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Exact Heavy To Light Meson Form Factors In The Chiral Limit Of Planar 1+1 QCD*

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ABSTRACT

The form factors of the flavor changing vector current between a $\bar{q}Q$ -meson and the lightest $\bar{q}q$ (pseudoscalar-)meson are computed exactly and explicitly in the 't Hooft model (planar QCD in 1 + 1 dimensions) in the limit that the mass of q -quark vanishes.

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EXACT HEAVY TO LIGHT MESON FORM FACTORS IN THE CHIRAL LIMIT OF PLANAR 1+1 QCD.

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ABSTRACT

The form factors of the flavor changing vector current between a $\bar{q}Q$ -meson and the lightest $\bar{q}q$ (pseudoscalar-)meson are computed exactly and explicitly in the 't Hooft model (planar QCD in 1 + 1 dimensions) in the limit that the mass of q -quark vanishes.

1. Introduction

Little is known about form factors of local operators between a heavy meson like the \bar{B} — with quantum numbers of a single heavy quark Q and a single light antiquark \bar{q} — and light pseudoscalar mesons like the π - K - η octet. Isgur and Wise have shown that heavy quark symmetries¹ relate several form factors², but nothing is known about their shape. Thus far all theoretical attempts to describe them are based on particular models of hadrons.

Surprisingly one can calculate the shape of these form factors exactly in the chiral limit of planar QCD in 1+1 dimensions. As we will see, the form factors f_{\pm} (defined below) are given by a single pole at the mass of the B with residue $\pm\mu_B^2 f_B/f_{\pi}$, where μ_X and f_X are the mass and decay constant of the X -meson, respectively. Whether this result is indicative of the behavior of the form factors in 3+1 QCD is at present a matter of pure speculation.

This talk reports on work done in collaboration with Paul Mende, first reported elsewhere.³ The shape of the form factor will be determined in section 2, and its normalization in section 3. These analytic results are compared with numerical computations in section 4.

2. Proof of Pole Dominance

The matrix element of the vector current $V_{\nu} = \bar{q}\gamma_{\nu}Q$ can be expressed in terms of two form factors f_{\pm} :

$$\langle \pi(p') | V_{\nu} | B(p) \rangle = (p + p')_{\nu} f_{+}(q^2) + (p - p')_{\nu} f_{-}(q^2) . \quad (1)$$

where $q = p - p'$ and throughout the paper p and p' always denote the momentum of the B and π mesons, respectively. In 1+1 dimensions by “ B and π mesons” we mean

that they are the lightest states with quantum numbers $Q\bar{q}$ and $q\bar{q}$, respectively. In what follows we will take the quark Q to have a fixed mass M and the quark q to have mass m , and we will consider the limit $m \rightarrow 0$.

In planar (ie, large N_c) QCD the form factors in (1) are saturated by couplings of the flavor changing current V_ν to the $Q\bar{q}$ resonances in that channel.^{4,5,6} We therefore can write

$$\langle \pi | V_\nu | B \rangle = \sum_n \frac{\langle 0 | V_\nu | B_n \rangle \langle \pi B_n | B \rangle}{q^2 - \mu_n^2}. \quad (2)$$

Here B_n denote the states with quantum number $\bar{q}Q$ ordered by increasing mass μ_n .

For odd parity states $|B_n\rangle$,

$$\langle 0 | V_\nu | B_n \rangle = \epsilon_{\nu\lambda} q^\lambda f_n. \quad (3)$$

We can describe the interactions giving the matrix element $\langle \pi B_n | B \rangle$ conveniently in terms of a hadron lagrangian, dual to the fundamental lagrangian, which couples the mesons via terms

$$\mathcal{L}_{\text{int}} = \sum_{abc} \hat{g}_{abc}(q^2) \epsilon_{\lambda\nu} \partial_\lambda \phi^a \partial_\nu \phi^b \phi^c. \quad (4)$$

Similarly, for even parity,

$$\langle 0 | V_\nu | B_n \rangle = q_\nu f_n, \quad (5)$$

with couplings

$$\mathcal{L}_{\text{int}} = \sum_{abc} \hat{g}_{abc}(q^2) \phi^a \phi^b \phi^c. \quad (6)$$

