Title
Lectures on the Theory of Competitive Equilibrium

Author
Arrow, Kenneth J

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Lectures on

THE THEORY OF COMPETITIVE EQUILIBRIUM

Kenneth J. Arrow

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Editors: Ross M. Starr, Wei Ying

University of California, San Diego (UCSD)

(Editors' e-mail contacts: rstarr@ucsd.edu, wying@ucsd.edu)

1 Lecture I

We postulate that all economic agents act at all times under (or as if under) perfect competition, in the sense that all agents take the various prices as given, and that no one agent is able to influence any price through either transacting or refusing to transact on either side of the market. In this first lecture, we will demonstrate the existence of a competitive equilibrium, representing economic activities in terms of individual excess demand functions.

Let $X_{ij}(p_1, .., p_n)$ = the excess demand by agent j for commodity i when the prices of the n commodities are $p_1, ..., p_n$. It is understood that this may be of either sign, positive for excess demand, negative for excess supply, and that certain of the commodities may be designated as various types of inputs (e.g., labor) without altering the above formulation.

Let p be the vector of prices $(p_1, ..., p_n)$ and denote by the inner product of two vectors, $p \cdot q$, the sum $\sum_{i=1}^{n} p_i \cdot q_i$. We may sum excess demands for each commodity over all economic agents to obtain the market excess demand for each commodity, thus $X_i(p) = \sum_j X_{ij}(p)$ is the market excess demand for the i^{th} commodity given the vector of prices p. Similarly $X(p)$ is the vector of market excess demands $(X_1(p), X_2(p), ..., X_n(p))$.

We provisionally seek a price vector p such that $X(p) = 0$, which is an equilibrium condition for all markets. This insures that the quantity supplied of each commodity equals the quantity demanded, so that each commodity market individually is cleared, and hence all are cleared. We are assigning the role of the equilibrating mechanism in these markets to the price vector p. We have a system of n equations ($X_i(p) = 0$ for $i = 1, ..., n$) and n unknowns $(p_1, ..., p_n)$ so that the possibility of a solution to the system is at least suggested.

If the excess demand functions are homogeneous of degree zero in prices, then we know that all prices may be multiplied by some (positive) constant without disturbing the equilibrium of the system. Therefore, we shall make the following assumption:

ASSUMPTION H: $X(p)$ is positively homogeneous of degree zero.

Thus $X(\lambda p) = X(p)$ for $\lambda > 0$ and $p \geq 0$.

Note: We wish $\lambda \neq 0$ since if $X(p^1) \neq X(p^2)$ then $X(\lambda p^1) \neq X(\lambda p^2)$ by homogeneity. However, as $\lambda \to 0$, both sides of the above would approach the same value $X(0)$ if it were defined; these are two different limits and so $X(p)$ is not defined at $p = 0$ and $X(\lambda p)$ is not defined when $\lambda = 0$.

Notation: By $p > 0$ we mean $\begin{Bmatrix} p \geq 0 \\ p \neq 0 \end{Bmatrix}$, that is, at least one of the prices must be non-zero.

By $p \gg 0$ we mean $p_i > 0$ for all i.

Since we wish to maintain the symmetric role of each of the commodities with respect to any other commodity, we will not select a numeraire; instead we will impose the further restriction on any set of prices chosen so that $\sum_{i=1}^{n} p_i = 1$. That is, for any price vector $p > 0$ that we might choose, we can select a value λ such that $\sum_{i=1}^{n} \lambda p_i = 1$, and by the homogeneity assumption, if p is an equilibrium price vector, then so is $\lambda p = (\lambda p_1, ..., \lambda p_n)$, which we can designate as $p_1^*, ..., p_n^*$, where $\sum_i p_i^* = 1$.

Notation: Define the set $\mathrm{S} = \{p \mid p \geq 0, \sum_{i=1}^{n} p_i = 1\}$ as the set of price vectors p which are non-negative and whose elements sum to one.

We can thus restrict the search for a $\bar{p}$ such that $X(\bar{p}) = 0$ to the set of price vectors p belonging to S.

We shall make a further assumption, an identity known as Walras' Law:

ASSUMPTION W: $\sum_{i=1}^{n} p_i X_i(p) \equiv 0$

From this, if $n-1$ excess demands are known to be zero, then the n^{th} is known to be zero; in the same fashion, given the values of $n-1$ prices, the n^{th} is known since $\sum_{i=1}^{n} p_i = 1$.

This procedure is analogous to the selection of a composite numeraire or market-basket. We know that if we were to select a single commodity as numeraire, its price must be non-zero; but we cannot tell in advance which commodities will have zero price at equilibrium.

Notice also that if supply exceeds demand at zero price, there can be no equating of excess demand to zero for this commodity, since the price must not be negative. This leads us to the following definitions of competitive equilibrium.

Definition: COMPETITIVE EQUILIBRIUM

Formally, an equilibrium is a vector $\bar{p} > 0$ such that:

(1) Primary Definition: for each commodity i, either

$$\text{(a) } X_i(\bar{p}) = 0, \text{ or}$$
$$\text{(b) } X_i(\bar{p}) < 0 \ \ \underline{\text{and}} \ \ \overline{p}_i = 0.$$

(2) Secondary Definition (equivalent to (1) under Walras' Law):

$$X(\bar{p}) \leq 0$$

THEOREM 1: Under assumption (W) above, definitions (1) and (2) of a competitive equilibrium are equivalent.

Proof: If definition (1) holds, then (2) obviously holds. If (2) holds, then we know

$$\sum_{i=1}^{n} \overline{p}_i X_i(\bar{p}) = 0$$
$$X_i(\bar{p}) \leq 0$$
$$\overline{p}_i \geq 0$$
$$\text{Consequently, } \ \overline{p}_i X_i(\bar{p}) \leq 0$$

But if $\sum_{i=1}^{n} \overline{p}_i X_i(\bar{p}) = 0$ and each of its terms must be less than or equal to zero, then each item individually must be zero. Thus $\overline{p}_i X_i(\bar{p}) = 0$ for all i, and so either

$$\text{(a) } \ X_i(\bar{p}) = 0 \text{ or}$$
$$\text{(b) } \ X_i(\bar{p}) < 0 \ \ \underline{\text{and}} \ \ \overline{p}_i = 0.$$

In the above, we have implicitly assumed that the excess demand functions $X_i(p)$ were single-valued. We shall retain this and add two further assumptions:

ASSUMPTION C: $X(p)$ is continuous: $\lim_{n\to\infty} X(p^n) = X(p^o)$ if $p^n \to p^o$.

We do not exclude infinite values so long as they are approached by every price sequence $p^n \to p^o$; thus $X(\bar{p})$ might be infinite.

ASSUMPTION B: $X(p)$ is bounded from below for excess demand (or above for excess supply).

In a pure exchange commodity no individual can supply more than he has; hence B clearly holds. In a production economy, at given prices, any firm may wish to supply indefinitely large amounts of a good as its price increases, but this would require inputs to rise correspondingly, and this cannot in fact happen since the supply of resources is finite.

We are considering the above system as a purely stationary economy, although an alternative consideration would be a one-period model with absolutely no carryover of commodities or intangibles to future periods.

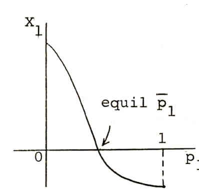
THEOREM 2: In a two-commodity model ($n = 2$), assumptions H, W, C, B imply the existence of a competitive equilibrium.

Proof: we give first a graphic interpretation and then a general proof:

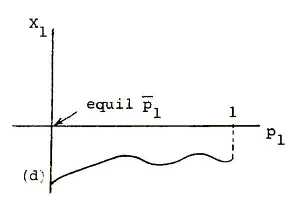
Graphic interpretation

Since $p_1 + p_2 = 1$, $p_1 \geq 0$, $p_2 \geq 0$, we may write $X_1(p_1, 1 - p_1)$ for any $0 \leq p_1 \leq 1$.

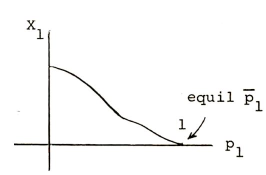
Case (a) - Excess demand positive for some p_1, negative for other p_1.



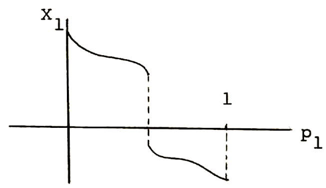
Case (b) - Excess demand negative for all p_1
At (d) $p_1 = 0$, $p_2 = 1$,
and $X_2(p_1, 1 - p_1) = 0$.



Case (c) - Excess demand positive for all p_1,
$0 < p_1 < 1$.
Case (c) is the same as case (b) with the role of the two commodities reversed.



Assumption C is of primary importance in the above; it insures that we cannot have an indeterminate case such as the following:



General Proof:

$$\text{Either case (a): } X_1(\overline{p}_1, 1-\overline{p}_1) = 0 \text{ for some } \overline{p}_1,\ 0 < \overline{p}_1 < 1,$$
$$\text{or case (b): } X_1(p_1, 1-p_1) < 0 \text{ for all } p_1,\ 0 < p_1 < 1,$$
$$\text{or case (c): } X_1(p_1, 1-p_1) > 0 \text{ for all } p_1,\ 0 < p_1 < 1.$$

By assumption C these three cases are mutually exclusive and exhaustive. By assumption B, excess demand for either commodity may be infinite only at either end ($p_1 = 0$ or $p_1 = 1$) and not in between, since the excess demand for the other commodity if both prices are positive would have to be $-\infty$ by Walras' Law (W), and this would contradict (B).

Case (a):

$$X_1(\overline{p}_1, 1-\overline{p}_1) = 0 \quad \text{for some } \overline{p}_1,\ 0 < \overline{p}_1 < 1$$
$$\overline{p}_1 X_1(\overline{p}_1, \overline{p}_2) + \overline{p}_2 X_2(\overline{p}_1, \overline{p}_2) = 0,\ \overline{p}_2 = 1 - \overline{p}_1 > 0.$$
$$\text{Since } \overline{p}_1 > 0, \text{ then } X_2(\overline{p}_1, \overline{p}_2) \text{ must equal zero.}$$

Case (b):

$$X(p_1, 1-p_1) < 0 \quad \text{for all } p_1,\ 0 < p_1 < 1.$$

Let $p_1 \to 0$, then $-\infty < X_1(0,1) \leq 0$ by (C) and (B). Also, $X_2(p_1, 1-p_1) = \frac{-p_1}{1-p_1} X_1(p_1, 1-p_1) > 0$ for $0 < p_1 < 1$, and so, as $p_1 \to 0$, $\frac{-p_1}{1-p_1} X_1(p_1, 1-p_1) \to 0$ and therefore $X_2(p_1, 1-p_1) \to 0$.

Therefore $X_2(0,1) = 0$ when $p_1 = 0$. Therefore, $\overline{p}_1 = 0$, $\overline{p}_2 = 1$ satisfies the definition of equilibrium.

Case (c):

$$X_1(p_1, 1-p_1) > 0 \quad \text{for all } p_1,\ 0 < p_1 < 1.$$

Now by (W), $p_1X_1(p_1, 1-p_1)+p_2X_2(p_1, 1-p_1) = 0$, and $p_1 > 0$, $p_2 > 0$, $X_1(p_1, 1-p_1) > 0$; therefore $X_2(p_1, 1-p_1) < 0$ for $0 < p_1 < 1$. Since $0 < p_1 = 1-p_2 < 1$, $X_2(p_2, 1-p_2) < 0$ for all p_2, $0 < p_2 < 1$. Therefore, case(c) reduces to case (b).

This concludes the proof of the existence of an equilibrium for the case $n = 2$.

THEOREM 3: Under assumptions H, W, C, B there exists a competitive equilibrium.

