Title
Degeneracy loci in Grassmannians

Author
Anzaldo, Leesa Bantad

Publication Date
2015

UNIVERSITY OF CALIFORNIA,
IRVINE

Degeneracy Loci in Grassmannians

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

Leesa Bantad Anzaldo

Dissertation Committee:
Professor Vladimir Baranovsky, Chair
Professor Karl Rubin
Professor Martin Zeman

2015

DEDICATION

To my parents, husband, sister, and grandparents.

TABLE OF CONTENTS

ACKNOWLEDGMENTS

I would like to thank my advisor for his time, patience, and kindness these past four years. I cannot emphasize enough how grateful I am for the opportunity to be his student. He has always supported me, especially when there were academic and personal obstacles I had to overcome.

I would also like to thank my family and friends for their love and support, and in particular, my parents, sister, and grandparents, who have always pushed me to further my education. Without my parents' sacrifices, I would not have made it this far.

I am thankful to the UCI math department staff, especially Donna McConnell, for always being helpful and generous with their time. I also thank the math department for their financial support during all of my years as a graduate student.

I would like to thank Dr. Alessandra Pantano, Dr. Catherine Famiglietti, Dr. De Gallow, Dr. Dennis Eichhorn, and Dr. Sarah Eichhorn for serving as my teaching role models. Their passion and commitment to undergraduate education is inspiring, and I have learned so much from working with them.

I would like to thank Professor Marc Rieffel, Professor Thomas Scanlon, and Professor Jeffrey Remmel for their guidance when I was an undergraduate. I also thank Professor Yenamandra Venkat Rao for his support when I decided to return to school to finish the PhD.

Lastly, I owe my greatest gratitude to my fully faithful functor: my husband. He helped me face my fears about math and has challenged me more than anyone else. His love and encouragement, despite the miles separating us, were invaluable.

CURRICULUM VITAE

Leesa Bantad Anzaldo

EDUCATION

Doctor of Philosophy in Mathematics **2015**
University of California, Irvine — Irvine, California

Master of Science in Mathematics **2010**
University of California, Irvine — Irvine, California

Bachelor of Arts in Mathematics, Philosophy **2007**
University of California, Berkeley — Berkeley, California

TEACHING EXPERIENCE

Teaching Assistant **2008–2009, 2011–2015**
University of California, Irvine — Irvine, California

Teaching Associate **2013–2014**
University of California, Irvine — Irvine, California

Adjunct Faculty Member **2010–2011**
Bunker Hill Community College — Boston, Massachusetts

Part-Time Lecturer **2010–2011**
Northeastern University — Boston, Massachusetts

HONORS AND AWARDS

Pedagogical Fellowship **2014–2015**
University of California, Irvine

Outstanding Mathematics Teaching Assistant Award **2013–2014**
University of California, Irvine

SEMINAR TALKS

Assessment in Mathematics Courses **Nov. 2014**
UCI Math Graduate Seminar — Irvine, California

Some Examples of Schubert Calculus **Dec. 2013**
Loyola Marymount University Math Seminar — Los Angeles, California

Initial Classification of Lie Algebras **Jan. 2013**
UCI Algebra Seminar — Irvine, California

ABSTRACT OF THE DISSERTATION

Degeneracy Loci in Grassmannians

By

Leesa Bantad Anzaldo

Doctor of Philosophy in Mathematics

University of California, Irvine, 2015

Professor Vladimir Baranovsky, Chair

The Thom-Porteous formula allows one to compute the cohomology class of a degeneracy locus of maps between vector bundles, given that certain codimension conditions are satisfied. It is known that the Hilbert scheme on projective space may be expressed as a degeneracy locus in a Grassmannian, and in a similar fashion, so can the Quot scheme. Here we determine cases in which the expected codimension agrees with the actual codimension and evaluate the cohomology class. We also give exact conditions for the existence of isotropic subspaces of Schur modules.

Introduction

Given a map between vector bundles over some variety, a degeneracy locus is the collection of points such that the rank of the map is bounded by a given integer. We examine two different topics in this thesis:

The first part of this dissertation examines the Hilbert scheme and Quot scheme of projective space. We can realize them as degeneracy loci in a Grassmannian, and under certain codimension conditions outlined by the Thom-Porteous formula, we have a formula for their cohomology class in terms of Chern classes. Moreover, we can use tools from the combinatorics of symmetric functions to more easily compute these classes. Here, we give some results for the computation of the class of these schemes, as well as ask the question of whether the expected codimension, as stated in the Thom-Porteous formula, agrees with the actual codimension. (However, it is still useful in applications to assume that the codimension requirement is always satisfied.) See Theorem 3.1.5 and Theorem 3.2.2.

The second part examines a particular case of the degeneracy locus, namely the zero set of a section of bundle. In E.A. Tevelev's paper [21] on isotropic subspaces of polylinear forms, he asks when isotropic subspaces of a finite dimensional vector space V over an algebraically closed field of a certain dimension exist. We define a subspace W with respect to some $s \in S^dV^*$ or Λ^dV^* to be isotropic if $s|_W = 0$. He gives a necessary and sufficient condition for the existence of such subspaces.

Here, we generalize his result to the case of Schur modules. Since symmetric polylinear forms correspond to single row partitions and skew-symmetric polylinear forms correspond to single column partitions, it is natural to ask what happens in the case of general partitions, i.e. for Schur modules. This also provides a geometric application to flag varieties: for a generic $s \in S_\lambda V^*$, when does there exist a subspace W of V such that $Flag_\lambda(W) \subset Flag_\lambda(V)$ is in the zero locus of s? Another interpretation of our result is a nice way of computing whether the top Chern class of certain vector bundles is nonzero. See Theorem 4.0.5, Corollary 4.0.7, and Corollary 4.0.9.

Chapter 1

Grassmannians and Degeneracy Loci

Let V be an n-dimensional complex vector space. The *Grassmannian*, denoted $Gr(k, V)$ or $Gr(k, n)$, is the set of all k-dimensional linear subspaces of V. We can represent an element in $Gr(k, n)$ by a $k \times n$ matrix A of rank k, and this representation is unique up to the action of $\mathbf{GL}_n$ acting by column operations.

Let I be a size k subset of $\{1, \ldots, n\}$ and V_{I^o} be the subspace of $\mathbf{C}^n$ spanned by $\{e_i \colon i \notin I\}$, and define $U_I = \{\Lambda \in Gr(k, n) \colon \Lambda \cap V_{I^o} = \{0\}\}$. Then U_I is the set of $\Lambda \in Gr(k, n)$ that can be represented by $\Lambda^I \in Gr(k, n)$ where the Ith $k \times k$ minor is I_k.

Example 1.0.1. Take $I = \{1, 3\}$ for $Gr(2, 4)$. Any $\Lambda \in Gr(2, 4)$ can be represented by either $\begin{bmatrix} 1 & * & 0 & * \\ 0 & * & 1 & * \end{bmatrix}$, $\begin{bmatrix} 1 & * & 0 & * \\ 0 & * & 0 & * \end{bmatrix}$, or $\begin{bmatrix} 0 & * & 0 & * \\ 0 & * & 0 & * \end{bmatrix}$, so $\Lambda \cap V_{I^o} = \{0\}$ iff the Ith 2×2 minor is I_2. $\square$

Since every $k \times n$ matrix whose Ith $k \times k$ minor is I_k represents a unique element of U_I, the map $\varphi_I \colon U_I \to \mathbf{C}^{k(n-k)}$ is a bijection. These maps give an atlas of charts making $Gr(k,n)$ a complex manifold [7, p.193]. Moreover, every $Gr(k, \mathbf{C}^n)$ is isomorphic to $Gr(n-k, (\mathbf{C}^n)^*)$ by the map that sends a k-dimensional subspace $W \subset \mathbf{C}^n$ to $(\mathbf{C}^n/W)^* \subset (\mathbf{C}^n)^*$.

1.1 Plücker Embedding, Plücker Relations

The *Plücker embedding* is the map $p \colon Gr(k,n) \to \mathbf{P}(\wedge^k \mathbf{C}^n) = \mathbf{P}^{\binom{n}{k}-1}$ defined by $p(\Lambda) = [v_1 \wedge \cdots \wedge v_k]$ where $v_1, \ldots, v_k$ form a basis for Λ (or $p(\Lambda) = [d_1 : \cdots : d_{\binom{n}{k}}]$ where the d_i's, called the *Plücker coordinates*, are the determinants of the $k \times k$ submatrices of A representing Λ). Since $p(\Lambda)$ is independent of the choice of basis, the map is well-defined, and since the span of v_i's in an element of $\mathbf{P}(\wedge^k \mathbf{C}^n)$ will generate a unique subspace in $Gr(k,n)$, the map is injective [8, p.63–64].

The image of $Gr(k,n)$ under p is a projective variety defined as the zero set of polynomials, called the *Plücker relations* of $Gr(k,n)$. Consider the case for $Gr(2,n)$. The following map gives a one-to-one correspondence between $Gr(2,n)$ and the set of $n \times n$ skew symmetric matrices of rank 2 modulo the equivalence relation $A = \lambda A$ for $\lambda \neq 0$: take $[e_k \wedge e_l]$ to the $n \times n$ matrix (a_{ij}) where $a_{kl} = 1$, $a_{lk} = -1$, and $a_{ij} = 0$ otherwise; extend linearly to obtain the image for $\Lambda \in Gr(2,n)$ with $p(\Lambda) = [v_1 \wedge v_2]$. For skew symmetric matrices, the determinant is a square and we call the square root its *Pfaffian* [13, p.586–589].

Example 1.1.1. $Gr(2,4)$

A general $\Lambda \in Gr(2,4)$ can be represented by $A = \begin{bmatrix} 0 & x_{12} & x_{13} & x_{14} \\ -x_{12} & 0 & x_{23} & x_{24} \\ -x_{13} & -x_{23} & 0 & x_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0 \end{bmatrix}$. One can compute that $\det A = (x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23})^2$, so $\operatorname{Pf} A = x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}$. Since $\operatorname{rank} A = 2$, $\operatorname{Pf} A = 0$ and the Plücker relation of $Gr(2,4)$ is $x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0$. $\square$

Example 1.1.2. $Gr(2,5)$

A general $\Lambda \in Gr(2,5)$ can be represented by $A = \begin{bmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} \\ -x_{12} & 0 & x_{23} & x_{24} & x_{25} \\ -x_{13} & -x_{23} & 0 & x_{34} & x_{35} \\ -x_{14} & -x_{24} & -x_{34} & 0 & x_{45} \\ -x_{15} & -x_{25} & -x_{35} & -x_{45} & 0 \end{bmatrix}$.

The Plücker relations are precisely the Pfaffians of the minors M_{ii} of A, namely

$$\operatorname{Pf} M_{1,1} = x_{23}x_{45} - x_{24}x_{35} + x_{25}x_{34}$$

$$\operatorname{Pf} M_{2,2} = x_{13}x_{45} - x_{14}x_{35} + x_{15}x_{34}$$

$$\operatorname{Pf} M_{3,3} = x_{12}x_{45} - x_{14}x_{25} + x_{15}x_{24}$$

$$\operatorname{Pf} M_{4,4} = x_{12}x_{35} - x_{13}x_{25} + x_{15}x_{23}$$

$$\operatorname{Pf} M_{5,5} = x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}$$

$\square$

In general, the rank 2 skew-symmetric matrices are defined by the 4×4 Pfaffians. The general form of the Plücker relations are

$$x_{i_1,\dots,i_d}x_{j_1,\dots,j_d} - \sum x_{i'_1,\dots,i'_d}x_{j'_1,\dots,j'_d} = 0$$

where the sum is taken over all $j'_1,\dots,j'_d$ obtained by replacing $j_1,\dots,j_d$ by a size k subset of $i_1,\dots,i_d$, and likewise for $i'_1,\dots,i'_d$.

1.2 Cell Decomposition

If $V = (V_1 \subset V_2 \subset \cdots \subset V_n)$ is a flag in $\mathbf{C}^n$, then the *Schubert cycles* of $Gr(k,n)$ are the subvarieties of the form

$$\sigma_a(V) = \{\Lambda\colon \dim(\Lambda \cap V_{n-k+i-a_i}) \geq i\}$$

for $a = (a_1, \ldots, a_k)$ where a is a nonincreasing sequence of integers between 0 and $n-k$. If $\sigma_a = \sigma_{a,0,0,\ldots}$, then σ_a is a *special Schubert cycle*. These generate $H_*(Gr(k,n), \mathbf{Z})$ and give $Gr(k,n)$ the structure of a *CW-complex* with only cells of even dimension.

Example 1.2.1. $Gr(2,4)$

The Schubert cycles of $Gr(2,4)$ are as follows.

$$\begin{aligned}
\sigma_{0,0}(V) &= \{\Lambda\colon \dim(\Lambda \cap V_3) \geq 1, \dim(\Lambda \cap V_4) \geq 2\} = Gr(2,4), \text{ codim } 0\\
\sigma_{1,0}(V) &= \{\Lambda\colon \dim(\Lambda \cap V_2) \geq 1, \dim(\Lambda \cap V_4) \geq 2\} = \{\Lambda\colon \dim(\Lambda \cap V_2) \geq 1\},\\
&\qquad \text{codim } 1\\
\sigma_{1,1}(V) &= \{\Lambda\colon \dim(\Lambda \cap V_2) \geq 1, \dim(\Lambda \cap V_3) \geq 2\} = \{\Lambda\colon \Lambda \subset V_3\}, \text{ codim } 2\\
\sigma_{2,0}(V) &= \{\Lambda\colon \dim(\Lambda \cap V_1) \geq 1, \dim(\Lambda \cap V_4) \geq 2\} = \{\Lambda\colon V_1 \subset \Lambda\}, \text{ codim } 2\\
\sigma_{2,1}(V) &= \{\Lambda\colon \dim(\Lambda \cap V_1) \geq 1, \dim(\Lambda \cap V_3) \geq 2\} = \{\Lambda\colon V_1 \subset \Lambda \subset V_3\},\\
&\qquad \text{codim } 3\\
\sigma_{2,2}(V) &= \{\Lambda\colon \dim(\Lambda \cap V_1) \geq 1, \dim(\Lambda \cap V_2) \geq 2\} = \{\Lambda\colon V_1 \subset \Lambda = V_2\},\\
&\qquad \text{codim } 4
\end{aligned}$$

□

1.2.1 Cellular Homology

Let X be a topological space and Δ^n be the standard n-simplex, i.e. an n-dimensional polytope which is the convex hull of $n+1$ vertices $e_0, \ldots, e_n$. Then a *singular n-simplex* is a continuous map from Δ^n to X; the set of all such maps generates a free abelian group denoted by $C_n(X)$. Let $\partial_n \colon C_n \to C_{n-1}$ be defined by

$$\partial_n \sigma_n(\Delta^n) = \sum_{k=1}^{n} (-1)^k [\sigma_n(e_0), \ldots, \sigma_n(e_{k-1}), \sigma_n(e_{k+1}), \ldots, \sigma_n(e_n)]$$

where $\sigma_n \in C_n$. Then $(C_\bullet(X), \partial_\bullet)$ is the *singular chain complex*. If $A \subset X$, then we have a short exact sequence $0 \to C_\bullet(A) \to C_\bullet(X) \to C_\bullet(X)/C_\bullet(A) \to 0$ and we call $H_n(C_\bullet(X)/C_\bullet(A))$ the *relative homology*, denoted by $H_n(X, A)$ [11, p.115].

Since cells in complex Grassmannians have only even dimension, $H_*(Gr(k,n), \mathbf{Z})$ has a basis which is in one-to-one correspondence with the Schubert cells. By Poincaré duality, which is defined below, the Schubert cycles form a basis for $H^*(Gr(k,n), \mathbf{Z})$ [11, p.137–140].

