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ABSTRACT

An approximation scheme for the one-nucleon Green's functions previously put forward by the authors is renormalized. The experimental mass and the constants Z_1 and Z_2 are rigorously expressed as free-particle limits of integrals over the kernels appearing in the scheme. The mass and wave function renormalization are carried out rigorously; the vertex renormalization is performed by a slight redefinition of the approximation scheme, without greatly altering the physical assumptions peculiar to each approximation. General prescriptions for renormalization are written down, and the first three approximations are explicitly shown to be finite.

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I. Introduction

Recently the authors[1] have proposed a covariant approximation scheme for the treatment of the coupled Green's functions equations of meson-nucleon systems. The procedure led to the replacement of the infinite set of coupled equations for the rigorous kernels by a finite set of approximate equations, involving Green's functions which describe processes with no more than a fixed number of external meson lines.

In (I) the question of renormalization was ignored. It is of course not known whether the usual infinities of pseudoscalar meson theory with pseudoscalar coupling are due to the use of the perturbation expansions in which they appear; however, whether the theory is finite or not, a renormalization has to be carried out. In the approximation scheme, whose validity may only be motivated in the low-energy region, it is expected that such high-frequency phenomena as the self-energy, etc., will not be described correctly, and the existence of infinities are a not unexpected feature. Nevertheless the lack of a correct description in the high-energy domain does not prevent one from performing a renormalization. For example, when a subset of perturbation graphs is summed rigorously,[2] the

1. R. Arnowitt and S. Gasiorowicz, Phys. Rev. 95, 538 (1954), to be referred to as I.
2. S. F. Edwards, Phys. Rev. 90, 284 (1953).

radical difference in the high-energy behavior of the sum and the individual terms of the series does not prevent the renormalization of the latter by perturbation methods.

In this paper a nonperturbation renormalization of the approximation scheme is carried out, i.e., equations involving the renormalized Green's functions, with finite masses and coupling constants, are derived. Although it is of course necessary to solve the resulting equations to see whether the solutions are finite, it will be shown that these equations generate the renormalized perturbation series, when expanded in powers of the coupling constant.

As already suggested in (I), it is hoped that neglecting vacuum polarization will not strongly affect the low-energy results. Thus the meson propagation function $\Delta_+(\xi - \xi')$ will be assumed to be a given function (namely the free particle kernel) of the experimental meson mass μ.

In the following section the conditions to be satisfied by the finite equations are stated. In subsequent sections rigorous expressions for the renormalization constants Z_1, Z_2, and m' are derived, the role of the overlapping divergences is discussed, and the approximation scheme for the renormalized equations is set up. In section V the second approximation is renormalized in detail, and in VI, the procedure for renormalizing the third approximation is outlined. While the general case is not discussed, the work of these two sections makes the extrapolation reasonably clear.

II. Preliminary Discussion

In I a rigorous set of equations coupling Green's functions involving one nucleon and an arbitrary number of mesons was derived. The approximation made there, which involves a decomposition of the last Green's function (appearing in a finite subset of equations) into a sum of products of lower Green's functions, was labeled by the number of "thick lines" in a particular time ordering. While this labeling was adequate in the unrenormalized equations, it was found to be ambiguous in dealing with the problem of renormalization, owing to the necessity of successively substituting the kernels back into earlier equations. These equations require an integration over some of the "thick line" variables, thus destroying the particular time ordering chosen. An equivalent convention, which we will adopt here, is to count the number of "strong interactions" between the meson and nucleon. Thus, for example, in the first approximation, writing $G(\xi\xi') \cong G\,\Delta_+(\xi-\xi')$ involves no strong interactions, while the second approximation, $G(\xi\xi'\xi'') \cong G(\xi)\,\Delta_+(\xi'-\xi'') + \ldots$ involves one strong meson interaction, namely the one appearing in $G(\xi)$. In general the nth approximation will allow (n - 1) strong interactions.

Introducing the Fourier transform of the Green's functions,

$$G(xx';\xi_1\ldots\xi_m) = (2\pi)^{-4(1+[m/2])}\int e^{ip(x-x')}\prod_{i=1}^{m} e^{ik_i(\xi_i-x)}\,G(p,k_1\ldots k_m) \times d^4p\,d^4k_1\ldots d^4k_m \qquad (2.1)$$

where $[m/2]$ is the integral part of $m/2$, the rigorous Eqs. (I 2.7), (I 2.8), and (I 2.9) become

$$[\gamma p+m]\,G(p) = 1 - g\int \gamma\, G(p,k)\,dk \tag{2.2}$$

$$[\gamma(p-k)+m]\,G(p,k) = ig\,(2\pi)^{-4}\int \gamma\, G(p,kk')\,dk' \tag{2.3}$$

$$[\gamma(p-k-k')+m]\,G(p,kk') = \Delta(k)\,\delta(k+k') - g\int \gamma\, G(p,kk'k'')\,dk'' \tag{2.4}$$

The particular choice of transform variables in (2.1) corresponds diagrammatically to a nucleon of momentum p emitting (in any order whatever) m mesons with momenta $k_1, k_2, \ldots, k_m$.

To exhibit some of the conditions which we wish to impose on the renormalization procedure, and to illustrate some of the difficulties which arise in a non-perturbation renormalization, let us briefly consider the renormalization of the equations in the first approximation.

