Title
Where Process- and Measurement Models Meet: Evaluation of States in Problem Solvinq

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Author
Drolser, Jan

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WHERE PROCESS- AND MEASUREMENT MODELS MEET:

Evaluation of states in problem solving

Jan Drösler, Universität Regensburg, Germany.

Summary:

Any process model of problem solving trying to capture goal directed progress needs an evaluation function of the states. Maximation of its gradient at each step serves as a decision rule in chosing among different legally possible moves. The present analysis gives sufficient conditions for the existence of an evaluation of states which is applicable to customary laboratory problems in the study of problem solving. The evaluation, moreover, is shown to establish a foundation for the measurement of "insight", the anticipatory aspect of human problem solving capacity.

Representation of states

In this study, the sets of states occuring in a problem solving task are represented by a linear module over a ring. This reduces the set of states to be discussed to those, which can be represented by vectors. In the study of problem sol''ing this happens more often than not. Luchins' water-jar task, defines states by triples of numbers, the components of which stand for the fillings in three jars. The missionary-cannibal task represents its states as triples, containing the numbers of missionaries, of cannibals, and of the boat's disposal. For other tasks in use like the "Tower of Hanoi" numerical coding is not standard, but easily achievable. This even holds for proof problems from the propositional calculus.

Vectorial coding only makes sense if it serves a psychological purpose like permitting computation of some sort. The most simple calculation is addition. This raises the question, whether representation of states by vectors makes vector addition a meaningful operation, the result of which reflects empirical facts. At first sight it does not look that way: If in the context of the Luchins water-jar task filling is interpreted as addition, then the jars are likely to run over. Water is spilled and problem solving within the rules yet to be discussed appears to be impossible. Likewise with the missionary-cannibal task addition can lead to an uncontrolled increase of the number of missionaries or cannibals destroying the problem setting. The same appears to hold for other tasks.

Within the present study these risks are abolished by a simple device: It consists in a representation of states by finite sets. These are, e.g., with the Luchins water-jar task the integers {0, 1, 2, 3, 4, 5} or with the missionary-cannibal task the integers {0, 1, 2}. with the proof problems the integers {0, 1}. This confinement has two decisive effects for the analysis of problem solving tasks. For once in the finite case anything that can happen may be analyzed. On the other hand, an addition of states can be introduced as a meaningful representation as soon as a boundary condition is introduced. It consists in identifying the largest integer with zero. It is agreed upon, that, e.g., any three missionaries meeting waive the crossing, and that a jar filled with six units of liquid triggers a mechanism which automatically empties the jar. Under these conditions the finite set of integers assumes the structure of a residual classes ring, which is the minimal structure necessary for the introduction of vectors.

Rules as equations and inequalities

The definitions given above further the analysis of the problem solving task because the drawing rules which are usually only given verbally, now can be formulated as equations or inequalities. This possibility is never given a priori, but presupposes structural foundations like those discussed.

The proposition, calling for a constant sum of filling levels in the Luchins water-jar task only makes sense, after at least a linear module has been defined. Then filling can be represented by vector addition: Any filling is given by the vector difference of two states represented by the vectors of their triples of filling levels. Such vector difference exists for any two states, even if they are not transformable into one another by the rules. It is a system of equations, which defines the legal moves and differentiates them from the illegal ones. It can be shown, for the tasks mentioned, that this is always possible, even for the proof task from the propositional calculus, algebraization of which was introduced by Stone (1936). Addition represents filling with the water-jar task, crossing with the missionary-cannibal task and the logical exclusive "or" with the proof problems.

Because of the finite set of states, for any equation representing a drawing rule the solution set of states fulfilling this equation can be found. This set comprises single states, e.g., for one rule in the Luchins task those which have a constant sum of filling levels. It may comprise pairs of states, the second member of which can be reached from the first in one move under the rule in question.
The purpose served by algebraization of the task is two-fold: For once the discourse is no longer confined to a certain, e.g., water-jar task but rather to the unlimited set of possible tasks of this kind. The realizations, presently used in the problem solving laboratory, are then special cases with certain parameters. Another purpose, however, is reflected by the fact, that the solution sets of states fulfilling the individual rules again allow computation of useful results.

The calculus of relations

Any binary relation can be represented by a matrix over a Boolean ring. In the present case the rows (as well as the columns) stand for the finite set of states. Entries of "one" in any cell means, that the row state can be transformed into the column state in one move according to the rule under study.