1.2.2 Cup Product

If $\alpha \in C^n(X), \beta \in C^m(X)$ are cochains, then the *cup product* of $\alpha \smile \beta \in C^{n+m}(X)$ is the cochain defined by $(\alpha \smile \beta)(\sigma) = \alpha(\sigma|_{[e_0,\ldots,e_n]})\beta(\sigma|_{[e_n,\ldots,e_{n+m}]})$ for $\sigma \in C_{n+m}(X)$. If α and β are cocycles and γ is a coboundary map, then one can show

$$\gamma(\alpha \smile \beta) = \gamma\alpha \smile \beta + (-1)^n \alpha \smile \gamma\beta.$$

This implies that the cup product of two cocycles is a cocycle and the cup product of a cocycle and a coboundary (or vice-versa) is a coboundary. Then we get the induced map of the cup product from $H^n(X) \times H^m(X)$ to $H^{n+m}(X)$. Therefore, the cohomology $H^*(Gr(k,n), \mathbf{Z})$ of the Grassmannian has a ring structure [11, p.206–207].

1.3 Poincaré Duality

An *oriented* manifold is one with a fixed orientation, i.e. an atlas such that at every point of the manifold, the Jacobian of the transition map is positive. Complex manifolds are always oriented because for any linear map $f\colon \mathbf{C}^n \to \mathbf{C}^n$, using the one-to-one correspondence $(x_1, \ldots, x_n, y_1, \ldots, y_n) \leftrightarrow (x_1 + iy_1, \ldots, x_n + iy_n)$ we have the linear map $g\colon \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ which is orientation-preserving since $\det g = |\det f|^2$.

Let X be a space and R be a ring. For $\alpha\colon \Delta^m \to X$ and $\beta \in C^n(X;R)$ such that $m \geq n$, define the *cap product* $\frown\colon C_m(X;R) \times C^n(X;R) \to C_{m-n}(X;R)$ by $\alpha \frown \beta = \beta(\alpha|_{[e_0,\ldots,e_n]})\alpha|_{[e_n,\ldots,e_m]}$. Since

$$\delta(\alpha \frown \beta) = (-1)^n(\delta\alpha \frown \beta - \alpha \frown \delta\beta),$$

the cap product takes a cycle and a cocycle to a cycle, a cycle and a coboundary to a boundary, and a boundary and a cocycle to a boundary. So we get the induced cap product from $H_m(X;R) \times H^n(X;R)$ to $H_{m-n}(X;R)$ [11, p.239–240].

If M is an n-dimensional oriented, compact, and connected manifold, then

$$H_n(M;\mathbf{Z}) \cong \mathbf{Z}.$$

Since $H_m(M;\mathbf{Z}) = 0$ for $m \notin \{0, 1, \ldots, n\}$, we call a generator (which corresponds to a choice of orientation) for the top homology $H_n(M;\mathbf{Z})$ a *fundamental class* for M, denoted $[M]$. So Poincaré duality can be phrased as the following:

Theorem 1.3.1 (Poincaré duality, [11, p.236,241])**.** *Let M be an n-dimensional oriented compact connected manifold with fundamental class $[M] \in H_n(M; \mathbf{Z})$. For all k, the map*

$$\begin{aligned} D \colon H^k(M; \mathbf{Z}) &\to H_{n-k}(M; \mathbf{Z}) \\ \alpha &\mapsto [M] \frown \alpha \end{aligned}$$

is an isomorphism.

1.4 Vector Bundles

1.4.1 Sheaves

If X is a topological space, then a *presheaf* $\mathcal{F}$ of abelian groups on X is given by the following:

- an abelian group $\mathcal{F}(U)$ for every open $U \subset X$, and
- a morphism of abelian groups $\rho_{UV} \colon \mathcal{F}(U) \to \mathcal{F}(V)$ for every inclusion of open sets $V \subset U \subset X$.

The following conditions must be satisfied:

- $\mathcal{F}(\varnothing) = 0$;
- $\rho_{UU} \colon \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map;
- and $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$ if $W \subset V \subset U \subset X$ are all open.

A presheaf $\mathcal{F}$ is a *sheaf* if for any open set $U \subset X$ and open cover $\{V_i\}$ of U, then the following condition is satisfied:

- if for each i there exists $s_i \in \mathcal{F}(V_i)$ such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ for all i, j, then there exists a unique $s \in \mathcal{F}(U)$ satisfying $s|_{V_i} = s_i$ for all i; by $s|_{V_i}$ we mean $\rho_{UV_i}(s)$ [10, p.61].

1.4.2 Vector Bundles

If $\mathcal{E}$ and M are topological spaces, then a *complex vector bundle of rank n over M* is $\mathcal{E}$ with $\pi\colon \mathcal{E} \to M$, a continuous surjective map, such that $\pi^{-1}(p)$ is an n-dimensional complex vector space for every $p \in M$; and *local triviality* is satisfied, i.e. for every $p \in M$, there exists a neighborhood $U \subset M$ of p and homeomorphism $\varphi\colon \pi^{-1}(U) \to U \times \mathbf{C}^n$ such that $\pi = \pi_1 \circ \varphi$, where $\pi_1\colon U \times \mathbf{C}^n \to U$ is the projection onto U, and $\varphi|_{\pi^{-1}(q)}\colon \pi^{-1}(q) \to \{q\} \times \mathbf{C}^n$ is a linear isomorphism for all $q \in U$. The *total space* is $\mathcal{E}$ and the *base space* is M. A rank 1 vector bundle is called a *line bundle* [14, p.103–104].

If we start with a smooth manifold M and an open cover $\{U_\alpha\}_{\alpha \in A}$, then we have a smooth rank n vector bundle $\pi\colon \mathcal{E} \to M$, where π is surjective, if there exist smooth homeomorphisms $\varphi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbf{C}^n$ such that $\pi = \pi_1 \circ \varphi_\alpha$ for each $\alpha \in A$, whose transition functions are $\varphi_\alpha \circ \varphi_\beta^{-1} = \tau_{\alpha,\beta}\colon U_\alpha \cap U_\beta \to GL_n$ satisfying $\tau_{\alpha,\beta}(p)\tau_{\beta,\gamma}(p) = \tau_{\alpha,\gamma}(p)$ for all $p \in U_\alpha \cap U_\beta \cap U_\gamma$ [14, p.121].

Equivalently, a rank n vector bundle $\mathcal{E}$ of M consists of an open cover $\{U_\alpha\}_{\alpha \in A}$ and isomorphisms $\rho_{\alpha,\beta}\colon \mathbf{C}^n \times (U_\alpha \times U_\beta) \to \mathbf{C}^n \times (U_\beta \times U_\alpha)$ for each $\alpha, \beta \in A$ such that $\rho_{\alpha,\beta}$ fixes fibers; $\rho_{\alpha,\beta} = \rho_{\beta,\alpha}^{-1}$; and $\rho_{\alpha,\beta}\rho_{\beta,\gamma}|_{U_\alpha \cap U_\beta \cap U_\gamma} = \rho_{\alpha,\gamma}|_{U_\alpha \cap U_\beta \cap U_\gamma}$ for all α, β, and γ in A.

We'll deal with vector bundles where $\mathcal{E}$ and M are algebraic varieties and π is a morphism. If $U \subset M$ is open, then a *section of $\mathcal{E}$ over U* is a morphism $s\colon U \to \mathcal{E}$

such that $\pi \circ s = id_U$. We denote the set of all sections of $\mathcal{E}$ over U by $\mathcal{E}(U)$. The *zero section* is $s\colon M \to \mathcal{E}$ where $s(p) = \varphi^{-1}(p, 0)$ for any neighborhood U of p and homeomorphism φ defined above, or in other words, s takes $x \in U$ to the zero element in $\pi^{-1}(x)$. A *global section* of $\mathcal{E}$ is a section of $\mathcal{E}(M)$ of $\mathcal{E}$ over M. The space of all global sections is denoted $H^0(M, \mathcal{E})$.

Here are some examples of vector bundles:

1. The *trivial vector bundle* is given by $\pi\colon M \times \mathbf{C}^n \to M$ where $\pi(p, \alpha_1, \dots, \alpha_n) = p$.

2. For any vector bundle $\pi\colon \mathcal{E} \to M$, we can define the *dual vector bundle* to be $\pi^*\colon \mathcal{E}' \to M$ where $\mathcal{E}' = \cup_{x \in M}[\pi^{-1}(x)]^*$ such that $[\pi^{-1}(x)]^* \mapsto x$.

3. If $X \subset \mathbf{P}^n$ is a projective variety, then the *tautological bundle* $\mathcal{O}_X(-1)$ over X is B with $\pi\colon B \to X$ where $B = \{(x, l)\colon l \in X, x \in l\} \subset \mathbf{C}^{n+1} \times \mathbf{P}^n$ and π is projection onto X.

4. By taking the dual of the tautological line bundle, we get the *hyperplane bundle* $\mathcal{H}$ on variety $X \subset \mathbf{P}^n$. When $X = \mathbf{P}^n$, the sheaf of the hyperplane bundle on $\mathbf{P}^n$ is denoted by $\mathcal{O}_{\mathbf{P}^n}(1)$ and its space of global sections is exactly the set of all linear polynomials in $\mathbf{C}[x_0, \dots, x_n]$. More generally, for the kth power of a hyperplane bundle $\mathcal{H}^{\otimes k}$, the space of all global sections $\mathcal{O}_{\mathbf{P}^n}(k)$ is the set of all degree k homogeneous polynomials [19, p.122–127].

5. If $\pi\colon \mathcal{E} \to M$ is a vector bundle and $f\colon N \to M$ is continuous, then the *pullback bundle* or the *induced bundle* on N is $f^*(\mathcal{E}) = \{(x, v)\colon x \in N, v \in \pi^{-1}(f(x))\}$.

6. If $\pi\colon \mathcal{E} \to M$, $\pi'\colon \mathcal{F} \to M$ are vector bundles and $\mathcal{E} \hookrightarrow \mathcal{F}$, then the *quotient bundle* $\rho\colon \mathcal{F}/\mathcal{E} \to M$ is $\{(x, v)\colon x \in M, v \in \pi'^{-1}(x)/\pi^{-1}(x)\}$ where ρ is again projection onto M.

7. Let $\pi\colon \mathcal{E} \to M$, $\pi'\colon \mathcal{F} \to M$ be vector bundles of rank n, m, respectively, and $\mathcal{G} = \{(e,f) \in \mathcal{E} \times \mathcal{F} \colon \pi(e) = \pi'(f)\}$. Then $\pi''\colon \mathcal{G} \to M$ where $(e,f) \mapsto \pi(e) = \pi'(f)$ is the rank $(n+m)$ *Whitney sum* vector bundle denoted by $\pi \oplus \pi'$.

8. The *tensor product bundle* is defined similarly: for vector bundles $\pi\colon \mathcal{E} \to M$, $\pi'\colon \mathcal{F} \to M$, we have $\mathcal{E} \otimes \mathcal{F} = \{(x,v)\colon x \in M, v \in \pi^{-1}(x) \otimes \pi'^{-1}(x)\}$ where $\pi''\colon \mathcal{E} \otimes \mathcal{F} \to M$ is projection onto M. Likewise, the *symmetric power bundle* is $S^k(\mathcal{E}) = \{(x,v)\colon x \in M, v \in S^k(\pi^{-1}(x))\}$ and the *exterior power bundle* is $\wedge^k(\mathcal{E}) = \{(x,v)\colon x \in M, v \in \wedge^k(\pi^{-1}(x))\}$.

1.4.3 Universal Bundles

Take the trivial vector bundle $\pi\colon Gr(k,n) \times \mathbf{C}^n \to Gr(k,n)$ over $Gr(k,n)$ and let $\pi'\colon S \to Gr(k,n)$ be the subbundle such that $\pi'^{-1}(\Lambda) = \Lambda \times \Lambda$. We call S the *universal subbundle* or the *tautological subbundle* of $Gr(k,n)$, which explicitly is $S = \{(\Lambda, v)\colon v \in \Lambda, \Lambda \in Gr(k,n)\}$. We call the quotient bundle $Q = (Gr(k,n) \times \mathbf{C}^n)/S$ the *universal quotient bundle* or the *tautological quotient bundle* on $Gr(k,n)$. When $k = 1$, the universal subbundle is the same as the tautological line bundle [7, p.207].

1.4.4 Fiber Bundles

A *fiber bundle* on M consists of spaces $\mathcal{E}$ and F with a continuous surjective map $\pi\colon \mathcal{E} \to M$ such that there exist an open cover $\{U_\alpha\}_{\alpha \in A}$ of M and homeomorphisms of the form $\varphi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times F$ such that $\pi = \pi_1 \circ \varphi_\alpha$; we call F the *fiber*. Therefore, a vector bundle is a fiber bundle where F is a vector space. We can also define fiber bundles using the alternative definitions for vector bundles from above by replacing $\mathbf{C}^n$ with any space F. Here are a few examples of fiber bundles:

1. A *projective bundle* is a fiber bundle whose fiber is projective space. For any rank n vector bundle $\mathcal{E}$ on M, the projective bundle $P(\mathcal{E})$ on M is a fiber bundle with fiber $\mathbf{P}^{n-1}$ where $P(\mathcal{E})$ is the space of all elements in $\mathbf{P}^{n-1}$ in all fibers of $\mathcal{E}$, i.e. the space obtained by gluing the patches $U_\alpha \times \mathbf{P}^{n-1}$ using the transition functions. In fact, any projective bundle on M is equivalent to $P(\mathcal{E}) = \{(x, l) \colon x \in M, l \in \mathbf{P}(\pi^{-1}(x))\}$ for some vector bundle $\pi \colon \mathcal{E} \to M$ [11, p.375].

   The set of complete flags of vector spaces in $\mathbf{C}^n$ form a projective variety called a *flag variety*, denoted by $\mathbf{F}^n$ [17, p.132–133]. We can generalize this to any n-dimensional vector space E, and for integers $0 < d_1 < \cdots < d_k < n$, we define a *partial flag variety* to be the set $Flag(d_1, \ldots, d_k, E) = \{W_{d_1} \subset \cdots \subset W_{d_k} \subset E \colon \dim W_{d_i} = d_i\}$. Some partial flag varieties are isomorphic to projective bundles of quotient bundles: let $X_k = Flag(1, \ldots, k, E)$, define $R_k = \{(e, W_1 \subset \cdots \subset W_k) \in E \times X_k \colon e \in W_k\}$, and take the quotient bundle $Q^k = (E \times X_k)/R_k$. Since the fiber of $W_1 \subset \cdots \subset W_k$ in Q^k is the fiber of $W_1 \subset \cdots \subset W_k$ in $E \times X_k$ mod the fiber of $W_1 \subset \cdots \subset W_k$ in R_k, then

$$\begin{aligned} P(Q^k) = \{(W_1 \subset \cdots \subset W_k, l) \colon & W_1 \subset \cdots \subset W_k \in X_k, \\ & l \in \mathbf{P}(\pi^{-1}(W_1 \subset \cdots \subset W_k))\} \\ = \{(W_1 \subset \cdots \subset W_k, l) \colon & W_1 \subset \cdots \subset W_k \in X_k, \\ & l \in \mathbf{P}((E \times \{W_1 \subset \cdots \subset W_k\})/(W_k \times \{W_1 \subset \cdots \subset W_k\}))\} \\ \cong \{(W_1 \subset \cdots \subset W_k, l) \colon & W_1 \subset \cdots \subset W_k \in X_k, l \in \mathbf{P}(E/W_k)\} \end{aligned}$$

   If $p \colon E \to E/W_k$ and $l \subset E/W_k$, then $\dim p^{-1}(l) = k + 1$, so $W_1 \subset \cdots \subset W_k \subset p^{-1}(l) \in X_{k+1}$. The map $W_1 \subset \cdots \subset W_k \subset p^{-1}(l) \mapsto (W_1 \subset \cdots \subset W_k, l)$ gives the desired isomorphism, i.e. $X_{k+1} \cong P(Q^k)$.