Decomposing

$$G(p, k\,k') \cong G(p)\,\Delta(k)\,\delta(k+k') \tag{2.5}$$

in Eq. (2.3) and substituting into Eq. (2.2), we obtain the equation for the one-nucleon Green's function,

$$\left[\gamma p + m + ig^2(2\pi)^{-4}\int \gamma\, G_0(p-k)\,\gamma\,\Delta(k)\,dk\right] G(p) = 1 \tag{2.6}$$

As is well known, the integral appearing in Eq. (2.6) may be renormalized by subtracting from it the first two terms in a Taylor expansion about $\gamma p + m = 0$. Thus Eq. (2.6) becomes

$$[(\gamma p + m')(1 + \int_1{}') - \delta m \int_1{}' + R(p)]\, G(p) = 1 \tag{2.7}$$

where

$$m' = m + \delta m = m + ig^2(2\pi)^{-4} \left(\int \gamma G_0(p-k)\gamma \Delta(k)\, dk\right)_{\gamma p + m = 0} \equiv m + \int_0{}' \tag{2.8}$$

$$\int_1{}' = ig^2(2\pi)^{-4} \left[\frac{\partial}{\partial \gamma p} \int \gamma G_0(p-k)\gamma \Delta(k)\, dk\right]_{\gamma p + m = 0} \tag{2.9}$$

and

$$R(p) = ig^2(2\pi)^{-4} \int \gamma G_0(p-k)\gamma \Delta(k)\, dk - \int_0{}' - (\gamma p + m)\int_1{}' \tag{2.10}$$

While R(p) is indeed finite, it is only true to order g^2 that $\delta m \int_1{}'$ may be neglected and the factor $(1 + \int_1{}')$ utilized to renormalize the Green's function. Furthermore, R(p) is a function of m rather than the renormalized mass m'. Thus if perturbation theory is not made use of Eq. (2.7) is effectively still unrenormalized.

For a satisfactory program, only the renormalized masses, coupling constants, and Green's functions should appear in the final equations, without any use being made of perturbation theory, though of course the limitations of the approximation to a given number of "strong interactions" must be imposed upon the infinite constants. A necessary condition for a successful renormalization is that the renormalized equations, upon solution in powers of the coupling constant, yield a series of renormalized graphs.

III. The Renormalization Constants

The renormalization procedure to be described below makes no use of perturbation theory. That multiplicative renormalization can be carried out will be explicitly assumed. Thus, in the notation of Matthews and Salam[3] we define the infinite constants Z_1 and Z_2 by

$$\psi(x) = Z_2^{1/2}\,\psi_1(x) \tag{3.1}$$

$$g = (Z_1/Z_2)\,g_1 \tag{3.2}$$

where $\psi(x)$ represents the second-quantized nucleon operator and the subscript "1" will denote the renormalized (finite) quantities. From Eq. (3.1) it follows that

$$G(p, k_1 \ldots k_m) = Z_2\,G_1(p, k_1 \ldots k_m) \tag{3.3}$$

In terms of the renormalized quantities, Eqs. (2.2) to (2.4) take the form

$$[\gamma p + m]\,Z_2 G_1(p) = 1 - g_1 Z_1 \int \gamma\, G_1(p, k)\, dk \tag{3.4}$$

$$[\gamma(p-k) + m]\,Z_2 G_1(p, k) = i g_1 (2\pi)^{-4} Z_1 \int \gamma\, G_1(p, k k')\, dk' \tag{3.5}$$

$$[\gamma(p-k-k') + m]\,Z_2 G_1(p, k k') = \Delta(k)\,\delta(k+k') - g_1 Z_1 \int \gamma\, G_1(p, k k' k'')\, dk'' \tag{3.6}$$

3. P. T. Matthews and A. Salam, Revs. Modern Phys. 23, 311 (1951).

Defining $\bar{G}_1(p, k_1 \ldots k_m)$ by

$$\bar{G}_1(p, k_1 \ldots k_m)\, G_1(p) \equiv G_1(p, k_1 \ldots k_m) \tag{3.7}$$

Eq. (3.4) may be written as

$$\left[\gamma p + m + g_1 (Z_1/Z_2) \int \gamma \bar{G}_1(p, k)\, dk\right] Z_2\, G_1(p) = 1 \tag{3.8}$$

Invoking the usual boundary condition on $G_1(p)$, that in the free particle limit

$$G_1(p) \longrightarrow [\gamma p + m']^{-1} \tag{3.9}$$

where m' is the renormalized mass, Eq. (3.8) may easily be recast into the form

$$\left[\gamma p + m' + g_1 Z_1 \int_R \gamma \bar{G}_1(p, k)\, dk\right] G_1(p) = 1 \tag{3.10}$$

where

$$m' = m + g_1(Z_1/Z_2)\left(\int \gamma \bar{G}_1(p, k)\, dk\right)_{\gamma p + m' = 0} = m + g_1(Z_1/Z_2)\int_0 \tag{3.11}$$

$$Z_2 = 1 - g_1 Z_1 \left(\frac{\partial}{\partial \gamma p} \int \gamma \bar{G}_1(p, k)\, dk\right)_{\gamma p + m' = 0} = 1 - g_1 Z_1 \int_1 \tag{3.12}$$

and

$$\int_R = \int - \int_0 - (\gamma p + m') \int_1 \tag{3.13}$$

Although Eq. (3.10) appears to be infinite, owing to the explicit presence of Z_1, this constant is needed to renormalize the additional overlapping vertex infinities arising from the k-integration. This will be discussed below. Z_1 of course renormalizes the vertex operator, as may easily be seen from its definition,

$$\Gamma(\xi) = \left(\delta/\delta g\, \phi(\xi)\right) G^{-1} = Z_1^{-1}\left(\delta/\delta g_1\, \phi(\xi)\right) G_1^{-1} = Z_1^{-1}\, \Gamma_1(\xi) \tag{3.14}$$

