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Design of Mechanisms to Draw Trigonometric Plane Curves

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This paper describes a mechanism design methodology that draws plane curves which have finite Fourier series parameterizations, known as trigonometric curves. We present three ways to use the coefficients of this parameterization to construct a mechanical system that draws the curve. One uses Scotch yoke mechanisms for each of the terms in the coordinate trigonometric functions, which are then added using a belt or cable drive. The second approach uses two-coupled serial chains obtained from the coordinate trigonometric functions. The third approach combines the coordinate trigonometric functions to define a single-coupled serial chain that draws the plane curve. This work is a version of Kempe's universality theorem that demonstrates that every plane trigonometric curve has a linkage which draws the curve. Several examples illustrate the method including the use of boundary points and the discrete Fourier transform to define the trigonometric curve. [DOI: 10.1115/1.4035882]

1 Introduction

The design of mechanisms to draw plane curves has found recent application in the construction of mechanical characters [1,2]. Nolle [3,4] and Koetsier [5,6] described the history of mechanism design to draw curves. An important result was Kempe's proof of the existence of a mechanism to trace any plane algebraic curves [7,8], which was verified by Jordan and Steiner [9] and Kapovich and Millson [10]. Demonstrations of Kempe's universality theorem have been presented by Kobel [11] and Saxena [12]. Also, see Liu and McCarthy [13].

In this paper, we show how to obtain a mechanical system that draws curves which have x and y coordinates defined by finite Fourier series, known as trigonometric curves. The class of trigonometric curves includes well-known curves, such as the limacon of Pascal, the cardioid, trifolium, hypocycloid, and Lissajous figures, as well as many other examples [14]. Artobolevskii [15] presented specialized linkages for drawing curves including the linkage in Fig. 1 to draw the trifolium. Our results can be viewed as a version of Kempe's universality theorem [7,9,10], which shows that linkages exist to draw a particular class of plane curves.

We present three ways to draw these curves. First, we use the mechanical Fourier synthesis system described by Miller [16]. Next, we follow Nie and Krovi [17] and obtain coupled serial chains defined by the coordinate Fourier series. Finally, we present a way to define a single-coupled serial chain that draws a given trigonometric plane curve. The methodology is illustrated with several examples, which include the use of boundary points and the discrete Fourier transform [18] to define the trigonometric curve.

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2 Trigonometric Curves

A trigonometric plane curve, $\boldsymbol{P} = (x(\theta), y(\theta))$, is a parameterized curve with coordinate functions that are finite Fourier series, that is,

$$\boldsymbol{P} = \begin{Bmatrix} x(\theta) \\ y(\theta) \end{Bmatrix} = \begin{Bmatrix} \sum_{k=0}^{m} a_k \cos k\theta + b_k \sin k\theta \\ \sum_{k=0}^{m} c_k \cos k\theta + d_k \sin k\theta \end{Bmatrix} \quad (1)$$

where a_k, b_k, c_k, and d_k, $k = 0, \ldots, m$, are the real coefficients and $\theta \in [0, 2\pi]$. In what follows, we present three ways to obtain mechanical systems that draw curves given by equations of this form. Since the equations in Eq. (1) both have finite Fourier series, so the mechanism we designed can draw the trigonometric curve exactly. The movement simulation of the mechanical systems is conducted under *SolidWorks Motion Analysis*.

2.1 Component Scotch Yoke Mechanisms. Consider the x component of Eq. (1) and rewrite the equation in the form

$$x(\theta) = \sum_{k=0}^{m} L_k \cos(k\theta - \psi_k) \quad (2)$$

where

$$L_k = \sqrt{a_k^2 + b_k^2}, \quad \psi_k = \arctan\frac{b_k}{a_k} \quad (3)$$

This equation can be viewed as the sum of the output of m Scotch yoke mechanisms each with an input crank length of L_k and initial

