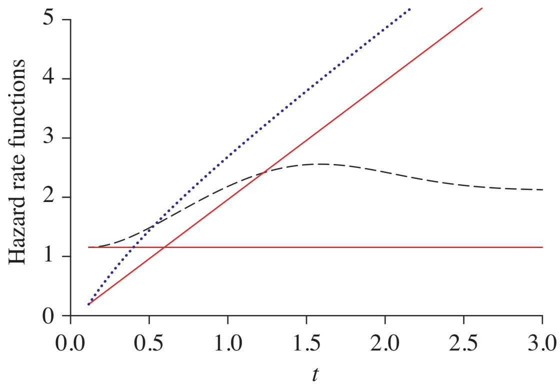


FIGURE 8: Plots of the hazard rate functions of the components (*solid*) and the systems in Example 4.1 ($T_1$ (*dashed*), $T_2$ (*dotted*)) when the components of type A have an exponential distribution with mean 1 (constant hazard rate) and those of type B have the Weibull reliability function $\overline{F}_B(t) = \exp(-t^2)$ for $t \geq 0$. Note that $T_1 \leq_{HR} X_A$ and $T_2 \leq_{HR} X_B$ but the other HR orderings do not hold.

which is decreasing in $x$ and increasing in $y$ in $(0, 1)^2$. Therefore, they are not HR-ordered for all $F_A$, $F_B$. For example, in Figure 8, we see that they are not HR-ordered when the components from A are exponential and those from B are Weibull.

This procedure can also be used to compare systems with a different number of components from each group. For example, to compare in the HR ordering the system in Figure 1 with a single component of type A, we consider the ratio

$$R(x, y) = \frac{\overline{Q}_1(x, y)}{x} = \frac{x^2 + xy - x^2y}{x} = x + y - xy,$$

which is increasing in $x$ and $y$ in $(0, 1)^2$. Therefore, $T_1 \leq_{HR} X_A$, where $X_A$ represents the lifetime of a component of type A (this is an expectable property due to the structure of this system). Obviously, in Figure 8, we see that they are HR-ordered. The conditions in Theorem 4.1(iv) and 4.1(v) do not hold and the existence of the LR ordering between these two systems remains to be determined.

Analogously, to compare $T_1$ with a single component of type B, we consider the ratio

$$R(x, y) = \frac{\overline{Q}_1(x, y)}{y} = \frac{x^2 + xy - x^2y}{y} = y^{-1}x^2 + x - x^2,$$

which is increasing in $x$ and decreasing in $y$ in $(0, 1)^2$. Therefore, they are not HR-ordered (for all $F_A$, $F_B$) as can be seen in Figure 8. In a similar way, for the other system, we obtain

$$R(x, y) = \frac{\overline{Q}_2(x, y)}{x} = \frac{2xy - x^2y}{x} = y(2 - x)$$

and

$$R(x, y) = \frac{\overline{Q}_2(x, y)}{y} = \frac{2xy - x^2y}{y} = 2x - x^2$$

and, therefore, $T_2 \leq_{HR} X_B$ (an expected property due to the structure of this system) but $T_2$ and $X_A$ are not HR-ordered (for all $F_A$, $F_B$) as can be seen in Figure 8. The conditions in Theorem 4.1(iv) and 4.1(v) do not hold, and so the existence of the LR ordering between these two systems also remains to be determined.