To perform the mass and Green's function renormalizations in the higher equations, we break up $G_1(p, k_1, \ldots k_m)$:

$$G_1(p, k_1 \ldots k_m) = G_{1r}(p, k_1 \ldots k_m) + \eta\, \overline{G}_1(p - k_1 - \cdots - k_{m-1}, k_m)\, G_1(p, k_1 \ldots k_{m-1}) \tag{3.15}$$

where $\eta = 1$ for $m =$ odd; $\eta = i(2\pi)^4$ for $m =$ even, and define a generalized $\int_R$ when more than one meson variable is present:

$$\int_R \gamma G_1(p, k_1 \ldots k_m)\, dk_m = \int \gamma\, G_{1r}(p, k_1 \ldots k_m)\, dk_m + \eta\left(\int_R \gamma\, \overline{G}_1(p - k_1 - \cdots - k_{m-1}, k_m)\, dk_m\right) G_1(p, k_1 \ldots k_{m-1}) \tag{3.16}$$

The second term on the r.h.s. of Eq. (3.15) is that part of the Green's function in which the k_m meson is emitted last and its vertex is not coupled to any of the other vertices. Using Eq. (3.16), and the definitions Eqs. (3.11) to (3.13), Eqs. (3.5) and (3.6) take the form

$$[\gamma(p-k)+m']G_1(p,k)=ig_1(2\pi)^{-4}Z_1\int_R \gamma\, G_1\,(p,k,k')\,dk' \qquad (3.17)$$

$$[\gamma(p-k-k')+m']G_1(p,kk')=\Delta(k)\delta(k+k')-g_1Z_1\int_R \gamma\, G_1(p,kk'k'')\,dk'' \qquad (3.18)$$

We next turn to the rigorous definition of Z_1. Comparing the mass operator in Eq. (3.10) with the usual expression for that quantity,[4] one sees that

$$\overline{G}_1(p,k)=ig_1(2\pi)^{-4}\,G_1(p-k)\,\Gamma_1(p-k,p)\,\Delta(k) \qquad (3.19)$$

An expression for Z_1 may now be obtained by invoking the boundary conditions on $\Gamma_1(p-k,p)$ in the free particle limit, namely[5]

$$\lim_{k^2+\mu^2\to 0} \overline{\psi}_1^{\,\circ}(p-k)\,\Gamma_1(p-k,p)\,\psi_1^{\circ}(p) = \overline{\psi}_1^{\,\circ}(p-k)\,\gamma\,\psi_1^{\circ}(p) \qquad (3.20)$$

where $\psi_1^{\circ}(p)$ is the renormalized plane wave spinor, a function of m'. Comparing with Eq. (3.19), the free particle limit of $\overline{G}_1^{*}(p,k)\equiv\overline{G}_1(p,k)\Delta^{-1}(k)$

4. J. Schwinger, Proc. Natl. Acad. Sci. U.S. 37, 452, 455 (1951).

5. An alternate definition would be to impose Eq. (3.20) in the limit of $k_\mu\to 0$. This was adopted by N. Kroll and M. Ruderman, Phys. Rev. 93, 194 (1954). It is purely a matter of convention whether one wants to treat the meson field as the static electromagnetic field or as the nucleon field. For the purposes of this paper, the choice of definition makes no difference.

is

$$\bar{G}_1^*(p,k) \rightarrow i g_1 (2\pi)^{-4} \left[\gamma(p-k)+m'\right]^{-1}\gamma \qquad (3.21)$$

Introducing the "reaction matrix,"

$$R_1(p,kk') = G_1(p,kk') - G_1(p)\Delta(k)\delta(k+k') \qquad (3.22)$$

Eq. (3.17) in the free particle limit becomes, after slight rearrangement,

$$Z_1\gamma = \gamma - \left(\int_R Z_1\gamma \bar{R}_1^*(p,kk')\,dk'\right)_0 \qquad (3.23)$$

where $(\ )_0$ denotes the free particle limit defined in Eq. (3.20).

Before we proceed to a more detailed discussion of Z_1 it might pay to reexamine the renormalization of the first approximation, in which $Z_1 = 1$, since $\int_R \gamma R_1(p,kk')\,dk' = 0$ there. Using the decomposition of the integral on the r.h.s. of Eq. (3.17), one finds that

$$G_1(p,k) = G_1^{(0)}(p-k)\, i g_1 (2\pi)^{-4}\gamma\Delta(k)G_1(p) \qquad (3.24)$$

where $G_1^{(0)}(p) \equiv [\gamma p + m']^{-1}$, which when substituted into Eq. (3.10) yields the finite equation

$$\left[\gamma p + m' + i g_1^2 (2\pi)^{-4}\int_R \gamma G_1^{(0)}(p-k)\gamma\Delta(k)\,dk\right]G_1(p) = 1 \qquad (3.25)$$

in which only renormalized quantities enter.[6]

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6. The correct Eq. (3.25) could of course be obtained in a trivial fashion from Eq. (2.7) by simply dropping the undesired terms. However, in the higher approximations, it is not clear which terms are to be dropped.