Heuristic Argument: A possible procedure to find an equilibrium would be to choose an arbitrary price vector p (satisfying the requirement $p \in S$) and check the resulting excess demands; where an excess demand was positive, we would be tempted to raise the corresponding commodity price, and where an excess demand was negative, to lower its price. As a first approximation, we might raise or lower all prices by some constant portion, β, of the excess demand, thus

$$p_i \to p_i + \beta X_i(p).$$

However, this rule might tell us to set some prices at a negative value, so we add the further modification

$$p_i \to Max[0,\ p_i + \beta X_i(p)]$$

to insure that the new prices are still all non-negative. But we wish $\sum_{i=1}^{n} p_i = 1$ for both the initial set and the new prices obtained from the application of the above rule. Therefore, the final transformation is as follows:

$$\text{for } p \in S, \text{ let } \theta(p) = \frac{1}{\sum_{i=1}^{n} Max[0,\ p_i+\beta X_i(p)]}$$

and then $p_i \to \theta(p) \cdot Max[0,\ p_i + \beta X_i(p)] = T_i(p)$ is the transformation of the i^{th} price. In vector form, $p \to T(p)$ is the transformation we will use.

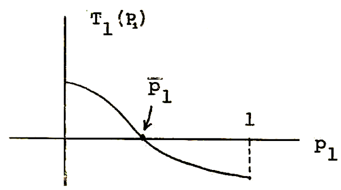
Further,

$$\sum_{i=1}^{n} T_i(p) = 1,\ T_i(p) \geq 0, \text{ and } T(p) \in S,$$
the unit simplex.

Consider again for the moment the case $n = 2$. Here we know $0 \leq T_1(p_1, 1-p_1) \leq 1$. Therefore,

$$T_1(p_1, 1-p_1) - p_1 \begin{cases} \geq 0 \text{ if } p_1 = 0 \\ \leq 0 \text{ if } p_1 = 1. \end{cases}$$

Graphically:



Here $T_1(\overline{p}_1, 1-\overline{p}_1) = \overline{p}_1$; since $T_1(p_1,p_2)+T_2(p_1,p_2) = 1 = p_1+p_2$, $T_2(\overline{p}_1, 1-\overline{p}_1) = 1-\overline{p}_1 = \overline{p}_2$. Thus at some point (at least one point) the price vector is transformed into itself — it is a fixed point, and this fixed point, as we shall see, is the equilibrium price vector.

Then to prove the existence of a competitive equilibrium we need to demonstrate that a fixed point exists in the n-dimensional transformation $T(p)$.

BROUWER FIXED-POINT THEOREM:

If $T(p)$ is a continuous transformation of a simplex into itself, then it has at least one fixed point $T(p) = p$.

More generally, the theorem is valid for a transformation of any closed, bounded convex set into itself. For proofs, see S. Lefschetz, Introduction to Topology; E. Burger, Einführung in die Spieltheorie.

To show that $T(p)$ is continuous requires us to show:

(a) $Max[0,\ p_i + \beta X_i(p)]$ is a continuous function.

(b) $\theta(p)$ is continuous, which requires for $\theta(p) = \frac{1}{f(p)}$

that $f(p) > 0$ and continuous.

(c) For f and g two continuous functions, $f \cdot g = h$ is a continuous function which is well known.

Now, $Max[0,\ p_i + \beta X_i(p)]$ is continuous if $p_i + \beta X_i(p)$ is continuous. By assumption C, $X_i(p)$ is continuous, β is a positive constant, and p_i is a continuous variable. Therefore, $Max[0,\ p_i + \beta X_i(p)]$ is continuous.

Lemma 1: If $X(p)$ satisfies assumption W for $p \in S$, then for each $p \in S$, $p_i + \beta X_i(p) > 0$ for some i.

Proof:

Suppose it is false for some $p \in S$. Then

(1) $p_i + \beta X_i(p) \leq 0$ for all i. Since $p \geq 0$, $X_i(p) \leq 0$ for all i. Thus p is an equilibrium price factor.

(2) From the definition of equilibrium, we have $p_i = 0$ for those i for which $X_i(p) < 0$, while, from (1), $p_i \leq 0$ if $X_i(p) = 0$. Then $p_i \leq 0$ for all i, which is impossible since $p > 0$.

From Lemma 1, $Max[0,\ p_i + \beta X_i(p)] > 0$ for some i. Thus $\sum_{i=1}^{n} Max[0,\ p_i + \beta X_i(p)] = \frac{1}{\theta(p)} > 0$ for all $p \in S$. Therefore $\theta(p) > 0$ for all $p \in S$, and is continuous since it is the reciprocal of a non-zero continuous function. This information is summed up by the following lemma.

Lemma 2: $T(p)$ is a continuous transformation.

We are now assured by the Brouwer Fixed-Point Theorem that a fixed point exists, and to complete the proof of equilibrium, we need only show that the fixed point is indeed a competitive equilibrium.

[1] $\bar{p} = T(\bar{p})$

$\overline{p}_i = T_i(\bar{p}) = \theta(\bar{p}) Max[0,\ \overline{p}_i + \beta X_i(\bar{p})]$.

[2] Let $A = \{i \mid \overline{p}_i + \beta X_i(\bar{p}) > 0\}$.

[3] Let $B = \{i \mid \overline{p}_i + \beta X_i(\bar{p}) \leq 0\}$. Thus $Max[0,\ \overline{p}_i + \beta X_i(\bar{p})] = 0$

for $i \in B$.

The division of indices into two subsets, A and B, amounts to the separation of free from non-free goods. We wish to show that for the free goods ($i \in B$), $X_i(\bar{p}) \leq 0$ and $\overline{p}_i = 0$, and for the non-free goods ($i \in A$) that $X_i(\bar{p}) = 0$.

[4] $\overline{p}_i = 0$ for $i \in B$ by [1], [3].

[5] $X_i(\bar{p}) \leq 0$ for $i \in B$, and we know by assumption (W) that

$$0 = \sum_{i \in A} \overline{p}_i X_i(\bar{p}) + \sum_{i \in B} \overline{p}_i X_i(\bar{p})$$

From [5] and assumption (B), we know that $X_i(\bar{p})$ is finite for $i \in B$.

From [4], $\overline{p}_i X_i(\bar{p}) = 0$ for $i \in B$.

[6] Thus $\sum_{i \in A} \overline{p}_i X_i(\bar{p}) = 0$.

Also $Max[0,\ \overline{p}_i + \beta X_i(\bar{p})] = \overline{p}_i + \beta X_i(\bar{p})$ for $i \in A$

From [1] $\overline{p}_i = \theta(\bar{p})[\overline{p}_i + \beta X_i(\bar{p})]$ for $i \in A$

[7] Then $[1 - \theta(\bar{p})]\overline{p}_i = \theta(\bar{p})\beta X_i(\bar{p})$ for $i \in A$.

Multiply both sides of [7] by $X_i(\bar{p})$ to obtain

$$[1 - \theta(\bar{p})]\overline{p}_i X_i(\bar{p}) = \theta(\bar{p})\beta[X_i(\bar{p})]^2 \text{ for } i \in A.$$

Then sum over $i \in A$

[8] $[1 - \theta(\bar{p})] \sum_{i \in A} \overline{p}_i X_i(\bar{p}) = \theta(\bar{p})\beta \sum_{i \in A}[X_i(\bar{p})]^2$ for $i \in A$.

But $\sum_{i \in A} \overline{p}_i X_i(\bar{p}) = 0$ for $i \in A$ by [6]; $\theta(\bar{p}) > 0$, $\beta > 0$.

Therefore, $\sum_{i \in A}[X_i(\bar{p})]^2 = 0$, which implies

[9] $X_i(\bar{p}) = 0$ for $i \in A$.

Thus, for $i \in B$, $\overline{p}_i = 0$ and $X_i(\bar{p}) \leq 0$ from [4], [5] and, for $i \in A$, $X_i(\bar{p}) = 0$ from [9].

The purpose of steps [5] through [8] is to set up a mathematical situation in which a zero on the left hand side of an equation guarantees that $X_i(\bar{p})$ can only be zero for $i \in A$ on the right hand side.

This concludes the proof of the existence of a competitive equilibrium.

2 Lecture II

In this second and in subsequent lectures, we will go behind the model outlined in Lecture I to explain the behavior of economic agents in more detail. We will do away with single-valued supply functions.

We shall begin with a brief outline of historical developments, beginning with the model of G. Cassel (The Theory of Social Economy, 1903).

Cassel model:

a_{ij} = the fixed coefficient of amount of input i required to produce a unit of final good j,

m final goods	v = vector of fixed factor supply
n factors	x = vector of final goods
	r = vector of factor prices
	p = vector of final goods prices

We have three groups of equations:

(1) Production $\quad v_i = \sum_j a_{ij} x_j \quad$ (n equations)

(2) Zero profit $\quad \sum_i a_{ij} r_i = p_j \quad$ (m equations)

(3) Closed system (demand) $\quad x = d(p, r \cdot v) \quad$ (m equations)

Notation: here j indexes commodities $(1, ..., j, ..., m)$
and i indexes factors $(1, ..., i, ..., n)$.

We have $(2m + n)$ equations and $(2m + n)$ unknowns, thus the possibility of an equilibrium solution is suggested.

Some thirty years later, Hans Neisser pointed out that nothing in the above model guaranteed that all p_i and r_i were non-negative. Von Stackelberg indicated that for $m < n$, in

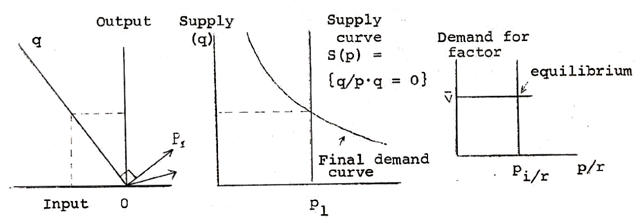
(1) there are n equations in m unknowns while in (2) there are m equations in n unknowns, and so a solution seems unlikely.

He drew the inference that fixed coefficients are unacceptable, but this actually doesn't follow. Schlesinger in his introduction to the Wald paper pointed out that, following Zeuthen's approach, "free goods" could be handled through inequalities rather than strict equations. The introduction of inequalities leads to the following revision:

$$(1)\ v_i \geq \sum_j a_{ij}x_j \text{ and } r_i = 0 \text{ if } v_i > \sum_j a_{ij}x_j$$
$$(2)\ \sum_i a_{ij}r_i \geq p_j, \text{ with no production } (x_j = 0) \text{ if } \sum_i a_{ij}r_i > p_j.$$

The following diagrams provide illustrations of the concepts of "supply" and "demand."

First example:

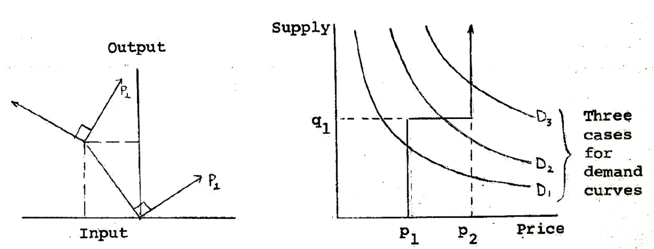


Price $= \frac{price\ of\ output}{price\ of\ input} = \frac{p}{r}$

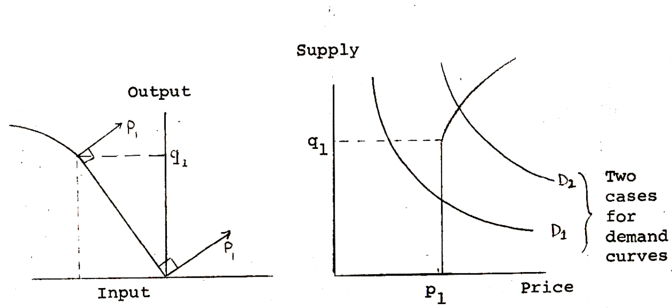
Choose a p_1 such that $p_1 \cdot q = 0$.