**Example 4.2.** Let us consider again the systems given in Figures 1 and 2 studied in Example 2.1 assuming that they have independent components. Remember that these systems are ST-ordered from the survival signatures. A straightforward calculation from (2.3) proves that the dual distortion functions of these systems are

$$\overline{Q}_1(x, y) = x^2 + xy - x^2y, \qquad \overline{Q}_2(x, y) = x + xy - x^2y.$$

Note that

$$D(x, y) = \overline{Q}_2(x, y) - \overline{Q}_1(x, y) = x(1 - x) \geq 0$$

and so $T_1 \leq_{\mathrm{ST}} T_2$ holds for all $F_{\mathrm{A}}, F_{\mathrm{B}}$. Analogously, the ratio $R = \overline{Q}_2/\overline{Q}_1$ is

$$R(x, y) = \frac{x + xy - x^2y}{x^2 + xy - x^2y} = \frac{1 + y - xy}{x + y - xy}$$

which is decreasing in $x$ and $y$ in $(0, 1)^2$. Therefore, they are HR-ordered, that is, $T_1 \leq_{\mathrm{HR}} T_2$ for all $F_{\mathrm{A}}, F_{\mathrm{B}}$. For example, in Figure 9, we see their hazard rate functions when the components from A are exponential and those from B are Weibull.

We have seen in Example 4.1, that $T_1$ is HR-worse than a single component of type A for all $F_{\mathrm{A}}, F_{\mathrm{B}}$ and that $T_1$ and a single component of type B are not HR-ordered for all $F_{\mathrm{A}}, F_{\mathrm{B}}$; see, e.g. Figure 9. In a similar way it can be seen by using Theorem 4.1(ii) that $T_2$ is not HR-ordered neither with a single component of type A nor with a single component of type B for all $F_{\mathrm{A}}, F_{\mathrm{B}}$. For example, as

$$\frac{\overline{Q}_2(x, y)}{y} = \frac{x + xy - x^2y}{y} = (1 + y^{-1})x - x^2$$

is increasing in $x$ and decreasing in $y$ in $(0, 1)^2$, $T_2$ is not HR-ordered with a single component of type B for all $F_{\mathrm{A}}, F_{\mathrm{B}}$. However, in Figure 9, we see that $T_2$ is HR-better than a single component of type B for these specific distributions.

Our final example shows that this technique can also be used for systems with dependent components and also shows that the conditions for the LR ordering stated in Theorem 4.1 might hold.

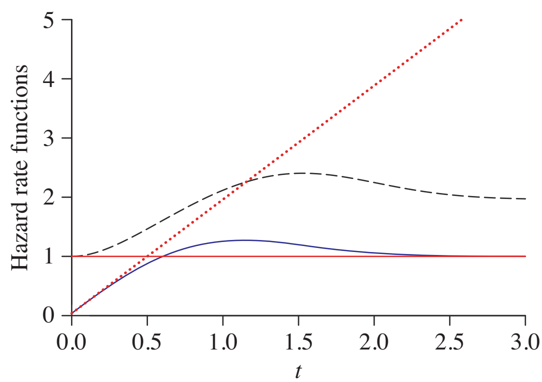


FIGURE 9: Plots of the hazard rate functions of the components (*solid*) and the systems in Example 4.2 ($T_1$ (*dashed*), $T_2$ (*dotted*)) when the components of type A have an exponential distribution with mean 1 (constant hazard rate) and those of type B have the Weibull reliability function $\overline{F}_{\mathrm{B}}(t) = \exp(-t^2)$ for $t \geq 0$. Note that $T_1 \leq_{\mathrm{HR}} T_2$ and $T_1 \leq_{\mathrm{HR}} X_{\mathrm{A}}$. These are a general property for all $F_{\mathrm{A}}, F_{\mathrm{B}}$. Moreover, in this example, we also have $X_{\mathrm{B}} \leq_{\mathrm{HR}} T_2$ for these specific distributions.