IV. Vertex Renormalization

The rigorous Eqs. (3.10), (3.17), (3.18) ... appear to be asymmetric in that the meson line corresponding to the integrated variable on the right-hand side always goes to the top of the diagram with a lowest-order vertex connection. This asymmetry is only apparent, as the rigorous equations contain all possible perturbation graphs. It is in fact possible to derive from the adjoint equation of motion,

$$\bar{\Psi}\left[-\gamma \overleftarrow{p} + m + g\gamma\phi\right] = 0 \tag{4.1}$$

an "adjoint"[7] set of equations, in which the integrated meson line ends at the bottom of the diagram, viz.:

$$G_1^{+}(p)(\gamma p + m') = 1 - g_1 \int_R G_1^{+}(p+k, k)\gamma Z_1^{+} dk \tag{4.2}$$

$$G_1^{+}(p+k,k)[\gamma(p+k) + m'] = i g_1 (2\pi)^{-4} \int_R G_1^{+}(p+k+k', kk')\gamma Z_1^{+} dk' \tag{4.3}$$

$$G_1^{+}(p+k+k', kk')[\gamma(p+k+k') + m'] = \Delta(k)\delta(k+k') - g_1 \int_R G_1^{+}(p+k+k'+k'', kk'k'')\gamma Z_1^{+} dk'' \tag{4.4}$$

where $G^{+}(p, k_1 \ldots k_m) \equiv G_1(p, k_1 \ldots k_m)$.[8] Proceeding as in Section III,

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7. Any "adjoint" quantity will henceforth be denoted by $\dagger$. This is not to be confused with the Hermitian adjoint which does not appear in this paper.

8. These two sets of equations are precisely related to the two ways of writing the mass operator: $T\rho\, \gamma G \Gamma \Delta \equiv T\rho\, \Delta \Gamma G \gamma$

one may write down rigorous expressions for $Z_2^+ (\equiv Z_2)$ and $Z_1^+ (\equiv Z_1)$, in terms of integrals over the G_1^+'s.

Owing to this equivalence between the adjoint and the "normal" (nonadjoint) quantities, one may generate an infinite variety of equivalent sets of rigorous equations by replacing any of the G_1's and Z_1's by their adjoints. However, once one cuts off the set of equations by means of the breakup approximation, the apparent differences between the various sets of equations become real and each approximate set, though still having the same general physical validity, generates a somewhat different set of graphs. One is thus presented with an infinite number of approximation schemes. The requirement of renormalizability narrows down the possibilities. The particular set chosen here is closely related to the normal set and is defined by replacing Z_1 by Z_1^+ in Eqs. (3.10), (3.17), (3.18) ... in those parts of the right-hand-side integrals in which the integrated meson line is connected to the nucleon line (a similar change being made in the adjoint set). For example, using Eq. (3.22),

$$Z_1 \int \gamma G_1(p;kk')dk' \rightarrow Z_1 \int \gamma G_1(p)\Delta(k)\delta(k+k')dk' + Z_1^+ \int \gamma R_1(p,kk')dk' \tag{4.5}$$

This convention has the consequence of symmetrizing the free particle limits of the two vertex points in the mass operator,[8] since now the adjoint quantities will generate graphs that are the mirror images of the "normal" graphs. While it appears that the introduction of the adjoint quantities (albeit only in their free-particle limit) into the normal equations couples the two sets, it turns out in practice that since the $Z_1^{\dagger}$ necessary in the normal set can always be obtained directly from Z_1 via the mirror property mentioned above without solving the adjoint set, the apparent coupling does not exist.

We now consider the equations in a given approximation with the Z_1's and their adjoints appropriately inserted. Since Z_1 is the free-particle part of a Green's function appearing in the scheme, its presence in an equation implies that a rigorous factorization, in addition to the decomposition peculiar to the approximation, has already taken place. Thus the $Z_1 \gamma$ which appears in the right-hand integrals of the rigorous equations acts as a free-particle limit of a strong interaction (at that vertex). Since we are restricted, in any approximation, to a fixed number of strong interactions, we must extract the relevant parts of Z_1 to fit in with this limitation. Thus, in the last equation, in which the largest kernel on the right-hand side is decomposed, the Green's function appearing in the factored form already contains the largest number of interactions allowed (by definition), and therefore the $Z_1 \gamma$ appearing in this equation must represent a zero interaction vertex, i.e., $Z_1 \gamma \cong \gamma$. When the resultant expression is substituted back into the next equation, one obtains an integral equation for the Green's function with the largest number of interactions allowed. In this case one simply counts the number of strong interactions in each term, and supplies the Z_1 necessary to yield the maximum number. Thus in this equation Z_1's defined by lower approximations (and hence containing fewer interactions) will appear. In determining the relevant parts of Z_1 for the earlier equations, complication appears in that fewer than the maximum number of interactions allowed appears on the left. Thus a question arises as to how to count the interactions appearing in the right-hand integrals. It turns out that the Z_1 needed here is the one that is obtained by counting one interaction more than the maximum, the meson line integration again being taken not to affect the interaction weight of a particular Green's function.[9,10] Using these conventions, we will show in the succeeding sections that the equations in the second and third approximation are renormalized. The extension to higher approximations seems in principle straightforward, but an explicit proof of convergence would involve tedious algebraic manipulation.