Demand for final goods $= \frac{r \cdot v}{p} = \frac{v}{(p/r)}$, which is a rectangular hyperbola [cf. eq.(3)]

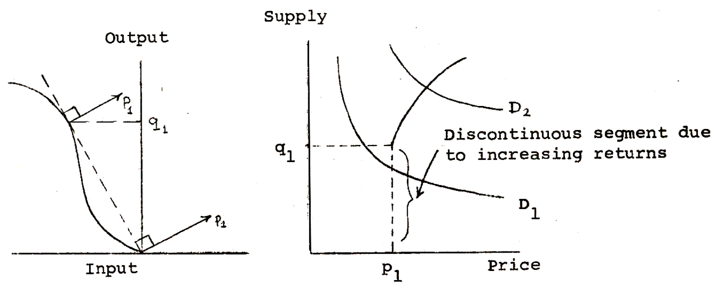
Second example:



Third example:

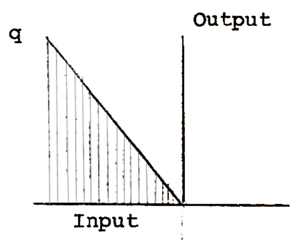


Fourth example:



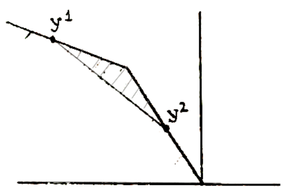
Here, demand curve D_1 does not lead to an equilibrium solution.

The first example illustrates production carried out under one process having an absolutely fixed coefficient a_{ij} whose magnitude is given by the slope of $\overline{0q}$. Supply is multi-valued at $p = p_1$. In the second example, we have two processes, one of which is more efficient but can only be operated up to a limit (q_1); beyond that point the less efficient must be used. Beyond the point q_1, the price required to call forth the additional supply rises from p_1 to p_2. In the third example we have a fixed coefficient process up to a limit q_1, and beyond this a gradual, continuous falling off of efficiency; i.e., constant returns up to q_1 and diminishing returns thereafter. The market price must rise by greater and greater amounts to call forth the additional supply. In the fourth example we have increasing returns up to q_1, and diminishing returns thereafter. We have a discontinuous supply set, since for any quantity less than q_1, a producer must use relatively more input per unit output than to produce at q_1. This is the result of a non-convex production-possibility set. Usually, we have free disposal, so that we have a production possibility set (the hatched area).



If any point on q is available then so is any point vertically below it.

Definition: A set is convex if, when y^1 and y^2 belong to it so does $\lambda y^1 + (1-\lambda)y^2$ for $0 \leq \lambda \leq 1$.



$\lambda y^1 + (1-\lambda)y^2$ traces out all points on the straight line segment connecting y^1 and y^2 as λ varies from 0 to 1 .

Notation: We will have occasion to refer to sequences (of vectors), say p^n, and to consider what happens as n increases (the sequence takes on further terms); we will designate terms of a sequence, $p^1, p^2, ..., p^n, ...$ approaching a limit p^o as n increases by $p^n \to p^o$ (equivalently, $\lim_{n\to\infty} p^n = p^o$). The superscript zero designates the limiting value in such cases.

Definition: A correspondence from a set A to a set B is a function with domain in A and range in the set of all subsets of B (a set-valued function).

Definition: Upper semi-continuity of a correspondence. A correspondence $\phi(p)$ defined on the set of all $p \in P$ is upper semi-continuous when, if $p^n \to p^o$, $x^n \in \phi(p^n)$ and $x^n \to x^o$, then $x^o \in \phi(p^o)$.

Let us now specify the general model:

(1) Production is organized in firms; i.e., in production-possibility sets Y^j where the j indexes firms. [Note: change of notation here from that of Cassel model.]

Production Assumptions:

P.I Y^j is convex for each j.

P.II $0 \in Y^j$.

P.III Y^j is closed.

Definition: A set is closed if and only if $y^n \in Y$ and $y^n \to y^o$ together imply $y^o \in Y$.

P.IV Y^j is bounded. (P.IV will later be dropped.)

Definition: A set is bounded if, for some N,

$$|y| \leq N \text{ for all } y \in Y.$$

Definition: A vector sum $Y = \sum_j Y^j$ is the set,

$$\{y \mid y = \sum_j y^j \text{ for some } y^j \in Y^j\}.$$

The social production-possibility set is $Y = \sum_j Y^j$.

The replacement for P.IV is

P.IV′: (a) if $y \in Y$ and $y \neq 0$, then $y_k < 0$ for some k. This is the "no free lunch" postulate.

(b) if $y \in Y$ and $y \neq 0$, then $-y \notin Y$. This is the irreversibility postulate — there exists no way to re-transform an output back to the original quantities of all inputs.

Now, taking p as given, for each j the firm "chooses" y^j such that $p \cdot y^j = Max\ p \cdot y^j$ for all $y^j \in Y^j$ which may not yield a unique solution.

Let $Y^j(p) = \{y^* \mid y^* \in Y^j,\ p \cdot y^* \geq p \cdot y^j \text{ for all } y^j \in Y^j\}$ be the excess supply correspondence of the firm.

Lemma 3: $Y^j(p)$ is non-null (from P.III and P.IV).

Lemma 4: $Y^j(p)$ is convex.

Proof: Let $y^1 \in Y^j(p)$ and $y^2 \in Y^j(p)$.

For fixed p, $p \cdot y^1 = p \cdot y^2 \geq p \cdot y^j$ for all $y^j \in Y^j$.

For $0 \leq \lambda \leq 1$, consider $p[\lambda y^1 + (1-\lambda)y^2] = \lambda p \cdot y^2 + (1-\lambda)p \cdot y^2 = p \cdot y^1 = p \cdot y^2 \geq p \cdot y^j$ for all $y^j \in Y^j$.

But any $\lambda y^1 + (1-\lambda)y^2 \in Y^j$ by P.I.

We also wish to prove that:

(1) the correspondence $Y^j(p)$ is upper semi-continuous and

(2) for any fixed p, the set $Y^j(p)$ is closed. These two are not equivalent, and we require both.

To prove (1), we assume $p^n \to p^o$, $y^n \in Y^j(p^n)$ and $y^n \to y^o$, and wish to show $y^o \in Y^j(p^o)$

Proof: $p^n \cdot y^n \geq p^n \cdot y$ for $y \in Y^j$, and $y^o \in Y^j$ since $y^n \in Y^j$ for all n and Y^j is closed (P.III).

If n approaches infinity, we have $p^o \cdot y^o \geq p^o \cdot y$ for all $y^o \in Y^j(p)$ and y^o is profit maximizing.

(2) We must show $y^n \in Y^j(p)$ and $y^n \to y^o$ imply $y^o \in Y^j(p)$.

From P. III, we know $y^o \in Y$.

By the definition of $Y^j(p)$

$$p \cdot y^n \geq p \cdot y \text{ for } y \in Y^j$$

$$\lim_{n \to \infty} p \cdot y^n = p \cdot y^o \geq p \cdot y \text{ for } y \in Y^j,$$

so $y^o \in Y^j(p)$ and $Y^j(p)$ is closed.

We sum up the above as the following theorem:

THEOREM 4: Under P.I - IV, for each p, $Y^j(p)$ is non-null, convex, closed, and bounded, and $Y^j(p)$ is a bounded upper semi-continuous correspondence.

Definition: Profit Function

Let $\pi_j(p) = \max_{y \in Y^j}(p \cdot y)$ be the profit function.

We wish now to show that $\pi_j(p)$ is continuous in p. Since $0 \in Y^j$ by P.II, then $\pi_j(p) \geq p{\cdot}0 = 0$. Let $p^n \to p^o$; choose $y^n \in Y^j(p^n)$ for $y^n \in Y^j$. The sequence $\{y^n\}$ is bounded because Y^j is bounded (P.III). By the Bolzano-Weierstrass Theorem, every bounded infinite sequence of this kind has a subsequence that converges to a limit. That is, from the sequence $\{y^n\}$, one may select certain elements from the sequence to form a new (sub-)sequence that converges to a limit. Without loss of generality, then, assume that $\{y^n\}$ is itself convergent. Then $y^n \to y^o$ and $y^o \in Y^j(p^o)$ by Theorem 4.

Then $(p^n - p^o) \cdot y^n \to 0$ as $p^n \to p^o$, and

$$p^n \cdot y^o - (p^n - p^o) \cdot y^n \to p^o \cdot y^o \text{ as } p^n \to p^o,\ y^n \to y^o.$$

Since y^o maximizes $p^o \cdot y^j$ for $y^j \in Y^j$ and y^n maximizes $p^n \cdot y^j$ for $y^j \in Y^j$,

$$p^o \cdot y^o \geq p^o \cdot y^n = p^n \cdot y^n - (p^n - p^o) \cdot y^n \geq p^n \cdot y^o - (p^n - p^o) \cdot y^n,$$

or,
$$p^o \cdot y^o + (p^n - p^o) \cdot y^n \geq p^n \cdot y^n \geq p^n \cdot y^o.$$

As n approaches infinity, $p^n \cdot y^o \to p^o \cdot y^o$, hence

$$\underline{\lim}_{n\to\infty} p^n \cdot y^n = p^o \cdot y^o, \text{ or}$$
$$\underline{\lim}_{n\to\infty} \pi_j(p^n) = \pi_j(p^o).$$

Again, to sum up:

THEOREM 5: Under P.I -IV, $\pi_j(p)$ is a continuous, non-negative function of p.

Definition: Market Excess Supply Correspondence

Let $Y(p) = \sum_j Y^j(p)$ be the market excess supply correspondence.

We now wish to show that Y, the social production possibility set, has all the properties of Y^j; that it satisfies all of P.I - P.IV. This is obvious for all but P.III. We wish then to show that Y is closed; for this purpose, P.IV [Y^j is bounded] is essential. We have to prove that $y^n \in Y$ and $y^n \to y^o$ imply that $y^o \in Y$.

Let $y^n = \sum_j y^{nj}$ for $y^{nj} \in Y^j$. As in the proof of Theorem 5, for each sequence $\{y^{n1}\}$, $\{y^{n2}\}$, ... there exists a convergent subsequence, since each is bounded. Without loss of generality assume it is $\{y^{nj}\}$. Then since $\{y^{nj}\}$ is bounded for each j, assume $y^{nj} \to y^{oj}$; summing over j, $y^n = \sum_j y^{nj} \to \sum_j y^{oj} = y^o$, and since $y^{oj} \in Y^j$, then $y^o \in Y$, as was to be shown.

Thus,

<u>THEOREM 6:</u> If P.I-IV hold for each Y^j, they hold for Y.

<u>Definition:</u> $Y^*(p)$, for any p, is the output which maximizes the value of total production.

$$Y^*(p) = \{y^* \mid y^* \in Y,\ p \cdot y^* \geq p \cdot y \text{ for } y \in Y\}.$$

We will show that $Y^*(p) = Y(p)$. Let $y^* \in Y^*(p)$, $y^* = \sum_j y^{*j}$, $y^{*j} \in Y^j$.

Let $y = \sum_{j \neq k} y^{*j} + y^k$ for some $y^k \in Y^k$.

Now $y \in Y$ and $p \cdot y^* \geq p \cdot y$ by definition of $Y^*(p)$; therefore

$$p \cdot (\textstyle\sum_{j \neq k} y^{*j} + y^{*k}) \geq\ p \cdot (\textstyle\sum_{j \neq k} y^{*j} + y^k).$$

Therefore $p \cdot y^{*k} \geq p \cdot y^k$ for all $y^k \in Y^k$. But $y^{*k} \in Y^k$ and so $y^{*k} \in Y^k(p)$. Thus if $y^* \in Y^*(p)$, $y^* \in Y(p)$.

We will also show that if $y^* \in Y(p)$, then $y^* \in Y^*(p)$. Assume $y^* \in Y(p)$, $y^* = \sum_j y^{*j}$, $y^{*j} \in Y^j(p)$, and $p \cdot y^{*j} \geq p \cdot y^j$ for all $y^j \in Y^j$.

Summing over j, $p \cdot y^* \geq p \cdot y$ for all $y \in Y$ and so $y^* \in Y^*(p)$.

<u>THEOREM 7:</u> Under P.I - IV, for each p, $Y(p)$ is non-null, convex, closed, and bounded, and $Y(p)$ is a bounded upper semi-continuous correspondence.

<u>Proof:</u> Follows from Theorems 4, 6, 7.