2. A *Grassmann bundle* is a fiber bundle where the fiber is a Grassmannian. For any vector bundle $\pi\colon \mathcal{E} \to M$, we can construct a Grassmann bundle as follows: there exists an open cover $\{U_\alpha\}_{\alpha \in A}$ and isomorphisms $\rho_{\alpha,\beta}\colon \mathbf{C}^n \times (U_\alpha \cap U_\beta) \to \mathbf{C}^n \times (U_\beta \cap U_\alpha)$, so for any $\alpha, \beta \in A$, we can define the isomorphism $f_{\alpha,\beta}\colon Gr(k,n) \times (U_\alpha \cap U_\beta) \to Gr(k,n) \times (U_\beta \cap U_\alpha)$ such that $(W, x) \mapsto (\rho_{\alpha,\beta}(W), x)$ where $\rho_{\alpha,\beta}(W) = \{(\pi_1 \circ \rho_{\alpha,\beta})(v,x)\colon (v,x) \in W \times (U_\alpha \cap U_\beta)\}$. The Grassmann bundle is equivalent to the set $Gr(k,\mathcal{E}) = \{(x,l)\colon x \in M, l \in Gr(k, \pi^{-1}(x))\}$ and the Grassmann bundle $\rho\colon Gr(k,\mathcal{E}) \to M$ is projection map.

   Given a vector bundle $\pi\colon \mathcal{E} \to M$, the pullback bundle

$$\begin{aligned}\rho^*(\mathcal{E}) &= \{((x,l),v)\colon (x,l) \in Gr(k,\mathcal{E}), v \in \pi^{-1}(\rho(x,l))\} \\ &= \{((x,l),v)\colon (x,l) \in Gr(k,\mathcal{E}), v \in \pi^{-1}(x)\}\end{aligned}$$

   is a vector bundle on the Grassmann bundle $\rho\colon Gr(k,\mathcal{E}) \to M$. This gives an example the analog of trivial vector bundles to Grassmann bundles. The analog of tautological subbundles for Grassmann bundles is $R^{\mathcal{E}} = \{((x,l),v)\colon v \in l \subset \pi^{-1}(x)\}$. Since $R^{\mathcal{E}} \subset \rho^*(\mathcal{E})$, $Q^{\mathcal{E}} = \rho^*(\mathcal{E})/R^{\mathcal{E}}$ is the analog of the tautological quotient bundle to Grassmann bundles.

   All partial flag varieties are isomorphic to Grassmann bundles of quotient bundles, i.e. $Flag(d_1, \ldots, d_k, d_k + a, E) \cong Gr(a, Q^{d_k})$, by an argument similar to the one above.

3. A *flag bundle* is a fiber bundle where the fiber is a flag variety, and a *partial flag bundle* is defined similarly. Given a rank n vector bundle $\pi\colon \mathcal{E} \to M$ and $k \leq n$, the partial flag bundle is equivalent to $FL(d_1, \ldots, d_k, \mathcal{E}) = \{(x,l)\colon x \in M, l \in Flag(d_1, \ldots, d_k, \pi^{-1}(x))\}$. Given a vector bundle $\pi\colon \mathcal{E} \to M$, the analog of the trivial vector bundle to a partial flag bundle $\rho\colon FL(d_1, \ldots, d_k, \mathcal{E}) \to M$ is again the pullback bundle $\rho^*(\mathcal{E})$, and the analogs of the tautological subbundles are

defined like above where $R^{\mathcal{E},d_i} = \{((x, l_{d_1} \subset \cdots \subset l_{d_k}), v) \colon v \in l_{d_i}, l_{d_1} \subset \cdots \subset l_{d_k} \in Flag(d_1, \dots, d_k, \pi^{-1}(x))\}$. Notice $R^{\mathcal{E},d_1} \subset \cdots \subset R^{\mathcal{E},d_k} \subset \rho^*(\mathcal{E})$, so the analog of tautological quotient bundles are $\rho^*(\mathcal{E})/R^{\mathcal{E},d_i}$ for $d_1, \dots, d_k$.

1.4.5 Maps to $\mathbf{P}^n$

If $X \subset \mathbf{P}^n$ and L is a complex line bundle on X, then a *linear system on* X is a vector space spanned by a set of linearly independent sections of the vector space of the global sections of L. A *complete linear system* $|L|$ on X is a linear system containing all global sections of L. If $|L|$ is a complete linear system generated by $s_0, \dots, s_n$, then let $\widetilde{s_i}(x)$ be the projection of $\varphi(s_i(x))$ on $\mathbf{C}$ where $x \in U$ and $\varphi \colon \pi^{-1}(U) \to U \times \mathbf{C}$ is the local trivialization. The numbers $\widetilde{s_i}(x) \in \mathbf{C}$ depend on the choice of φ, but for any another local trivialization φ' with corresponding projection $\widetilde{s_i}'$, the ratios $\widetilde{s_i}(x)/\widetilde{s_i}'(x)$ are constant for all i. Therefore, we can define a partial map $X \dashrightarrow \mathbf{P}^n$ by $x \mapsto [\widetilde{s_0}(x) \colon \dots \colon \widetilde{s_n}(x)]$.

Conversely, for any partial map from X to $\mathbf{P}^n$, we can find a complete linear system and a line bundle on X to define the maps as above. One can take the line bundle to be the pullback of H and the linear system generated by the pullbacks of the coordinate functionals $x_0, \dots, x_n$ [19, p.130–133].

This map is not always defined on all of X since it might occur that $\widetilde{s_0}(x) = \cdots = \widetilde{s_n}(x) = 0$ for some x. If no such x exists, we call the linear system *base-point free* and the line bundle is *globally generated*. Notice that there is a natural map from the space of all sections $H^0(X, L) \times X \mapsto L$ via $s \times x \mapsto s(x)$ and this map is surjective if and only if the linear system is base-point free. Therefore, maps from X to $\mathbf{P}^n$ are in one-to-one correspondence with surjective maps $\mathbf{C}^{n+1} \times X \to L$ where L is a line bundle [19, p.131].

1.4.6 Maps to Grassmannians

More generally, maps $X \to Gr(k,n)$ are in one-to-one correspondence with rank k vector bundles $\mathcal{E}$ on X with an n-dimensional subspace $W \subset H^0(X,\mathcal{E})$ such that the sections of W generate $\mathcal{E}$ at every $x \in X$. This is because on $Gr(k,n)$, we have the inclusion $S \hookrightarrow \mathbf{C}^n \times Gr(k,n)$ and taking the dual, we have $\mathbf{C}^n \times Gr(k,n) \to S^*$. By taking the pullback along the map $f\colon X \to Gr(k,n)$, we get $\mathbf{C}^{n*} \times Gr(k,n) \to f^*S^*$ and hence maps from $X \to Gr(k,n)$ are in one-to-one correspondence with surjective maps $\mathbf{C}^n \times X \to \mathcal{E}$ where $\mathcal{E}$ is a rank k vector bundle.

1.5 Chern Classes

If X is a projective variety and L is a complex line bundle on X, then the *first Chern class* is the unique element $c_1(L) \in H^2(X,\mathbf{Z})$ satisfying the following properties:

- $c_1(f^*L) = f^*c_1(L)$ for continuous $f\colon Y \to X$
- $c_1(L \otimes M) = c_1(L) + c_1(M)$ for complex line bundles L, M on X
- if h is the *hyperplane class*, i.e. the generator of $H^2(\mathbf{P}^n)$, then $c_1(\mathcal{O}_{\mathbf{P}^n}(-1)) = -h$.

The first Chern class of the trivial line bundle is 0. Since the $L \otimes L^*$ is the trivial bundle, additivity implies that $c_1(L^*) = -c_1(L)$.

If $\mathcal{E}$ is a complex vector bundle of rank k on X, then the *total Chern class* is $c(\mathcal{E}) \in H^*(X)$ such that

- $c(f^*\mathcal{E}) = f^*c(\mathcal{E})$ for $f\colon Y \to X$ continuous
- $c(\mathcal{E} \oplus \mathcal{F}) = c(\mathcal{E}) \cup c(\mathcal{F})$ for complex vector bundles $\mathcal{E}, \mathcal{F}$
- $c(L) = 1 + c_1(L)$ for any complex line bundle L.

The $2n$th component of $c(\mathcal{E})$ in $\mathrm{H}^{2n}(X, \mathbf{Z})$ is defined to be the *nth Chern class* $c_n(\mathcal{E})$. If E has a complete flag $0 = F_0 \subset F_1 \subset \cdots \subset F_m = \mathcal{E}$, set $L_i = F_i/F_{i-1}$. Then the *Chern roots* of $c_n(\mathcal{E})$ are $x_i := c_1(L_i)$, and by additivity, $c_n(\mathcal{E})$ is the nth elementary function $e_n(x_1, \ldots, x_m)$

Lemma 1.5.1. *If* $0 \to \mathcal{E} \to \mathcal{E}' \to \mathcal{E}'' \to 0$, *then* $c(\mathcal{E}') = c(\mathcal{E})c(\mathcal{E}'')$.

Lemma 1.5.2 (Gysin morphisms, projection formula, [17, p.156])**.** *If* $f\colon Y \to X$ *is a continuous map between compact, connected, complex varieties,* $\dim Y = m$, *and* $\dim X - n$, *then the Gysin morphism* f_* *is defined by the composition*

$$f_*\colon H^q(Y) \cong H_{m-q}(Y) \xrightarrow{f_*} H_{m-q}(X) \cong H^{q-(m-n)}(X)$$

where the isomorphisms are given by Poincaré duality and f_* *is the induced map on cohomology, and the projection formula states that*

$$f_*(f^*\alpha \cup \beta) = \alpha \cup f_*(\beta).$$

Theorem 1.5.3. *If* Q *is the tautological quotient bundle on a Grassmannian and* σ_k *is the special Schubert variety of codimension* k, *then* $c_k(Q) = \sigma_k$.

Proof. Notice we can write $c_k(Q) = \sum_{|\lambda|=k} a_\lambda \sigma_\lambda$ since the σ_λ form a basis for $H^{2k}(X, \mathbf{Z})$. Since $\sigma_\lambda \smile \sigma_\mu = \delta_{\mu,\hat{\lambda}}$ for any partitions λ and μ which are both contained in an $m \times n$ rectangle such $|\lambda| + |\mu| = mn$, it suffices to show $c_k(Q) \smile \sigma_\lambda$ is 1 if $\lambda = \lambda(1, k)$, where $\lambda(1, k)$ is the $m \times n$ rectangle with the $1 \times k$ rectangle removed, and 0 otherwise.

Let $\lambda \neq \lambda(1, k)$ be a partition fitting in a $m \times n$ rectangle such that $|\lambda| = mn - k$. Then $\lambda_m \geq n - k + 1$, so $n + m - \lambda_m \leq m + k - 1$. This implies that for any flag in $\mathbf{C}^{m+n}$, if $\Lambda \in \sigma_\lambda(V) = \{\Lambda \in Gr(m, m+n)\colon \dim(\Lambda \cap V_{n+i-\lambda_i}) \geq i\}$, then $\dim(\Lambda \cap V_{n+m-\lambda_m}) \geq m$, so $\Lambda \in V_{m+k-1}$. Therefore, $\sigma_\lambda(V) \subset Gr(m, m+k-1)$.

If we now work in $Gr(m, m+k-1)$, notice that for any $\Lambda \in \sigma_\lambda$, we have the exact sequence $0 \to V_{m+k-1}/\Lambda \to \mathbf{C}^{m+n}/\Lambda \to \mathbf{C}^{m+n}/V_{m+k-1} \to 0$. These give us fiber bundles, where the V_{m+k-1}/Λ's give a rank $k-1$ fiber bundle and $\mathbf{C}^{m+n}/V_{m+k-1}$ is constant and hence gives the trivial bundle on $Gr(m, m+k-1)$. Let Q be the universal quotient bundle on $Gr(m, m+n)$ and let Q' be Q restricted to $Gr(m, m+k-1)$. Then $Q' = \{\mathbf{C}^{m+n}/\Lambda \colon \Lambda \in Gr(m, m+k-1)\}$. Since $\mathbf{C}^{m+n}/V_{m+k-1}$ is trivial, its Chern roots are all 0, which implies that $c_k(Q') = 0$ because at most $k-1$ factors in each degree k monomial is non-zero. Let f be the inclusion map of $Gr(m, m+k-1)$ into $Gr(m, m+n)$. By the projection formula, $\alpha \smile f_*(\beta) = f_*(f^*\alpha \smile \beta)$, so $c_k(Q) \smile \sigma_\alpha = f_*(f^*c_k(Q) \smile [\sigma_\lambda]) = f_*(c_k(Q') \smile [\sigma_\lambda]) = f_*(0 \smile [\sigma_\lambda]) = 0$.

Now let $\lambda = \lambda(1, k)$. Then there is an isomorphism $g \colon \sigma_\lambda \to \mathbf{P}(V_{m+k}/V_{m-1})$. Let Q'' be the universal quotient bundle on $\mathbf{P}(V_{m+k}/V_{m-1})$. For any $\Lambda \in \sigma_\lambda$, there is an exact sequence $0 \to \Lambda/V_{m-1} \to V_{m+k}/V_{m-1} \to Q'' \to 0$. Then again by the projection formula, $c_k(Q) \smile \sigma_\lambda = g_*(g^*c_k(Q) \smile [\sigma_\lambda]) = g_*(c_k(Q'')) = 1$ [17, p.108,121–124]. $\square$

1.5.1 Splitting Principle

If $\mathcal{E} \to X$ is a complex vector bundle of rank n, then there exists space Y and a continuous map $f \colon Y \to X$ such that

- the pullback bundle $f^*\mathcal{E}$ on Y has a filtration, that is, there is a full flag $0 = F_0 \subset F_1 \subset \cdots \subset F_n = f^*\mathcal{E}$
- $f^* \colon H^*(X) \to H^*(Y)$ is injective [17, p.122].

Let $L_i = F_i/F_{i-1}$ be a line bundle on X using the notation above. Then $c_1(L_i) \in H^2(Y, \mathbf{Z})$. Since $0 \to F_{i-1} \to F_i \to L_i \to 0$, then by Lemma 1.5.1, $c(F_i) = c(F_{i-1})c(L_{i-1})$ for $i = 1, \ldots, n$ and hence $c(f^*\mathcal{E}) = \prod_{i=1}^n c(L_i)$. If we denote $c_1(L_i)$

by x_i, then $c_i(f^*\mathcal{E}) = e_i(x_1, \ldots, x_n)$ where e_i is the elementary symmetric function defined below. Furthermore, $c_i(f^*\mathcal{E}) = f^*(c_i(\mathcal{E}))$, so the elementary functions are in the image of f^*.

By the Splitting Principle, since f^* is injective, we don't lose information when making calculations in $H^*(Y)$. This is convenient because we can now work with Chern roots in our calculations.

1.6 Symmetric Functions

A *symmetric polynomial* is of the form $f(x_1, \ldots, x_n)$ in n variables such that

$$f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

for any $\sigma \in S_n$.

A *monomial symmetric function* is of the form

$$m_{(a_1,\ldots,a_n)}(x_1, \ldots, x_n) = \sum_{\lambda} x^{\lambda}$$

where the sum ranges over all distinct permutations λ of $(a_1, \ldots, a_n)$.

Any monomial symmetric function where the nonzero parts are one, i.e. is of the form

$$m_{(1,\ldots,1,0,\ldots,0)}(x_1, \ldots, x_n) = e_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k},$$

is called a *basic elementary symmetric function*. The general elementary symmetric function is defined by $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_i}$ for each partition λ.

A monomial symmetric function where the only nonzero part is k is

$$m_{(k,0,\ldots,0)}(x_1, \ldots, x_n) = p_k(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^k$$

called a *basic power-sum symmetric polynomial*. The general power-sum symmetric function is defined by $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_i}$ for each partition λ.

Taking the sum of all monomials in n variables of degree $k \geq 1$ gives another type of symmetric polynomial,

$$h_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 \leq \cdots \leq i_n \leq n} x_{i_1} \cdots x_{i_n},$$

called a *basic complete homogeneous symmetric function*. The general complete homogeneous symmetric function is defined by $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_i}$ for each partition λ.