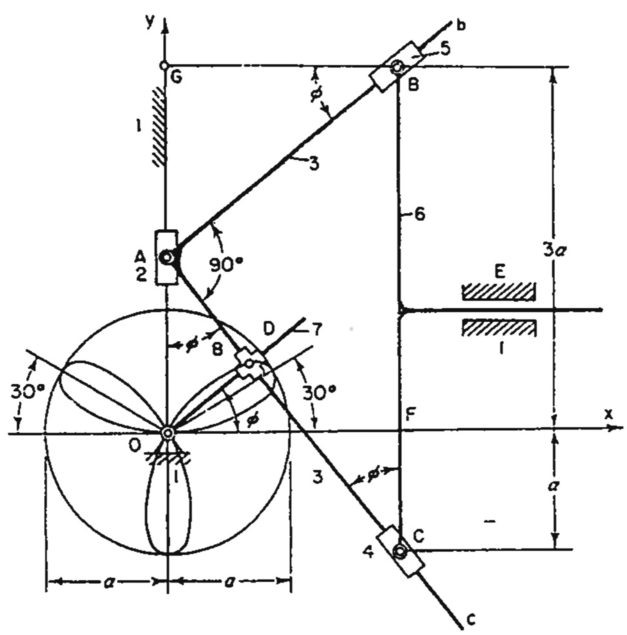

Fig. 1 Artobolevskii [15] designed this mechanism to draw the trifolium (three-petal) curve

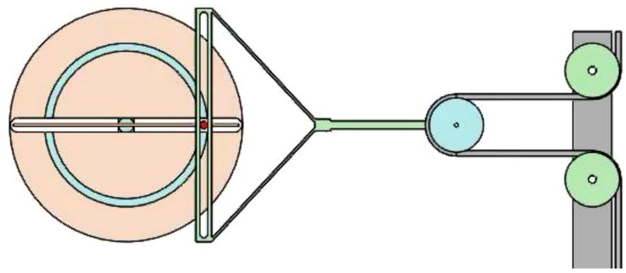

Fig. 2 A Scotch yoke mechanism that transforms the rotation of a crank into a cosine curve

angle ψ_k. Figure 2 shows the construction of one unit of Scotch yoke mechanism. The system is driven such that the angle of the crank k is given by

$$\phi_k = k\theta - \psi_k \tag{4}$$

The output of the set of m Scotch yoke linkages is added by acting on a belt or cable drive to generate the $x(\theta)$ component of the curve P. The initial configuration of the system is defined by the phase angles ψ_k.

The y component of the curve P yields a similar relationship

$$y(\theta) = \sum_{k=0}^{m} M_k \cos(k\theta - \eta_k) \tag{5}$$

where

$$M_k = \sqrt{c_k^2 + d_k^2}, \quad \eta_k = \arctan\frac{d_k}{c_k} \tag{6}$$

This equation defines a set of Scotch yoke mechanisms that generate the y component of the curve P.

All of the Scotch yoke mechanisms are connected to a single input, thus the whole mechanical system has a single input. Use the output of the two sets of Scotch yoke mechanisms to drive in the x and y of a cursor to draw the desired curve. Figure 3 shows the system of a set of Scotch yoke mechanisms that can draw a hypocycloid curve.

2.2 Component-Coupled Serial Chains. The Scotch yoke mechanisms of Sec. 2.1 can be replaced by the projection of serial chains that have their joints driven by belt or cable drives coupled to a single input joint angle.

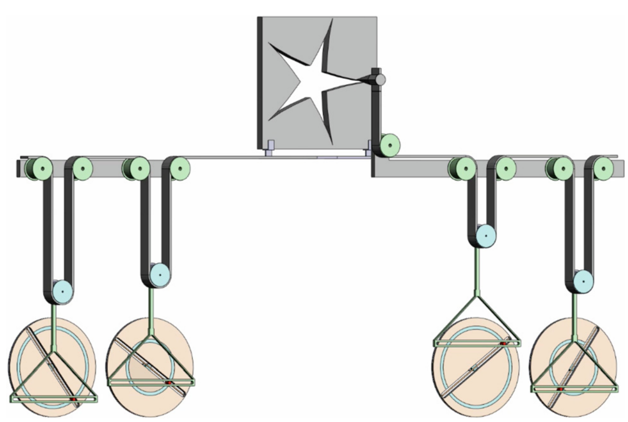

Fig. 3 One set of Scotch yoke mechanisms drives the *x*-component and another set drives the *y*-component of a cursor to draw a plane curve

Let the x component in Eq. (2) be the x projection of the end-point of a serial chain formed from m links of length L_k that are each positioned at an angle α_k relative to the x-axis, given by

$$\alpha_k = k\theta - \psi_k \tag{7}$$

These angles are driven by a single input angle θ through a cable drive that has a decreasing pulley diameters at each link in order to increase the rotation angle required by $k\theta$. The input angles of the kth link and $(k-1)$th link are related by the ratio of the pulley diameters $D_k/D_{k-1} = (k-1)/k$.

