Title
A WEIGHTED LEAST SQUARES SOLUTION FOR INCOMPLETE DATA BY THURSTONE' S CASE V OF THE LAW OF COMPARATIVE JUDGMENTS WITH A GENERALIZATION TO THE METHOD OF TRIADS FOR MULTIDIMENSIONAL SCALING

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A WEIGHTED LEAST SQUARES SOLUTION FOR INCOMPLETE DATA

by Thurstone's Case V of the Law of Comparative Judgments

with a Generalization to the Complete Method of Triads

for Multidimensional Scaling

Ledyard R. Tucker and Lee G. Cooper

Abstract

A general solution for incomplete data is proposed. For the special case of incomplete data resulting from the complete method of triads a procedure is proposed which provides a solution for the interpoint distances which never requires finding more than the inverse of a 3 x 3 matrix, no matter how many stimuli are being scaled.

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In a stimulus comparison situation the basic data from which the scale values are determined are the observed proportion, P_{jk} , the proportion of time that stimulus k is judged as being in some sense preferable to stimulus j . Irregular data in such a scaling situation may arise by happenstance or design. It may happen that a stimulus is never preferred over another stimulus or that one is always preferred over another. The extreme proportions which result are not transformable into acceptable normal deviates. If a subject fails to respond to an item his missing data causes difficulty.

One case in which irregular data result from design is in the complete method of triads. In this data collection design the subject is presented the stimuli in triads. One stimulus is considered the key and the subject must choose which of the other two stimuli is more comparable to the key. If stimulus 1 is the key and it is paired with stimuli 2 and 3 , choosing stimulus 2 indicates a preference for pair 1,2 over 1,3 . For example, with four stimuli, there is no way of comparing pair 1,2 with itself or with pair 3,4 . With pair 1,3 there is no comparison with itself and with pair 2,4 , etc. This results in an incomplete data matrix.

A solution is desired in which the irregular data are given weights of zero and the other data are given weights of one. Given x_{jk} , the normal deviate corresponding to the observed proportion P_{jk} , and weights w_{jk} , we want a solution for the scale values s_j which minimizes the discrepancy between the model, Thurstone's Case V , and the observed data.

The model indicates,

$$\hat{x}_{jk} = s_k - s_j \quad . \qquad 1.$$

The error in this case is

$$e_{jk} = x_{jk} - \hat{x}_{jk} \quad . \qquad 2.$$

Let us define a function Q

$$Q = \sum_j \sum_k e_{jk}^2 w_{jk} \quad . \qquad 3.$$

We, therefore, want the values of s_j and s_k which minimize Q . These are found by taking the partial derivative of Q with respect to the scale values; setting this value to zero and solving for the scale values.

For a scale value s_h ,

$$\frac{\partial Q}{\partial s_h} = 2 \sum_{j \neq h} e_{jh} \, (-1) w_{jh} + 2 \sum_{k \neq h} e_{hk} \, (-1)(-1) w_{hk} + 2 e_{hh} (-1)(0) w_{hh}$$

$$= 2 \sum_{j \neq h} (x_{jh} - s_h + s_j) w_{jh} + 2 \sum_{k \neq h} (x_{hk} - s_k + s_h) w_{hk}$$

$$= 2 \Big\{ - \sum_{j \neq h} x_{jh} w_{jh} + s_h \sum_{j \neq h} w_{jh} - \sum_{j \neq h} s_j w_{jh} + \sum_{k \neq h} x_{hk} w_{hk}$$

$$- \sum_{k \neq h} s_k w_{hk} + s_h \sum_{k \neq h} w_{hk} \Big\} \quad . \qquad 4.$$

Setting $\frac{\partial Q}{\partial s_h} = 0$ gives

$$s_h \Big(\sum_{j \neq h} w_{jh} + \sum_{k \neq h} w_{hk} \Big) - \sum_{j \neq h} s_j \, w_{jh} - \sum_{k \neq h} s_k \, w_{hk} =$$

$$\sum_{j \neq h} x_{jh} \, w_{jh} - \sum_{k \neq h} x_{hk} \, w_{hk} \quad . \qquad 5.$$

Consider a matrix A which has diagonal elements a_{hh} ,

$$a_{hh} = \sum_{j \neq h} w_{jh} + \sum_{k \neq h} w_{hk} \quad , \qquad 6.$$

and off-diagonal elements a_{hm}

$$a_{hm} = - (w_{mh} + w_{hm}) \quad \text{for } m \neq h \quad . \qquad 7.$$

Also consider a column vector Y where

$$y_h = \sum_{j \neq h} x_{jh} - \sum_{k \neq h} x_{hk} \, w_{hk} \quad . \qquad 8.$$

Then,

$$\sum_m a_{hm} \, s_m = y_h \quad , \qquad 9.$$

or, in matrix notation

$$A \; S = Y \qquad 10.$$

where S is a column vector.