A *Schur function* is of the form

$$s_{(a_1,\ldots,a_n)}(x_1, \ldots, x_n) = \frac{\det\,(x_i^{a_j+n-j})}{\det\,(x_i^{n-j})}$$

where $(a_1, \ldots, a_n)$ is a partition. The denominator is equivalent to the *Vandermonde determinant*, $\prod_{1\leq i<j\leq n}(x_i - x_j)$ [20, p.334].

The symmetric polynomials $\Lambda_n = \oplus_{k=0}^{n} \Lambda_n^k$ in n variables, where Λ_n^k is the set of homogeneous symmetric polynomials in n variables of degree k, form a ring over $\mathbf{Z}$. The monomial symmetric functions, the basic elementary symmetric functions, the basic complete homogeneous symmetric functions, and the Schur functions each forms a basis for Λ_n (but the power-sum symmetric functions only form a basis for Λ_n after tensoring with $\mathbf{Q}$) [17, p.7–10].

For $m \geq n$, define the map $\rho^k_{m,n} \colon \Lambda^k_m \to \Lambda^k_n$ such that $x_i \mapsto 0$ if $i > n$ and $x_i \mapsto x_i$ otherwise. Take Λ^k to be the inverse limit $\varprojlim_n \Lambda^k_n$, that is, the set $\{(f_i) \colon f_i \in \Lambda^k_i, \rho^k_{i,i-1}(f_i) = f_{i-1}\}$. The ring of *symmetric functions* is $\Lambda = \oplus_{k \geq 0} \Lambda^k$ [15, p.10–12].

Theorem 1.6.1. *The integral cohomology of $Gr(k, n)$ is isomorphic to a quotient of symmetric functions via the surjective map $\Lambda \to Gr(k, n)$ where $s_\lambda \mapsto \sigma_\lambda \in H^*(Gr(k, n), \mathbf{Z})$ if λ fits in the $k \times n - k$ rectangle, and $s_\lambda \mapsto 0$ otherwise [4, p.152].*

1.6.1 Pieri's Formula

If $a = a, 0, 0, \ldots$, then for any λ,

$$(\sigma_a \cdot \sigma_\lambda) = \sum_{\lambda_i \leq c_i \leq \lambda_{i-1}, \sum_i c_i = a + \sum \lambda_i} \sigma_c$$

[7, p.203]

Example 1.6.2. In $Gr(3, 8)$, we can compute $\sigma_3 \cdot \sigma_{4,3,1}$ using Pieri's forumula. Since $4 \leq c_1 \leq 5, 3 \leq c_2 \leq 4, 1 \leq c_3 \leq 3$ and $\sum_i c_i = 11$, then $\sigma_3 \cdot \sigma_{4,3,1} = \sigma_{4,4,3} + \sigma_{5,3,3} + \sigma_{5,4,2}$. $\square$

1.6.2 Giambelli's Formula

$$\sigma_{a_1,\ldots,a_d} = \begin{vmatrix} \sigma_{a_1} & \sigma_{a_1+1} & \sigma_{a_1+2} & \cdots & \sigma_{a_1+d-1} \\ \sigma_{a_2-1} & \sigma_{a_2} & \sigma_{a_2+1} & \cdots & \sigma_{a_2+d-2} \\ \vdots & & & & \vdots \\ \sigma_{a_d-d+1} & \sigma_{a_d-d+2} & \sigma_{a_d-d+3} & \cdots & \sigma_{a_d} \end{vmatrix}$$

[7, p.205]

Example 1.6.3. To multiply $\sigma_{2,1} \cdot \sigma_{2,1}$ in $Gr(3,8)$, we use Giambelli's formula to express $\sigma_{2,1}$ in terms of special Schubert cycles, then apply Pieri's rule. By Giambelli's formula

$$\sigma_{2,1} = \begin{vmatrix} \sigma_2 & \sigma_3 & \sigma_4 \\ \sigma_0 & \sigma_1 & \sigma_2 \\ 0 & 0 & \sigma_0 \end{vmatrix} = \sigma_0 \begin{vmatrix} \sigma_2 & \sigma_3 \\ \sigma_0 & \sigma_1 \end{vmatrix} = \begin{vmatrix} \sigma_2 & \sigma_3 \\ \sigma_0 & \sigma_1 \end{vmatrix} = \sigma_2\sigma_1 - \sigma_3\sigma_0$$

Then by Pieri's formula,

$$\begin{aligned} \sigma_{2,1} \cdot \sigma_{2,1} &= (\sigma_2\sigma_1 - \sigma_3\sigma_0)\sigma_{2,1} \\ &= \sigma_2\sigma_1\sigma_{2,1} - \sigma_3\sigma_0\sigma_{2,1} \\ &= \sigma_2(\sigma_{3,1} + \sigma_{2,2} + \sigma_{2,1,1}) - \sigma_3\sigma_{2,1} \\ &= \sigma_{3,3} + \sigma_{4,2} + 2\sigma_{3,2,1} + \sigma_{2,2,2} + \sigma_{4,1,1} \end{aligned}$$

□

Example 1.6.4. Lines intersecting 4 generic lines in $\mathbf{P}^3$

Let L_1, L_2, L_3, and L_4 be 4 generic lines in $\mathbf{P}^3$. Then the set of lines intersecting L_i is $\sigma_1(L_i)$. Since the lines are generic, the cohomology class of the set of lines intersecting every L_i is

$$\sigma_1^4 = \sigma_1^2(\sigma_1 \cdot \sigma_1) = \sigma_1^2(\sigma_2 + \sigma_{1,1}) = \sigma_1(\sigma_1\sigma_2 + \sigma_1\sigma_{1,1}) = \sigma_1(2\sigma_{2,1}) = 2\sigma_{2,2}$$

Since $\int 2\sigma_{2,2} = 2$, then there are 2 lines intersecting L_1, L_2, L_3, and L_4 [7, p.206]. $\square$

1.7 Degeneracy Formulas for Maps of Vector Bundles

1.7.1 Degeneracy Locus

Let $\varphi\colon \mathcal{E} \to \mathcal{F}$ be a morphism of vector bundles $\mathcal{E}, \mathcal{F}$ on X with ranks e, f, respectively. The *degeneracy locus* of φ is

$$D_r(\varphi) = \{x \in X \colon \varphi(x)\colon \mathcal{E}(x) \to \mathcal{F}(x) \text{ has rank } \leq r\}$$

for $r \leq f$. This is a closed subvariety of X.

Example 1.7.1. Let $\mathbf{C}^m \hookrightarrow \mathbf{C}^n \to Q$ and $\varphi = f \circ g$ be a morphism of these vector bundles on $X = Gr(k, n)$. Since the kernel of φ is $\mathbf{C}^m \cap S$, we have

$$\begin{aligned} D_0(\varphi) &= \{x \in X \colon \mathbf{C}^m(x) \subset S(x)\} \\ &= \{\Lambda \in Gr(k,n)\colon \dim(\Lambda \cap V^m) \geq n\} \\ D_1(\varphi) &= \{x \in X \colon \varphi(x)\colon \mathbf{C}^m(x) \to Q(x) \text{ has rank } \leq 1\} \\ &= \{\Lambda \in Gr(k,n)\colon \dim(\Lambda \cap V^m) \geq n-1\} \end{aligned}$$

and in general,

$$D_r(\varphi) = \{\Lambda \in Gr(k,n)\colon \dim(\Lambda \cap V^m) \geq n-r\}$$

So the degeneracy loci of φ give us Schubert varieties of $Gr(k, n)$. $\square$

Let $f\colon X \to Y$ be C^∞ and Z be a subvariety of Y. We say f is *transverse* to Z if $\operatorname{codim}(df(T_xX) \cap T_{f(x)}Z) = \operatorname{codim} df(T_xX) + \operatorname{codim} T_{f(x)}Z$. If E is a complex vector bundle on X of rank n and s is a family $s_1, \ldots, s_n$ of global sections of E, then the *degeneracy locus* is the set $D_{i,j}(s) = \{x \in X\colon \dim\langle s_1(x), \ldots, s_i(x)\rangle \leq i - j\}$ for $0 \leq i - j < n$ [17, p.125–127].

1.7.2 Gauss-Bonnet Formula

If s is a global section of $\mathcal{F}$, transverse to the zero section, and if $X_s = s^{-1}(0)$, then $c_f(\mathcal{F}) = [X_s]$ where $f = \operatorname{rank}\mathcal{F}$ [17, p.126].

Example 1.7.2. Lines on a generic cubic surface in $\mathbf{P}^3$

Let $F(x_1, x_2, x_3, x_4) = 0$ be a cubic surface in $\mathbf{P}^3$ and let $V = \mathbf{C}^4$. Since F is a homogeneous degree 3 polynomial, $F \in S^3(V^*)$. Similarly, for any subspace $W \subset V$, $F|_W \in S^3(W^*)$. Since $S^3(R^*) = \{(w, f)\colon w \in Gr(2, V), f \in S^3(W^*)\}$ is a vector bundle on $Gr(2, V)$ where R is the tautological subbundle on $Gr(2, V)$ and $(w, f) \mapsto w$, the map from $Gr(2, V)$ to $S^3(R^*)$ where $W \mapsto (W, F|_W)$ is a global section of $S^3(R^*)$. Since the dimension of the space of degree 3 homogeneous polynomials in 2 variables is 4, $\dim S^3(W^*) = 4$ and hence the rank of $S^3(R^*)$ is 4.

Let $g\colon Gr(2, V) \to S^3(R^*)$ where $W \mapsto F|_W$. Then $X_g = \{W \in Gr(2, V)\colon F|_W \equiv 0\}$, the set of lines on the surface defined by $F = 0$. For a generic choice of F, $\dim X_g = \dim Gr(2, V) - \operatorname{rank} S^3(R^*) = 0$. (All smooth surfaces are generic in this sense.) By the Gauss-Bonnet formula, $[X_g] = c_4(S^3(R^*))$. By 1.5.1, we can define $(L_i^*)^j = L_i^{-j}$

where the L_i's are line bundles. Then

$$\begin{aligned} c(S^3(R^*)) &= c(L_1^{-3} + L_1^{-2}L_2^{-1} + L_1^{-1}L_2^{-2} + L_2^{-3}) \\ &= (1 - 3c_1(L_1))(1 + (-2c_1(L_1) - c_1(L_2))) \\ &\quad \cdot (1 + (-c_1(L_1) - 2c_1(L_2)))(1 - 3c_1(L_2)) \end{aligned}$$

Denote the Chern roots of R^* by $c_1(L_i) = [L_i]$. Then the top Chern class is

$$\begin{aligned} c_4(S^3(R^*)) &= -3[L_1](-2[L_1] - [L_2])(-[L_1] - 2[L_2])(-3[L_2]) \\ &= 18[L_1]^3[L_2] + 45[L_1]^2[L_2]^2 + 18[L_1][L_2]^3 \\ &= 18([L_1]^3[L_2] + [L_1][L_2]^3) + 45[L_1]^2[L_2]^2 \\ &= 18[L_1][L_2](([L_1] + [L_2])^2 - 2[L_1][L_2]) + 45([L_1][L_2])^2 \\ &= 18e_2([L_1],[L_2])(e_1^2([L_1],[L_2]) - 2e_2([L_1],[L_2])) + 45e_2^2([L_1],[L_2]) \\ &= 18c_2(R^*)(c_1^2(R^*) - 2c_2(R^*)) + 45c_2^2(R^*) \end{aligned}$$

Since $Gr(2,V) \cong Gr(2,V^*)$, R^* on $Gr(2,V)$ is isomorphic to Q on $Gr(2,V^*)$. The kth Chern class of Q is σ_k, so the kth Chern class of R^* is $\sigma_{1,\ldots,1}$ since the induced map on cohomology from the isomorphism $R^* \cong Q$ transposes the partition, i.e. $\sigma_\lambda \mapsto \sigma_{\lambda^T}$. Therefore,

$$c_4(S^3(R^*)) = 18\sigma_{1,1}(\sigma_1^2 - 2\sigma_{1,1}) + 45\sigma_{1,1}^2 \quad = 18\sigma_{2,2} - 36\sigma_{2,2} + 45\sigma_{2,2} = 27\sigma_{2,2}$$

Since $\int 27\sigma_{2,2} = 27$, there are 27 lines on the surface. □

Example 1.7.3. Lines on a generic quintic surface in $\mathbf{P}^4$

Let $F(x_1, x_2, x_3, x_4, x_5) = 0$ be a quintic surface in $\mathbf{P}^4$ and let $V = \mathbf{C}^5$. Let $g\colon Gr(2,V) \to S^5(R^*)$ where $W \mapsto F|_W$. As in the previous example, if R is the tautological subbundle on $Gr(2,V)$, then the rank of $S^5(R^*)$ is 6, so for generic choice of F, $\dim X_g =$

$\dim Gr(2,V) - \operatorname{rank} S^5(R^*) = 0$. By the Gauss-Bonnet formula $[X_g] = c_6(S^5(R^*))$, so using similar notation as above, we have

$$\begin{aligned}
c(S^5(R^*)) &= c(L_1^{-5} + L_1^{-4}L_2^{-1} + L_1^{-3}L_2^{-2} + L_1^{-2}L_2^{-3} + L_1^{-1}L_2^{-4} + L_2^{-5}) \\
&= (1 - 5c_1(L_1))(1 + (-4c_1(L_1) - c_1(L_2)))(1 + (-3c_1(L_1) - 2c_1(L_2))) \\
&\cdot (1 + (-2c_1(L_1) - 3c_1(L_2)))(1 + (-c_1(L_1) - 4c_1(L_2)))(1 - 5c_1(L_2))
\end{aligned}$$

$$\begin{aligned}
c_6(S^5(R^*)) &= 5[L_1](4[L_1] + [L_2])(3[L_1] + 2[L_2])(2[L_1] + 3[L_2])([L_1] + 4[L_2])5[L_2] \\
&= 25e_2([L_1],[L_2])(e_1([L_1],[L_2]) + 3[L_1])(2e_1([L_1],[L_2]) + [L_1]) \\
&\cdot (2e_1([L_1],[L_2]),[L_2])(e_1([L_1],[L_2]),3[L_2]) \\
&= 25e_2([L_1],[L_2])(24e_1^4([L_1],[L_2]) + 58e_1^2([L_1],[L_2])e_2([L_1],[L_2]) \\
&\quad + 9e_2^2([L_1],[L_2])) \\
&= 25c_2(R^*)(24c_1^4(R^*) + 58c_1^2(R^*)c_2(R^*) + 9c_2^2(R^*)) \\
&= 25\sigma_{1,1}(24\sigma_1^4 + 58\sigma_1^2\sigma_{1,1} + 9\sigma_{1,1}^2) \\
&= 25(48\sigma_{3,3} + 58\sigma_{3,3} + 9\sigma_{3,3}) \\
&= 2875\sigma_{3,3}
\end{aligned}$$

Since $\int 2875\sigma_{3,3} = 2875$, there are 2875 lines on the surface. □

1.7.3 Thom-Porteous Formula

If $\varphi\colon \mathcal{E} \to \mathcal{F}$ is a morphism from a rank e vector bundle $\mathcal{E}$ to a rank f bundle $\mathcal{F}$, then

i) If $D_k(\varphi) \neq \varnothing$, then any irreducible component has codimension $\leq (f-k)\times(e-k)$.

ii) If equality holds for all components, then we have

$$[D_k(\varphi)] = s_{(f-k)\times(e-k)}(\mathcal{F}-\mathcal{E}) = \det\left(c_{f-k-i+j}(\mathcal{F}-\mathcal{E})\right)_{1\leq i,j\leq e-k}.$$

Here, we define $c(\mathcal{F}-\mathcal{E}) = c(\mathcal{F})/c(\mathcal{E})$ [17, p.127].