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9. This convention also holds for terms that are identifiable with the right-hand-side integrals of "earlier" equations, when these appear in the last equation, in which the decomposition is made.

10. This prescription appears to admit an extra interaction in the renormalization constant in the earlier equations. A similar phenomenon of the inclusion of a higher approximation structure to renormalize one of lower order occurs in the conversion of a subtractive renormalization into a multiplicative one in perturbation theory. Thus for the second-order vertex operator one has

$$\Gamma = \gamma + e^2 \Gamma_1 = \gamma + e^2 A + e^2 \Gamma_c \cong (1 + e^2 A)(\gamma + e^2 \Gamma_c)$$

where A is the infinite constant and Γ_c the convergent part of Γ_1.

V. Renormalization of the Second Approximation

The equations of the second approximation are obtained by substituting the decomposition

$$G_1(p, k k' k'') \cong G_1(p, k)\Delta(k')\delta(k'+k'') + G_1(p, k')\Delta(k'')\delta(k''+k)$$
$$+ G_1(p, k'')\Delta(k)\delta(k+k') \qquad (5.1)$$

into Eqs. (3.10), (3.17), and (3.18). Following the conventions discussed in the previous section, Eq. (3.18) becomes

$$G_1(p, k k') = \Delta(k)\delta(k+k')G_1(p) - g_1 G_1^{(0)}(p-k-k')\gamma G_1(p, k)\Delta(k')$$
$$- g_1 G_1^{(0)}(p-k-k')\gamma G_1(p, k')\Delta(k) \qquad (5.2)$$

where, in obtaining the first term on the right, use was made of equation (3.10). Substituting this result into Eq. (3.17), one obtains

$$[\gamma(p-k)+m']G_1(p, k) = ig_1(2\pi)^{-4} Z_1^{(2)}\gamma\Delta(k)G_1(p)$$
$$- ig_1^2(2\pi)^{-4}\left(\int_R \gamma G_1^{(0)}(p-k-k')\gamma\Delta(k')dk'\right)G_1(p, k)$$
$$- ig_1^2(2\pi)^{-4}\int \gamma G_1^{(0)}(p-k-k')\gamma\Delta(k)G_1(p, k')dk' \qquad (5.3)$$

The subscript "R" does not appear on the second integral, as this term corresponds to the G_{1r} in Eq. (3.16) for this approximation. The first integral is just the correctly renormalized mass operator of the first approximation. No Z_1 appears in the integrals, as $G_1(p, k)$ and $G_1(p, k')$

already contain one strong interaction. The first term, on the other hand, contains the rigorous Z_1 ($\equiv Z_1^{(2)}$) of this approximation, defined by Eqs. (3.23) and (5.2):

$$Z_1^{(2)} \gamma = \gamma + g_1 \left(\int_R \gamma G_1^{(0)}(p-k-k') \gamma \bar{G}_1(p,k') dk' \right)_0 \quad (5.4)$$

(again on the right-hand-size Z_1 of Eq. (3.23) is set equal to unity by our convention). Equation (5.3) may be rewritten in the form

$$\begin{aligned} \bar{G}_1(p,k) &= i g_1 (2\pi)^{-4} G_1^{(1)}(p-k) \gamma \Delta(k) \\ &\quad - i g_1^2 (2\pi)^{-4} G_1^{(1)}(p-k) \Big[\int \gamma G_1^{(0)}(p-k-k') \gamma \bar{G}_1(p,k') dk' \\ &\quad - \left(\int_R \gamma G_1^{(0)}(p-k-k') \gamma \bar{G}_1(p,k') dk' \right)_0 \Big] \Delta(k) \end{aligned} \quad (5.5)$$

where $G_1^{(1)}(p)$ is the one-nucleon propagator of the first approximation. Equation (3.10) becomes

$$[\gamma p + m'] G_1(p) = 1 - g_1 \left(\int_R Z_1^{(2)+} \gamma \bar{G}_1(p,k) dk \right) G_1(p) \quad (5.6)$$

Here the rigorous $Z_1^{(2)+}$ appears, since we wish to include an extra interaction as discussed in the previous section. Writing

$$\bar{G}_1(p,k) = i g_1 (2\pi)^{-4} G_1^{(1)}(p-k) \Gamma_1^{(2)}(p-k, p) \Delta(k) \quad (5.7)$$

Eq. (5.5) becomes an integral equation for the vertex operator $\Gamma_1^{(2)}(p,k)$. A straightforward perturbation expansion shows that Eq. (5.5) yields the graphs of Fig. 1 correctly renormalized.[11] The proof of the finiteness of Eq. (5.6) involves showing that the mass operator $g_1 \int_R Z_1^{(2)+} \gamma \bar{G}_1 (p,k)\, dk$ is finite. This can be done directly by examining the perturbation solution of the equations, and indeed it is found that Eq. (5.6) is divergence free. However a simpler proof, depending only on the symmetry[12] of the mass operator, is available, and it has the advantage of greatly simplifying the proof in higher approximations.

