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In a recent paper,[1] Taylor discussed low-pressure equilibrium solutions of the hydromagnetic equations

$$\underset{\sim}{j} \times \underset{\sim}{B} = \nabla \cdot \underset{\sim}{P} \qquad (1)$$

$$4\pi \underset{\sim}{j} = \nabla \times \underset{\sim}{B} \qquad (2)$$

and

$$\nabla \cdot \underset{\sim}{B} = 0 , \qquad (3)$$

with $\underset{\sim}{P} = P_{\perp} \underset{\sim}{I} + (P_{\parallel} - P_{\perp}) \underset{\sim}{nn}$, where $\underset{\sim}{I}$ is the idem tensor and $\underset{\sim}{n}$ is the unit vector along the magnetic field. He showed that if $P_{\perp}$ and $P_{\parallel}$ are functions of the magnitude B of the field and also satisfy the condition $(dP_{\parallel}/dB) - (P_{\parallel} - P_{\perp})/B) = 0$, the equations are satisfied at low pressure

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(that is, low "β"). This latter is the condition that the parallel component of the right-hand side of Eq. (1) vanish. A property of these solutions is that $j_{\parallel}$, the component of current parallel to $\underset{\sim}{B}$, vanishes everywhere. Since vacuum fields with closed surfaces of constant B exist (so-called "minimum-B fields"[2]), contained low-β equilibria can be found.

In general, static plasma containment is one of two types: (a) the plasma is tied to field lines and these field lines are contained within a closed volume (this means that the plasma surface is also a magnetic surface which will be of toroidal shape if $\underset{\sim}{B}$ is nowhere zero), or (b) the plasma is tied to field lines, but these leave the plasma volume and containment is achieved by the mirror effect. Taylor confined his remarks to mirror-type fields of class (b),[1] and in the following the same restriction is made. Figure 1 shows schematically how such an equilibrium might appear.

In this letter we show that solutions with $P_{\parallel}$ and $P_{\perp}$ functions of B are also valid at high β and retain the property that $j_{\parallel}$ must be everywhere zero. These high-β solutions are studied first by way of the moment equations, and then by the general theory of adiabatic particle motion. By the latter method, more general high-β equilibria are found, and those where $P_{\perp}$ and $P_{\parallel}$ are functions of B are recognized as special cases. For these, the particle density is also a function of B.

Equations (1) and (2) can be transformed to

$$\frac{1}{4\pi} \mathbf{B} \times \nabla \times \nu \mathbf{B} = \nabla P_{\parallel} - \frac{1}{2} \mu \nabla B^2 \tag{4}$$

where $\mu = (P_{\parallel} - P_{\perp})/B^2$ and $\nu = \mu - 1$. It is apparent now that the assumption that $\mathbf{P} = \mathbf{P}(B)$ for finite pressure is consistent with the equilibrium equations. With $\mathbf{P} = \mathbf{P}(B)$, Eq. (4) becomes

$$\frac{1}{4\pi} \mathbf{B} \times \nabla \times \nu \mathbf{B} = \left[\frac{dP_{\parallel}(B)}{dB} - \mu B\right] \nabla B, \tag{5}$$

so that either $\mathbf{B}\cdot\nabla B = 0$, or $dP_{\parallel}(B)/dB - \mu B = 0$. The first leads to trivial solutions such as fields with combined azimuthal and cylindrical symmetry. The second leads to $\mathbf{B} \times (\nabla \times \nu \mathbf{B}) = 0$, or $\nabla \times \nu \mathbf{B} = k\mathbf{B}$, where k is a scalar such that $\mathbf{B}\cdot\nabla k = 0$. Then we have

$$4\pi \mathbf{j} = \nabla \times \mathbf{B} = -\frac{1}{\nu} \nabla \nu \times \mathbf{B} + \frac{k}{\nu} \mathbf{B} . \tag{6}$$

The surface of the plasma is the constant-B surface at which $P_{\perp}$ and $P_{\parallel}$ vanish. The current density normal to this surface must vanish. (Only continuously varying pressures are considered, so that there are no surface currents on the bounding surface. Discontinuous derivatives of the pressures are permitted however, so that $\mathbf{j}$ may be non zero at the surface.) Since $\nabla\nu$ is normal to the boundary, the component of $\mathbf{j}$ normal to the boundary is k times the normal component of $\mathbf{B}$. Thus k must vanish at the boundary, and inside the plasma also, since $\mathbf{B}\cdot\nabla k = 0$. Then we have

$$\underset{\sim}{j} = \frac{\nabla(P_{\perp} + P_{\parallel}) \times \underset{\sim}{B}}{4\pi(P_{\parallel} - P_{\perp}) - B^2} \quad . \tag{7}$$

At the plasma boundary we have

$$\underset{\sim}{j} = - \frac{\nabla P_{\perp} \times \underset{\sim}{B}}{B^2} \quad , \tag{8}$$

since $\nabla P_{\parallel}$ vanishes there via $dP_{\parallel}/dB - \mu B = 0$, with $\mu = 0$. The current density is perpendicular to $\underset{\sim}{B}$ therefore, and lies in the surface of constant B , both at the plasma boundary and within the plasma. Thus even though $\underset{\sim}{B}$ is not normal to the constant-B surface, $j_{\parallel}$ and the component of $\underset{\sim}{j}$ normal to the surface vanish simultaneously. If $\underset{\sim}{B}$ happens to be normal to the constant-B surface at some point within the plasma or on its boundary, then $\underset{\sim}{j}$ is zero there. For a configuration such as in Fig. 1, which has spheroidal B surfaces, there is a line along which $\underset{\sim}{j}$ vanishes.