To this point the solution is essentially equivalent to the Gulliksen (1956) solution. Gulliksen suggests simply inverting A and obtaining the scale values in S. However, the matrix A resulting from situations involving irregular data will often have no inverse. It may be possible to set up a matrix B of constants such that (A+B) has an inverse. Since the scale values sum to zero,

$$B \; S = 0 \qquad 11.$$

Therefore, for any matrix of constants,

$$(A + B)\ S = Y \qquad 12.$$

In the case of incomplete data resulting from the complete method of triads it is always possible to find a matrix B such that $(A_n + B)$, where the subscript n indicates the number of stimuli, has an inverse. In the four stimulus case the matrices w_n and A_n are as shown below.*

W_4

	1·2	1·3·	1·4	2·3	2·4	3·4
1·2	0	1	1	1	1	0
1·3	1	0	1	1	0	1
1·4	1	1	0	0	1	1
2·3	1	1	0	0	1	1
2·4	1	0	1	1	0	1
3·4	0	1	1	1	1	0

A_4

	1·2	1·3	1·4	2·3	2·4	3·4
1·2	8	-2	-2	-2	-2	0
1·3	-2	8	-2	-2	0	-2
1·4	-2	-2	8	0	-2	-2
2·3	-2	-2	0	8	-2	-2
2·4	-2	0	-2	-2	8	-2
3·4	0	-2	-2	-2	-2	8

figure 1. Matrices W and A for $n = 4$.

If we choose 2 as the constant in the matrix B, then $(A_4 + B)$ is the matrix shown below.

	1·2	1·3	1·4	2·3	2·4	3·4
1·2	10	0	0	0	0	2
1·3	0	10	0	0	2	0
1·4	0	0	10	2	0	0
2·3	0	0	2	10	0	0
2·4	0	2	0	0	10	0
3·4	2	0	0	0	0	10

figure 2. $(A_4 + B)$

* It should be remembered that with the method of triads we solve for the interpoint distances between pairs of stimuli rather than the scale values.

and $(A_4 + B)^{-1}$ is simply,

	1·2	1·3	1·4	2·3	2·4	3·4
1·2	$\frac{10}{96}$	0	0	0	0	$\frac{-2}{96}$
1·3	0	$\frac{10}{96}$	0	0	$\frac{-2}{96}$	0
1·4	0	0	$\frac{10}{96}$	$\frac{-2}{96}$	0	0
2·3	0	0	$\frac{-2}{96}$	$\frac{10}{96}$	0	0
2·4	0	$\frac{-2}{96}$	0	0	$\frac{10}{96}$	0
3·4	$\frac{-2}{96}$	0	0	0	0	$\frac{10}{96}$

figure 3. $(A_4 + B)^{-1}$

An additive constant must still be determined for the six interpoint distances before the four scale values may be determined.

When there are five or more stimuli the solution is somewhat more difficult, but never involves more than finding the inverse of a 3x3 matrix. To show this it is convenient to work with $1/2\ (A_n + B)$ and to partition it as below for the five stimulus case.