Chapter 2

Hilbert Schemes and Quot Schemes

2.1 Hilbert Polynomials

Let M be a finitely generated graded $S = K[x_0, \ldots, x_n]$-module where K is a field. Then the *Hilbert function* H_M of M is

$$H_M(i) = \dim_k M_i$$

for $i \in \mathbf{Z}$. For example, $H_S(i) = \binom{i+n}{n}$ for $i \geq 0$. There is a unique polynomial $P_M(t) \in \mathbf{Q}[t]$, called the *Hilbert polynomial* of M, such that $P_M(t) = H_M(t)$ for $t \gg 0$. Let $\mathfrak{m} = \langle x_0, \ldots, x_n \rangle$ and let I be a homogeneous ideal of S. The *saturation* of I is

$$(I : \mathfrak{m}^\infty) = \langle f \in S : f\mathfrak{m}^i \subset I \text{ for some } i > 0 \rangle.$$

I is *saturated* if $I = (I : \mathfrak{m}^\infty)$. [16, p.1–2]

2.2 Gotzmann's regularity theorem

Let I be a homogeneous ideal of S and

$$0 \to \mathbf{F}_l \to \cdots \to \mathbf{F}_1 \to S \to S/I \to 0$$

be the minimal free resolution of S/I where $\mathbf{F}_i = \oplus_j S(-j)^{\oplus \beta_{i,j}}$ and $S(-i)$ denotes the grading shift $S(-i)_j = S_{j-i}$. If $k \geq \max_{i,j}\{j-i \colon \beta_{i,j} \neq 0\}$, then S/I is k**-regular**. If S/I is k-regular, then I is generated in degree $\leq k+1$ and $H_{S/I}(k) = P_{S/I}(k)$.

Theorem 2.2.1 (Gotzmann's regularity theorem)**.** *Let P be a Hilbert polynomial, so there exist $a_1 \geq \cdots \geq a_D \geq 0$ such that*

$$P(t) = \sum_{j=1}^{D} \binom{t + a_i - i + 1}{a_i}.$$

If I is a saturated ideal such that $P_{S/I}(t) = P(t)$, then S/I is $(D-1)$-regular. We call D the Gotzmann number of P.

For $n, d \in \mathbf{N}$, there exist unique $k_j > k_{j+1} \geq 0$ such that

$$n = \sum_{j=0}^{t} \binom{k_j}{d-j}.$$

The *Macaulay upper boundary* of n with respect to d is

$$n^{\langle d \rangle} = \sum_{j=0}^{t} \binom{k_j + 1}{d - j + 1}.$$

Theorem 2.2.2 (Macaulay)**.** *Let I be a homogeneous ideal of S and $k \in \mathbf{N}$ such that all minimal generators of I have degree $< k$. Then $H_{S/I}(k+1) \leq H_{S/I}(k)^{\langle k \rangle}$.*

Theorem 2.2.3 (Gotzmann's persistence theorem)**.** *Let I be a homogeneous ideal of*

S and $k \in \mathbf{N}$ such that all minimal generators of I have degree $< k$. If $H_{S/I}(k+1) = H_{S/I}(k)^{\langle k \rangle}$, then $H_{S/I}(t+1) = H_{S/I}(t)^{\langle t \rangle}$ for $t \geq k$.

In particular, if D is the Gotzmann number of P, then

$$P(t+1) = P(t)^{\langle t \rangle} \text{ for all } t \geq D.$$

Corollary 2.2.4. *If I is a homogeneous ideal of S generated in degrees $\leq D$ such that $H_{S/I}(D) = P(D)$ and $H_{S/I}(D+1) = P(D+1)$, then $P_{S/I} = P$.*

2.3 Hilbert schemes

The *Hilbert scheme* $Hilb_P(\mathbf{P}^n)$ is a space that parameterizes subschemes of $\mathbf{P}^n$ with Hilbert polynomial P. There is a 1-1 correspondence between $Hilb_P(\mathbf{P}^n)$ and $X/\sim$ where

$$X = \{I \colon I \text{ is a homogeneous ideal of } S \text{ and } P_{S/I} = P\}$$

$$I \sim J \text{ if } (I \colon \mathfrak{m}^\infty) = (J \colon \mathfrak{m}^\infty).$$

Let P be a Hilbert polynomial with Gotzmann number D and let $\mathcal{G}_D = Gr(H_S(D) - P(D), S_D)$. There is a 1-1 correspondence between $X/\sim$ and the following set

$$\{L \in \mathcal{G}_D \colon \dim(S_1 L) \leq H_S(D+1) - P(D+1)\}$$

given by the maps

$$L \mapsto \langle L \rangle,\ I \mapsto I_D.$$

Therefore, $Hilb_P(\mathbf{P}^n)$ is a subscheme of a Grassmannian with a certain dimension condition.

Let $\mathcal{E}$ be the tensor product bundle of the trivial bundle $S_1 \times \mathcal{G}_D$ with the tautological subbundle of $\mathcal{G}_D$, $\mathcal{F}$ be the trivial bundle $S_{D+1} \times \mathcal{G}_D$, and $\varphi\colon \mathcal{E} \to \mathcal{F}$ be the morphism defined by multiplication. Then

$$Hilb_P(\mathbf{P}^n) = D_{H_S(D+1)-P(D+1)}(\varphi).$$

Let j be the embedding of $Hilb_P(\mathbf{P}^n)$ into $Gr(k,n)$. If $Hilb_P(\mathbf{P}^n)$ has the appropriate codimension and $\alpha_i = j^*(\beta_i)$ for cohomology classes α_i, β_i of $Hilb_P(\mathbf{P}^n)$, $Gr(k,n)$, respectively, then

$$\int_{Hilb(\mathbf{P}^n)} \alpha_1 \ldots \alpha_k = \int_{Gr(k,n)} \beta_1 \ldots \beta_k \cdot [D_k(\varphi)]$$

where $[D_k(\varphi)]$ is evaluated using the Thom-Porteous formula. Even if $Hilb_P(\mathbf{P}^n)$ does not have the required codimension, the result from applying the Thom-Porteous formula is still useful in applications.

2.4 Quot Schemes

Let $S = k[x_0, \ldots, x_n]$ where k is a field and $\mathcal{O} = \mathcal{O}_{\mathbf{P}^1}$. Let $Quot_P(\mathcal{O}^r)$ be the space of quotients of $\mathcal{O}^r$ with Hilbert polynomial P. Let $0 \to N \to S^{\oplus N} \to M \to 0$. The construction of the quotient scheme is similar to that of the Hilbert scheme. By [1, p.4–9], quotients with Hilbert polynomial P have a similar type of Gotzmann number D by regularity. A coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n$ is d-regular if for all $i > 0$, $H^i(\mathbf{P}^n, \mathcal{F}(d-i)) = 0$.

Theorem 2.4.1 (Dellaca)**.** *If F is a rank r free S-module with module generators having degree at most 0, N is a graded submodule of F, and $M = F/N$, with Gotzmann representation*

$$P_M(t) = \sum_{i=1}^{D} \binom{t + a_i - i + 1}{a_i}$$

then the associated sheaf $\widetilde{N}$ is D-regular.

By a similar persistence theorem for this sheaf, the dimensions stabilize for large enough values.

Corollary 2.4.2 (Dellaca)**.** *If $\mathcal{F}$ is a coherent sheaf on $\mathbf{P}^n$ and $a \in \mathbf{Z}^{\geq 0}$ such that $\mathcal{F}(a)$ is generated by global sections and $\mathcal{F}(a)$ has Gotzmann regularity D, then $\mathcal{F}$ is $D + a$-regular.*

So we can also realize $Quot_P(\mathcal{O}^r)$ as a subscheme of a Grassmannian with certain dimension conditions.

Theorem 2.4.3 (Dellaca)**.**

$$Quot_P(\mathcal{O}^r) \cong \{F \in Gr(S_D^r, P(D)) \colon \operatorname{codim} F \cdot S_1 = P(D+1)\}$$

Chapter 3

Cohomology Classes of $Quot_P(\mathbf{P}^1)$ and $Hilb^2(\mathbf{P}^2)$

3.1 Quot Scheme of $\mathbf{P}^1$

3.1.1 Thom-Porteous Formula for $Quot_P(\mathbf{P}^1)$

By the work of Dellaca, if $Quot_P(\mathcal{O}^r)$ has Hilbert polynomial $P(t) = k(t+1)+m$, then the Thom-Porteous formula gives $Quot_P(\mathcal{O}^r)$ an expected dimension of $(r-k)k+rm$, where the degeneracy locus is of $Gr((r-k)(\frac{r(r+1)}{2}+m+1)-m, r(\frac{r(r+1)}{2}+m+1))$. To compute the actual dimension of $Quot_P(\mathcal{O}^r)$, a subsheaf of $\mathcal{O}^r$ is split into the sum of standard line bundles, $\oplus\mathcal{O}(-t_i)$ where $\sum t_i = m$. The dimension of $Hom(\oplus\mathcal{O}(-t_i), \mathcal{O}^r)$ is $r(r-k)-rm$. However, the identification of $Quot_P(\mathcal{O}^r)$ with a subset of $Hom(\oplus\mathcal{O}(-t_i), \mathcal{O}^r)$ is not unique, so one needs to mod out by $Aut(\oplus\mathcal{O}(-t_i))$. The dimension of $Aut(\oplus\mathcal{O}(-t_i))$ is bounded below by $(r-k)^2$ and hence the dimension of $Quot_P(\mathcal{O}^r)$ is bounded above by $r(r-k)-rm-(r-k)^2 =$

$(r-k)k+rm$. In this case, the expected dimension agrees with the actual dimension. [1, p.9–11]

3.1.2 Computing the class $Quot_P(\mathbf{P}^1)$

We first recall two facts:

Theorem 3.1.1. *[15, p.72] If $s_\lambda(x,y)$ is a Schur function corresponding to a partition λ in the variables $x_1, x_2, \ldots, y_1, y_2 \ldots$, then*

$$s_\lambda(x,y) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(y) s_\nu(x).$$

Theorem 3.1.2 (Schubert's Duality Theorem). *[4, p.149] For Schubert cycles $\sigma_\lambda, \sigma_\mu$ satisfying $|\lambda| + |\mu| = rn$,*

$$\sigma_\lambda \sigma_\mu = \begin{cases} 1 & \text{if } \lambda_i + \mu_{r+1-i} = n \text{ for all } 1 \le i \le r \\ 0 & \text{if } \lambda_i + \mu_{r+1-i} > n \text{ for any } i \end{cases}$$

Consider the Quot scheme $Quot_P(\mathcal{O}^r{}_{\mathbf{P}^1})$ with Hilbert polynomial $P(t) = s(t+1)+k$. Let $0 \to \mathcal{K} \to \mathcal{O}^r \to \mathcal{F} \to 0$ and twist by large enough n, so $\dim H^0(\mathcal{O}^r_{\mathbf{P}^1}(n)) = r(n+1)$, $\dim H^0(\mathcal{F}(n)) = s(n+1)+k$, and $\dim H^0(\mathcal{K}(n)) = (r-s)(n+1)-k$. In $Gr((r-s)(n+1)-k, r(n+1))$, let φ be the map between universal and trivial vector bundles $\mathcal{R} \otimes S_1 \to \mathcal{O} \otimes S^r_{n+1}$ with ranks $e = 2(r-s)(n+1)-2k$ and $f = r(n+2)$, respectively. Given the condition $\operatorname{rank}\varphi \le (r-s)(n+2)-k$ (and call this rank l), the Quot scheme is a degeneracy locus in $Gr((r-s)(n+1)-k, r(n+1))$ by Dellaca's theorem [1, p.9–11]. If Q is the quotient bundle of the Grassmannian, then

$c(\mathcal{R}) = c(Q)^{-1}$ (by 1.5.1), so by the Thom-Porteous formula,

$$
\begin{aligned}
[D_l(\varphi)] &= s_{(f-l)\times(e-l)}(\mathcal{F}-\mathcal{E}) \\
&= s_{(f-l)\times(e-l)}(Q \oplus Q) \\
&= \sum_{\mu,\nu} c_{\mu\nu}^{(f-l)\times(e-l)} \sigma_\mu \sigma_\nu && (3.1.3) \\
&= \sum_{\mu,\nu} \sum_{\lambda} c_{\mu\nu}^{(f-l)\times(e-l)} c_{\mu\nu}^{\lambda} \sigma_\lambda \\
&= \sum_{\mu} \sum_{\lambda} c_{\mu\mu^c}^{\lambda} \sigma_\lambda. && (3.1.4)
\end{aligned}
$$

where λ is of size $(f-l)(e-l)$ and fits in a $((r-s)(n+1)-k) \times (s(n+1)+k)$ box, and μ^c denotes the complement of μ in the rectangle $(f-l)\times(e-l)$. Line (3.1.3) and line (3.1.4) are applications of Theorem 3.1.1 and Theorem 3.1.2, respectively. Therefore, we have the following theorem:

Theorem 3.1.5. *Using the notation above, the cohomology class of $Quot_P(\mathcal{O}^r_{\mathbf{P}^1})$ as a degeneracy locus in $Gr((r-s)(n+1)-k, r(n+1))$, where $P(t) = s(t+1)+k$, is given by*

$$\sum_{\mu} \sum_{\lambda} c_{\mu\mu^c}^{\lambda} \sigma_\lambda.$$

Example 3.1.6. To compute $[Quot_P(\mathcal{O}_{\mathbf{P}^1})]$ in $Gr(2,4)$ where $P(t) = 2$, then $r = 1$, $s = 0$, and $k = 2$. Since the Gotzmann number is $\frac{r(r+1)}{2} + k$ by [1, p.10], taking $n = 3$ is large enough. The only partitions that fit in a 2×2 box and are of size $(f-l)(e-l) = 2$ are $\lambda_1 = (2)$ and $\lambda_2 = (1,1)$, so we need only compute $c^{\lambda_i}_{\varnothing(1,1)}$ and $c^{\lambda_i}_{(1)(1)}$. Therefore,

$$[Quot_P(\mathcal{O}_{\mathbf{P}^1})] = \sigma_2 + 3\sigma_{1,1}. \qquad \square$$

3.2 $Hilb^2(\mathbf{P}^2)$

Consider the Veronese embedding $v_n\colon \mathbf{P}(V) \to \mathbf{P}(Sym^n V)$ where $\dim V = 3$, and let $N = \binom{n+2}{2}$. We can define a map f on $Hilb^2(\mathbf{P}^2)$ in the following way: for an ideal I corresponding to a length 2 subscheme of $\mathbf{P}^2$, either I vanishes at two distinct points in $\mathbf{P}^2$ or I vanishes at a single point with multiplicity 2. (For example, we could take $I = \langle x_0x_1, x_2\rangle$ in the former case and $I = \langle x_0, x_1^2\rangle$ in the latter.) Then there is an embedding of $Hilb^2(\mathbf{P}^2)$ into $Gr(2, Sym^n V)$ defined as follows:

- If I vanishes at two distinct points x and y, then consider the unique line passing through both points $v_n(x)$ and $v_n(y)$; this is a line in $\mathbf{P}(Sym^n V)$ and hence an element of $Gr(2, Sym^n V)$
- Otherwise, notice $\mathfrak{m}_x^2 \subset I \subset \mathfrak{m}_x$. Since $\dim(\mathcal{O}_x/I_x) = 2$ and $\dim(\mathcal{O}_x/\mathfrak{m}_x) = 1$, $\dim(\mathfrak{m}_x/I_x) = 1$. The dimension of the cotangent space $\mathfrak{m}_x/\mathfrak{m}_x^2$ is 2, and therefore $\dim(I_x/\mathfrak{m}_x^2) = 1$. Thus, the kernel of the quotient map $(\mathfrak{m}_x/\mathfrak{m}_x^2)^* \to (I_x/\mathfrak{m}_x^2)^*$ is a line in $T_x\mathbf{P}^2$. The image of this line under the embedding from $T_x\mathbf{P}^2$ into $T_{v_n(x)}\mathbf{P}(Sym^n V)$ gives a line in $\mathbf{P}(Sym^n V)$.