Consider the rigorous mass operator $g^2 \int_R \gamma G \Gamma \Delta = g_1^2 Z_1^2 \int_R \gamma G_1 \Gamma_1 Z_1^{-1} \Delta$. As is well known, the $\int_R$ in the above expressions $\sim Z_1^{-2}$, since the above expression is finite. One of the Z_1^{-1} comes from Γ, the other one results from the overlapping divergences caused by the integration.[13] This can be seen more explicitly by considering the alternate form for the mass operator, $g_1^2 Z_1^2 \int_R \Gamma_1 Z_1^{-1} G_1 \gamma \Delta$, which shows that in the rigorous case the upper interaction point has all the structure of the complete vertex operator. In any approximation scheme, the mass operator will still be of the form $\int_R \gamma G^a \Gamma^a \Delta$, which will explicitly go as $(Z_1^a)^{-1}$

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11. Aside from the presence of $G_1^{(1)}$, Eq. (5.5) is identical with that obtained by B. P. Nigam. (Proceedings of the Rochester Conference on High Energy Physics, 1954).

12. By symmetry we mean that the same diagrams are present when all the graphs are turned upside down.

13. A. Salam, Phys. Rev. 84, 217 (1951).

(this factor coming from Γ^a). Furthermore, one can always find quantities $\Gamma^{a'}$, $G^{a'}$ such that $\int_R \Gamma^{a'} G^{a'} \gamma \Delta$ generates the same set of diagrams, so that the approximate $\int_R$ also goes as $(Z_1^{a'})^{-1}$. Consequently[12] $\int_R$ must be proportional to $(Z_1^a Z_1^{a'})^{-1}$. If the mass operator is symmetric, it is clear that $G^{a'} = G^a$ and $\Gamma^{a'} = \Gamma^{a+}$, and therefore the dependence of $\int_R$ on Z_1 is $(Z_1^a \, Z_1^{a+})^{-1}$. Hence in this case the overlapping divergences produced by the integration in $\int_R \gamma \bar{G}_1(p,k)\,dk$ yields a proportionality factor $(Z_1^{a+})^{-1}$. Thus $g_1 Z_1^{a+} \int_R \gamma \bar{G}_1(p,k)\,dk$ is finite. As can be seen from Fig. 1, the mass operator (obtained by joining the external meson line to the top of the diagram) is indeed symmetric,[14] and hence the factor $Z_1^{(2)+}$ renormalizes Eq. (5.6).[15]

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14. The symmetry of the mass operator seems to be a general feature of the approximation scheme.

15. The mass operator in Eq. (5.6), though convergent, appears as the product of two infinite quantities. Presumably, however, a treatment via the Feynman cutoff method would yield a unique finite limit as the cutoff approached infinity.

VI. Renormalization of the Third Approximation

In this approximation, the highest Green's function appearing on the right-hand side of

$$[\gamma(p-k-k'-k'')+m']G_1(p,kk'k'') = ig_1(2\pi)^{-4}\int \gamma\, G_1(p,kk'k''k''')\,dk''' \tag{6.1}$$

is decomposed in accordance with our general scheme (cf. I, Eq. (2.20), to yield

$$\begin{aligned} G_1(p,kk'k'') = {} & ig_1(2\pi)^{-4}G_1^{(0)}(p-k-k'-k'')\gamma\,[R_1(p,kk')\Delta(k'') \\ & + R_1(p,k'k'')\Delta(k) + R_1(p,k''k)\Delta(k')] \\ & + \Delta(k)\delta(k+k')G_1(p,k'') + \Delta(k')\delta(k'+k'')G_1(p,k) \\ & + \Delta(k'')\delta(k''+k)G_1(p,k') \end{aligned} \tag{6.2}$$

Use was made of Eq. (3.17) to reduce some of the structures to yield the $G(p, k)$ terms. Substituting this into Eq. (3.18) we obtain the integral equation for the unknown quantity $R_1(p\,k\,k')$:

$$\begin{aligned} R_1(p,kk') = {} & -g_1G_1^{(0)}(p-k-k')[Z_1^{(2)}\gamma G_1(p,k)\Delta(k') + Z_1^{(2)}\gamma G_1(p,k')\Delta(k)] \\ & - ig_1^2(2\pi)^{-4}G_1^{(0)}(p-k-k')\int\gamma\, G_1^{(0)}(p-k-k'-k'')\gamma \\ & \times\{R_1(p,kk')\Delta(k'') + R_1(p,k'k'')\Delta(k) + R_1(p,k''k)\Delta(k')\}\,dk'' \end{aligned} \tag{6.3}$$

the Z_1's being chosen in accordance with our convention of Section IV. The reaction matrix, $R_1(p\ k\ k')$, represents a two-meson emission process, and may be divided into four disjoint parts:

$$R_1(p, kk') = (2\pi)^4 i\, \overline{Q}_1(p-k', k)\, G_1(p, k') + T_1(p, kk') + (2\pi)^4 i\, \overline{Q}_1(p-k, k')\, G_1(p, k) + S_1(p, kk') \tag{6.4}$$