Somewhat modifying a matrix calculus introduced by Copilowish (1918) permits algebraic manipulation of such matrices. Any two relations, each representing a rule, thus lead to a new relation. It represents, what is given if both rules are obeyed. The resulting relation, e.g., with the Luchins water-jar task, characterizes transformation of row state into column state in simultaneous accordance with all rules. In this connection the unitary relations first have to be transformed into binary relations by logical product, if, e.g., both, the giving and the receiving vessel have to conform to the condition of constant sum of filling levels.

Reachability of states

Analysis now turns to the question whether any column state can be reached by the given row state in a certain number of legal moves. This amounts to logical analysis whether the state can be reached in one or in two, etc. legal moves. The representation chosen here permits this analysis to be performed algebraically for arbitrarily large matrices. The result is found out for any given number of steps which, however, need not be larger than the number of states minus one. The resulting binary matrices contain entries "one" if the column state is reachable from the row state in so and so many legal steps as were analyzed in that matrix. The final step of the task analysis consists in combining these matrices into a single one the cells of which tell, in at least how many steps a row state can be transformed into the column state. This is done by pairwise comparison of successive matrices of the first kind. If the matrix of n-1 steps does not yet show transformability of the row state into the column state but the matrix in n steps at that cell carries a "one", then this n is identified as the minimal number of legal steps in question. The evaluation of each state with respect to any goal state can be read off this "reachability matrix in n steps" as the entries of the column vector associated with the goal state. This completes analysis of sufficient conditions for the existence of a state evaluation function for arbitrary goal states in a problem solving task such as they are used in the psychological laboratory. Fig. 1 gives an example for a missionary-cannibal task with first component in the triple designating the number of missionaries, the second the number of cannibals, and the third, if 1, availability of the boat, all at the left bank. There are 2 missionaries, 2 cannibals and a boat capacity of 2. The computation of the combined drawing rule eliminates 6 of the $3 \cdot 3 \cdot 2 = 18$ states as not reachable.

	221	210	211	200	201	110	111	020	021	010	011
210	1										
211	2	3									
200	1	2	1								
201	6	7	4	5							
110	1	2	1	2	5						
111	4	5	2	3	2	3					
020	1	2	3	2	7	2	5				
021	4	5	2	3	2	3	2	5			
010	3	4	1	2	3	2	1	4	1		
011	6	7	4	5	2	5	2	7	2	3	
000	5	6	3	4	1	4	1	6	1	2	1

Fig. 1. Symmetric reachability matrix in n steps for (2,2,2) missionary-cannibal problem computed from the rules within the module representation. Entries give minimal number of steps from row to column state.

The measurement of "insight"

For each goal, the evaluations establish a binary relation over the set of states. It can be interpreted as "... closer to the goal than ..." and is connected and transitive. It is, therefore, a weak ordering relation which plays a central rôle in measurement theory (cf. Krantz et al., 1971). The weak ordering together with some other postulates permits measurement in the technical sense of goal distance. Now, this magnitude is not in need of measurement because it can be calculated under the conditions specified. Still the cue can be taken up for the analysis of psychological data which consist of sequences of the states actually visited in the course of an experiment in problem solving. Each step can be interpreted as the result of a preference reaction over the legal steps possible at that choice point. This opens up the possibility of scaling the subjective evaluation by means of a measurement model, e.g., of ordinal measurement. Variation of the parametrization of a task creates problems of differing but controlled complexity. The problem solver's "insight" is the amount of complexity, expressed as given minimal length of solution path, the laboratory data of which do not invalidate the assumptions at the base of the measurement scaling procedure used. Empirical work by Sydow (1970) with the "Tower of Hanoi" shows, that the subjective evaluation is a function of both goal distance as well as distance from the starting point.

References:

COPILOWISH, I.M. Matrix development of the calculus of relations. J. Symb. Logic, 1948, 13, 193-203.

KRANTZ, D.H., LUCE, R.D., SUPPES, P., & TVERSKY, A. Foundations of measurement, Vol. 1. New York: Academic Press, 1971.

LUCHINS, A.S. Mechanization in Problem Solving. Psychological Monographs, 1942, 54 (whole nr. 248), 1-95.

STONE, M.H. The representation theorem for Boolean algebra. Trans. Am. Math. Soc., 1936, 40, 37-111.

SYDOW, H. Zur metrischen Erfassung von subjektiven Problemzuständen und zu deren Veränderung im Denkprozeß. Z. Psychol., 1970, 177, 145-198; 178, 1-50.