Lemma 3.2.1. *Let $\mathbf{P}^2 = \mathbf{P}(V)$. Then the map $f\colon Hilb^2(\mathbf{P}^2) \to Gr(2, Sym^n V)$ defined above is an embedding.*

Proof. The map f is injective: first we claim that $v_n(\mathbf{P}^2)$ contains no lines. If not, let $L \in v_n(\mathbf{P}^2)$ and H be a hyperplane in $\mathbf{P}^{N-1}$. Let $C = H \cap v_n(\mathbf{P}^2)$; then $v_n^{-1}(C)$ is a degree n curve in $\mathbf{P}^2$ for some $n > 1$. If the degree of $v_n^{-1}(L)$ is r, then $v_n^{-1}(L) \cap v_n^{-1}(C)$ contains rn points. Then v_n maps rn points to one point since C and L intersect at exactly one point in $v_n(\mathbf{P}^2)$. This is a contradiction since v_n is an embedding.

Thus, if L is a chord of $v_n(\mathbf{P}^2)$, then $L \cap v_n(\mathbf{P}^2)$ and has multiplicity at least 2. Since the ideal $I(v_n(\mathbf{P}^2))$ is generated by quadrics, $L \cap v_n(\mathbf{P}^2)$ is the common zero set of these quadrics restricted to L. Then $L \cap v_n(\mathbf{P}^2)$ has multiplicity at most 2, and hence exactly 2. This implies chords of $v_n(\mathbf{P}^2)$ intersect at a 0-dimensional subscheme of multiplicity 2, so f is injective.

Moreover, by [18, Theorem 18.29], if I is the ideal sheaf associated with a subscheme in $Hilb^2(\mathbf{P}^2)$, then the tangent space of $Hilb^2(\mathbf{P}^2)$ at I is $Hom_{\mathcal{O}_{\mathbf{P}^2}}(I, \mathcal{O}_{\mathbf{P}^2}/I)$. By construction, the tangent space of the image of I in $Gr(2, Sym^n V)$ is

$$Hom_{\mathbf{C}}(H^0((\mathcal{O}_{\mathbf{P}^2}/I)(n))^*, H^0(I(n))^*) \cong Hom_{\mathbf{C}}(H^0(I(n)), H^0((\mathcal{O}_{\mathbf{P}^2}/I)(n))).$$

Since I is generated by polynomials of degree at most 2, we see that the restriction of a nonzero map $I \to \mathcal{O}_{\mathbf{P}^2}/I$ to any degree $n \geq 2$ is also nonzero (some generator of I has nonzero image; multiply it by a suitable power of a linear function to get a nonzero element in the image in degree n). So the map on tangent spaces is also injective. □

Theorem 3.2.2. *Let* $\mathbf{P}^2 = \mathbf{P}(V)$ *and* $N = \binom{n+2}{2}$. *Using the map* f *in the Lemma 3.2.1*

$$[Hilb^2(\mathbf{P}^2)] = (a - 3b + c)\sigma_{\lambda_1} + (b - c)\sigma_{\lambda_2} + c\sigma_{\lambda_3}$$

where σ_{λ_i} *are Schubert classes in* $Gr(Sym^n(V^*), 2) = Gr(2, Sym^n V)$ *defined by*

$$\lambda_1 = (2 \times (N-2)) \setminus (4), \lambda_2 = (2 \times (N-2)) \setminus (3,1), \lambda_3 = (2 \times (N-2)) \setminus (2,2)$$

and

$$a = 3(n^4 - 4n^2 + 4n - 1), \quad b = (n^2-1)^2 - \frac{(n-1)(n-2)}{2}, \quad c = \frac{n^2(n^2-1)}{2}.$$

Proof. $[Hilb^2(\mathbf{P}^2)]$ can be written as a linear combination of Schubert classes:

$$[Hilb^2(\mathbf{P}^2)] = \alpha\sigma_{\lambda_1} + \beta\sigma_{\lambda_2} + \gamma\sigma_{\lambda_3}.$$

By Poincaré duality, we can find these coefficients by computing:

$$[Hilb^2(\mathbf{P}^2)] \cdot \sigma_1^4 = a, \quad [Hilb^2(\mathbf{P}^2)] \cdot \sigma_1^2\sigma_{1,1} = b, \quad [Hilb^2(\mathbf{P}^2)] \cdot \sigma_{1,1}^2 = c.$$

Recall $\sigma_1 = [\{\Lambda\colon \dim(\Lambda \cap V_{N-2}) \geq 1\}]$ and $\sigma_{1,1} = [\{\Lambda\colon \Lambda \subset V_{N-1}\}]$, where V_{N-1}, V_{N-2} are any subspaces of $Sym^n V$ of dimension $N-1, N-2$, respectively. Alternatively

$$\sigma_1 = [\{L\colon L \text{ is a line in } \mathbf{P}^{N-1}, L \cap \mathbf{P}(V_{N-2}) \neq \varnothing\}]$$
$$\sigma_{1,1} = [\{L\colon L \text{ is a line in } \mathbf{P}^{N-1}, L \in \mathbf{P}(V_{N-1})\}].$$

To compute a, consider the maps

$$\begin{tikzcd} \widetilde{\mathbf{P}^2 \times \mathbf{P}^2} \arrow[r, "j"] \arrow[d, "\pi"'] & Hilb^2(\mathbf{P}^2) \arrow[r] & Gr(2, Sym^n V) \arrow[r] & \mathbf{P}^{N-1} \\ \mathbf{P}^2 \times \mathbf{P}^2 & & & \end{tikzcd}$$

where $\widetilde{\mathbf{P}^2 \times \mathbf{P}^2}$ is the blowup along the diagonal Δ. If $\mathcal{O}(1)$ is the Plücker line bundle, then $\sigma_1 = c_1(\mathcal{O}(1))$. Consider the surjective map $i^*\colon H^*(\mathbf{P}^2 \times \mathbf{P}^2) \to H^*(\Delta)$. If $x = [\ell \times \mathbf{P}^2], y = [\mathbf{P}^2 \times \ell]$ are hyperplanes and we let $w = \sigma_1$, then i^* is the map $\mathbf{Z}[x,y]/\langle x^3, y^3\rangle \to \mathbf{Z}[w]/\langle w^3\rangle$ that takes $x, y \mapsto w$. By [12, Theorem 1, p.571],

$$H^*(\widetilde{\mathbf{P}^2 \times \mathbf{P}^2}) \cong \frac{H^*(\mathbf{P}^2 \times \mathbf{P}^2)[t]}{(P(t), t \cdot \ker i^*)}$$

where $P(t)$ is a polynomial whose constant term is $[\Delta]$ and whose restriction to $H^*(\Delta)$

is the Chern polynomial of the normal bundle $\mathcal{N}_\Delta(\mathbf{P}^2 \times \mathbf{P}^2)$, i.e.

$$i^*P(t) = t^2 + tc_1(\mathcal{N}) + c_2(\mathcal{N}).$$

The isomorphism is induced by $\pi^*\colon H^*(\mathbf{P}^2 \times \mathbf{P}^2) \to H^*(\widetilde{\mathbf{P}^2 \times \mathbf{P}^2})$ by sending $-t$ to $[E]$ where E is the exceptional divisor.

$[\Delta] = dx^2 + exy + fy^2$ for some d, e, f, so we can solve for the coefficients by computing $[\Delta]y^2, [\Delta]xy, [\Delta]x^2$. Notice $x^3 = y^3 = x^4 = y^4 = 0$, so $[\Delta]y^2 = dx^2y^2, [\Delta]xy = ex^2y^2, [\Delta]x^2 = fx^2y^2$. Since $[\Delta]y^2, [\Delta]xy, [\Delta]x^2$, and x^2y^2 are all a class of a point, then $d = e = f = 1$ and hence $[\Delta] = x^2 + xy + y^2$.

To evaluate $c_1(\mathcal{N})$, notice $\mathcal{N} = T_{\mathbf{P}^2} = Q(1)$, where Q is the quotient bundle. If x_1, x_2 are the Chern roots of Q, then the Chern roots of $Q(1)$ are $x_1 + (x_1 + x_2)$ and $x_2 + (x_1 + x_2)$. Then $c_1(\mathcal{N}) = c_1(Q(1)) = 3(x_1 + x_2) = 3w$. Therefore

$$i^*P(t) = t^2 + 3wt + i^*[\Delta] \quad \text{so} \quad P(t) = t^2 + 3xt + x^2 + xy + y^2.$$

Since $\ker(i^*) = \langle x - y \rangle$,

$$H^*(\widetilde{\mathbf{P}^2 \times \mathbf{P}^2}) = \mathbf{Z}[x, y, t]/\langle t^2 + 3xt + x^2 + xy + y^2, tx - ty, x^3, y^3 \rangle.$$

Notice j is a $2\colon 1$ map and $\sigma_1 = c_1(\mathcal{O}(-E) \otimes \mathcal{O}_{\mathbf{P}^2}(n, n)) = t + nx + ny$. Since a point in $\widetilde{\mathbf{P}^2 \times \mathbf{P}^2}$ corresponds to x^2y^2, $[Hilb^2(\mathbf{P}^2)]\sigma_1^4$ is $\frac{1}{2}$ times the coefficient of x^2y^2 in $(t + nx + ny)^4$. Using Macaulay2,

```
R=frac(QQ[n])[x,y,t]

I=ideal(x^3,y^3,t^2+(3*x)*t+(x^2+x*y+y^2),t*x-t*y)

S=R/I
```

```
(t+n*x+n*y)^4

x^2*y^2
```

we find that the coefficient of x^2y^2 is $6(n^4-4n^2+4n-1)$. Therefore, $a = 3(n^4 - 4n^2+4n-1)$.

To compute b, first multiply $[Hilb^2(\mathbf{P}^2)]$ by $\sigma_{1,1}$ by choosing some hyperplane H in $\mathbf{P}^{N-1}$. The intersection of $Hilb^2(\mathbf{P}^2)$ with $\{L\colon L \text{ is a line in } \mathbf{P}^{N-1}, L\in H\}$ is the set of chords of $H\cap v_n(\mathbf{P}^2)=C$. Then, to multiply by σ_1^2, choose two codimension 2 subspaces W_1, W_2 of $\mathbf{P}^{N-1}$ that do not intersect C, and take the intersection of the set of chords in C with $\{L\colon L \text{ is a line in } \mathbf{P}^{N-1}, L\cap W_1\neq\varnothing\}\cap\{L\colon L \text{ is a line in } \mathbf{P}^{N-1}, L\cap W_2\neq\varnothing\}$. This is just the set of chords in C intersecting both W_1 and W_2. The intersection of H with $v_n(\mathbf{P}^2)$ is a degree n curve C of genus $\frac{(n-1)(n-2)}{2}$ [7, 220].

Let $\mathbf{P}^1$ be the set of all hyperplanes containing W_1 and $\widehat{\mathbf{P}^1}$ be the set of all hyperplanes containing W_2. If $H_i(x)$ is the hyperplane containing the point x and W_i, then there is a bihomogeneous map $F\colon C\to\mathbf{P}^1\times\widehat{\mathbf{P}^1}$ defined by $x\mapsto(H_1(x),H_2(x))$. Then a chord between points x,y in C intersects both W_1 and W_2 if and only if $H_i(x)=H_i(y)$ for $i=1,2$. Then counting these chords is the same as counting pairs (x,y) such that $F(x)=F(y)$; in order words, we want to count the singularities of $F(C)$. We can count this number by computing the arithmetic genus of $F(C)$:

Notice that the number of points in $F(C)\cap(\mathbf{P}^1\times\{H_2(x)\})$ is equivalent to the number of points in $C\cap H_2(x)$; this number is n^2 because the degree of the Veronese map is n^2 in this case. Then, F has bidegree (n^2,n^2), and hence the arithmetic genus of $F(C)$ is $(n^2-1)(n^2-1)$. Therefore, $b=(n^2-1)(n^2-1)-\frac{(n-1)(n-2)}{2}$.

To compute c, we multiply $[Hilb^2(\mathbf{P}^2)]$ by $\sigma_{1,1}^2$ by choosing two hyperplanes H_1, H_2 in $\mathbf{P}^{N-1}$. Then the intersection of $Hilb^2(\mathbf{P}^2)$ with $\{L\colon L \text{ is a line in } \mathbf{P}^{N-1}, L\in H_1\}\cap$

$\{L\colon L$ is a line in $\mathbf{P}^{N-1}, L \in H_2\}$ is the set of chords of $H_1 \cap H_2 \cap v_n(\mathbf{P}^2)$. Since $H_1 \cap v_n(\mathbf{P}^2) = C_1$ and $H_2 \cap v_n(\mathbf{P}^2) = C_2$ are both degree n curves, they intersect at n^2 points. Thus, the number of chords of $C_1 \cap C_2$ is $\frac{n^2(n^2-1)}{2}$.

Finally, we can solve for α, β, γ by solving the system of equations obtained by multiplying $[Hilb^2(\mathbf{P}^2)] = \alpha\sigma_{\lambda_1} + \beta\sigma_{\lambda_2} + \gamma\sigma_{\lambda_3}$ by each of $\sigma_1^4, \sigma_1^2\sigma_{1,1}, \sigma_{1,1}^2$. Since $[Hilb^2(\mathbf{P}^2)]\sigma_1^4 = a$, $[Hilb^2(\mathbf{P}^2)]\sigma_1^2\sigma_{1,1} = b$, $[Hilb^2(\mathbf{P}^2)]\sigma_{1,1}^2 = c$, and

$$\begin{aligned} \sigma_{\lambda_1}\sigma_1^4 = 1, \quad \sigma_{\lambda_2}\sigma_1^4 = 3, \quad \sigma_{\lambda_3}\sigma_1^4 = 2 \\ \sigma_{\lambda_1}\sigma_1^2\sigma_{1,1} = 0, \quad \sigma_{\lambda_2}\sigma_1^2\sigma_{1,1} = 1, \quad \sigma_{\lambda_3}\sigma_1^2\sigma_{1,1} = 1 \\ \sigma_{\lambda_1}\sigma_{1,1}^2 = 0, \quad \sigma_{\lambda_2}\sigma_{1,1}^2 = 0, \quad \sigma_{\lambda_3}\sigma_{1,1}^2 = 1 \end{aligned}$$

then $\alpha = a - 3b + c$, $\beta = b - c$, and $\gamma = c$. □

Example 3.2.3. When $n = 3$, one can compute $a = 168$, $b = 63$, and $c = 36$, so

$$[Hilb^2(\mathbf{P}^2)] = 15\sigma_{\lambda_1} + 27\sigma_{\lambda_2} + 36\sigma_{\lambda_3}$$

where $Hilb^2(\mathbf{P}^2)$ is embedded in $Gr(2, Sym^3V)$. □

Chapter 4

Isotropic Subspaces of Schur Modules

Let $V = \mathbf{C}^n$ and $\mathcal{R}$ be the tautological subbundle of $Gr(k, V)$. Recall that for a partition λ, the Schur module $S_\lambda M$ is a functor with respect to a module M, namely the image of the Schur map [5, 76]. For $s \in H^0(Gr(k, V), S_\lambda \mathcal{R}^*)$, then a k-dimensional subspace W of V is **isotropic with respect to** s if $s(W) = 0$. Moreover, we say s is k**-isotropic** if there exists a subspace W of V that is isotropic with respect to s. Recall the following theorem (see [22, Corollary 4.1.9])

Theorem 4.0.1 (Borel-Weil)**.** *If λ is a partition, then as representations of* $\mathbf{GL}(V)$*,*

$$H^0(Gr(k, V), S_\lambda \mathcal{R}^*) = (S_\lambda V)^*.$$

Tevelev used the Borel-Weil theorem to generalize the notion of isotropic subspace for $Sym^2 V^*$ and $\Lambda^2 V^*$: given $s \in Sym^d V^*$ or $s \in \Lambda^d V^*$, a subspace W of V is **isotropic with respect to** s if $s|_W = 0$. We generalize the definition even further for Schur modules (which is compatible with the definition for polylinear forms):