The first two terms correspond to the emission of k' followed by the emission of k from two uncoupled vertices and from a compound vertex respectively. The second two terms interchange k and k'. Substituting this into Eq. (6.3) and separating the four disjoint processes, one obtains the following integral equations for $\overline{Q}_1$ and S_1:

$$\overline{Q}_1(p-k', k) = i g_1 G_1^{(1)}(p-k-k')\, Z_1^{(2)}\, \gamma\, \Delta(k) - i g_1^2 (2\pi)^{-4} G_1^{(1)}(p-k-k') \int \gamma\, G_1^{(0)}(p-k-k'-k'')\, \gamma\, \overline{Q}_1(p-k', k'')\, \Delta(k) \tag{6.5}$$

$$S_1(p, kk') = -i g_1^2 (2\pi)^{-4} G_1^{(1)}(p-k-k') \int \gamma\, G_1^{(0)}(p-k-k'-k'')\, \gamma\, \Delta(k') \times \left\{ (2\pi)^4 i\, \overline{Q}_1(p-k'', k)\, G_1(p, k'') + T_1(p, kk'') + S_1(p, kk'') \right\} dk'' \tag{6.6}$$

and a similar equation for T_1. Comparing Eq. (6.5) with Eq. (5.3), we see that

$$\overline{Q}_1(p,k) = \overline{G}_1^{(2)}(p,k) \tag{6.7}$$

and hence $Z_1^{(2)}$ in Eq. (6.5) correctly renormalizes that equation. In Eq. (6.6) it is clear that since S_1 (and T_1) contain only coupled vertices, the integrated meson line k'' goes past at least two vertex points, and thus all graphs generated are of the "finite self-energy" type. Thus Eq. (6.6) is also finite.

In terms of the quantities appearing in Eq. (6.4), Eq. (3.17) has the form

$$\begin{aligned}
[\gamma(p-k)+m'] \overline{G}_1^*(p,k) &= ig_1(2\pi)^{-4} Z_1^{(3)} \gamma \\
&+ ig_1^2(2\pi)^{-4} \Big[(2\pi)^4 i \int Z_1^{(2)+} \gamma \overline{G}_1^{(2)}(p-k',k) \overline{G}_1^*(p,k')dk' \\
&+ \int Z_1^{(2)+} \gamma \overline{T}^*(p,kk')dk' + \int Z_1^{(2)+} \gamma \overline{S}_1^*(p,kk')dk' \\
&+ (2\pi)^4 i \left(\int Z_1^{(2)+} \gamma \overline{G}_1^{(2)}(p-k,k')dk' \right) \overline{G}_1^*(p,k)
\end{aligned} \tag{6.8}$$

where $Z_1^{(3)}$ is the rigorous vertex renormalization constant of this approximation:

$$\begin{aligned}
Z_1^{(3)}\gamma = \gamma - \Big(\int_R Z_1^{(2)+} \gamma \Big[i(2\pi)^4 \overline{G}_1^{(2)}(p-k',k) \overline{G}_1^*(p,k') \\
+ \overline{T}^*(p,kk') + \overline{S}^*(p,kk') \Big] dk' \Big).
\end{aligned} \tag{6.9}$$

Again the $Z_1^{(2)+}$ appears on the right-hand side of Eqs. 6.8) and (6.9) in accordance with the rule allowing one extra strong interaction in earlier equations.[16] We first note that the last term on the right can be identified with the mass operator of the second approximation (cf. Eq. 5.6) multiplied by $\bar{G}_1^*(p, k)$. This was shown in Section V to be finite. We next show that the $Z_1^{(2)+}$ in the remaining right-hand-side structures correctly renormalize the overlapping divergences produced by the k' integration. In Eq. (6.6), a series of generalized graphs for $S_1(p\ k\ k')$ may be generated by using the term proportional to $G_1(p\ k'')$ as an inhomogeneous term and iterating (ignoring the presence of the $T_1(p\ k\ k'')$ term). These graphs (Fig. 2a,b), when substituted into the $S_1(p\ k\ k')$ term in Eq. (6.8), combine with the first integral to yield the unrenormalized vertex structure $\Gamma^{(2)+}$ at the top (Fig. 2c). Thus to renormalize the overlaps of this combination, it is clear that a $Z_1^{(2)+}$ is required at the top of the diagram. Similarly the remaining diagrams of S_1 can be obtained by using the T_1 structure in Eq. (6.6) as an inhomogeneous term. When the k' integration in Eq. (6.8) is carried out for these graphs, and they are combined with the second term on the right-hand side of that equation, the vertex $\Gamma^{(2)+}$ is again obtained at the top (Fig. 2d), and the $Z_1^{(2)}$ therefore renormalizes this structure. When $Z_1^{(3)}\gamma$ is replaced by its definition Eq. (6.9), Eq. (6.8) reduces to the term $i g_1 (2\pi)^{-4}\gamma$ together with the terms representing Figs. (2c and 2d), with their free-particle limits subtracted off. Since all vertices had previously

16. Note that in this approximation, the equation defining $Z_1^{(3)}$ is the free-particle limit of an "earlier" equation, which requires the use of an extra interaction on the right. The second approximation was anomalous in that the $Z_1^{(2)}$ equation was the free-particle limit of the integral equation of that approximation, and therefore this convention did not apply there.

been renormalized, this final subtraction renders the graphs of Figs. (2c) and (2d) finite.

Finally, to show that the mass operator is symmetric, and therefore that Eq. (3.10) is finite (Sec. V), we note that all terms on the right of Eq. (6.8) have a $G_1(p\ k)$ structure at the bottom of the graph,[17] and thus a straightforward iteration of the type leading to Fig. 2 shows that $\int \gamma \overline{G}_1^{(3)}(p, k)\, dk$ is symmetric, and hence that $Z_1^{(3)+} \int \gamma \overline{G}_1(p, k)\, dk$ is finite.