In the same way, the y component of P can be generated by a serial chain constructed from m links of length M_k that are each positioned at an angle β_k relative to the y-axis, given by

$$\beta_k = k\theta - \eta_k \tag{8}$$

The coupled serial chains for the x and y components have the same inputs, thus this mechanical system has degree-of-freedom one. The two serial chains can be coupled so they move a cursor along the x and y coordinates to draw the curve P. The system of coupled serial chains that draw a hypocycloid is shown in Fig. 4.

2.3 Single-Coupled Serial Chain. In this section, we show how to obtain a single-coupled serial chain that draws trigonometric curves (1). Introduce the complex form of this curve as

$$P(\theta) = x(\theta) + iy(\theta) = \sum_{k=0}^{m}(a_k \cos k\theta + b_k \sin k\theta) + \sum_{k=0}^{m} i(c_k \cos k\theta + d_k \sin k\theta) \tag{9}$$

Introduce the identities

$$\cos\phi = \frac{1}{2}(e^{i\phi} + e^{-i\phi}), \quad \sin\phi = \frac{1}{2i}(e^{i\phi} - e^{-i\phi}) \tag{10}$$

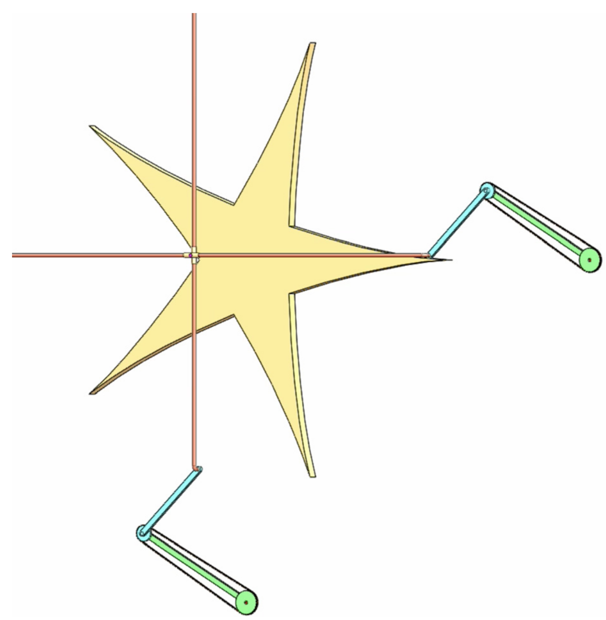

Fig. 4 A coupled serial chain drives the *x*-component and a separate coupled serial chain drives the *y*-component of a cursor to draw a plane curve

to obtain

$$P(\theta) = \frac{1}{2}\sum_{k=0}^{m}((a_k + d_k) + i(c_k - b_k))e^{ik\theta} + ((a_k - d_k) + i(c_k + b_k))e^{-ik\theta} \tag{11}$$

Next, we convert the complex coefficients to exponential form.

Here, we use the same notation as in Sec. 2.1, though now the terms are defined differently. The magnitudes L_k and M_k are given by

$$L_k = \frac{1}{2}\sqrt{(a_k + d_k)^2 + (c_k - b_k)^2}$$
$$M_k = \frac{1}{2}\sqrt{(a_k - d_k)^2 + (c_k + b_k)^2} \tag{12}$$

and the associated phase angles are

$$\psi_k = \arctan\left(\frac{c_k - b_k}{a_k + d_k}\right), \quad \eta_k = \arctan\left(\frac{c_k + b_k}{a_k - d_k}\right) \tag{13}$$

Thus, the equation of the curve P becomes

$$P(\theta) = \frac{1}{2}\sum_{k=0}^{m} L_k e^{i(k\theta - \psi_k)} + M_k e^{-i(k\theta + \eta_k)} \tag{14}$$

Separate the x and y components of this equation to obtain

$$\boldsymbol{P}(\theta) = \left\{ \begin{array}{c} \sum_{k=0}^{m} L_k \cos(k\theta - \psi_k) + M_k \cos(-k\theta - \eta_k) \\ \sum_{k=0}^{m} L_k \sin(k\theta - \psi_k) + M_k \sin(-k\theta - \eta_k) \end{array} \right\} \tag{15}$$

This is the equation of set of links in a serial chain, where those denoted L_k rotate counterclockwise and those denoted M_k rotate clockwise. This is because the links denoted L_k have inputs $k\theta$, while the links denoted M_k have inputs $-k\theta$.

All of the links are connected to the same input angle θ and are driven by cable or belt drives at increasing speeds by pulleys of decreasing radius. The phase angles ψ_k and η_k define the initial configuration of the system. Figure 5 shows that the single-coupled serial chain which has two links can draw a hypocycloid curve.