	1·2	1·3	1·4	1·5	2·3	2·4	2·5	3·4	3·5	4·5
1·2	7	0	0	0	0	0	0	1	1	1
1·3	0	7	0	0	0	1	1	0	0	1
1·4	0	0	7	0	1	0	1	0	1	0
1·5	0	0	0	7	1	1	0	1	0	0
2·3	0	0	1	1	7	0	0	0	0	1
2·4	0	1	0	1	0	7	0	0	1	0
2·5	0	1	1	0	0	0	7	1	0	0
3·4	1	0	0	1	0	0	1	7	0	0
3·5	1	0	1	0	0	1	0	0	7	0
4·5	1	1	0	0	1	0	0	0	0	7

figure 4. $1/2\ (A_5 + B)$

The typical off diagonal element of $1/2\ (A_n + B)$ will be

$$\frac{1}{2}\ (a_{hm} + 2) = \frac{1}{2}\left\{-\ (w_{mh} + w_{hm}) + 2\right\} \qquad 13.$$

as can be seen from Equation 7, and it is known that

$$b_{hm} + 2\ . \qquad 14.$$

Thus, the matrix $1/2\ (A_n + B)$ will have for off diagonal elements zero where W has a one, and a one where W has zero. In the diagonal we have from Equation 8 that

$$\frac{1}{2}\ (a_{hh} + 2) = \frac{1}{2}\left\{\sum_{j\neq h} w_{jh} + \sum_{k\neq h} w_{jk} + 2\right\} \qquad 15.$$

The typical diagonal element for n stimuli is therefore (2n - 3). Also note that the sum of each row in $1/2\ (A_n + B)$ is $\frac{n\ (n-1}{2}$.

The supermatrix representation of $1/2\ (A+B)$ would be as shown below.

f_{11}	f_{12}	f_{13}
f_{21}	f_{22}	f_{23}
f_{31}	f_{32}	f_{33}

figure 5 . $1/2\ (A_n + B)$

This method of partitioning is such that in general the orders of the submatrices are as follows:

f_{11} is $1x1$

f_{12} is $1x2\ (n-2)$

f_{13} is $1x\ \frac{(n-2)\ (n-3)}{2}$

f_{21} is $2\ (n-2)\ x\ 1$

f_{22} is $2\ (n-2)\ x\ 2\ (n-2)$

f_{23} is $2\ (n-2)\ x\ \frac{(n-2)\ (n-3)}{2}$

f_{31} is $\frac{(n-2)\ (n-3)}{2}\ x\ 1$

f_{32} is $\frac{(n-2)\ (n-3)}{2}\ x\ 2\ (n-2)$

f_{33} is $\frac{(n-2)\ (n-3)}{2}\ x\ \frac{(n-2)\ (n-3)}{2}$

This partitioning has very interesting properties. When the stimulus pairs are ordered as in figures 1-4, the single element in f_{11} is $(2n-3)$; each of the elements in f_{12} is zero; each element in f_{13} is one, which makes the sum of the elements in f_{13} $\frac{(n-2)(n-3)}{2}$.

Each row in f_{21} is zero; each row in f_{22} has a diagonal element $(2n-3)$ and $(n-3)$ off diagonal elements each equal to one, which makes the sum in each row of f_{22} equal to $3(n-2)$; each row in f_{23} has $\frac{(n-3)(n-4)}{2}$ elements each equal to one. Each row in f_{31} is equal to one; in f_{32} each row has $2(n-4)$ elements equal to one; each row of f_{33} has a diagonal element equal to $(2n-3)$ and $\frac{(n-4)(n-5)}{2}$ elements equal to one, making the sum of each row of f_{33} equal to $\left[\frac{(n-2)(n-3)}{2} + 4\right]$.

Consider the first column of the inverse of $1/2\,(A_n + B)$. If we partition it conformally to the first row of $1/2\,(A_n + B)$, we may represent the product of $1/2\,(A_n + B)$ and the column vector, C, as below.

$$\left[\begin{array}{c|c|c} f_{11} & f_{12} & f_{13} \\ \hline f_{21} & f_{22} & f_{23} \\ \hline f_{31} & f_{32} & f_{33} \end{array}\right] \left[\begin{array}{c} c_1 \\ \hline c_2 \\ \hline c_3 \end{array}\right] \left[\begin{array}{c} 1 \\ \hline 0 \\ \hline 0 \end{array}\right] \qquad 16.$$

This matrix equation can be broken down into three equations.

$$f_{11}\,c_1 + f_{12}\,c_2 + f_{13}\,c_3 = 1 \qquad 17.$$

$$f_{21}\,c_1 + f_{22}\,c_2 + f_{23}\,c_3 = 0 \qquad 18.$$

$$f_{31}\,c_1 + f_{32}\,c_2 + f_{33}\,c_3 = 0 \qquad 19.$$

Since we know that if a solution exists it is unique, assume that the solution is such that all the elements in c_2 are equal to each other and that the elements in c_3 are all equal to each other. If this assumption leads to a proper solution we know that the assumption was correct, since we already know that some solution exists.