**Example 4.3.** Let us consider a series system with two dependent components of type A and an independent component of type B, $T_1 = \min(X_1^{\mathrm{A}}, X_2^{\mathrm{A}}, X_1^{\mathrm{B}})$ and another series system with two dependent components of type A, $T_2 = \min(Y_1^{\mathrm{A}}, Y_2^{\mathrm{A}})$. The components of type A have reliability $\overline{F}_{\mathrm{A}}$ and those of type B have reliability $\overline{F}_{\mathrm{B}}$. We assume that the dependency of components of type A is modelled by a Clayton survival copula

$$K(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta} \quad \text{for } \theta \geq 1.$$

Then the reliability of the first system is

$$\overline{F}_{T_1}(t) = \mathbb{P}(X_1^{\mathrm{A}} > t, X_2^{\mathrm{A}} > t, X_1^{\mathrm{B}} > t) = \overline{F}_{\mathrm{B}}(t)K(\overline{F}_{\mathrm{A}}(t), \overline{F}_{\mathrm{A}}(t)) = \overline{Q}_1(\overline{F}_{\mathrm{A}}(t), \overline{F}_{\mathrm{B}}(t)),$$

where $\overline{Q}_1(x, y) = yK(x, x) = y(2x^{-\theta} - 1)^{-1/\theta}$. Analogously, the reliability of the second system is

$$\overline{F}_{T_2}(t) = \mathbb{P}(X_1^{\mathrm{A}} > t, X_2^{\mathrm{A}} > t) = K(\overline{F}_{\mathrm{A}}(t), \overline{F}_{\mathrm{A}}(t)) = \overline{Q}_2(\overline{F}_{\mathrm{A}}(t), \overline{F}_{\mathrm{B}}(t)),$$

where $\overline{Q}_2(x, y) = K(x, x) = (2x^{-\theta} - 1)^{-1/\theta}$. Then the gamma function defined in Theorem 4.1(iv) is

$$\gamma(x, y, v_2) = \frac{x^{-\theta-1}(2x^{-\theta} - 1)^{-1-1/\theta}}{yx^{-\theta-1}(2x^{-\theta} - 1)^{-1-1/\theta} + v_2(2x^{-\theta} - 1)^{-1/\theta}}.$$

Hence,

$$\frac{1}{\gamma(x, y, v_2)} = y + v_2(2x - x^{\theta+1}).$$

Therefore, $\gamma$ is decreasing in $y$ and $v_2$ and increasing (decreasing) in $x$ when $\theta > 1$ ($\theta = 1$) in the set $(0, 1)^2 \times (0, \infty)$. Thus, if $\theta = 1$ and $F_{\mathrm{A}} \leq_{\mathrm{LR}} F_{\mathrm{B}}$, then, from Theorem 4.1(iv), we have $T_1 \leq_{\mathrm{LR}} T_2$. In general, if the Clayton copula is replaced by an arbitrary survival copula $K$, then the same property holds when

$$\frac{(\partial/\partial x)K(x, x)}{K(x, x)}$$

is decreasing. For example, this property holds when the components of type A are independent, i.e. the copula is the product copula and $K(x, x) = x^2$.

Analogously, if we want to use Theorem 4.1(v), then the delta function satisfies

$$\frac{1}{\delta(x, y, v_1, v_2)} = y + y\frac{v_2}{v_1}\frac{K(x, x)}{x(\partial/\partial x)K(x, x)}$$

and hence $\delta$ is decreasing in $y$ and $v_2$ and increasing in $v_1$. Moreover, $\delta$ is decreasing in $x$ if

$$\frac{x(\partial/\partial x)K(x, x)}{K(x, x)}$$

is decreasing. If this last condition holds, then from Theorem 4.1(v), we have $T_1 \leq_{\mathrm{LR}} T_2$ for all IHR $F_{\mathrm{A}}$ and all DHR $F_{\mathrm{B}}$. This condition is not satisfied by the Clayton copula since

$$\frac{x(\partial/\partial x)K(x, x)}{K(x, x)} = \frac{2}{2 - x^{\theta}}$$

is increasing for all $\theta \geq 1$. Of course, this condition is satisfied by the product copula since

$$\frac{x(\partial/\partial x)K(x, x)}{K(x, x)} = 2.$$

## 5. Comparisons based on RR-plots

The purpose of this section is to provide a more detailed study of the ST ordering of two systems. We restrict this study to the case in which we only have two type of components (i.e. $r = 2$ with the notation used in the preceding sections). Thus, we give the following definition.