The form factors of Eq. (1) can then be written

$$f_+ = \sum_{\text{even parity}} \frac{-f_n(q^2) \hat{g}_{\pi B_n}(q^2) q^2}{q^2 - \mu_n^2} \quad (7)$$

$$f_- = \sum_{\text{odd parity}} \frac{f_n(q^2) \hat{g}_{\pi B_n}(q^2)}{q^2 - \mu_n^2} + \sum_{\text{even parity}} \frac{f_n(q^2) \hat{g}_{\pi B_n}(q^2) (\mu_B^2 - \mu_\pi^2)}{q^2 - \mu_n^2}, \quad (8)$$

which is obtained with the help of the useful two-dimensional formula

$$\epsilon_{\lambda\nu} q^\nu = \left[-q^2 (p + p')_\lambda + (\mu_B^2 - \mu_\pi^2) (p - p')_\nu \right] / 2\epsilon^{\rho\sigma} p_\rho p'_\sigma. \quad (9)$$

Note that the expansions (8) have momentum dependent numerators, proportional to the off-shell decay constants $f_n(q^2)$ and three point couplings, $\hat{g}_{\pi B_n}(q^2)$. If the form factors vanish as $|q^2| \rightarrow \infty$, we can replace the momentum dependent numerators by on-shell numerators,

$$f_+ = \sum_{\text{even parity}} \frac{-f_n \hat{g}_{\pi B_n}(\mu_n^2) \mu_n^2}{q^2 - \mu_n^2} \quad (10)$$

$$f_- = \sum_{\text{odd parity}} \frac{f_n \hat{g}_{\pi B_n}(\mu_n^2)}{q^2 - \mu_n^2} + \sum_{\text{even parity}} \frac{f_n \hat{g}_{\pi B_n}(\mu_n^2) (\mu_B^2 - \mu_\pi^2)}{q^2 - \mu_n^2}, \quad (11)$$

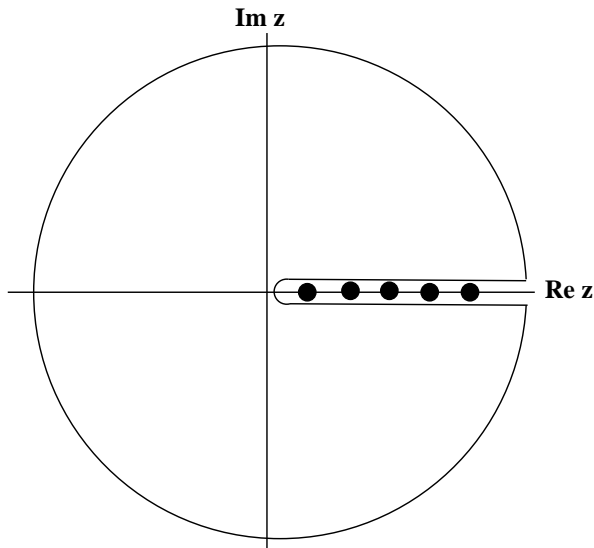


Fig. 1. Contour of integration for the integral in Eq. 12. The filled circles denote the location of poles of the form factors, $z = \mu_0^2, \mu_1^2, \mu_2^2, \dots$

To verify this, consider Cauchy's theorem for the form factors,

$$f_{\pm}(q^2) = \frac{1}{2\pi i} \int_C dz \frac{f_{\pm}(z)}{z - q^2} \quad (12)$$

where the contour C consists of a circle $|z| = R \rightarrow \infty$, excluding the positive real axis, $z = R$, and a line just below and above the positive real axis extending from $z = \mu_B^2 \pm i\epsilon$ to $z = R \pm i\epsilon \rightarrow \infty \pm i\epsilon$ (and enclosing μ_B^2 on the left); see Fig. 1. Now, if, as assumed, $|f_{\pm}(z)| \rightarrow 0$ as $|z| \rightarrow \infty$, the integral over the circle at infinity vanishes. The integral can thus be traded for a sum over integrals over infinitesimal contours around each pole, $z = \mu_n^2$. Using the explicit form of f_{\pm} in Eq. (8), we see that only the n -th term in the sum has a pole at $z = \mu_n^2$. Applying Cauchy's theorem again gives the on-shell numerators.

