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A PROOF OF THE MANDELSTAM REPRESENTATION IN PERTURBATION THEORY

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I wish to report a proof that the Mandelstam representation[1] for a scattering amplitude is valid for every term in the perturbation series for the amplitude. The proof applies to any system of interacting particles that does not have anomalous thresholds. The latter requirement can be expressed by the mass conditions obtained in fourth order. In this letter I will outline the main steps of the proof, using the equal-mass case as an illustration. The details are contained in two papers[2,3] in the course of publication and a third[4] shortly to be submitted.

The invariant energies squared will be denoted s, t, and u; a term in the expansion of the amplitude $A(s, t)$ will be written $F(s, t)$. The main steps include the following:

(1) The physical branch of F in a physical scattering region has the representation

$$F(s, t) = \lim_{\epsilon \to 0} c \int_0^1 d\alpha_1 \ldots d\alpha_n \, \frac{\delta(1 - \Sigma\alpha_i)[C(\alpha)]^{n-2\ell-1}}{[D_\epsilon(\alpha, s, t)]^{n-2\ell}}$$

where we have (I, Section 6),[2]

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$$D_\epsilon(\alpha, s, t) = sf(\alpha) + tg(\alpha) - m^2 K(\alpha) + i\epsilon\, C(\alpha)\, \Sigma\alpha_i$$

$$= D(\alpha, s, t) + i\epsilon\, C(\alpha)\, \Sigma\,\alpha_i \; .$$

Singularities (i.e. branch points) of F will occur when D_ϵ has end-point zeros or coincident zeros (pinching the integration contour) in each α variable as ϵ approaches 0. End-point zeros can be re-interpreted by using reduced diagrams.

(2) $D(\alpha, s, t)$ is negative for real positive α in the region $s > 0$, $t > 0$, $u > 0$. Hence F is real in this region (I, Sections 4 and 8 (F)).

(3) The only straight lines of singularities of the physical branch of F are normal thresholds, since they must intersect a physical scattering region (I, Section 5).

(4) Curves of singularities $\Gamma(s, t)$ of F in the real s, t plane have slope (I, Section 7):

$$\frac{dt}{ds} = -\frac{f(\alpha)}{g(\alpha)} \; ,$$

and they have normal thresholds as asymptotes.

(5) From (2), (3) and (4) it can be shown (III)[4] that there are no singularities in the region $s < 4m^2$, $t < 4m^2$, $u < 4m^2$, and that D is negative and F real in this region. It also follows that no curve of singularities enters the real s, t plane in the region $-4m^2 < t < 4m^2$.

(6) A single-variable dispersion relation in s can be proved (III) for t real and with $(-4m^2 < t < 4m^2)$, by noting that for

$s > 4m^2$ and real α we have

$$D(\alpha, s + i\epsilon, t) \neq 0 .$$

We obtain the correct physical branch of $F(s, t)$ in the physical region $s > 4m^2$ by defining it from

$$F(s, t) = \lim_{\epsilon \to 0} c \int_0^1 d\alpha_1 \ldots d\alpha_n \frac{\delta(1 - \Sigma \alpha_i)[C(\alpha)]^{n-2l-1}}{[D(\alpha, s + i\epsilon, t)]^{n-2l}} .$$

No distortion of the real α contours is necessary except in the limit as ϵ approaches 0. This proves that $F(s, t)$ has no singularities in the upper-half s plane when t is real and in $(-4m^2 < t < 4m^2)$. It also defines the physical sheet of the complex variable s.

(7) If a diagram, or reduced diagram, has its first thresholds at $s = a$, $t = b$, $u = c$, the limits $\pm 4m^2$ in (5) and (6) can be extended to suitable combinations of a, b, and c. The discussion below can then be carried through by using these limits.

(8) The curves of singularities $\Gamma(s, t)$ defined by $t = t(s)$, with s and t real will lie on surfaces of singularities $\Sigma(s, t)$. These will be singularities of some analytic continuation of F. Our problem is to prove that none of these surfaces correspond to singularities on the physical sheets of s, t and u. This can be done by proving that (a) in the region $s > 4m^2$, $t > 4m^2$, all curves of singularities have negative slope, and (b) there are no disconnected complex singularities in the physical sheet (II).[3]

(9) There are two alternative methods which give these results. I will illustrate the first (III) by examining the continuation of $F(s, t)$ past $t = 4m^2$ along any path in the upper half of the s plane and with $t = t_1 + i\epsilon$. A path of this type leads to the physical branch of F in a physical region, and it lies on the physical sheet. If no such path meets a singularity of F, our required result is true. If the path does meet a singularity, it may be either a disconnected complex singularity or a complex singularity coming from a spurious turning point. By continuity, the latter must occur at a minimum of t along a curve of singularities, since these curves cannot disappear from the physical sheet except asymptotically through normal thresholds. If any path meets either type of singularity, choose the singularity on the physical sheet for which t_1 is nearest to $4m^2$, if possible. But we can then trace the singularity along its related singular surface Σ on the path $s_1 + is_2 = s(t_1 + i\epsilon)$ to smaller values of t_1 until t_1 is less than $4m^2$. This path does not meet either $t = 4m^2$ or $s = 4m^2$, since for finite t_1 the function $F(s_1 + is_2, t_1 + i\epsilon)$ is analytic near these thresholds. Letting ϵ tend to zero, we obtain a contradiction of our single-variable dispersion relation below $t = 4m^2$. This proves our required result that there are no spurious turning points and no disconnected complex singularities.

(10) The second method (III) uses the elegant device of analytic completion.[3] The result of Section (6) is first extended to show that $F(s, t_1 \pm i\epsilon)$ is analytic for s in the upper half plane. Then we define, for $t_1 < 4m^2$,

$$F(s, t_1 + i\epsilon) = \frac{1}{2\pi i}\int_C \frac{F(z, t_1 + i\epsilon)dz}{z - s}$$

where C is a semicircle in the upper half plane. The contour C can now be displaced in the complex s, t space by continuously increasing t_1. Provided the contour does not meet any singularities, this procedure defines an analytic continuation in the upper half s plane for all real positive values of t_1. There are no anomalous thresholds, and the contour is not distorted near the normal thresholds since near these $F(s_1 + is_2, t_1 + i\epsilon)$ is analytic. Spurious turning points (minima in t) and disconnected complex singularities cannot exist, since they would lead to horns extending into the volume swept out by C. This proves that the analytic completion we require is possible. The curves on which $F(s_1 + i\epsilon', t_1 + i\epsilon)$ is singular, as ϵ and ϵ' tend to 0, will have normal thresholds as asymptotes and will have negative slope.

11. A similar discussion (III) shows that $F(s + i\epsilon, t - i\epsilon')$ analytically continued from the real region is not singular for $s > 4m^2$, $t > 4m^2$ except in the limit ϵ, $\epsilon' \to 0$. In this limit it is singular only at the normal thresholds and not on the curves of singularities. Thus where the slope of Γ is negative, it extends to Σ only at points $s + i\epsilon$, $t - i\epsilon'$ that lie in nonphysical sheets.

12. From (10) and (11) a double application of Cauchy's theorem is possible (II) and establishes the validity of the Mandelstam representation for every term F in the perturbation series for the amplitude A.

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