To summarize the moment equation approach, we find high-β minimum-B equilibria as solutions of

$$\nabla \times \nu \underset{\sim}{B} = 0 \, , \tag{9}$$

with $\nu = \nu(B)$, and these solutions have the property that $j_{\parallel}$ vanishes everywhere.

[illegible] starting point for [illegible] equilibria [illegible] the particle [illegible] the current density [illegible] plasma,[3] The starting point for finding equilibria by the particle method is the solution to $\nabla \cdot \underset{\sim}{j} = 0$ to lowest order in $m/e \equiv \epsilon$, the adiabatic-expansion parameter.[3] The result is

$$n_o(\underset{\sim}{r}, K, M) = \frac{B}{v_{\parallel o}} Q(K, J, M) \tag{10}$$

where n_o is the (lowest order) density of guiding centers corresponding to particles with magnetic moment M and energy K; $v_{\parallel o}$ is the zeroth-order parallel velocity, given by $\left[\frac{2}{m}(K - MB)\right]^{1/2}$; and Q is the number of guiding centers per unit magnetic flux for particles of magnetic moment M and longitudinal invariant J; K is a function of J, M, and the field line. It was also proven in reference 3 that Q depends on position only through the spatial dependence of K for a given J and M. Therefore in Eq. (10) Q is shown as a function of (K, J, M). Eq. (10) is the condition that the current of guiding centers vanish. However, the divergence of the total current vanishes under the same conditions, since the two currents differ by the curl of the magnetism.

The guiding-center density and pressure tensor written in terms of Q are

$$\begin{aligned} N &= \iint n_0 \, dK \, dM = B \iint dK \, dM \, Q(K, J, M) \left[\frac{2}{m}(K-MB)\right]^{-\frac{1}{2}} \\ P_\perp &= \iint n_0 \, M \, B \, dK \, dM = B^2 \iint dK \, dM \, M \, Q \left[\frac{2}{m}(K-MB)\right]^{-\frac{1}{2}} \\ P_\parallel &= \iint n_0 \, M \, v_{\parallel o}^{2} \, dK \, dM = mB \iint dK \, dM \, Q \left[\frac{2}{m}(K-MB)\right]^{\frac{1}{2}} . \end{aligned} \tag{11}$$

where J is to be taken as a function of K,M and the field line. The parallel equilibrium condition from Eq. (1) that $\underset{\sim}{n}\cdot\nabla\cdot\underset{\sim}{P} = 0$ is satisfied by Eq. (11). If Q is chosen independent of J, the only dependence of N, $P_{\perp}$, and $P_{\parallel}$ on position is through B. These are the solutions obtained in the moment-equation section and by Taylor for the low-β case.

In summary, we have obtained by the particle approach more general high-β equilibria and have also shown that if constant-B surfaces are constant-pressure surfaces, they are also surfaces of constant guiding-center density.

ACKNOWLEDGMENTS

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FIGURE CAPTION

Fig. 1. Minimum-B field configuration.

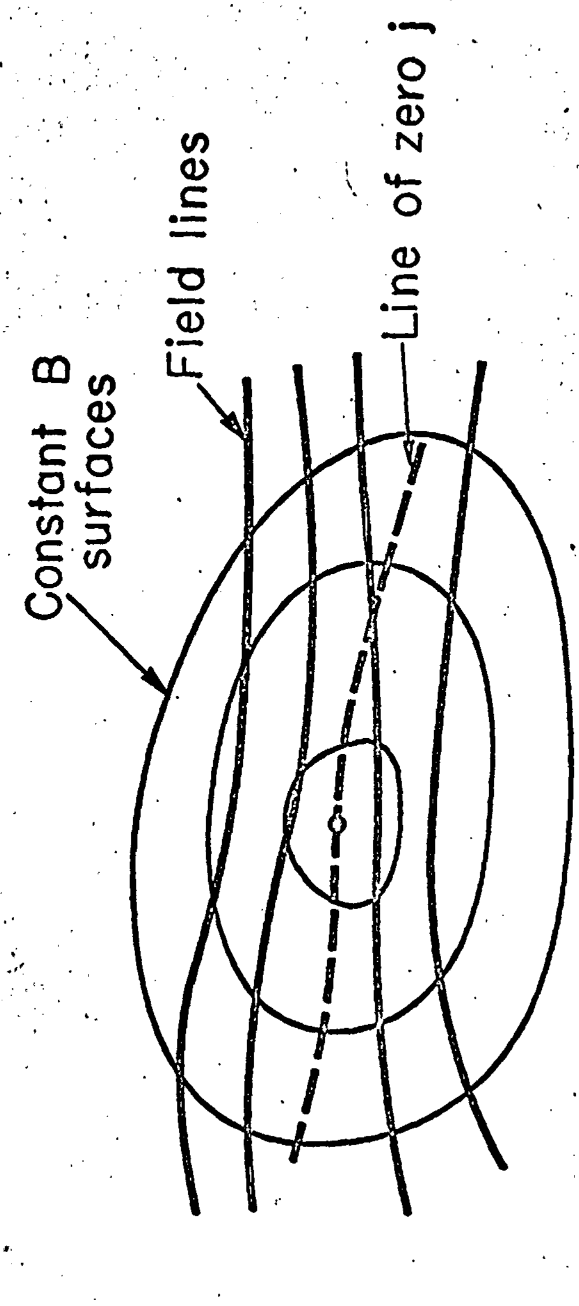

Fig. 1.