Definition 4.0.2. Let λ be a partition. For $s \in (S_\lambda V)^*$, a subspace W of V is **isotropic with respect to** s if $s|_{S_\lambda W} = 0$. $\square$

In this chapter, we answer the following question: for generic $s \in (S_\lambda V)^*$, when does there exist an isotropic subspace W of V with respect to s? Tevelev gives necessary and sufficient conditions for the existence of isotropic subspaces with respect to symmetric or skew-symmetric polylinear forms in his theorem below (4.0.4). One could answer this question using the fact that $s \in H^0(Gr(k,V), S_\lambda \mathcal{R}^*)$ is k-isotropic if and only if $c_{top}(S_\lambda \mathcal{R}^*) \neq 0$, but computing the top Chern class in general is hard:

Example 4.0.3. Let $V = \mathbf{C}^7$ and $k = 5$, and take $s \in \Lambda^3 V^*$. By the splitting principle, there exist line bundles $L_1, \ldots, L_5$ from the flag bundle associated with the tautological subbundle of $Gr(5,7)$ such that

$$c(\Lambda^3 \mathcal{R}^*) = c\left(\sum_{1 \le i < j < k \le 5} L_i^{-1} L_j^{-1} L_k^{-1}\right).$$

If the Chern roots of $\mathcal{R}^*$ are denoted by x_i's, then

$$c_{top}(\Lambda^3 \mathcal{R}^*) = \prod_{1 \le i < j < k \le 5} (x_i + x_j + x_k).$$

The Borel presentation of the cohomology ring of $Gr(5,7)$ gives us

$$H^*(Gr(5,7)) \otimes \mathbf{Q} = \mathbf{Q}[x_1, \ldots, x_7]^{S_5 \times S_2}/I$$

which is the ring of invariant polynomials where S_5 acts on $x_1, \ldots, x_5$ and S_2 acts on x_6, x_7, which we then mod out by the ideal I of all positive degree symmetric functions [17, 138] Therefore, it is enough to determine whether $c_{top}(\Lambda^3 \mathcal{R}^*)$ is in $\langle p_1, \ldots, p_7 \rangle$, the ideal generated by the power sum symmetric polynomials in $x_1, \ldots, x_7$. This is easily answered using Macaulay2, but difficult by hand:

```
QQ[x_1..x_7]
p = k -> sum(apply(7,i->x_(i+1)^k))
f = product(apply(subsets(toList(1..5), 3), s->x_(s_0) + x_(s_1)
 + x_(s_2)));
I = ideal(f)
J = ideal(apply(7, i-> p(i+1)));
isSubset(I,J)
```

The output is true, and hence the top Chern class is 0, so there does not exist a 5-dimensional isotropic subspace of V with respect to a generic s. We arrive at the same conclusion using Tevelev's theorem below: $\square$

Theorem 4.0.4 (Tevelev). *Let $s \in S^d V^*$ or $s \in \Lambda^d V^*$ be a form in general position. The space V contains a k-dimensional isotropic subspace with respect to s if and only if*

$$n \geq \frac{\binom{d+k-1}{d}}{k} + k \quad \text{or} \quad n \geq \frac{\binom{k}{d}}{k} + k, \quad \textit{respectively,}$$

with the following exceptions:

1. *if $s \in S^2 V^*$ or $s \in \Lambda^2 V^*$ is a form in general position, then V contains a k-dimensional isotropic subspace if and only if $n \geq 2k$;*

2. *if $s \in \Lambda^{n-2} V^*$ is in general position and n is even, then V contains a k-dimensional isotropic subspace if and only if $k \leq n - 2$;*

3. *if $s \in \Lambda^3 V^*$ is in general position and $n = 7$, then V contains a k-dimensional isotropic subspace if and only if $k \leq 4$.*

We give a similar criterion for all partitions λ not included in Tevelev's theorem (except for $\lambda = \varnothing$, which is not interesting).

Theorem 4.0.5. *Let V be an n-dimensional vector space, λ be a partition, and take $k \geq 3$ such that $2 \leq \ell(\lambda) < k$ and $\lambda_1 \geq 2$. Then a generic $s \in (S_\lambda V)^*$ is k-isotropic if and only if*

$$n \geq \frac{\dim(S_\lambda \mathbf{C}^k)}{k} + k.$$

Notice that when rearranged, the inequality can be written as

$$\dim(Gr(k,n)) \geq \dim(S_\lambda \mathbf{C}^k).$$

Example 4.0.6. Let $V = \mathbf{C}^6$ and $k = 3$, and take $s \in S_{(2,1)}V^*$. By the splitting principle, we can write

$$c(S_\lambda \mathcal{R}^*) = c\left(\sum_T L_1^{-T(1)} L_2^{-T(2)} L_3^{-T(3)}\right)$$

where the L_i's are line bundles, T is a semistandard Young tableau of shape λ with entries in $\{1,2,3\}$ (see 4.0.11 for the definition), and $T(i)$ is the number of boxes in T labeled with i. If the Chern roots of $\mathcal{R}^*$ are denoted by x_i's, then we can find out if the top Chern class is 0 by determining whether the product

$$(2x_1 + x_2)(2x_1 + x_3)(x_1 + 2x_2)(x_1 + x_2 + x_3)^2(x_1 + 2x_3)(2x_2 + x_3)(x_2 + 2x_3)$$

is in the ideal generated by the power sum symmetric polynomials in $x_1, \ldots, x_6$: $\langle p_1, \ldots, p_6 \rangle$.

This can be answered using Macaulay2:

```
QQ[x_1..x_6];
p = k -> sum(apply(6,i->x_(i+1)^k));
f = (2*x_1+x_2)*(2*x_1+x_3)*(x_1+2*x_2)*(x_1+x_2+x_3)^2
     *(x_1+2*x_3)*(2*x_2+x_3)*(x_2+2*x_3);
I = ideal(f);
J = ideal(apply(6, i-> p(i+1)));
isSubset(I,J)
```

Our final output is false and hence there exists a 3-dimensional isotropic subspace with respect to a generic s. This verifies the conclusion of Theorem 4.0.5 since

$$\dim(Gr(3,6)) = 9 \geq 8 = \dim(S_{(2,1)}\mathbf{C}^3). \qquad \square$$

For a geometric interpretation of Theorem 4.0.5, recall that the more general version of the Borel-Weil theorem [22, Theorem 4.1.8] says $H^0(Flag_\lambda(V), \mathcal{L}(\lambda)) = (S_\lambda V)^*$, where $\mathcal{L}(\lambda)$ is a line bundle. Then the zero locus of s, denoted by $Z(s)$, is a subvariety of $Flag_\lambda(V)$. Therefore, we have the following consequence:

Corollary 4.0.7. *Let V be an n-dimensional vector space and λ be a partition. For a generic $s \in (S_\lambda V)^*$, there exists a k-dimensional subspace W of V such that $Flag_\lambda(W) \subset Flag_\lambda(V)$ is in the zero locus of s if and only if*

$$n \geq \frac{\dim(S_\lambda \mathbf{C}^k)}{k} + k.$$

The forward direction of Theorem 4.0.5 is implied by the following general fact, denoted here by Lemma 4.0.8, which Tevelev also uses in his proof. In the case of our theorem, X is our Grassmannian and $\mathcal{E} = S_\lambda \mathcal{R}^*$ in the lemma below. Notice $S_\lambda \mathcal{R}^*$ is generated by global sections, i.e. for any $W \in Gr(k,n)$, the map

$H^0(Gr(k,n), S_\lambda\mathcal{R}^*) \to (S_\lambda W)^*$, where $s \mapsto s|_{S_\lambda W}$, is surjective. This is true because $H^0(Gr(k,n), S_\lambda\mathcal{R}^*) = (S_\lambda\mathbf{C}^n)^*$ and $S_\lambda W \hookrightarrow S_\lambda\mathbf{C}^n$ is injective.

Lemma 4.0.8. *Let X be a connected variety of dimension n and $\mathcal{E}$ be a rank r vector bundle on X. Assume $\mathcal{E}$ is generated by global sections. If $r > n$, then $Z(s) = \varnothing$ for almost all $s \in H^0(X, \mathcal{E})$.*

Proof. Define $Z = \{(s,x) \in H^0(X,\mathcal{E}) \times X \colon s(x) = 0\}$, and let $\pi_1 \colon Z \to H^0(X,\mathcal{E})$, $\pi_2 \colon Z \to X$ be projection maps. Let $ev_x \colon H^0(X,\mathcal{E}) \to \mathcal{E}_x$ take $s \mapsto s(x)$ for $x \in X$. By definition, for any $s \in H^0(X,\mathcal{E})$ and $x \in X$,

$$\pi_1^{-1}(s) \cong Z(s)$$

$$\pi_2^{-1}(x) = \{x \in H^0(X,\mathcal{E}) \colon s(x) = 0\} = \ker ev_x.$$

Since $\mathcal{E}$ is generated by global sections, ev_x is surjective and hence

$$\dim \pi_2^{-1}(x) = (\dim H^0(X,\mathcal{E})) - r.$$

Since $(0,x) \in Z$ for all $x \in X$, π_2 is a surjective map between irreducible varieties,

$$\dim Z = \dim X + \max_{x \in X}\{\dim \pi_2^{-1}(x)\} = n + (\dim H^0(X,\mathcal{E})) - r$$

so $\dim Z < \dim H^0(X,\mathcal{E})$. This implies π_1 is not surjective, and hence $\overline{\pi_1(Z)}$ is a closed proper subvariety of $H^0(X,\mathcal{E})$. Hence, if s is in the open subset $H^0(X,\mathcal{E}) \setminus \overline{\pi_1(Z)}$, then $Z(s) = \pi_1^{-1}(s) = \varnothing$. $\square$

Notice that in the proof of Lemma 4.0.8, π_2 is a projective map because it can be factored as $Z \to \mathbf{P}^n \times X \to X$ where the first map is an isomorphism of Z onto a closed subvariety of $\mathbf{P}^n \times X$, and the second map is the projection of $\mathbf{P}^n \times X$ onto X. Therefore, $\pi_2(Z)$ is closed, so we have the following corollary:

Corollary 4.0.9. *Under the same assumptions as Theorem 4.0.5, if* $n \geq \frac{\dim(S_\lambda \mathbf{C}^k)}{k} + k$, *then every* $s \in (S_\lambda V)^*$ *is* k*-isotropic.*

To prove the reverse direction of Theorem 4.0.5, we use a general version of a lemma by Tevelev [21, p.849].

Lemma 4.0.10 (Tevelev)**.** *Let* V *be an* n*-dimensional vector space and* λ *be a partition. If*

$$\dim(S_\lambda \mathbf{C}^{k-i}) \leq (k-i)(n-k-i)$$

for all $i = 0, \ldots, \min\{k, n-k\}$*, then for generic* $s \in (S_\lambda V)^*$*,* V *contains a* k*-dimensional isotropic subspace with respect to* s*.*

In order to show that the inequalities above are satisfied, we compute $\dim(S_\lambda \mathbf{C}^{k-i})$ by evaluating a Schur polynomial s_λ in $(1, 1, \ldots, 1)$ and applying tools from combinatorics.

Definition 4.0.11. For a partition λ, a **semistandard Young tableau** is a Young diagram of shape λ filled with some positive integers so that rows are weakly increasing from left to right and columns are strictly increasing from top to bottom. If λ is a partition, then $s_\lambda(1^n)$ is the number of semistandard Young tableaux with the shape λ and filled with entries from $\{1, \ldots, n\}$.

If b is any box in λ, then the **content of** b is $j - i$ if b is in the ith row from top to bottom and the jth column from left to right; this is denoted by $c(b)$. The **hook length at** b is the number of squares below and to the right of b, including b once, denoted by $h(b)$. □

Example 4.0.12. These are all possible semistandard Young tableaux of shape $\lambda = (2,1)$ filled with entries from $\{1,2,3\}$:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \; .$$

Therefore, $s_{(2,1)}(1,1,1) = 8$. The values for hook length and content for boxes of λ are filled in below:

$$h\colon \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad c\colon \begin{array}{|c|c|}\hline 0 & 1 \\ \hline -1 \\ \cline{1-1}\end{array}.$$ □

Theorem 4.0.13 (Hook-Content Formula)**.** *Let λ be a partition and b be any box in λ. Then*

$$s_\lambda(1^n) = \prod_{b\in\lambda} \frac{n + c(b)}{h(b)}.$$

We use the following well-known result [5, p.77]:

Theorem 4.0.14. *Let λ be a partition. Then*

$$\dim\left(S_\lambda(\mathbf{C}^n)\right) = s_\lambda(1^n).$$

We can prove that $n \geq \frac{\dim(S_\lambda \mathbf{C}^k)}{k} + k$ implies $\dim\left(S_\lambda \mathbf{C}^{k-i}\right) \leq (k-i)(n-k-i)$ for most values of $i \in \{0, \ldots, \min\{k, n-k\}\}$ by showing that

$$\frac{\dim\left(S_\lambda \mathbf{C}^{k-i}\right)}{k-i} \geq \frac{\dim\left(S_\lambda \mathbf{C}^{k-i-1}\right)}{k-i-1} + 1. \tag{4.0.15}$$

Tevelev uses induction to prove (4.0.15); for example, he gives the following lemma used for $Sym^d V^*$ where $d \geq 3$:

Lemma 4.0.16. *If $d \geq 3$ and $\alpha \geq 2$, then*

$$\frac{\binom{d+\alpha-1}{d}}{\alpha} \geq \frac{\binom{d+\alpha-2}{d}}{\alpha-1} + 1.$$

However, this quickly becomes difficult for general partitions. This can be seen in the following examples of hooks and rectangular partitions because $\dim(S_\lambda \mathbf{C}^n)$ is no longer a single binomial coefficient. One can perform a painful induction in particular cases, but it is hard to generalize.

Example 4.0.17. If $\lambda = (d, 1)$ where $d \geq 2$, then by the Hook-Content Formula,

$$\begin{aligned} s_\lambda(1^n) &= \frac{n}{d+1} \cdot \frac{n+1}{d-1} \cdot \frac{n+2}{d-2} \cdots \frac{n+d-1}{1} \cdot \frac{n-1}{1} \\ &= \frac{(n+d-1)(n+d-2)\cdots(n-1)}{(d+1)(d-1)!} \\ &= \frac{d(n-1)}{d+1}\binom{n+d-1}{d}. \end{aligned} \qquad \square$$

Example 4.0.18. If $\lambda = (d, d)$ where $d \geq 2$, then

$$s_\lambda(1^n) = \frac{n+d-1}{(n-1)(d+1)}\binom{n+d-2}{d}^2.$$

More generally, if $\lambda = (d, \dots, d)$ have l parts where $d, l \geq 2$, then

$$s_\lambda(1^n) = \left((l-1)!\binom{n+d-l}{d}\right)^l \prod_{j=1}^{l-1} (n-j)^{j-l}\left(\frac{n+d-j}{j(d+l-j)}\right)^j. \qquad \square$$

Now, we give inequalities that will assist in proving (4.0.15).