3 Lecture III

The firm excess supply correspondence $Y^j(p)$ is that set which maximizes profits. Because of the boundedness assumption, there must exist at least one profit-maximizing point for each firm. We sum over the firms to obtain the market excess supply correspondence $Y(p)$, which, for each p, is closed, convex, bounded, and non-null, and is a bounded upper semi-continuous correspondence Note that P.IV′ applies to $Y(p)$, the aggregate relation. This is a stronger postulate than if defined over individual firms. That (a version of) P.IV′ holds for each individual firm does not insure that it holds for society. We are about to drop P.IV and replace it with P. IV′.

Let ξ be the vector of total initial resources or endowments. Because of P.IV′ (a), there can never be an infinite production (since ξ is finite).

Definition: y is attainable in Y if: $y + \xi \geq 0$ for $y \in Y$.

The set of attainable vectors y is bounded, and it is among these that an equilibrium vector is to be found (if it exists).

THEOREM 9: Under P.I, P.II, P.III, P.IV′, the set of attainable vectors in Y is bounded.

Proof: Suppose it is not; then there is a sequence of vectors $\{y^n\}$ such that:

(1) $|y^n| \to +\infty$ as n becomes large

(2) $y^n \in Y$

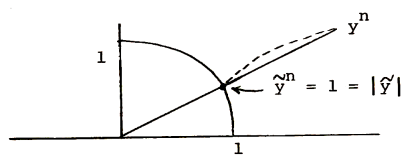
(3) $y^n + \xi \geq 0$

We shall show that these imply a contradiction.

Let $\widetilde{y}^n = \frac{y^n}{|y^n|}$; since $\lim_{n\to\infty} |y^n| = +\infty$, we know $|y^n| \geq 1$ for n large. Note that $\widetilde{y}^n = \frac{1}{|y^n|} y^n + (1 - \frac{1}{|y^n|})0$.

Under P.II, $0 \in Y$ and under P.I, $\widetilde{y}^n \in Y$, since the above is a convex linear combination of y^n and 0, both in Y, when $|y^n| \geq 1$.

Thus, we can make $|\widetilde{y}^n| = 1$; that is, we can project the y^n sequence back to the unit circle:



Since $\{\widetilde{y}^n\}$ is bounded, we can, without loss of generality, let $\widetilde{y}^n \to \widetilde{y}$ and $|\widetilde{y}| = 1$. From P.III, $\widetilde{y} \in Y$ and from (3) above

$$\frac{y^n}{|y^n|} + \frac{\xi}{|y^n|} \geq 0. \quad \text{or} \quad \widetilde{y}^n + \frac{\xi}{|y^n|} \geq 0.$$

Now, by (1) above, $\lim_{n\to\infty}(\frac{\xi}{|y^n|}) = 0$; therefore $\lim_{n\to\infty}(\widetilde{y}^n + \frac{\xi}{|y^n|}) \geq 0$, or $\widetilde{y} \geq 0$, while $\widetilde{y} \neq 0$, since $|\widetilde{y}| = 1$, which contradicts P.IV′(a).

Any particular individual firm vector might not satisfy $y + \xi \geq 0$. Thus we say that y^j is attainable in Y^j if there exists a y^k for each of the $k \neq j$ firms such that $\sum_{i=1}^m y^i$ is attainable for all $i = 1, 2, ..., m$ firms. We wish to show that this implies boundedness for the firms' production possibilities.

<u>THEOREM 10:</u> Under P.I, P.II, P.III, P.IV′ the set of attainable vectors in Y^j is bounded.

<u>Proof:</u> Suppose it is not. Then there exists a sequence $\{y^{nj}\}$ such that:

(1) $|y^{nj}| \to +\infty$

(2) $y^{nj} \in Y^j$

(3) $y^n = y^{nj} + \sum_{k\neq j} y^{nk}$ is attainable and thus bounded.

We show that this contradicts P.IV′(b).

Let $\widetilde{y}^{nj} = \frac{y^{nj}}{|y^{nj}|}$ and $y^{*nj} = \frac{\sum_{k\neq j} y^{nk}}{|y^{nj}|}$. Also, $y^{nj} = y^{nj} + \sum_{k\neq j} 0$ is contained in Y, and $\sum_{k\neq j} y^{nk} = 0 + \sum_{k\neq j} y^{nk}$ is contained in Y.

For n large, $\widetilde{y}^{nj} \in Y$ and $y^{*nj} \in Y$.

$$\frac{y^n}{|y^{nj}|} = \widetilde{y}^{nj} + y^{*nj} \text{ from (3) above.}$$

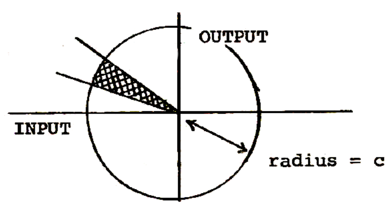
$\{\widetilde{y}^{nj}\}$ is a bounded sequence, so, by an argument used previously, we can assume without loss of generality that it is convergent, thus $\widetilde{y}^{nj} \to \widetilde{y}^j$.

Then $y^{*nj} = \frac{y^n}{|y^{nj}|} - \widetilde{y}^{nj}$ and $\lim_{n\to\infty}(\frac{y^n}{|y^{nj}|}) = 0$, so that $\lim_{n\to\infty}(y^{*nj}) = -\widetilde{y}^j$. Since $\widetilde{y}^j \in Y$, $|\widetilde{y}^j| = 1$ and $\widetilde{y}^j \neq 0$ [Note: if $|\widetilde{y}^j| = 1$, $\widetilde{y}^j$ certainly cannot be zero], then $\lim_{n\to\infty}(y^{*nj}) \in Y$ which contradicts P.IV′(b), since $\widetilde{y}^j$ and $-\widetilde{y}^j$ cannot both be in Y.

If we find the maximum of a certain set, then apply a restriction to the set (as with attainability) we may find that maximizing subject to the constraint yields a different maximum vector than the unconstrained maximum. We have to show that if the constraint of attainability is removed, we still have the same profit maximizing vector. Since the firm doesn't "recognize" our mathematical attainability restriction, we must show that the presence of the restriction does not affect the equilibrium the firm chooses given the prices.

<u>THEOREM 11:</u> Assume P.I, P.II, P.III, P.IV′. Choose c such that $|y^j| < c$ for all attainable vectors in y^j. Let $\widetilde{Y}^j = Y^j \cap \{y^j \mid |y^j| \leq c\}$.

Graphically:



Let $\widetilde{Y}^j(p) = \{y^{*j} \mid p \cdot y^{*j} \geq p \cdot y^j \text{ for all } y^j \in \widetilde{Y}^j, y^{*j} \in \widetilde{Y}^j\}$. Then:
(a) $\widetilde{Y}^j(p)$ for each p is closed, convex, bounded, and non-null, and $\widetilde{Y}^j(p)$ is a bounded upper semi-continuous correspondence;
(b) if y^j is attainable and $y^j \in \widetilde{Y}^j(p)$, then $y^j \in Y^j(p)$

<u>Proof:</u> For part (a), it is evident that P.I-P.IV are satisfied for $\widetilde{Y}^j$, since the intersection of two closed, convex sets is closed and convex; 0 belongs to both Y^j and $\{y^j \mid |y^j| \leq c\}$, and the intersection of two sets, one of which is bounded, must be bounded. Thus $\widetilde{Y}^j(p)$

satisfies Theorem 8.

For part (b), suppose y^j attainable and $y^j \in \widetilde{Y}^j(P)$ but $y^j \notin Y^j(p)$. Then $p \cdot \overline{y}^j > p \cdot y^j$ for some $\overline{y}^j \in Y^j$ and also $p \cdot [\alpha \overline{y}^j + (1-\alpha) y^j] > p \cdot y^j$ for any α, $0 < \alpha \leq 1$.

Furthermore, $\alpha \overline{y}^j + (1-\alpha) y^j \in Y^j$.

Now $|y^j| < c$, so we can choose some $\alpha > 0$ so that $|\alpha \overline{y}^j + (1-\alpha) y^j| \leq c$ for this α. But then $\alpha \overline{y}^j + (1-\alpha) y^j \in \widetilde{Y}^j$, so $y^j \notin \widetilde{Y}^j(p)$, as was assumed.

This completes the analysis of firms, and we turn next to the analysis of households.

In the theory of households to be presented below, we will choose to consider leisure as a good to be consumed rather than as a factor (a certain type of labor) to be supplied, so that the choice set is non-negative. Let there be r types of labor offered, $\ell_1, ..., \ell_r$, and define the standard unit as one "day." Then define r "leisures" $x_1, ..., x_r$ as the difference between a standard unit of each type of labor and the amount of each type of labor actually supplied. Thus:

$$
\begin{aligned}
x_1 &= 1 - \ell_1 \\
x_2 &= 1 - \ell_2 \\
&\vdots \\
x_r &= 1 - \ell_r,
\end{aligned}
$$

where $\ell_i \geq 0$ for all i, $\sum_{i=1}^r \ell_i \leq 1$. Therefore, $x_i \geq 0$ for all i, and $\sum_{i=1}^r x_i = r - \sum_{i=1}^r \ell_i \leq r - 1$ (which insures no one works more than 24 hours per day). Now $0 \notin X^i$ so that the consumer chooses from a set bounded from below rather than from the entire non-negative orthant. The above is subsumed in the statement that the set of consumption vectors is convex and bounded from below.

We also assume a utility function $U^i(X^i)$ for each individual. [For a discussion of conditions leading to the existence of $U^i(X^i)$ see Debreu, Theory of Value, ch. 4, and Debreu, ch. XI in Thrall, Coombs, Davis, Decision Processes.].

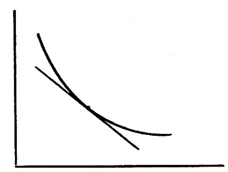
We shall specify during the analysis of households some or all of the following consumptions assumptions.

C.I: X^i is convex and closed.

C.II: $U^i(X^i)$ is continuous and semi-strictly quasi-concave.

Note: That is, $\{x^i \mid U^i(x^i) \geq u\}$ is a convex set for each u:

"quasi-concave" — may have flat regions
"semi-strictly" — if x_i is strictly preferred to x_2,
i.e., the indifference curves have no thickness.



Thus if $u^i(x^{1i}) > u^i(x^{2i})$ then $u^i[\alpha x^{1i} + (1-\alpha)x^{2i}] > u^i(x^{2i})$ for $0 < \alpha \leq 1$.

C.III: $M_i(p) = p \cdot \xi^i + \mu_i(p)$: for each individual,

income = initial endowment value + share of profits received, where $0 \leq \mu_i(p)$ is continuous, homogeneous of degree one and $\sum_i \mu_i(p) = \sum_j \pi^j(p)$. [We do not specify the method of distribution.]

C.IV: $\xi^i \gg \omega^i$ for some $\omega^i \in X^i$.

C.IV′: There exists an $\omega^i \in X^i$ such that $\xi^i_k > \omega^i_k$ for $\xi^i_k > 0$, and $\xi^i_k \geq \omega^i_k \geq 0$ otherwise.

We will use C.IV at this time. It states that each individual can supply a strictly positive amount of each kind of labor and possesses a strictly positive amount of each non-labor asset. C.IV′, on the contrary, is not an assumption but a restatement of the convention by which leisure is introduced as a commodity. Suppose an individual can supply r types of labor, say $1, ..., r$. We can write $\xi^i_k = 1$ $(k = 1, ..., r)$, while X^i satisfies the constraint, $\sum_{k=1}^r x^i_k \geq r-1$. Then we can certainly choose ω^i_k $(k = 1, ..., r)$ so that $\sum_{k=1}^r \omega^i_k \geq r-1$, $\omega^i_k < \xi^i_k$ (e.g., $\omega^i_k = \frac{r-1}{r}$). If we choose $\omega^i_k = 0$ for $k > r$, where k may present a type of labor not supplied by the individual or a non-labor asset, we see that C.IV′ is satisfied.

C.V: X^i is bounded.

Note: This will also be dropped later — it implies that no one can consume more than a given (finite) amount of any commodity.

This replacement for C.V will be

C.V′: $x^i \geq 0$ for $x^i \in X^i$ and $U^i(x^i)$ has no maximum in X^i (i.e., no satiation).