Keeping this assumption in mind let t_{ij} stand for the sum of the elements in a row of f_{ij} and let c_1^*, c_2^*, and c_3^* be the elements in c_1, c_2, and c_3 respectively. Then the matrix equations [Equations 17-19] become equivalent to the following algebraic equations:

$$t_{11}\ c_1^* + 0 + t_{13}\ c_3^* = 1 \quad , \qquad 20.$$

$$0 + t_{22}\ c_2^* + t_{23}\ c_3^* = 0 \quad , \qquad 21.$$

$$t_{31}\ c_1^* + t_{32}\ c_2^* + t_{33}\ c_3^* = 0\ . \qquad 22.$$

From Equation 20 we find that,

$$c_2^* = -\frac{t_{23}}{t_{22}}\ c_3^* \quad . \qquad 23.$$

Substituting this result into Equation 22 results in the following:

$$c_1^* - \left[\frac{t_{32}\ t_{23}}{t_{22}} - t_{33}\right] c_3^* = 0 \quad , \qquad 24.$$

which leads to,

$$c_1^* = \left[\frac{t_{32}\ t_{23} - t_{22}\ t_{33}}{t_{22}}\right]\ c_3^* \quad . \qquad 25.$$

Substituting this result into Equation 20 results in,

$$\left[\frac{t_{11}\ (t_{32}\ t_{23} - t_{22}\ t_{33})}{t_{22}} + t_{13}\right]\ c_3^* = 1 \quad , \qquad 26.$$

or that,

$$c_3^* = \frac{t_{22}}{\left[t_{11}\ (t_{32}\ t_{23} - t_{22}\ t_{33}) + t_{13}\ t_{22}\right]} \qquad 27.$$

Using this result it is easily shown that

$$c_1^* = \frac{(t_{32}\ t_{23} - t_{22}\ t_{33})}{\left[t_{11}\ (t_{32}\ t_{23} - t_{22}\ t_{33}) + t_{13}\ t_{22}\right]} \quad , \qquad 28.$$

and,

$$c_2^* = \frac{-t_{23}}{\left[t_{11}\ (t_{32}\ t_{23} - t_{22}\ t_{33}) + t_{13}\ t_{22}\right]} \quad . \qquad 29.$$

It is further assumed that these three terms c_1^* , c_2^* , and c_3^* are the only terms in the inverse of $1/2\ (A_n + B)$; that c_1^* appears in the inverse where ever the element $(2n - 3)$ appears in $1/2\ (A_n + B)$, (i.e., in the diagonals of $\{1/2\ (A_n + B)\}^{-1}$) ; that c_2^* appears in the inverse where ever a zero appears in $1/2\ (A_n + B)$; and that c_3^* appears in the inverse where ever a one appears in the original.

To show that the matrix specified is, in fact, an inverse of $1/2\ (A_n + B)$, it will first be shown that the diagonal elements of the product of $1/2\ (A_n + B)$ and its proposed inverse, are unities regardless of whether the proposed inverse pre- or post- multiplies $1/2\ (A_n + B)$.

For the first row of $1/2\ (A_n + B)$ the result has really already been shown; for in this case post-multiplication by the first column of the proposed inverse reduces simply to Equation 20 , which is, of course, satisfied by the proposed inverse. For pre-multiplication of $1/2\ (A_n + B)$ by the proposed inverse the same result will occur since both $1/2\ (A_n + B)$ and the proposed inverse are symmetric. But without loss of generality each row of $1/2\ (A_n + B)$ could be interchanged with the first row and the columns could be rearranged so that the new first row would have the same arrangement as the original first row. When the corresponding rearrangement is preformed on the proposed inverse the conditions of Equation 20 must still hold. Thus the diagonal elements of the product of $1/2\ (A_n + B)$ and the proposed inverse are one, as required.

The second part of the proof is to show that any row, i , of $1/2\ (A_n + B)$ times any column, j , of the proposed inverse (for $i \neq j$) has zero sum of products; and that any row, i , of the proposed inverse times any column, j , of $1/2\ (A_n + B)$ has zero sum of products.