Lemma 4.0.19. *Let λ be a nonempty partition.*

1. For $k \geq 2$*,*

$$\frac{s_\lambda(1^k)}{k} \geq \frac{s_\lambda(1^{k-1})}{k-1}. \tag{4.0.20}$$

2. If, in addition, $2 \leq \ell(\lambda) \leq k-1$*, then*

$$\frac{s_\lambda(1^k)}{k} \geq \frac{s_\lambda(1^{k-1})}{k-1} + \frac{1}{k}. \tag{4.0.21}$$

Proof. To prove the second part of the lemma, notice (4.0.21) is equivalent to

$$k(s_\lambda(1^k) - s_\lambda(1^{k-1})) \geq s_\lambda(1^k) + k - 1.$$

Let $g_\lambda(k)$ be the number of semistandard Young tableaux with shape λ with entries in $\{1, \ldots, k\}$ and labeled with at least one k. Since

$$s_\lambda(1^k) = g_\lambda(k) + s_\lambda(1^{k-1}),$$

we can prove the equivalent statement

$$(k-1)g_\lambda(k) \geq s_\lambda(1^{k-1}) + k - 1.$$

Let μ be the subpartition of λ obtained by removing the box in the last column of the last row of λ. Let ν be the partition obtained by adding a box to the end of the first row of μ. Then

$$\begin{aligned}(k-1)g_\lambda(k) &\geq (k-1)s_\mu(1^{k-1})\\ &= s_1(1^{k-1})s_\mu(1^{k-1})\\ &\geq s_\lambda(1^{k-1}) + s_\nu(1^{k-1})\\ &\geq s_\lambda(1^{k-1}) + k - 1.\end{aligned}$$

If we label a partition of shape μ with entries in $\{1, \ldots, k-1\}$, reattach a box to μ in order to obtain λ, and label this new box with k, then we obtain a semistandard Young tableau of shape λ with entries in $\{1, \ldots, k\}$; this proves the first line above. The second line is obvious. Since $\ell(\lambda) \geq 2$, $\lambda \neq \nu$, so the third line follows by Pieri's rule. Since $\ell(\lambda) \leq k-1$, this implies that $\ell(\nu) \leq k-1$, so $s_\nu(1^{k-1}) \neq 0$. Moreover, we obtain a semistandard Young tableau if the last box in the first row of ν is filled with any integer in $\{1, \ldots, k-1\}$; the remaining boxes in the first row are labeled with 1; and for the remaining rows, the boxes in the ith row are labeled with i. Therefore, we have at least $k-1$ semistandard Young tableaux of shape ν with entries in $\{1, \ldots, k-1\}$, proving the fourth line.

Now we prove the first part of the lemma. It is clearly true when $\lambda = (1)$. Otherwise, we again choose μ to be the subpartition of λ obtained by removing the box in the last column of the last row of λ. Notice that (4.0.21) is equivalent to

$$(k-1)g_\lambda(k) \geq s_\lambda(1^{k-1}).$$

Using a similar reasoning as above, we obtain

$$\begin{aligned}(k-1)g_\lambda(k) &\geq (k-1)s_\mu(1^{k-1}) \\ &= s_1(1^{k-1})s_\mu(1^{k-1}) \\ &\geq s_\lambda(1^{k-1}).\end{aligned}$$

□

Definition 4.0.22. Let λ be a partition. If μ is a subpartition of λ such that λ/μ is a skew shape whose columns contain at most one box each, then λ/μ is a **horizontal strip**. We denote the collection of all horizontal strips by HS. □

The following is a well-known fact:

Proposition 4.0.23. *For any partition λ,*

$$s_\lambda(1^k) = \sum_{\lambda/\mu \in HS} s_\mu(1^{k-1}).$$

Proof. We can partition the collection of all semistandard Young tableau of shape λ with entries in $\{1,\ldots,k\}$ into subsets based on the placement of k's. Since k can appear at most once in each column of λ, the size of such a subset is the same as the number of semistandard Young tableau of some unique $\mu \subset \lambda$ such that $\lambda/\mu \in HS$ and labeled with entries in $\{1,\ldots,k-1\}$. □

Lemma 4.0.24. *Let λ be a partition and take $k \geq 3$. If $1 \leq \ell(\lambda) \leq k-2$ and $\lambda \neq (1),(2),(1,1)$, then*

$$\frac{s_\lambda(1^k)}{k} \geq \frac{s_\lambda(1^{k-1})}{k-1} + 1.$$

Proof. We perform induction on k. The case when $k = 3$ corresponds to symmetric forms, which was proved by Tevelev's lemma 4.0.16.

Now let $k > 3$ and λ be a partition satisfying $1 \leq \ell(\lambda) \leq k-2$ and not equal to (1), (2), or $(1,1)$. By induction, we suppose that for any partition μ not equal to (1), (2), or $(1,1)$, and satisfying $1 \leq \ell(\mu) \leq k-3$, then

$$\frac{s_\mu(1^{k-1})}{k-1} \geq \frac{s_\mu(1^{k-2})}{k-2} + 1.$$

If $\ell(\lambda) = 1$, then we can use Lemma 4.0.16. Otherwise, we use Proposition 4.0.23 several times in the computation below.

$$
\begin{aligned}
\frac{s_\lambda(1^k)}{k} &= \frac{1}{k}\sum_{\lambda/\mu\in HS} s_\mu(1^{k-1}) \\
&= \frac{1}{k}\sum_{\substack{\lambda/\mu\in HS\\ 1\le \ell(\mu)\le k-3\\ \mu\neq(1),(2),(1,1)}} s_\mu(1^{k-1}) + \frac{1}{k}\sum_{\substack{\lambda/\mu\in HS\\ \ell(\mu)=k-2\\ \mu\neq(1,1)}} s_\mu(1^{k-1}) + \frac{1}{k}\sum_{\substack{\lambda/\mu\in HS\\ \mu=(1),(2),(1,1)}} s_\mu(1^{k-1}) \\
&= \frac{k-1}{k}\sum_{\substack{\lambda/\mu\in HS\\ 1\le \ell(\mu)\le k-3\\ \mu\neq(1),(2),(1,1)}} \frac{s_\mu(1^{k-1})}{k-1} + \frac{k-1}{k}\sum_{\substack{\lambda/\mu\in HS\\ \ell(\mu)=k-2\\ \mu\neq(1,1)}} \frac{s_\mu(1^{k-1})}{k-1} \qquad (4.0.25) \\
&\qquad + \frac{k-1}{k}\sum_{\substack{\lambda/\mu\in HS\\ \mu=(1),(2),(1,1)}} \frac{s_\mu(1^{k-1})}{k-1} \\
&\ge \frac{k-1}{k}\sum_{\substack{\lambda/\mu\in HS\\ 1\le \ell(\mu)\le k-3\\ \mu\neq(1),(2),(1,1)}} \left(\frac{s_\mu(1^{k-2})}{k-2}+1\right) + \frac{k-1}{k}\sum_{\substack{\lambda/\mu\in HS\\ \ell(\mu)=k-2\\ \mu\neq(1,1)}} \left(\frac{s_\mu(1^{k-2})}{k-2}+\frac{1}{k-1}\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.0.26) \\
&\quad + \frac{k-1}{k}\sum_{\substack{\lambda/\mu\in HS\\ \mu=(1),(2),(1,1)}} \frac{s_\mu(1^{k-2})}{k-2} \\
&= \frac{k-1}{k(k-2)}\sum_{\lambda/\mu\in HS} s_\mu(1^{k-2}) + \sum_{\substack{\lambda/\mu\in HS\\ 1\le \ell(\mu)\le k-3\\ \mu\neq(1),(2),(1,1)}} \frac{k-1}{k} + \sum_{\substack{\lambda/\mu\in HS\\ \ell(\mu)=k-2\\ \mu\neq(1,1)}} \frac{1}{k} \\
&= \frac{(k-1)^2}{k(k-2)}\frac{s_\lambda(1^{k-1})}{k-1} + \sum_{\substack{\lambda/\mu\in HS\\ 1\le \ell(\mu)\le k-3\\ \mu\neq(1),(2),(1,1)}} \frac{k-1}{k} + \sum_{\substack{\lambda/\mu\in HS\\ \ell(\mu)=k-2\\ \mu\neq(1,1)}} \frac{1}{k} \\
&\ge \frac{s_\lambda(1^{k-1})}{k-1} + \sum_{\substack{\lambda/\mu\in HS\\ 1\le \ell(\mu)\le k-3\\ \mu\neq(1),(2),(1,1)}} \frac{k-1}{k} + \sum_{\substack{\lambda/\mu\in HS\\ \ell(\mu)=k-2\\ \mu\neq(1,1)}} \frac{1}{k} \qquad (4.0.27) \\
&\ge \frac{s_\lambda(1^{k-1})}{k-1} + \frac{k-1}{k} + \frac{1}{k} \qquad (4.0.28) \\
&= \frac{s_\lambda(1^{k-1})}{k-1} + 1.
\end{aligned}
$$

Line (4.0.26) follows by the induction hypothesis on the first sum and Lemma 4.0.19 applied to the remaining sums. After rearranging terms and noticing $(k-1)^2 > k(k-2)$, we have line (4.0.27). In Line (4.0.27), the first sum has at least one summand because we can let μ be the subpartition of λ obtained by removing the last row of λ; and the second sum has at least one summand because we can take μ to be λ. This proves line (4.0.28). $\square$

We can now prove finish the proof of the main theorem:

Proof of Theorem 4.0.5. Since $s_\lambda(1^j) = 0$ for $j < \ell(\lambda)$, by Tevelev's Lemma 4.0.10 it suffices to show

$$\dim (S_\lambda \mathbf{C}^{k-i}) \leq (k-i)(n-k-i)$$

for all $i = 0, \ldots, k - \ell(\lambda)$. Since $i \neq k$, if we assume

$$n \geq \frac{\dim (S_\lambda \mathbf{C}^k)}{k} + k,$$

then using the Lemma 4.0.24,

$$\begin{aligned}
n &\geq \frac{s_\lambda(1^k)}{k} + k \\
&\geq \frac{s_\lambda(1^{k-1})}{k-1} + k + 1 \\
&\geq \frac{s_\lambda(1^{k-2})}{k-2} + k + 2 \\
&\vdots \\
&\geq \frac{s_\lambda(1^{\ell(\lambda)+2})}{\ell(\lambda)+2} + k + k - \ell(\lambda) - 2 \\
&\geq \frac{s_\lambda(1^{\ell(\lambda)+1})}{\ell(\lambda)+1} + k + k - \ell(\lambda) - 1.
\end{aligned}$$

This proves the inequality for $i = 0, \ldots, k - \ell(\lambda) - 1$.

If λ is a rectangle, then

$$\dim(S_\lambda \mathbf{C}^{\ell(\lambda)}) = s_\lambda(1^{\ell(\lambda)}) = 1 \leq \ell(\lambda)(n - \ell(\lambda))$$

because $\ell(\lambda) > 1$, so the inequality for $i = k - \ell(\lambda)$ is true.

If λ is not a rectangle, then let μ be the partition obtained by removing all columns of height $\ell(\lambda)$ from λ; therefore, $s_\lambda(1^{\ell(\lambda)}) = s_\mu(1^{\ell(\lambda)})$. If $\mu = (1)$, then

$$\dim(S_\lambda \mathbf{C}^{\ell(\lambda)}) = s_\mu(1^{\ell(\lambda)}) = \ell(\lambda) \leq \ell(\lambda)(n - \ell(\lambda))$$

because $\ell(\lambda) < k \leq n$. If $\mu = (2)$ or $(1,1)$, then our assumption, $n \geq \frac{\dim(S_\lambda \mathbf{C}^k)}{k} + k$, says

$$n \geq \frac{3k+1}{2}, \quad n \geq \frac{3k-1}{2},$$

which imply the desired inequalities

$$n \geq \frac{3\ell(\lambda)+1}{2} = \frac{s_\lambda(1^{\ell(\lambda)})}{\ell(\lambda)} + \ell(\lambda),$$

$$n \geq \frac{3\ell(\lambda)-1}{2} = \frac{s_\lambda(1^{\ell(\lambda)})}{\ell(\lambda)} + \ell(\lambda),$$

respectively. Otherwise, we can prove (4.0.15) for our remaining case

$$\begin{aligned}
\frac{s_\lambda(1^{\ell(\lambda)+1})}{\ell(\lambda)+1} &\geq \frac{s_\mu(1^{\ell(\lambda)+1})}{\ell(\lambda)+1} \\
&\geq \frac{s_\mu(1^{\ell(\lambda)})}{\ell(\lambda)} + 1 \\
&= \frac{s_\lambda(1^{\ell(\lambda)})}{\ell(\lambda)} + 1.
\end{aligned}$$

The first inequality is true because given a semistandard Young tableau of shape μ filled with entries from $\{1, \ldots, \ell(\lambda)+1\}$, one can obtain a semistandard Young tableau of shape λ filled with entries from $\{1, \ldots, \ell(\lambda)+1\}$ in the following way: adjoin a rectangle to the left of μ in order to obtain the shape λ, and label the entire ith row of the rectangle with i. Since $\ell(\mu) \leq \ell(\lambda) - 1$, we can apply Lemma 4.0.24 to obtain the second inequality. The last line follows because the rectangle removed from λ in order to obtain μ must be constant along rows when filled with integers $1, \ldots, \ell(\lambda)$, and this is done in exactly one way. Therefore, we've proved the case when $i = k - \ell(\lambda)$. □

Bibliography

[1] Dellaca, Roger. *Gotzmann regularity for globally generated coherent sheaves* (2014) arXiv:1410.8612v3.

[2] Fulton, William. *Flags, Schubert polynomials, degeneracy loci, and determinantal formulas.* Duke Math. J. 65 (1992), no. 3, 381–420.

[3] Fulton, William. *Intersection theory.* Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 2. Springer-Verlag, Berlin, 1998.

[4] Fulton, William. *Young tableaux. With applications to representation theory and geometry.* London Mathematical Society Student Texts, 35. Cambridge University Press, Cambridge, 1997.

[5] Fulton, William; Harris, Joe. *Representation theory. A first course.* Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer-Verlag, New York, 1991.

[6] Grayson, Daniel R. and Michael E. Stillman, Macaulay2, a software system for research in algebraic geometry, Available at `http://www.math.uiuc.edu/Macaulay2/`.

[7] Griffiths, Phillip; Harris, Joseph. *Principles of algebraic geometry.* Reprint of the 1978 original. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994.

[8] Harris, Joe. *Algebraic geometry. A first course.* Graduate Texts in Mathematics, 133. Springer-Verlag, New York, 1992.

[9] Harris, Joe; Tu, Loring W. *On symmetric and skew-symmetric determinantal varieties.* Topology 23 (1984), no. 1, 71–84.

[10] Hartshorne, Robin. *Algebraic geometry.* Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[11] Hatcher, Allen. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[12] Keel, Sean. *Intersection theory of moduli space of stable n-pointed curves of genus zero.* Trans. Amer. Math. Soc. 330 (1992), no. 2, 545–574.

[13] Lang, Serge. *Algebra.* Revised third edition. Graduate Texts in Mathematics, 211. Springer-Verlag, New York, 2002.

[14] Lee, John M. *Introduction to smooth manifolds.* Graduate Texts in Mathematics, 218. Springer-Verlag, New York, 2003.

[15] Macdonald, I. G. *Symmetric functions and Hall polynomials.* Second edition. With contributions by A. Zelevinsky. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.

[16] Maclagan, Diane (2007), Notes on Hilbert Schemes, available at `https://homepages.warwick.ac.uk/staff/D.Maclagan/papers/HilbertSchemesNotes.pdf`

[17] Manivel, Laurent. *Symmetric functions, Schubert polynomials and degeneracy loci.* Translated from the 1998 French original by John R. Swallow. SMF/AMS Texts and Monographs, 6. Cours Spcialiss [Specialized Courses], 3. American Mathematical Society, Providence, RI; Socit Mathmatique de France, Paris, 2001.

[18] Miller, Ezra; Sturmfels, Bernd. *Combinatorial commutative algebra.* Graduate Texts in Mathematics, 227. Springer-Verlag, New York, 2005.

[19] Smith, Karen E.; Kahanpää, Lauri; Kekäläinen, Pekka; Traves, William. *An invitation to algebraic geometry.* Universitext. Springer-Verlag, New York, 2000.

[20] Stanley, Richard P. *Enumerative combinatorics.* Vol. 2. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. Cambridge Studies in Advanced Mathematics, 62. Cambridge University Press, Cambridge, 1999.

[21] Tevelev, E. A. *Isotropic subspaces of polylinear forms.* (Russian) Mat. Zametki 69 (2001), no. 6, 925–933; translation in Math. Notes 69 (2001), no. 5-6, 845–852.

[22] Weyman, Jerzy. *Cohomology of vector bundles and syzygies.* Cambridge Tracts in Mathematics, 149. Cambridge University Press, Cambridge, 2003.