**Definition 5.1.** Suppose that two coherent systems with lifetimes $T_1$ and $T_2$ have components taken from two different types. Let $\overline{Q}_1$ and $\overline{Q}_2$ be the respective dual distortion functions obtained from (4.1). Then we define the domination region associated with these systems as

$$C = \{(x, y) \in [0, 1]^2 \colon D(x, y) \geq 0\},$$

where $D(x, y) = \overline{Q}_2(x, y) - \overline{Q}_1(x, y)$.

Then we have the following immediate result.

**Theorem 5.1.** *Suppose that two coherent systems with lifetimes $T_1$ and $T_2$ have components taken from two different types with distributions $F_1$ and $F_2$. Let $D$ be the domination region associated with these systems. Then $T_1 \leq_{\mathrm{ST}} T_2$ holds if and only if $(\overline{F}_1(t), \overline{F}_2(t)) \in D$ for all $t$.*

It follows that the ordering $T_1 \leq_{\mathrm{ST}} T_2$ holds for given distributions $F_1, F_2$ if the plot $(\overline{F}_1(t), \overline{F}_2(t))$ is inside the domination region $D$ for all $t$. This plot can be called an *RR-plot* (reliability-reliability plot) and can be used to determine if two particular systems are ordered for two given distributions. Of course, if the survival signatures are ordered (or if $\overline{Q}_1 \leq \overline{Q}_2$), then $D = [0, 1]^2$. In particular, if $(x, y) \in D$ for all $0 < x < y < 1$, then $T_1 \leq_{\mathrm{ST}} T_2$ holds whenever $\overline{F}_1 \leq_{\mathrm{ST}} \overline{F}_2$. In the RR-plots we can also add the level curves for the function $D(x, y)$ to see what is the difference between the reliability functions of both systems; see the next example and Figure 10. In the following example, we study systems with independent components. However, note that this technique can also be applied to systems with dependent components.

**Example 5.1.** Let us consider again the 6-component systems given in Figures 3 and 4 studied in Example 2.2. The respective dual distortion functions are

$$\overline{Q}_1(x, y) = x^3 + xy^2 + 2x^2y^2 - 3x^3y^2 - 2x^2y^3 + 2x^3y^3$$

and

$$\overline{Q}_2(x, y) = 9xy - 9xy^2 + 3xy^3 - 9x^2y + 9x^2y^2 - 3x^2y^3 + 3x^3y - 3x^3y^2 + x^3y^3.$$

The difference of the dual distortion functions obtained in (2.5) can be reduced to

$$D(x, y) = 9xy - 10xy^2 + 3xy^3 - 9x^2y + 7x^2y^2 - x^2y^3 + 3x^3y - x^3y^3 - x^3.$$

By plotting the level curves of the difference function $D(x, y)$, we obtain the domination region $D$ for these systems given in Figure 10 (the region above the level curve with level 0).

Since the border of this region is below the diagonal, we have $T_1 \leq_{\mathrm{ST}} T_2$ whenever $X_{\mathrm{A}} \leq_{\mathrm{ST}} X_{\mathrm{B}}$. Note that $X_{\mathrm{A}} \leq_{\mathrm{ST}} X_{\mathrm{B}}$ holds if and only if the RR-plot $(\overline{F}_1(t), \overline{F}_2(t))$ is above the diagonal for all $t$. Recall that this property was already obtained in Example 2.2. However, the RR-plot allows us to develop a more accurate analysis and to show that the ordering $T_1 \leq_{\mathrm{ST}} T_2$ may occur even if the condition $X_{\mathrm{A}} \leq_{\mathrm{ST}} X_{\mathrm{B}}$ does not hold. For example, if we want to see