Several observations are in order. It is large N_c which allows us to treat the resonances as stable without continuum couplings in Eq. (2). It selects the three point couplings to one-particle intermediate states. In using convergence as $|q^2| \rightarrow \infty$, we make an assumption about the large-momentum behavior of the interactions, information unavailable from a low-energy analysis or standard chiral lagrangian analysis. The shift of the numerators to the residues at the poles, familiar in dispersion theory, has a simple physical origin: in the “effective” meson field theory there is freedom to make arbitrary field redefinitions without changing the on-shell S -matrix. Here that freedom is used to replace the momentum dependent couplings (that is, the higher derivative operators) by constants at the expense of shifting the coefficients of higher point functions, which in turn are down by powers of $1/N_c$. Note the contrast with the use of field redefinitions by Georgi⁷. In his analysis higher point functions are

suppressed instead by powers of the cutoff. The required behaviour as $|q^2| \rightarrow \infty$ can be established by explicit analysis of the quark scattering matrix⁶.

Now in the chiral limit — the light quark mass $m \rightarrow 0$ and $\mu_\pi^2 \rightarrow 0$ — the on-shell three-point couplings $\langle \pi B_n | B \rangle$ vanish while the decay constants f_n remain finite. This is a direct consequence of chiral symmetry and leads immediately to a pole-dominated form factor, ie, $f(q^2) \sim 1/(q^2 - \mu_B^2)$.

To show this in detail, fix the state B_n and consider the matrix element of the *light-light* current $a_\lambda = \bar{q}\gamma_\lambda\gamma_5 q$:

$$\langle B_n | a_\lambda | B \rangle = \sum_\ell \frac{\langle 0 | a_\lambda | \pi_\ell \rangle \langle \pi_\ell B_n | B \rangle}{p'^2 - \mu_\ell^2} \quad (13)$$

Here the sum is over the tower of $\bar{q}q$ states, $|\pi_\ell(p')\rangle$. Again, since the form factors of a_λ vanish⁶ as $|p'^2| \rightarrow \infty$, we replace the numerators in the sum in (13) by their on-shell values.

Observe that axial current conservation implies $0 = \langle 0 | \partial \cdot a | \pi_\ell \rangle = \mu_\ell^2 f_\ell$. Therefore $f_\ell = 0$ unless $\mu_\ell = 0$, so the axial current couples only to the massless pion, $\pi = \pi_0$. Therefore in the chiral limit,

$$\langle B_n | a_\lambda | B \rangle \rightarrow \frac{f_\pi \langle \pi B_n | B \rangle}{p'^2} p'_\lambda \quad (14)$$

Again applying axial current conservation, we have

$$0 = \partial \cdot a \rightarrow \frac{f_\pi \langle \pi B_n | B \rangle}{p'^2} p'^2 \quad (15)$$

or, more explicitly,

$$0 = \begin{cases} f_\pi \hat{g}_{\pi B_n}(\mu_n^2) & \text{if } n \text{ is odd;} \\ (2\epsilon^{\lambda\sigma} p_\lambda p'_\sigma) f_\pi \hat{g}_{\pi B_n}(\mu_n^2) & \text{if } n \text{ is even.} \end{cases} \quad (16)$$

We emphasize that this equation is valid when all three states are on-shell. It immediately follows that

$$\hat{g}_{\pi B_n}(\mu_n^2) = 0, \quad n \neq 0. \quad (17)$$

so, except for the ground state, all the coupling constants in the effective lagrangian in (4) and (6) vanish on the mass-shell. The case $n = 0$ is singled out because the factor $2\epsilon^{\lambda\sigma} p_\lambda p'_\sigma = 0$, so it need not follow that $\hat{g}_{\pi BB}$ vanishes. Below we show that it does not.

Combining these results and introducing the coupling $g_{\pi BB} = \mu_B^2 \hat{g}_{\pi BB}(\mu_B^2)$ in analogy with the definition that is natural in four dimensions, the form factors are

$$f_+ = -f_- = -\frac{f_B g_{\pi BB}}{q^2 - \mu_B^2} \quad (18)$$

3. Normalization

The coupling $g_{\pi BB}$ can be fixed by an application of the Callan-Trieman relation, adapted to 1+1 QCD. To derive it, consider the following matrix element,

$$\mathcal{M}_{\mu\nu} = i \int d^2x e^{ip'x} \langle 0 | T(a_\mu(x) V_\nu(0)) | B(p) \rangle \quad (19)$$

In the chiral limit, the light-light axial current a_μ is conserved. The divergence $p'_\mu \mathcal{M}_{\mu\nu}$ gives therefore an equal time commutator of a_0 and V_ν . One has

$$p'_\mu \mathcal{M}_{\mu\nu} \rightarrow -i \langle 0 | A_\nu | B(p) \rangle = f_B p_\nu \quad (20)$$