Note: This insures that there can be no real hoarding — no one can have an income greater than his expenditure. Note that this contradicts C.V.

Let $X^i(p)$ be $\{x^i \mid x^i$ maximizes $U^i(x^i)$ subject to $p \cdot x^i \leq M_i(p),\ x^i \in X^i\}$.

THEOREM 12: Under C.I, C.II, C.III and C.V, $X^i(p)$ is non-null, closed, convex, and bounded.

Let $C^i(p) = \{x^i \mid p \cdot x^i \leq M_i(p),\ x^i \in X^i\}$. Since $\xi^i \in X^i$, and $p \cdot \xi^i \leq M_i(p)$ by C.III, $C^i(p)$ is the intersection of this set with the closed convex set, X^i, and so is closed and convex. Since $C^i(p)$ is bounded, closed, and non-null, the continuous function $U^i(x^i)$ attains a maximum on it. Hence $X^i(p)$ is non-null. It is clearly closed, and, as a subset of $C^i(p)$, bounded.

Finally, let x^1 and x^2 be two elements of $X^i(p)$, x a convex combination of them. Since $U^i(x^i)$ attains its maximum at both x^1 and x^2, $U^i(x^1) = U^i(x^2)$. From the quasi-concavity of $U^i(x^i)$, $U^i(x) \geq U^i(x^1) = U^i(x^2)$. But $x \in C^i(p)$, since the latter is convex. Then $U^i(x) \geq max_{x^i \in C^i(p)} U^i(x^i) = U^i(x^1)$; hence, $x \in X^i(p)$.

THEOREM 13: Under C.I, C.II, C.III and C.V, $X^i(p)$ is bounded and upper semi-continuous at any p^o for which $M_i(p^o) > p^o \cdot \omega^i$ for some $\omega^i \in X^i$.

This restricts consideration to the set of prices for which an individual's income is more than enough the buy the cheapest bundle permissible.

Proof: Suppose (1) $p^n \to p^o$

(2) $x^n \to x^o$

(3) $x^n \in X^i(p^n)$.

We show that this implies $x^o \in X^i(p^o)$. Now $x^n \in C^i(p^n)$ and $p^n \cdot x^n \leq M_i(p^n)$. As $n \to \infty$, $p^o \cdot x^o \leq M_i(p^o)$ because $M_i(p)$ is continuous by C.III.

Now $x^o \in X^i$, therefore

(4) $x^o \in C^i(p^o)$.

Let $x^1 \in C^i(p^o)$ and distinguish between the following two cases: $p^o \cdot x^1 < M_i(p^o)$ and $p^o \cdot x^1 = M_i(p^o)$.

Case (a): $p^o \cdot x^1 < M_i(p^o)$.

Then $p^n \cdot x^1 < M_i(p^n)$ for n large (by continuity) and thus $x^1 \in C^i(p^n)$; but x^n was optimal from (3) above. Thus $u^i(x^n) \geq u^i(x^1)$ and in the limit as $n \to \infty$, $u^i(x^o) \geq u^i(x^1)$.

Case (b): $p^o \cdot x^1 = M_i(p^o)$.

(5) $M_i(p^o) > p^o \cdot \omega^i$, by hypothesis

(6) $p^n \cdot x^1 \to p^o \cdot x^1 = M_i(p^o) > p^o \cdot \omega^i$

(as $n \to \infty$).

Now as p^n changes x^1 may or may not be in $C^i(p^n)$; however, $p^n \cdot \omega^i < M_i(p^n)$ for n large.

We want to find a λ_n such that

$$\lambda_n p^n \cdot x^1 + (1 - \lambda_n) p^n \cdot \omega^i \leq M_i(p^n);$$

let $$\lambda_n = min[\tfrac{M_i(p^n) - p^n \cdot \omega^i}{p^n \cdot x^1 - p^n \cdot \omega^i},\ 1].$$

Therefore, $0 < \lambda_n \leq 1$ for n sufficiently large, and then $p^n \cdot [\lambda_n x^1 + (1 - \lambda_n)\omega^i] \leq M_i(p^n)$.

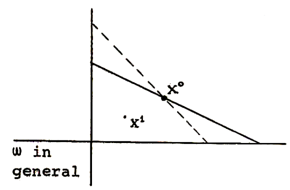
For x^n optimal,

$$U^i(x^n) \geq U^i[\lambda_n x^1 + (1-\lambda_n)\omega^i].$$

Now $M_i(p^n) \to M_i(p^o)$ and $p^n \cdot x^1 \to p^o$; $x^1 = M_i(p^o)$ as $n \to \infty$; therefore $\lambda_n \to 1$ as $n \to \infty$, and so $U^i(x^o) \geq U^i(x^1)$. Hence, $U^i(x^o) \geq U^i(x^1)$ for all $x^1 \in C^i(p^o)$, so that $x^o \in X^i(p^o)$.

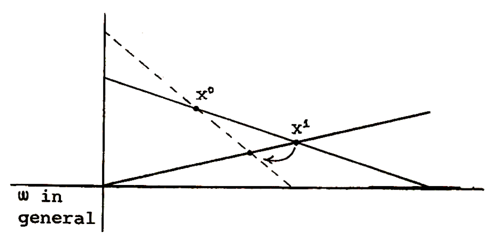
4 Lecture IV

Case(a) compares x^o and x^1 where x^1 does not lie on the budget line and is not excluded as prices change:



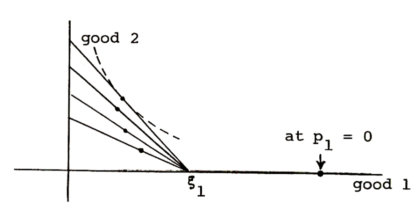
here the utilities of x^o and x^1 may be compared directly.

In case (b), however, x^1 is excluded, and the construction of λ_n serves to pull it back to within the consumption set as prices change.



Also notice that (in a two-commodity case, for simplicity) where only one commodity is held, as the price of that commodity falls to zero we may have a discontinuous jump to a point of satiation.

For $p_1 > 0$, $x^i(p_1) < \xi_1$ but for $p_1 = 0$, $x^i(p_1)$ may be unbounded (when C.V does not hold).



Corollary 13.1: Under C.I, C.II, C.III, C.IV, C.V, $X^i(p)$ is non-null, closed, convex, and bounded, for each p, and $X^i(p)$ is a bounded upper semi-continuous correspondence in p.

Proof: Under C.IV, $M_i(p^o) \geq p^o \cdot \xi^i > p^o \cdot \omega^i$ for all $p^o > 0$, since $\xi^i \gg \omega^i$.

Definition: $X(p) = \sum_i X^i(p)$ (household excess demand correspondence).

Corollary 13.2: Under C.I, C.II, C.III, C.IV, C.V, $X(p)$ has the same properties as $X^i(p)$ in Corollary 13.1.

Now x^i is attainable in X^i if $0 \leq x^i \leq y + \xi$ for some $y \in Y$, $x^i \in X^i$. This implies free disposal for the market, as firms can produce a little more than consumers desire; therefore there must be a costless way to dispose of the surplus.

THEOREM 14: The set of attainable $x^i \in X^i$ is bounded. (since $0 \leq x^i \leq y + \xi$ and y, ξ are bounded).

THEOREM 15: Assume C.I, C.II, C.III, C.IV, C.V$'$; choose a c such that $|x^i| < c$ for x^i attainable. Let $\widetilde{X}^i = X^i \cap \{x^i \mid |x^i| \leq c\}$ and let $\widetilde{C}_i(p) = \{x^i \mid p \cdot x^i \leq M_i(p),\ x^i \in \widetilde{X}^i\}$ and let $\widetilde{X}^i(p) = \{x^i \mid x^i \text{ maximize } U^i(x^i) \text{ in } \widetilde{C}^i(p)\}$.

Then (a) for each p, $\widetilde{X}^i(p)$ is bounded, closed, convex, and non-null, and $\widetilde{X}^i(p)$ is an upper semi-continuous correspondence (this analogous to Theorem 11); and (b) if x^i is attainable and $x^i \in \widetilde{X}^i(p)$ then $x^i \in X^i(p)$ — (this is analogous to Theorem 11, using semi-strictly quasi-concavity here as linearity of profit function was used in Theorem 11).

Definition: $\widetilde{X}(p) = \sum_i \widetilde{X}^i(p)$.

Corollary 15.1: $\widetilde{X}(p)$ is closed, convex, bounded, and non-null, and, for each p, $\widetilde{X}(p)$ is an upper semi-continuous correspondence.

Proof: Assumptions of Corollary 13.1 hold here.

THEOREM 16: Under C.I, C.II, C.III, C.IV, C.V$'$,

$$p \cdot x^i = M_i(p) \text{ for all } x^i \in X^i(p).$$

Proof: By C.V$'$, there exists $\overline{x}^i \in X^i$ such that

$$U^i(\overline{x}^i) > U^i(x^i).$$

Suppose $p \cdot x^i < M_i(p)$. Choose $\alpha > 0$ so that

$$p \cdot [\alpha \overline{x}^i + (1-\alpha)x^i] \le M_i(p).$$

Since X^i is convex, $\alpha \overline{x}^i + (1-\alpha)x^i \in X^i$ and hence to $C^i(p)$.

From semi-strict quasi-concavity, $U^i[\alpha \overline{x}^i + (1-\alpha)x^i] > U^i(x^i)$, which contradicts the fact that $x^i \in X^i(p)$.

Corollary 16.1: (a) if $x^{oi} \in X^i(p)$ and $U^i(x^{1i}) > U^i(x^{oi})$ then $p \cdot x^{1i} > p \cdot x^{oi}$; (b) if $x^{oi} \in X^i(p)$ and $U^i(x^{1i}) \ge U^i(x^{oi})$ then $p \cdot x^{1i} \ge p \cdot x^{oi}$.

A consumer may be both price- and utility-indifferent in the same way that firms may be profit-indifferent.

Definition: $\bar{p}$ is a competitive equilibrium if $x \le y + \xi$ for some $x \in X(\bar{p})$ and some $y \in Y(\bar{p})$, $\bar{p} > 0$.

Let $Z(p) = X(p) - Y(p) - \xi$ (a vector sum) be the market excess demand correspondence; then $\bar{p}$ is an equilibrium price if $\bar{z} \le 0$ for $\bar{z} \in Z(\bar{p})$.

THEOREM 17: Under P.I, P.II, P.III, P.IV and C.I, C.II, C.III, C.IV, C.V,

(a) $Z(p)$ is non-null, closed, convex and bounded, and for each p, $Z(p)$ is a bounded upper semi-continuous correspondence;

(b) $p \cdot z \le 0$ for all $z \in Z(p)$.

Proof of (b): For $x^i \in X^i(p)$, $p \cdot x^i \le p \cdot \xi^i + \mu_i(p)$. Therefore, $p \cdot x \le p \cdot \xi + \sum_i \mu_i(p) = p \cdot \xi + \sum_j \pi^j(p) = p \cdot \xi + p \cdot y$ for $y \in Y(p)$. Thus $p \cdot (x - \xi - y) \le 0$ when $x \in X(p)$, $y \in Y(p)$, or $p \cdot z \le 0$.

[Note: when C.V′ holds rather than C.V, $p \cdot z = 0$.]

THEOREM 18: Market Equilibrium

If $Z(p)$ is non-null, closed, convex, and bounded; if, for each p, $Z(p)$ is a bounded upper semi-continuous correspondence; and if $p \cdot z \leq 0$ for each $z \in Z(p)$, then there exists a competitive equilibrium.

Proof: To prove this, we will use two mappings — prices into excess demands, and excess demands into prices — and show that a fixed point of these transformations exists and represents an equilibrium.

First, choose an M such that $|z| \leq M$ for any $z \in Z(p)$ for all p. For p in the simplex S ($S = \{p \mid p \geq 0, \sum_i p_i = 1\}$). Let $P(z) = \{p^* \mid p^* \text{ maximizes } p \cdot z \text{ for } p \in S\}$. We have the two mappings, $p \to Z(p)$ and $z \to P(z)$. Thus we have the set $Q \begin{pmatrix} p \\ z \end{pmatrix} = \left\{ \begin{pmatrix} p^* \\ z^* \end{pmatrix} \middle| \begin{matrix} p^* \in P(z) \\ z^* \in Z(p) \end{matrix} \right\}$ for $|z| \leq M$ and $p \in S$.