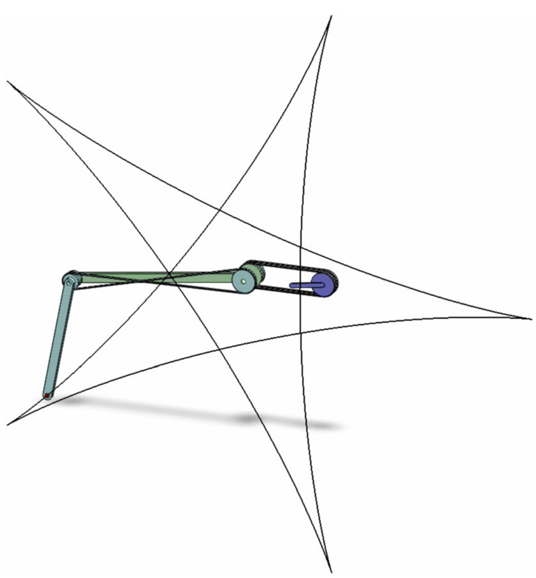

Fig. 5 The end-point of a single-coupled serial chain draws a plane curve

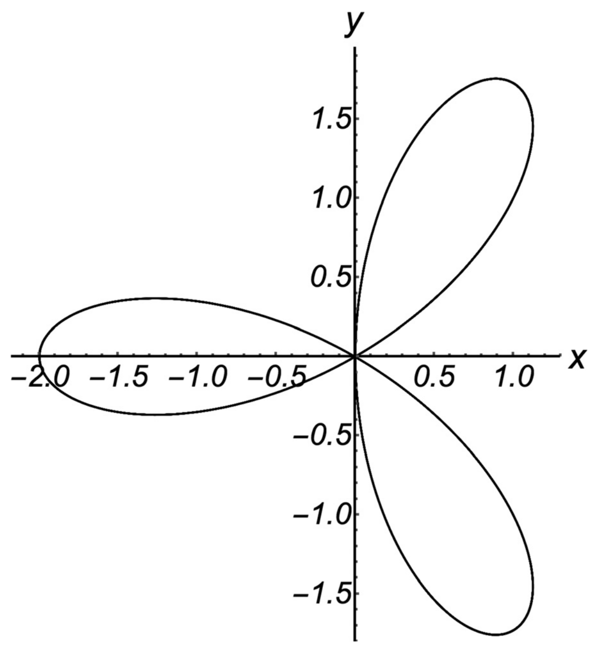

Fig. 6 Trifolium with the size of the petal set to $a = 2$

3 Mechanisms That Draw the Trifolium

In this section, we demonstrate the use of the equations in Sec. 2 to design mechanisms to draw the trifolium curve. Figure 6 shows the trifolium curve defined by Eq. (16) with α equal to 2.

3.1 Trifolium Using Scotch Yoke Mechanisms. The trifolium is often defined in radial coordinates by the formula [19,20]

$$P: \ \rho = -a\cos 3\theta \tag{16}$$

where a point P is defined by the length $\rho(\theta)$ of a radius vector at the angle θ to the x-axis. The constant a defines the size of the petals of the trifolium.

The coordinate functions for this curve are given by

$$\boldsymbol{P}_T = \left\{ \begin{array}{c} \rho\cos\theta \\ \rho\sin\theta \end{array} \right\} = \left\{ \begin{array}{c} x(\theta) \\ y(\theta) \end{array} \right\} = \left\{ \begin{array}{c} -a\cos 3\theta\cos\theta \\ -a\cos 3\theta\sin\theta \end{array} \right\} \tag{17}$$

Set the size of the petals to $a = 2$ and expand the products of sine and cosine functions to obtain the trigonometric form of this curve

$$\boldsymbol{P}_T = \left\{ \begin{array}{c} -\cos 2\theta - \cos 4\theta \\ \sin 2\theta - \sin 4\theta \end{array} \right\} \tag{18}$$