Alternatively, evaluate the matrix element and then contract with p' . The currents a_μ and V_ν create states in the π and B towers, respectively, and they couple to the B meson through the matrix element $\langle \pi_\ell B_n | B \rangle$, thus,

$$\sum_\ell \sum_n \frac{f_{\pi_\ell} p'_\mu}{p'^2 - \mu_\ell^2} \frac{f_{n\nu}}{q^2 - \mu_n^2} \langle \pi_\ell B_n | B \rangle \quad (21)$$

where by $f_{n\nu}$ we mean either of (3) or (5) according to whether B_n has odd or even parity, respectively. Now, contracting with p'_μ and letting $p' \rightarrow 0$ only the $\ell = 0$ term, that is the pion, remains in the sum over ℓ . Moreover, all the matrix elements $\langle \pi_\ell B_n | B \rangle$ must vanish linearly with p' , because goldstone bosons are always derivatively coupled. The only term that survives is the one with $n = 0$, that is, the B -meson term itself, because the denominator $q^2 - \mu_B^2$ also vanishes linearly with p' . Explicitly we have then

$$f_B p_\nu = \lim_{p' \rightarrow 0} \frac{[f_B(q^2) \epsilon_{\nu\lambda} q^\lambda] [2\epsilon^{\rho\sigma} p_\rho p'_\sigma \hat{g}_{\pi BB}(q^2)]}{q^2 - \mu_B^2} \quad (22)$$

This can be simplified with the use of Eq. (9). We note that in the limit $p' \rightarrow 0$ one has $q^2 = \mu_B^2$ and the couplings involved are automatically on-shell, so we obtain

$$-f_\pi \hat{g}_{\pi BB}(\mu_B^2) = 1 \quad (23)$$

Since $f_\pi = \sqrt{N_c/\pi}$, this completely determines the coupling $g_{\pi BB}$.

Our final results reads as follows

$$f_+ = -f_- = \frac{f_B/f_\pi}{q^2/\mu_B^2 - 1} \quad (24)$$

These are the *exact* form factors in the chiral limit.

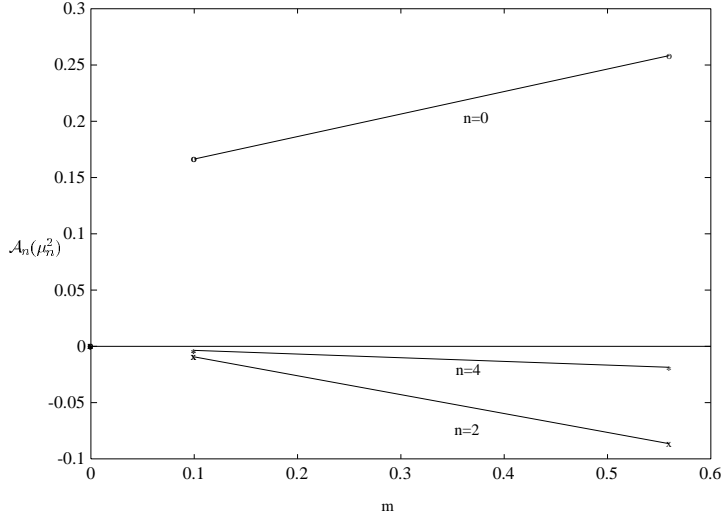


Fig. 2. The residue $\mathcal{A}_n(\mu_n^2)$ vs. m , the light quark mass. The value was computed numerically for $m = 0.1, 0.56$ and a line connecting the pairs of points drawn to guide the eye. For $n \neq 0$, the $\mathcal{A}_n \rightarrow 0$ in the chiral limit, $m \rightarrow 0$. Here $M^2 = 20000$.

4. Experimental Verification

In this section we will assume a unit system with $g^2 N_c / \pi = 1$.

The results of the previous sections have been verified through numerical investigations of the case of small, nonvanishing ‘light mass’ m .³ The form factors may be written as a sum over resonances,

$$f_+(q^2) = \sum_{\text{even } n} \frac{\mathcal{A}_n(q^2)}{1 - q^2/\mu_n^2} \quad (25)$$

with residues given in terms of the couplings introduced above by

$$\mathcal{A}_n(q^2) \equiv -\frac{f_n g_{\pi B n}(q^2)}{\mu_n^2}. \quad (26)$$

Figs. 2 and 3 show the residues $\mathcal{A}_n(\mu_n^2)$ plotted as functions of the light quark mass for $n = 0$ through 14. To perform this computation one first writes the residues $\mathcal{A}_n(\mu_n^2)$ as overlap integrals of ‘t Hooft wave functions³. The ‘t Hooft wave functions can be determined numerically, and the overlap integrals can also be done numerically. It is apparent that $\mathcal{A}_n(\mu_n^2) \rightarrow 0$ as $m \rightarrow 0$, except for $n = 0$, which approaches a limit consistent with Eq. (23). The decay constants are rather insensitive to m ⁸, so the couplings $\hat{g}_{\pi B n}$ are seen to vanish linearly with it.