KAKUTANI FIXED-POINT THEOREM:

If T is closed, convex, and bounded, and $\phi(x)$ is an upper semi-continuous correspondence from T to subsets of T, and if $\phi(x)$ is non-null, closed, convex for each x, then there exists an $\bar{x} \in T$ such that $\bar{x} \in \phi(\bar{x})$.

Now Q satisfies all of the above conditions, thus

$$\begin{pmatrix} \bar{p} \\ \bar{z} \end{pmatrix} \in Q \begin{pmatrix} \bar{p} \\ \bar{z} \end{pmatrix} \quad \text{or} \quad \begin{cases} (1) & \bar{p} \in P(\bar{z}) \\ (2) & \bar{z} \in Z(\bar{p}). \end{cases}$$

Then (1), (2), together with $p \cdot z \leq 0$ for each $z \in Z(p)$ is sufficient for a competitive equilibrium. From (1),

(a) $\bar{p} \cdot \bar{z} \geq p \cdot \bar{z}$ for $p \in S$; from (2) and hypothesis,

(b) $\bar{p} \cdot \bar{z} \leq 0$ and, from (a), $p \cdot \bar{z} \leq 0$ for any $p \in S$.

To hold for any $p \in S$, we must have $\bar{z} \leq 0$ which satisfies the conditions for a competitive equilibrium.

Note: (1) If there is at least one market in which quantity supplied equals quantity demanded, such that $\bar{z}_k = 0$ for some k, $p \cdot \bar{z} = 0$ for some $p \in S$, then $\bar{p} \cdot \bar{z} = 0$ from (a), (b) and assumption W.

(2) Suppose $\bar{z}_k \leq 0$ for all k. Then there exists a $\bar{p} > 0$ for which $\bar{p} \cdot \bar{z} < 0$ and this violates assumption W.

THEOREM 19: Under P.I, P.II, P.III, P.IV and C.I, C.II, C.III, C.IV, C.V, there exists a competitive equilibrium.

Proof: Merely a restatement of Theorems 17, 18.

We now wish to drop the boundedness assumptions P.IV and C.V, and replace them by P.IV′ and C.V′. Define

$$\widetilde{Z}(p) = \widetilde{X}(p) - \widetilde{Y}(p) - \xi.$$

By Theorems 11 and 15, the hypotheses of Theorem 18 hold for $\widetilde{Z}(p)$.
There exists a $\bar{p}$, $\bar{z}$ such that $\bar{p} > 0, \bar{z} \in \widetilde{Z}(p), \bar{z} \leq 0$.

Also $\bar{z} = \bar{x} - \bar{y} - \xi$ for $\bar{x} \in \widetilde{X}(p), \bar{y} \in \widetilde{Y}(p)$. Thus $\bar{y} + \xi = \bar{x} - \bar{z} \geq \bar{x} = \sum_i \overline{x}^i \geq 0$ for $\overline{x}^i \in \widetilde{X}^i(\bar{p})$ for each i, by C.V′.

Hence $\bar{y}$ is attainable and $\bar{y} = \sum_i \overline{y}^j$ for $\overline{y}^j \in \widetilde{Y}^j(\bar{p})$ for each j. Thus $\overline{y}^j$ is attainable, and $\overline{y}^j \in Y^j(\bar{p})$ by Theorem 11. Similarly, $\overline{x}^i \in X^i(\bar{p})$ by Theorem 15.

THEOREM 20: Under P.I, P.II, P.III, P.IV′ and C.I, C.II, C.III, C.IV, C.V′, there exists a competitive equilibrium with $\bar{p} \cdot \bar{z} = 0$.

Now we wish to relax assumption C.IV′ regarding the assets held by individuals.

$$\text{Let } \overline{X}^i(p) = \begin{cases} X^i(p) \text{ if } M_i(p) > p \cdot \omega^i \text{ for some } \omega^i \in X^i \\ \{\overline{x}^i \mid \overline{x}^i \text{ minimizes } p \cdot \overline{x}^i \text{ for } \overline{x}^i \in X^i\} \text{ otherwise.} \end{cases}$$

This definition implies if $p_k = 0$ for all the assets held by the individual, then $\overline{X}^i(p)$ is the set of all commodity bundles involving those assets and any other free goods in any quantities.

By Theorem 13, $\overline{X}_i(p)$ is upper semi-continuous at any p for which $M_i(p) > min_{x^i \in X^i}\, p{\cdot}x^i$. Now let $M_i(p^o) = min_{x^i \in X^i}\, p^o \cdot x^i$. Let $p^n \to p^o$, $x^n \in \overline{X}^i(p^n)$ and $x^n \to x^o$; we shall prove that $x^o \in \overline{X}^i(p^o)$.

Note that $x^n \in C^i(p^n)$.

Now $min_{x^i \in X^i}\, p^n \cdot x^i \leq p^n \cdot x^n \leq M_i(p^n) \to M_i(p^o) = min_{x^i \in X^i}\, p^o \cdot x^i$.

Take the limit of the above as $n \to \infty$; then

$$p^o \cdot x^o = min_{x^i \in X^i}\, p^o \cdot x^i. \text{ Hence } x^o \in \overline{X}^i(p^o).$$

Lemma 6: Under C.I, C.II, C.III, and C.V, $\overline{X}^i(p)$ is non-null, closed, convex, and bounded, and, for each p, $\overline{X}^i(p)$ is an upper semi-continuous correspondence.

Lemma 7: Under C.I, C.II, C.III, C.V′ and P.I, P.II, P.III, P.IV′, there exist $\bar{p}$, $\overline{x}^i$, $\bar{y}$ such that $\sum_i \overline{x}^i \leq \bar{y} + \xi$, $\bar{p} > 0$, for $\overline{x}^i \in \overline{X}^i(\bar{p})$, $\bar{y} \in Y(\bar{p})$.

Proof: If we replace C.V′ and P.IV′ by C.V and P.IV, this follows from Lemma 6 and Theorem 18. The extension to C.V′ and P.IV′ is the same as the proof of Theorem 20.

Corollary to Lemma 7: Under the hypotheses of Lemma 7, if $M_i(\bar{p}) > min_{x^i \in X^i}\, \bar{p} \cdot x^i$ for all i, then $\bar{p}$ is a competitive equilibrium.

Proof: Lemma 7, definition of $\overline{X}^i(p)$.

Definition: A feasible distribution is a set (x^i, y^j) for which

$$\textstyle\sum_i x^i \leq \sum_j y^j + \xi, \text{ for } x^i \in X^i,\ y^j \in Y^j.$$

Definition: Consumer a is said to be (directly) productively related to consumer b if, given any feasible distribution (x^{oi}, y^{oj}), there exists a $\bar{\xi} > 0$ with $\bar{\xi}_k = 0$ for all k for which $\xi^a_k = 0$; and $x^{1i} \in X^i$ for all i, $y^1 \in Y$, such that

(1) $\sum_i x^{1i} \leq y^1 + \xi + \bar{\xi}$

and (2) $u^i(x^{1i}) \geq u^i(x^{oi})$ for all $i \neq b$

and (3) $u^b(x^{1b}) > u^b(x^{ob})$

that is, if only one person (a) receives an additional endowment ($\bar{\xi}$) containing only goods he already has in some positive amount, and the additional endowment can be used (in production and/or distribution) in at least one way such that the utility levels of all persons except b are at least as great as before and individual b is made better off, then individual a is directly productively related to individual b.

Definition: Individual a is indirectly productively related to individual b if there exist individuals $i_0, i_1, ..., i_p$ (with individual $i_0 = \underline{a}$ and $i_p = \underline{b}$) with individual i_s directly productively related to i_{s+1} for $s = 0, 1, ..., p - 1$. [1]

Assumption K: Connectivity

Every individual is indirectly productively related to every other individual.

Then as replacements for C.IV, we shall substitute assumption K and the following postulate P.V.

P.V.: There exists a $y^* \in Y$ such that $y^*_k > 0$ for all k for which $\xi_k = 0$.

This simply says that if we are not endowed with a particular good, there exists a way to produce it from the endowments — a"good" is a good only if it exists in nature or can be produced.

[1] Editors' interpretation: original was "$s = 0, 1, ..., p = 1$", we recommend to read as "$s = 0, 1, ..., p - 1$". We thank Prof. Alexis Akira Toda for helping with this passage.

5 Lecture V

Lemma 8: Under P.I, P.II, P.V there exists a $\bar{\bar{y}}$ such that $\bar{\bar{y}} + \xi \gg 0$.

Proof: Take y^* as in P.V; consider the primary components of y^*, i.e., those for which $\xi_k > 0$. They are negative in general. We can choose a $\lambda > 0$ such that $\xi_k + \lambda y_k^* > 0$ for all k for which $\xi_k > 0$. Also $\lambda y_k^* > 0$ for all k for which $\xi_k = 0$; thus $\xi_k + \lambda y_k^* > 0$ for all k for which $\xi_k \geq 0$ by P.V. Now simply let $\bar{\bar{y}}_k = \lambda y_k^*$.

Lemma 9: Under C.III and C.IV′, $M_i(p) = \min_{x^i \in X^i} \; p \cdot x^i$ if and only if $M_i(p) = 0$.
(a) If $M_i(p) = 0$, then $0 = M_i(p) \geq p \cdot \xi^i \geq 0$. Thus $p \cdot \xi^i = 0$ and

$$p \cdot \xi^i \geq \min_{x^i \in X^i} \; p \cdot x^i \geq 0.$$

(b) Suppose $\min_{x^i \in X^i} \; p \cdot x^i = M_i(p) \geq p \cdot \xi^i \geq p \cdot \omega^i \geq \min_{x^i \in X^i} \; p \cdot x^i$, where ω^i is chosen as in C.IV′.

$$M_i(p) = p \cdot \xi^i = p \cdot \omega^i$$
$$\text{or} \quad p \cdot (\xi^i - \omega^i) = 0.$$

Then $p_k = 0$ if $\xi_k^i > \omega_k^i$ and thus $p_k = 0$ if $\xi_k^i > 0$ since if $\xi_k^i > 0$, then $\xi_k^i > \omega_k^i$ by C.IV′. Thus $p \cdot \xi^i = 0$ and $M_i(p) = p \cdot \xi^i = 0$.

We wish now to show that at least one person has a positive income at the price vector $\bar{p}$ in Lemma 7.

To prove $M_i(\bar{p}) > 0$ for some i, suppose $M_i(\bar{p}) = 0$ for all i. Under Lemma 7, we have profit maximization but zero incomes and consumers can "buy" only free goods. We now show that zero incomes for consumers are inconsistent with profit maximization.

Now $\bar{p} \cdot \xi^i = 0$ and summing over i, $\bar{p} \cdot \xi = 0$ and also $\mu_i(\bar{p}) = 0$. Thus there are no dividends or income. Then there must be no profit.

$$0 = \sum_i \mu_i(\bar{p}) = \sum_j \pi^j(\bar{p}) = \bar{p} \cdot \bar{y}.$$

Now $\bar{p} \cdot y \leq \bar{p} \cdot \bar{y}$ for all $y \in Y$, by definition of $Y(p)$. In particular, let $y = \bar{\bar{y}}$, as defined in Lemma 8.

$$0 < \bar{p} \cdot (\bar{\bar{y}} + \xi) = \bar{p} \cdot \bar{\bar{y}} + \bar{p} \cdot \xi \leq \bar{p} \cdot \bar{y} + \bar{p} \cdot \xi = 0.$$

But $0 < 0$ is a contradiction; therefore someone must have a positive income.

Consider now someone <u>directly productively related</u> to the individual who has a positive income; there must be at least one such related person by Assumption K. We show that if individual <u>a</u> is productively related to individual <u>b</u> who has a positive income, then <u>a</u> also has a positive income.