Table 1 Coefficients of the component trigonometric functions for the trifolium curve

k	a_k	b_k	c_k	d_k
0	0	0	0	0
1	0	0	0	0
2	−1	0	0	1
3	0	0	0	0
4	−1	0	0	−1

Table 2 Component dimensions for component Scotch yoke mechanisms and coupled serial chains to draw the trifolium

k	L_k	ψ_k	M_k	η_k
1	0	0	0	0
2	1	π	1	$\pi/2$
3	0	0	0	0
4	1	π	1	$-\pi/2$

Table 3 Dimensions of the single-coupled serial chain to draw the trifolium

k	L_k	ψ_k	M_k	η_k
1	0	0	0	0
2	0	0	1	π
3	0	0	0	0
4	1	π	0	0

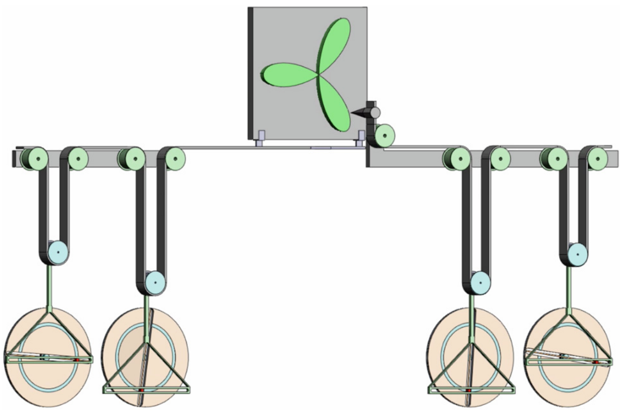

Fig. 7 A system of Scotch yoke mechanisms driven by a single input that draws the trifolium

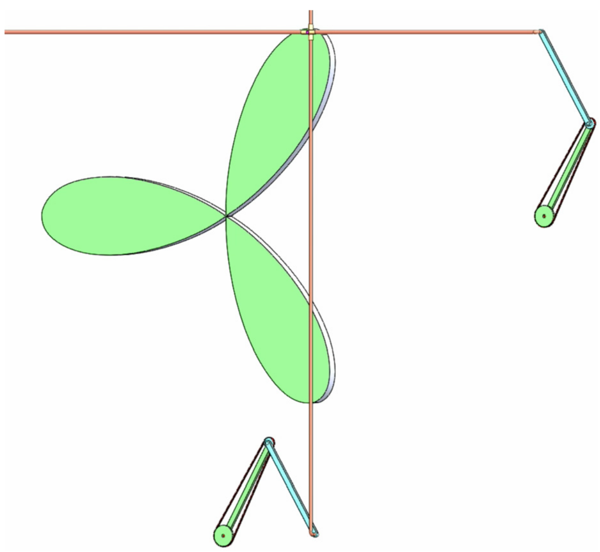

Fig. 8 A system of two-coupled serial chains driven by a single input that draws the trifolium

The coefficients of the component trigonometric equations listed in Table 1 can be used to determine the dimensions of a set of component Scotch yoke mechanisms, using Eqs. (3) and (6). These dimensions are listed in Table 2. The mechanical system that draws this trifolium curve is shown in Fig. 7.

3.2 Trifolium Using Component-Coupled Serial Chains. The analysis provided in Sec. 2 shows that the same dimensions used to construct the Scotch yoke mechanisms can be used to define a pair of component serial chains that position a cursor to draw a trigonometric curve like the trifolium. Figure 8 shows the mechanical system of coupled serial chains that draw the trifolium curve.

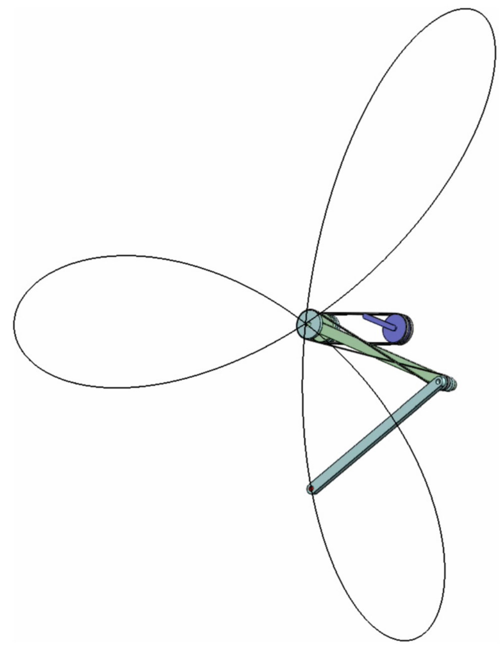

Fig. 9 A single constrained coupled serial chain with one input that draws the trifolium

3.3 Trifolium Using a Single-Coupled Serial Chain. In order to define the single-coupled serial chain that draws the trifolium curve, we use the coefficients of the trigonometric form of the equation, Table 1, and compute the dimensions, L_k and M_k, and initial angles, ψ_k and η_k, using Eqs. (12) and (13). These dimensions are listed in Table 3. Recall that for a single-coupled serial chain, those links denoted L_k rotate counterclockwise, while those links denoted M_k rotate clockwise and the angles ψ_k and η_k denote the initial configurations of the system. Figure 9 shows the single-coupled serial chain that draws the trifolium curve.