Consider the first row of $1/2\ (A_n + B)$. It must have zero sum of products with the columns (or rows since the proposed inverse is symmetric) of the second or third section of the proposed inverse. For rows in the second section, the first term in the sum of products is $f_{11} \cdot c_2^*$; the next $2(n-2)$ terms are zero since they correspond to f_{12} which is the null vector of order $2(n-2)$. Of these $2(n-2)$ terms there are $(n-3)$ terms containing c_3^* , since there are $(n-3)$ terms equal to one in the corresponding row in section f_{22} of $1/2\ (A_n + B)$; there are $(n-2)$ terms containing c_2^* , since there are $(n-2)$ terms equal to zero in the corresponding row in section f_{22} of $1/2\ (A_n + B)$; and there is one term involving c_1^* , corresponding to the diagonal element. The remaining $\frac{(n-2)\ (n-3)}{2}$ terms in the sum of products correspond to unit vector f_{13} times the appropriate section in the column of the proposed inverse. Since there are a total of $\frac{(n-2)\ (n-3)}{2}$ elements in a row of $1/2\ (A_n + B)$ equal to one, and (n-3) of them were in the f_{22} section of the row there must be $\frac{(n-4)\ (n-3)}{2}$ elements in the appropriate section of the column of the proposed inverse which have a value c_3^*. That leaves (n-3) terms for which the proposed inverse has a value of c_2^*.

Collecting these results gives the following expression for evaluation:

$$f_{11}\ c_2^* + 0 + (n-3)\ c_2^* + \frac{(n-4)\ (n-3)}{2}\ c_3^* = (3n-6)\ c_2^* + \frac{(n-4)\ (n-3)}{2}\ c_3^* \quad . \qquad 30.$$

Since the denominators of c_2^* and c_3^* are the same, (c.f. Equations 27 and 29), denote these denominators as d. Substituting the results of Equations 27 and 29 into Equation 30 gives the following:

$$3(n-2)\ c_2^* + \frac{(n-4)\ (n-3)}{2}\ c_3^* = \frac{-3(n-2)\ t_{23}}{d} + \frac{\frac{(n-4)\ (n-3)}{2}\ t_{22}}{d} \quad . \qquad 31.$$

It has previously been established that,

$$t_{23} = \frac{(n-3)\ (n-4)}{2} \quad , \qquad 32.$$

and

$$t_{22} = 3(n-2) \quad . \qquad 33.$$

Substituting Equations 32 and 33 into Equation 31 gives

$$\frac{-3(n-2)\ t_{23}}{d} + \frac{\frac{(n-4)\ (n-3)}{2}\ t_{22}}{d} + \frac{-3(n-2)\ (n-3)\ (n-4)/2}{d} + \frac{3(n-4)\ (n-3)\ (n-2)/2}{d} = 0 \quad . \qquad 34.$$

Equation 34 gives the desired result. Since both $1/2\ (A_n + B)$ and the proposed inverse are symmetric, this result will occur for pre-multiplication by the proposed inverse as it did for post-multiplication. Also, by rearranging rows and columns this result could be shown for multiplication of the first row of $1/2\ (A_n + B)$ by the corresponding columns in the third section in the proposed inverse. The proposed inverse has, thus, been shown to be the inverse of $1/2\ (A_n + B)$.

$1/2\ (A_n + B)^{-1}$ can be summarized by saying that for every diagonal element in $1/2\ (A_n + B)$ the diagonal element in $1/2\ (A_n + B)^{-1}$ is c_1^*. For every element equal to zero in $1/2\ (A_n + B)$ the inverse has the element c_2^* and for each element equal to one, the inverse has the element c_3^*. The results in Equation 27 - 29 may be stated solely as a function of the number of stimuli being scaled, n.

$$c_1^* = \frac{n^3 + n^2 - 8n + 12}{2n^2\,(n-1)^2} \qquad 35.$$

$$c_2^* = \frac{(n-3)\,(n-4)}{2n^2\,(n-1)^2} \qquad 36.$$

$$c_3^* = \frac{-3(n-2)}{n^2(n-1)^2} \qquad 37.$$

To illustrate this method a five stimulus example was taken from Torgerson (1958, 280-285). From the nine Munsell colors in his example, the data for colors 1,2,6,7, and 9, which were renumbered 1 through 5, were used in this illustration. He reports values corresponding to $\sqrt{2}$ times the normal deviate, and these values were used here without reconversion since the scaling isn't crucial.