By the definition of productive relatedness, we can find x^{1i}, y^1, $\bar{\xi}$ such that

(1) $U^i(x^{1i}) \geq U^i(\overline{x}^i)$ for all i,

(2) $U^b(x^{1b}) > U^b(\overline{x}^b)$,

(3) $\sum_i x^{1i} \leq y^1 + \xi + \bar{\xi}$, $y^1 \in Y$,

(4) $\overline{\xi}_k = 0$ if $\xi^a_k = 0$.

By Corollary 16.1, it follows from (1) and (2) that

(5) $\bar{p} \cdot x^{1i} \geq \bar{p} \cdot \overline{x}^i$ for all i for which $M_i(\bar{p}) > 0$,

(6) $\bar{p} \cdot x^{1b} > \bar{p} \cdot \overline{x}^b$.

(5) also holds obviously if $M_i(\bar{p}) = 0$

$$\bar{p} \cdot \textstyle\sum_i x^{1i} > \bar{p} \cdot \sum_i \overline{x}^i = \sum_i M_i(\bar{p}) = \bar{p} \cdot (\bar{y} + \xi).$$

Suppose $M_a(\bar{p}) = 0$; this implies $\bar{p} \cdot \xi^a = 0$ and $\overline{p}_k = 0$ if $\xi_k > 0$.
Then $\overline{p}_k = 0$ if $\overline{\xi}_k > 0$ by construction (Assumption K and definition of being productively related), so $\bar{p} \cdot \bar{\xi} = 0$. Then

$$\bar{p} \cdot (y^1 + \xi) = \bar{p} \cdot (y^1 + \xi + \bar{\xi}) \text{ since } \bar{p} \cdot \bar{\xi} = 0.$$

or $\quad \bar{p} \cdot (y^1 + \xi + \bar{\xi}) \geq \bar{p} \cdot (\sum_i x^{1i}) > \bar{p} \cdot (\bar{y} + \xi)$,

or $\quad \bar{p} \cdot y^1 > \bar{p} \cdot \bar{y}$,

which is a contradiction, since $\bar{y}$ maximizes profits at prices $\bar{p}$.

Therefore $M_a(\bar{p}) > 0$; individual <u>a</u>'s assets cannot have zero value if he is productively related to individual <u>b</u> who has a positive income.

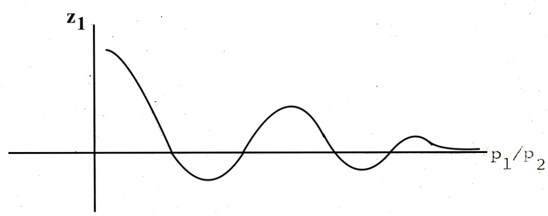
Thus, since each individual is indirectly productively related to all other individuals by assumption K, the chain of direct relationships implies that each individual must have a positive income.

<u>THEOREM 21:</u> Under P.I, P.II, P.III, P.IV′, P.V, C.I, C.II, C.III, C.IV′, C.V′, Postulate K, there exists a competitive equilibrium.

6 Lecture VI

In the preceding five lectures we have been concerned with the existence of a competitive equilibrium; in this final lecture we shall cover uniqueness and stability.

For the discussions of uniqueness and stability, we return to the model of the first lecture which employs single-valued excess demand and production functions. We will present two alternative conditions, either of which is sufficient for both uniqueness and stability. We know that under assumptions (W), (H), (B), and (C) together with the assumption of single-valued excess demand functions, we obtain a solution similar to that in the following diagram:[2]



(1) The first of the sufficient conditions for uniqueness and stability is that consumers collectively act as if there were only one consumer in the market; that is, the market (household) excess demand function satisfies the Samuelson weak axiom of revealed preference. we restate this axiom as the following assumption:

Assumption (RP): If $p^o \cdot x(p^o) \geq p^o \cdot x(p^1)$ and p^o is not a positive multiple of p^1, then

$$p^1 \cdot x(p^o) > p^1 \cdot x(p^1),$$

where $x(p^i)$ is the excess demand vector given price vector p^i.

THEOREM 22: (Wald) — If p^o is not a positive multiple of p^1, and (W), (H), (B), (C), and (RP) hold, then both p^o and p^1 cannot be equilibrium price vectors.

Proof: Suppose both p^o and p^1 are equilibrium vectors, with $\{p^o, x^o, y^o\}$ one equilibrium set and $\{p^1, x^1, y^1\}$ the other.

[2] Illegible in original. Editors' interpolation: "we obtain a solution similar to that in the following diagram:"

Now, $x^o \leq y^1 + \xi$, [3] so

(a) $p^o \cdot x^1 \leq p^o \cdot (y^1 + \xi) \leq p^o \cdot (y^o + \xi) = p^o \cdot x^o$ at equilibrium; however, by symmetry,

(b) $p^1 \cdot x^o \leq p^1 \cdot (y^o + \xi) \leq p^1 \cdot (y^1 + \xi) = p^1 \cdot x^1$ at equilibrium.

However, both (a) and (b) cannot hold simultaneously under (RP).

(2) The alternative assumption to (1) is that of gross substitutability borrowed from international trade theory [see e.g., Mosak, J. L., General Equilibrium Theory in International Trade, Bloomington, Ind., 1944 (p.45f.)]. This is a statement about aggregate excess demands, both for households and firms.

Assumption (S): $z_k(p)$ is a strictly increasing function of p_ℓ for $k \neq \ell$.

Note: By homogeneity, $z_k(\lambda p) = z_k(p)$ for $\lambda > 0$. Suppose $p_k = 0$ and increase the value of λ so that other prices increase; then, by (S), the excess demand for commodity k must be a strictly increasing function of λ – but the right-hand side indicates that it is not. The only way by which this can be resolved under gross substitutability is for $z_k(p) = +\infty$ if $p_k = 0$; but for p^o to be an equilibrium price vector, $z(p^o) \leq 0$. Therefore $p^o \gg 0$ is an implication of the assumption of gross substitutability.

Lemma 10: (a) If $z(p)$ satisfies assumptions (W), (H), (B), (C), and (S), then

$$\lim_{p_k \to 0} z_k(p) = +\infty \text{ and } z(p^o) = 0 \text{ together imply } p^o \gg 0;$$

(b) if $p^1 \gg 0$ and $p^2 \gg 0$ and $\frac{p^1_m}{p^2_m} = max_k \frac{p^1_k}{p^2_k}$ then $z_m(p^1) < z_m(p^2)$ unless p^1 is a positive multiple of p^2;

(c) if $p^1 \gg 0$ and $p^2 \gg 0$ and $\frac{p^1_m}{p^2_m} = min_k \frac{p^1_k}{p^2_k}$ then $z_m(p^1) > z_m(p^2)$ unless p^1 is a positive multiple of p^2.

Proof: (b) Let $\lambda = \frac{p^1_m}{p^2_m}$; then $\frac{p^1_k}{p^2_k} \leq \lambda$ for all k. Thus $p^1_k \leq \lambda p^2_k$ for all k, and $p^1_m = \lambda p^2_m$; therefore $p^1_k > p^2_k$ for some $k \neq m$. Thus $z_m(\lambda p^2) \geq z_m(p^1)$ and by assumption (H), $z_m(\lambda p^2) = z_m(p^2)$; therefore $z_m(p^2) > z_m(p^1)$ unless p^1 is a positive multiple of p^2.[4]

[3] Editors' interpretation: we recommend to read "$x^o \leq y^1 + \xi$" as "$x^o \leq y^o + \xi$ and $x^1 \leq y^1 + \xi$"

[4] Illegible in original. Editors' interpolation: "Thus $z_m(\lambda p^2) \geq z_m(p^1)$ and by assumption (H), $z_m(\lambda p^2) = z_m(p^2)$; therefore

(c) For part (c), reverse the inequalities and proceed in analogous fashion to the proof of part (b).[5]

THEOREM 23: If (W), (H), (B), (C), and (S) hold, then competitive equilibrium is unique up to a positive multiple of p.

Proof: Suppose both p^1 and p^2 are equilibrium price vectors with $p^1 \neq \lambda p^2$ for $\lambda > 0$; then $\begin{cases} 0 = z_k(p^1) \\ 0 = z_k(p^2). \end{cases}$, which are equilibrium conditions.

Let $\frac{p_m^1}{p_m^2} = max_k \frac{p_k^1}{p_k^2}$. By lemma 10, $z_m(p^1) < z_m(p^2)$, which is a contradiction $(0 < 0)$.

Remark: Assumption (RP) implies that excess demand is downward sloping in the neighborhood of a crossing (equilibrium point), as income effects cancel out at equilibrium. It does not imply downward sloping excess demand everywhere.

We now wish to show that either (RP) or (S) is sufficient for stability of the equilibrium as well as for uniqueness. Let the mechanism of adjustment of prices to excess demands be of the form $\frac{dp_k}{dt} = c_k z_k(p)$, where c_k is a coefficient of adjustment speed. By appropriate choice of units, we may make $c_k = 1$ for each k, which does not affect either (RP) or (S), each of which is independent of changes in the units.

Then $\frac{dp_k}{dt} = z_k(p)$ is a set of differential equations; a solution to this set, $p(t|p(0))$, is a price vector which is a function of time and which passes through a certain given point at an arbitrary time $t = 0$. We assume that there is only one such time path for each starting point; it follows that the path is not dependent on the location of the origin —

$$p(t|p(t_o)) = p(t + t_o|p(0)).$$

We assume further that $p(t|p(0))$ is a continuous function of $p(0)$.

By local stability we mean that if $p(0)$ is close to an equilibrium price vector $\bar{p}$, then

$z_m(p^2) > z_m(p^1)$ unless p^1 is a positive multiple of p^2"

[5] Illegible in original. Editors' interpolation: "(c) For part (c), reverse the inequalities and proceed in analogous fashion to the proof of part (b)"

$p(t|p(0))$ converges to $\bar{p}$.

By global stability we mean that $p(t|p(0))$ converges to the equilibrium price vector $\bar{p}$ from any arbitrary $p(0)$.

If it is required that prices be non-negative, the adjustment process has to be modified in the case of a negative excess demand at a zero price. we assume

$$\frac{dp_k}{dt} = \begin{cases} z_k(p) & \text{if } p_k > 0 \text{ \underline{or} if } z_k(p) > 0 \\ 0 & \text{if } p_k = 0 \text{ and } z_k(p) \leq 0. \end{cases}$$

The above represents a non-normalized system, one in which there is no numeraire. If we choose a numeraire then we must distinguish the numeraire commodity (say, the n^{th}) as follows:

Normalized system

$$\text{for } k \neq n,\ \frac{dp_k}{dt} = \begin{cases} z_k(p) & \text{if } p_k > 0 \text{ \underline{or} if } z_k(p) > 0 \\ 0 & \text{if } p_k = 0 \text{ and } z_k(p) \leq 0. \end{cases}$$

for $k = n$, $p_n(t) \equiv 1$.

Assumption N: $z_n(p) > 0$ if $p_n = 0$.

Lemma 11: If (N) holds, then for some number M,

$$z_n(p) > 0 \text{ for } p_n = 1, \text{ all } |p| \geq M.$$

Proof: Suppose not. Then, for every M, there is a price vector p such that $|p| \geq M$, $p_n = 1$, $z_n(p) \leq 0$. We can in particular choose a sequence $\{p^\nu\}$, such that $|p^\nu| \to \infty$, $p_n^\nu = 1$, $z_n(p_n^\nu) \leq 0$. Let

$$\overline{p}^\nu = \frac{p^\nu}{|p^\nu|}$$

Then

$$\overline{p}_n^\nu = \frac{1}{|p^\nu|} \to 0,$$
$$|\overline{p}^\nu| = 1.$$

By (H), $\quad z_n(\overline{p}^\nu) = z_n(p^\nu) \leq 0$.

Since $\{\overline{p}^\nu\}$ is bounded, some sub-sequence converges to $\bar{p}$, with $|\bar{p}| = 1$. Then, by (C),

$$z_n(\bar{p}) \leq 0,$$
$$\overline{p}_n = 0,$$
which contradicts (N).