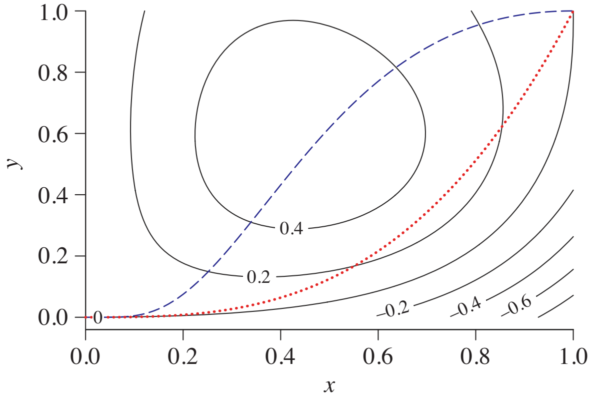


FIGURE 10: Domination region for the systems in Example 5.1 and RR-plots when the components of type A have an exponential distribution with mean 1 (i.e. $x = \overline{F}_1(t) = \exp(-t)$ for $t \geq 0$) and those of type B have the Weibull reliability function $y = \overline{F}_2(t) = \exp(-t^2)$ for $t \geq 0$ (*dashed*) or an exponential of mean $\frac{1}{3}$, i.e. $y = \overline{F}_2(t) = \exp(-3t)$ for $t \geq 0$ (*dotted*).

the performance of both systems when $\overline{F}_1(t) = \exp(-t)$ (exponential) and $\overline{F}_2(t) = \exp(-t^2)$ (Weibull), we may examine the RR-plot $(\overline{F}_1(t), \overline{F}_2(t))$; see the dashed line in Figure 10. There we see that this plot is inside the domination region $D$ and so we have $T_1 \leq_{\text{ST}} T_2$. Note that, in this case, $X_{\text{A}}$ and $X_{\text{B}}$ are not ST-ordered (the dashed line crosses the diagonal). Analogously, if $\overline{F}_1(t) = \exp(-t)$ and $\overline{F}_2(t) = \exp(-3t)$, we obtain the dotted line in Figure 10, obtaining again $T_1 \leq_{\text{ST}} T_2$. Note that in this case $X_{\text{A}} \geq_{\text{ST}} X_{\text{B}}$ (the dotted line is below the diagonal). The level curves tell us the approximate value of the difference of the reliability functions of both systems. In Figure 10, we see that in the first case (dashed line), the system 2 is much better than system 1 (specially for small values of $t$), than in the second case (dotted line) where the RR-plot is closer to the border of the domination region (the 0 level curve). Of course, we can find distributions for the components in which these systems are not ST-ordered (i.e. where the RR-plot crosses the border of $D$) or they are ST-ordered in the opposite direction (i.e. the RR-plot is below the border of $D$).

## 6. Discussion

In this paper we have studied the use of the survival signature for the comparison of systems made of components of different types, without taking lifetime distributions for components into account. An alternative approach has also been proposed by using generalized distorted distributions. As most real-world systems have multiple types of components, the new results are likely to have substantially more impact than existing comparison methods for systems with i.i.d. components.

In real applications, however, one may wish to compare different systems also with assumed distributions for the component lifetimes, or indeed with (possibly nonparametric) statistical inference on such distributions. In the first case, it may be natural to compare the systems' reliability by explicitly considering the probability that one system survives the other (that is, using the metric of 'stochastic precedence' as employed in [2], [9], and [15]). We note that this approach was suggested in [7], and performed for some nonparametric predictive inference methods using the signature in recent works by Coolen and Al-Nefaiee; see [1] and [6]. For the latter, the reader may wish to consider recent results on statistical inference for system reliability using the survival signature, within the Bayesian framework (see [3]) and within the nonparametric predictive inference framework (see [8]). These works also contain some further relevant theoretical results on the survival signature, e.g. its computation based on survival signatures of subsystems, and the changes to the survival signature incurred by replacement of one component. The study of possible extensions in these directions is left for future investigations.

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