That $\mathcal{A}_0(\mu_n^2)$ approaches the limit dictated by the Callan-Triemann relation can be seen more clearly in Fig. 4 which shows the dependence of $-\hat{g}_{\pi B n} = -g_{\pi B n}/\mu_n^2$ on the large mass M for fixed ‘light’ mass m . The dependence on the heavier mass M , for large M , of the decay constants⁹ and couplings is seen to be as expected.

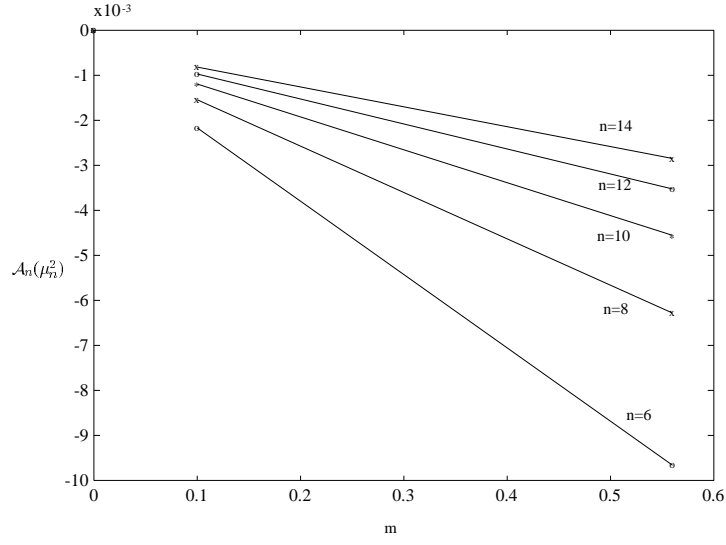


Fig. 3. The residue $\mathcal{A}_n(\mu_n^2)$ vs. m , the light quark mass, for more resonant states, $n = 6-18$.

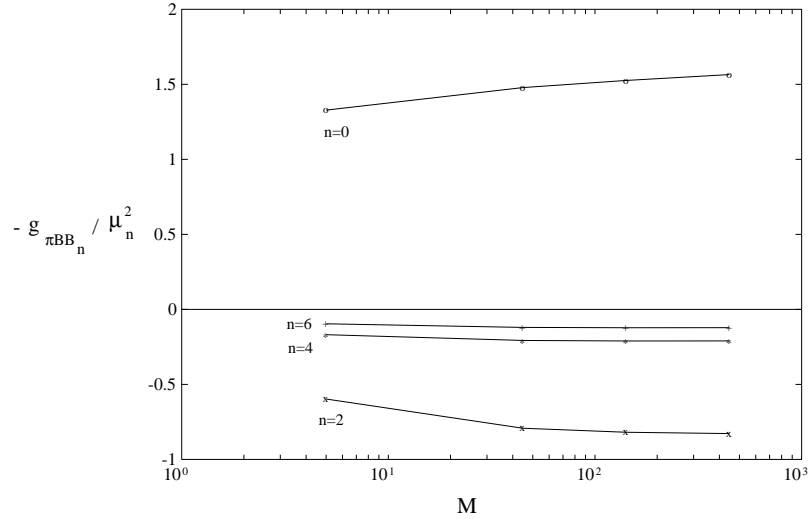


Fig. 4. Approach to the heavy quark limit: $-g_{\pi B n} / \mu_n^2$ vs. M for $m = 0.56$ ($\mu_\pi^2 = 3.09$) and $M^2 = 25, 2000, 20000, 200000$. Results for $m = 0.1$ are similar.

5. Conclusions

In the 't Hooft model the form factors of flavor changing currents for the decay of a meson into a “pion” are given by a single pole in the chiral limit. Our main result is summarized in Eq. (24).

Away from the chiral limit the form factors are no longer given by a single pole. However for quark masses $m^2 < g^2 N_c / \pi$ the corrections to a single pole form factor are small, and the form factors are effectively pole dominated over the whole physical region.

6. Acknowledgements

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