Consider in the non-normalized system

$$\tfrac{d}{dt}(\textstyle\sum_{k=1}^n p_k^2) = 2\sum_{k=1}^n p_k \tfrac{dp_k}{dt}.$$

If $p_k > 0$ or $z_k > 0$, $\frac{dp_k}{dt} = z_k$, so that $p_k \frac{dp_k}{dt} = p_k z_k$. If $p_k = 0$ and $z_k \leq 0$, then $\frac{dp_k}{dt} = 0 = p_k z_k$. Hence,

$$\tfrac{d}{dt}(\textstyle\sum_{k=1}^n p_k^2) = 2\sum_{k=1}^n p_k z_k = 0 \text{ by (W).}$$

That is, under Walras' Law, the sum of squares of prices is a constant, and price adjustments at any moment are such that they lie upon a (hyper)-sphere of given radius – i.e., the non-normalized price system is bounded.

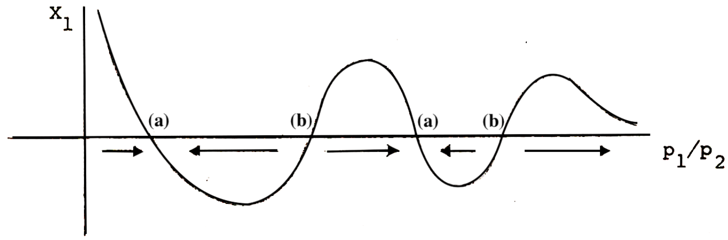
For the numeraire system,

$$\tfrac{d}{dt}(\textstyle\sum_{k=1}^n p_k^2) = 2\sum_{k=1}^{n-1} p_k \tfrac{dp_k}{dt} = 2\sum_{k=1}^{n-1} p_k \cdot z_k = -2\ p_n \cdot z_n < 0$$
$$\text{for } |p| \geq M$$

under Walras' Law, assumption N, and lemma 11. Again, the system is bounded. Thus:
THEOREM 24: The non-normalized system is always bounded; the normalized system is bounded if assumption (N) holds.

Definition: Global Stability

An equilibrium set S is globally stable if and only if, starting from any $p(0)$, $p(t|p(0))$ converges to some element $p \in S$. (Starting from any initial position, the system moves to an equilibrium.)



In the above diagram, points (a) are points of stable equilibria; points (b) are unstable equilibria. Also, since p_1 and p_2 are bounded, as $\frac{p_1}{p_2} \to \infty$, $p_2 \to 0$ necessarily, yielding a further equilibrium point (stable, in the above diagram).

THEOREM 25: With two commodities, we always have global stability.

Lemma 12: If (RP) holds for $x(p)$ it holds for $z(p)$.

Proof: Suppose $p^o \cdot z(p^o) \geq p^o \cdot z(p^1)$,

$$z(p^o) = x(p^o) - y(p^o) - \xi$$
$$z(p^1) = x(p^1) - y(p^1) - \xi.$$

Then $p^o \cdot [x(p^o) - y(p^o)] \geq p^o \cdot [x(p^1) - y(p^1)]$ and

$$p^o \cdot x(p^o) \geq p^o \cdot x(p^1) + p^o \cdot [y(p^o) - y(p^1)];$$

under profit maximization, the last term is non-negative, therefore

$$p^o \cdot x(p^o) \geq p^o \cdot x(p^1).$$

Thus by (RP),

$p^1 \cdot x(p^o) > p^1 \cdot x(p^1) \geq p^1 \cdot x(p^1) + p^1 \cdot [y(p^o) - y(p^1)]$, where the last term is non-positive by profit maximization. Thus

$$p^1 \cdot [x(p^o) - y(p^o) - \xi] > p^1 \cdot [x(p^1) - y(p^1) - \xi],$$
$$\text{or} \quad p^1 \cdot z^o > p^1 \cdot z^1.$$

To demonstrate global stability, we will use a method of proof due to Lyapunov: the so-called second method of Lyapunov [Problème Générale de la Stabilité du Mouvement, A.M.S. 17, Princeton, 1949]

Let $V = \sum_{k=1}^{n}(p_k - \overline{p}_k)^2$ and let $S = \{k \mid \frac{dp_k}{dt} = z_k\}$.
Then $\frac{dV}{dt} = 2\sum_{k=1}^{n}(p_k - \overline{p}_k)\frac{dp_k}{dt} = 2\sum_{k\in S}(p_k - \overline{p}_k)\cdot z_k$.
For $k \notin S$, $p_k = 0$ and $z_k \leq 0$. By (B), z_k is bounded from below, so that $p_k z_k = 0$ for $k \notin S$. Hence

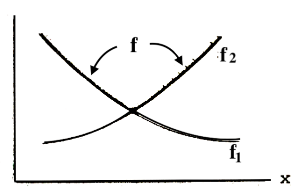
$$\sum_{k\in S} p_k z_k = \sum_{k=1}^{n} p_k z_k - \sum_{k\notin S} p_k z_k = \sum_{k=1}^{n} p_k z_k = 0 \text{ by (W).}$$
$$\frac{dV}{dt} = 2\sum_{k=1}^{n}(p_k - \overline{p}_k)\cdot z_k = -2\sum_{k\notin S}^{n}\overline{p}_k \cdot z \leq -2\sum_{k=1}^{n}\overline{p}_k \cdot z_k,$$
$$\text{since if } k \notin S,\ z_k \leq 0,\ \overline{p}_k \cdot z_k \leq 0, \text{ and } -2\sum_{k\notin S}\overline{p}_k \cdot z_k \geq 0$$

Now, in (RP), let $p^o = p(t)$ and $p^1 = \bar{p}$. For $z(\bar{p}) \leq 0$, $p \cdot z(\bar{p}) \leq 0 = p \cdot z(p)$ by (W); then by (RP), $\bar{p} \cdot z(\bar{p}) < \bar{p} \cdot z(p)$ and therefore $\frac{dV}{dt} < 0$. One can always buy the equilibrium excess demands at any set of prices so the equilibrium excess demands are revealed preferred to the ordinary excess demands for any price vector – that is, $\bar{p} \cdot z(p) > \bar{p} \cdot z(\bar{p}) = 0$. Since $\frac{dV}{dt} < 0$, if p is non-equilibria, we can get arbitrarily close to the equilibrium point.

<u>THEOREM 26:</u> (RP) is sufficient for (global) stability.

Before discussing stability under gross substitutability (S), consider the derivatives of the maximum (or minimum) of <u>n</u> functions of one variable $f(x) = max_i f_i(x)$.

At the point where $f_1(x) = f_2(x)$, $f(x)$ is obviously not differentiable, but it does have right- and left-hand derivatives.



Clearly also, $f^+(x)$, the right-hand derivative, is the larger of $f_1(x)$ and $f_2'(x)$ and $f^-(x)$ the smaller.[6] In general, at any point, let S be the set $\{i \mid f_i(x) = f(x)\}$, so that $f_i(x) = f_j(x)$ if i and j belong to S, $f_i(x) > f_j(x)$ if $i \in S$, $j \notin S$. Then $f^+(x) = max_{i\in S} f_i'(x)$, $f^-(x) = min_{i\in S} f_i'(x)$.

Let $\frac{dp_k}{dt} = z_k$ (since $z_k = +\infty$ if $p_k = 0$, under (S), the non-negativity condition becomes superfluous),

[6]Editors' Interpretation: we recommend to read "$f_1(x)$" as "$f_1'(x)$". We find the concluding sentence difficult to interpret.

$$\overline{V} = max_k \frac{p_k(t)}{\bar{p}_k}$$
$$\underline{V} = min_k \frac{p_k(t)}{\bar{p}_k}$$
$$\text{and let } M(t) = \{k| \ \frac{p_k(t)}{\bar{p}_k} = \overline{V}\}.$$

Now $z_k(p) < z_k(\bar{p})$ for $k \in M(t)$ by lemma 10.

$$\frac{d\overline{V}^+}{dt} = max_{k \in M(t)} \frac{\frac{d}{dt}(p_k(t))}{\bar{p}_k} = max_{k \in M(t)} \frac{z_k}{\bar{p}_k} < 0,$$
$$\frac{d\overline{V}^-}{dt} = min_{k \in M(t)} \frac{\frac{d}{dt}(p_k(t))}{\bar{p}_k} = min_{k \in M(t)} \frac{z_k}{\bar{p}_k} < 0$$

since $z_k(p) < z_k(\bar{p})$ for $k \in M(t)$, where the $+, -$ identify respectively right-hand and left-hand derivatives. Thus $\overline{V}(t)$ is strictly decreasing; conversely, by lemma 10, $\underline{V}(t)$ is strictly increasing, and thus

$$\overline{V}(0) \geq \overline{V}(t) \geq \frac{p_k(t)}{\bar{p}_k} \geq \underline{V}(t) \geq \underline{V}(0).$$

Thus by a standard argument on the convergence of monotone sequences,

$$\lim_{t \to \infty} \overline{V}(t) = \lim_{t \to \infty} \underline{V}(t) = V(\infty) = 1$$

so that

$$\frac{p_k(t)}{\bar{p}_k} \to 1, \text{ or } p_k(t) \to \bar{p}_k \text{ as } t \to \infty.$$

Thus assumption (S) is sufficient for global stability.

Remark: Future prices. Let p^p refer to present prices and p^f refer to expected future prices:

$$\frac{dp_k^p}{dt} = z_k^p(p^p, p^f) \text{ for } k = 1, ..., n.$$

Assume adaptive expectations

$$\frac{dp_k^f}{dt} = \beta_k(p_k^p - p_k^f) \text{ for } \beta_k > 0.$$

Assuming there exists a stationary equilibrium $\bar{p}(t) \equiv \bar{p}$ for which $\begin{cases} z^f(\bar{p}, \bar{p}) = 0 \\ z^p(\bar{p}, \bar{p}) = 0 \end{cases}$ and assumption (S) holds among all present and future commodities, then this system is always stable.

Editors' Notes

During the mid-1960s at Stanford University, the offices for economic theory and econometrics were at Serra House, a former retirement residence of the university president. There the typescript of the "Lectures on the Theory of Competitive Equilibrium" was available. The duplication format was particularly informal, known as "ditto," in coarse blue typeface on slick paper.

For editor Starr, the duplicated notes were a favorite guide to general equilibrium theory. More idiomatic and direct than the Arrow and Debreu (1954) article, than Debreu's *Theory of Value*, and Arrow and Hahn's *General Competitive Analysis* (1971). It helped guide the exposition in portions of Starr's *General Equilibrium Theory: An Introduction* (1997).

For editor Ying, it was such a fortune to engage in this project as an undergraduate. Understanding and transforming the General Equilibrium Theory content from Arrow was a joyful experience. Hope this could help to build a more thorough understanding to Arrow's and Starr's other works.

Editing of the Arrow lectures has focused on updating the mathematical notation from mid 1960s typewriter usage to 2020s style. Notably many forward slashes, /, in the original are replaced by vertical bars, |. There have been two other efforts: consistency with the original and correction of typographical errors. For example, "(YP90" was restated as "Y(p)". Some incompletely consistent notation has been retained from the original. For example, superior bar distinguishing equilibrium values, and then additional values of other variables also distinguished by superior bar. Every effort has been made for accuracy and reader accessibility. In some instances where mathematical consistency has not been clear, editors' footnotes appear noting the issue. Remaining errors are nevertheless our responsibility.

ACKNOWLEDGEMENTS

We thank the Arrow family for permission to publicly post edited notes with attribution to Kenneth J. Arrow. The original copy of the "Lectures" we use as a master copy comes from the archive of Arrow papers at Duke University (`https://library.duke.edu/rubenstein/findingaids/arrow/`). The text there includes the notation that it is from Prof. Eytan Sheshinski's Economics 215 course at Harvard in 1967. More formally, we gladly acknowledge the Kenneth J. Arrow Papers, David M. Rubenstein Rare Book & Manuscript Library, Duke University. We thank Sara Seten Berghausen, Associate Curator of Collections. David M. Rubenstein Rare Book & Manuscript Library. She has been most accommodating and helpful. We also acknowledge assistance from Ms. Adele Barsh (UCSD library), Prof. Theodore Bergstrom (University of California, Santa Barbara), Mr. Andrew Diep-Tran, Mr. Pietro Emilio Spini, Prof. Alexis Akira Toda(UCSD).