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17. That S_1 and T_1 have this property is evident from iterating Eq. (6.6) and the corresponding equation for T_1 (which we have not written down).

VII. Conclusion

In the preceding sections a method has been given for the renormalization of an approximation scheme for the meson-nucleon interaction problem. Starting with the assumption that the renormalization is multiplicative, we carry out the mass and amplitude renormalizations (Z_2) independently of the approximation. The vertex renormalization presented a more complicated problem, owing to the existence of overlapping divergences. The definition of the renormalization constant in terms of integrals over Green's functions was derived by imposing, upon the vertex, a boundary condition analogous to the one usually applied in quantum electrodynamics, rather than that of Deser, Goldberger, and Thirring.[18] In order to carry out the vertex renormalization it was necessary to consider the equations order by order. By redefining the approximation scheme to include the necessary diagrams forming Z_1 an unambiguous prescription for carrying out the renormalization was found. Actually this seems to confirm the fact that any covariant approximation scheme, which (order by order) approaches the rigorous solution, (i.e., eventually includes all Feynman graphs), can, by a suitable adjunction to what is included in an approximation, be renormalized in a consistent fashion, provided that the renormalized perturbation series can be rearranged and summed in any sequence. In this paper a redefinition of the approximation was made without significantly changing the physical content of each approximation. Such an approach would appear to be applicable to the renormalization of the Tamm-Dancoff method.

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18. S. Deser, M. Goldberger, and W. Thirring, Phys. Rev. 94, 711 (1954).

FIGURE CAPTIONS

Figure 1: Graphs generated by $G_1^{(2)}(p, k)$.

Figure 2a,b: Some of the graphs of $S_1(p, k, k')$.

Figure 2c,d: A graphical representation of the first three integrals on the right-hand side of Eq. (6.8).

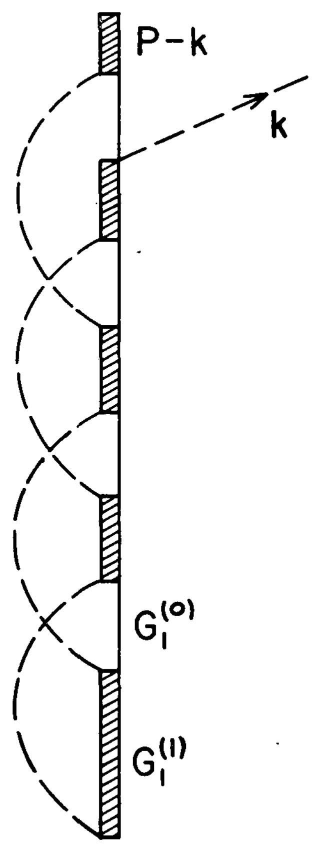

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Figure 1

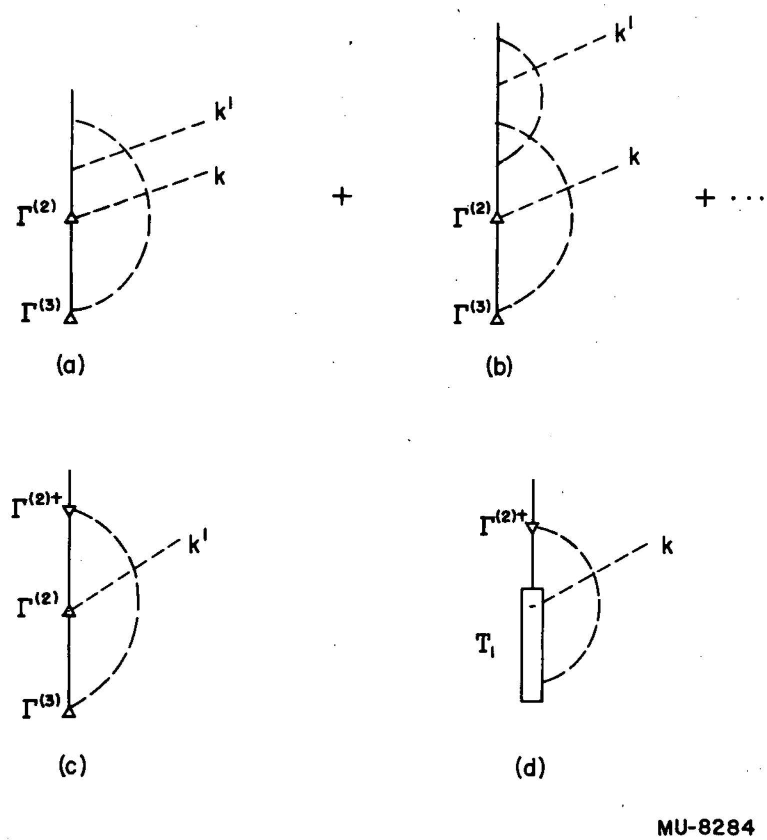
k'
k
$\Gamma^{(2)}$
$\Gamma^{(3)}$
(a)
+
k'
k
$\Gamma^{(2)}$
$\Gamma^{(3)}$
(b)
+ · · ·
$\Gamma^{(2)+}$
k'
$\Gamma^{(2)}$
$\Gamma^{(3)}$
(c)
$\Gamma^{(2)+}$
k
T_1
(d)
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Figure 2