The basic data are presented in Table 1. The application of Equation 8 to these data resulted in the vector Y presented in Table 2. The inversion of $1/2\ (A_5 + B)$, shown in figure 4, was accomplished using equations 35-37 , and the resulting inverse appears in Table 3.

Insert Tables 1,2, and 3 about here.

The comparative interpoint distances were obtained by premultiplying the column vector Y by the inverse of $1/2\ (A_5 + B)$. These results are presented in the first row of Table 4. The second row of Table 4 contains the corresponding comparative interpoint distances obtained by Torgerson. The correlation between these two sets of interpoint distances is .99 , which is as close to perfect as decimal accuracy would allow.

Insert Table 4 about here.

REFERENCES

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Torgerson, W. S. *Theory and Method of Scaling*. New York: John Wiley and Sons, Inc., 1958.

Table 1

The $\sqrt{2}$ times the normal deviate for each stimulus pair.

	1·2	1·3	1·4	1·5	2·3	2·4	2·5	3·4	3·5	4·5
1·2	--	3.16	3.76	4.00	3.46	2.00	2.29	--	--	--
1·3	-3.16	--	-.48	.48	-2.29	--	--	-1.14	.19	--
1·4	-3.76	.48	--	1.77	--	-2.74	--	.00	--	-1.77
1·5	-4.00	-.48	-1.77	--	--	--	-2.74	--	.09	-2.74
2·3	-3.46	2.29	--	--	--	-2.00	.28	-.48	1.14	--
2·4	-2.00	--	2.74	--	2.00	--	2.29	1.27	--	-1.77
2·5	-2.29	--	--	2.74	-.28	-2.29	--	--	.58	-2.27
3·4	--	1.14	.00	--	.48	-1.27	--	--	2.57	-1.58
3·5	--	-.19	--	-.09	-1.14	--	-.58	-2.57	--	-2.74
4·5	--	--	1.77	2.74	--	-1.77	2.27	1.58	2.74	--

Table 2

The Vector Y

Pair Number									
1,2	1,3	1,4	1,5	2,3	2,4	2,5	3,4	3,5	4,5
-37.52	12.80	12.04	23.28	4.46	-9.06	7.62	-2.68	14.62	-25.74

Table 3

$\left\{1/2\ (A_5 + B)\right\}^{-1}$

	1·2	1·3	1·4	1·5	2·3	2·4	2·5	3·4	3·5	4·5
1·2	.15250	.00375	.00375	.00375	.00375	.00375	.00375	-.02250	-.02250	-.02250
1·3	.00375	.15250	.00375	.00375	.00375	-.02250	-.02250	.00375	.00375	-.02250
1·4	.00375	.00375	.15250	.00375	-.02250	.00375	-.02250	.00375	-.02250	.00375
1·5	.00375	.00375	.00375	.15250	-.02250	-.02250	.00375	-.02250	.00375	.00375
2·3	.00375	.00375	-.02250	-.02250	.15250	.00375	.00375	.00375	.00375	-.02250
2·4	.00375	-.02250	.00375	-.02250	.00375	.15250	.00375	.00375	-.02250	.00375
2·5	.00375	-.02250	-.02250	.00375	.00375	.00375	.15250	-.02250	.00375	.00375
3·4	-.02250	.00375	.00375	-.02250	.00375	.00375	-.02250	.15250	.00375	.00375
3·5	-.02250	.00375	-.02250	.00375	-.00375	-.02250	.00375	.00375	.15250	.00375
4·5	-.02250	-.02250	.00375	.00375	-.02250	.00375	.00375	.00375	.00375	.15250

Table 4

Comparative interpoint distances

Pair Number

1,2	1,3	1,4	1,5	2,3	2,4	2,5	3,4	3,5	4,5
-5.22	2.62	1.09	3.65	0.41	-2.68	0.55	-0.23	3.08	-3.30
-2.37	1.56	1.09	2.23	0.80	-0.47	0.78	0.57	1.88	-1.30

The first row contains the interpoint distances obtained in this illustration. The second row contains the corresponding interpoint distances obtained by Torgerson. The correlation between the rows is .99.