This drawing linkage for the trifolium curve consists of two links, as compared to the exceedingly complex drawing linkage obtained by Kobel [11] using Kempe's method. This is also simpler than Artobolevskii's eight-bar linkage [15] shown in Fig. 1.

4 Mechanisms That Draw the Butterfly Curve

A version of the butterfly curve [21] can be generated by the formula

$$P_B : \rho(\theta) = 7 - \sin\theta + 2.3\sin 3\theta + 2.5\sin 5\theta - 2\sin 7\theta - 0.4\sin 9\theta + 4\cos 2\theta - 2.5\cos 4\theta \qquad (19)$$

where radius vector ρ lies at the angle θ relative to the x-axis. Figure 10 shows this butterfly curve.

The trigonometric form of the butterfly curve is obtained by expanding the coordinates, $\boldsymbol{P}_B = (\rho\cos\theta, \rho\sin\theta) = (p_{Bx}, p_{By})$, to obtain

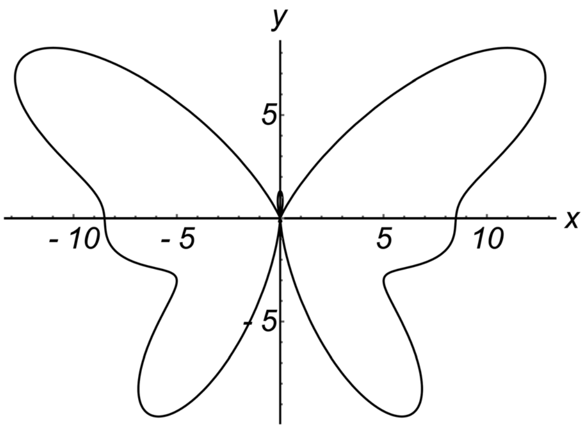

Fig. 10 The plot of a butterfly curve from its polar equation

Table 4 Coefficients of the component trigonometric functions for the butterfly curve

k	a_k	b_k	c_k	d_k
0	0	0	−0.5	0
1	9	0	0	5
2	0	0.65	1.65	0
3	0.75	0	0	3.25
4	0	2.4	0.1	0
5	−1.25	0	0	−1.25
6	0	0.25	−2.25	0
7	0	0	0	0
8	0	−1.2	0.8	0
9	0	0	0	0
10	0	−0.2	0.2	0

Table 5 Dimensions for the component Scotch yoke mechanisms to draw the butterfly curve

k	L_k	ψ_k	M_k	η_k
0	0	0	0.5	π
1	9	0	5	$\pi/2$
2	0.65	$\pi/2$	1.65	0
3	0.75	0	3.25	$\pi/2$
4	2.4	$\pi/2$	0.1	0
5	1.25	π	1.25	$-\pi/2$
6	0.25	$\pi/2$	2.25	π
7	0	0	0	0
8	1.2	$-\pi/2$	0.8	0
9	0	0	0	0
10	0.2	$-\pi/2$	0.2	0

$$
\begin{aligned}
p_{Bx} &= 9\cos\theta + 0.65\sin 2\theta + 0.75\cos 3\theta + 2.4\sin 4\theta \\
&\quad -1.25\cos 5\theta + 0.25\sin 6\theta - 1.2\sin 8\theta - 0.2\sin 10\theta \\
p_{By} &= -0.5 + 5\sin\theta + 1.65\cos 2\theta + 3.25\sin 3\theta + 0.1\cos 4\theta \\
&\quad -1.25\sin 5\theta - 2.25\cos 6\theta + 0.8\cos 8\theta + 0.2\cos 10\theta
\end{aligned} \tag{20}
$$

The coefficients of these equations are listed in Table 4.

4.1 Butterfly Using Scotch Yoke Mechanisms. The coefficients of the component trigonometric equations listed in Table 4 are used to calculate the dimensions of the component Scotch yoke mechanisms, using Eqs. (3) and (6). These dimensions are listed in Table 5. These dimensions can also be used to construct the component-coupled serial chains that draw this curve.

Table 6 Dimensions of the single-coupled serial chain to draw the butterfly curve

k	L_k	ψ_k	M_k	η_k
0	0.25	$-\pi/2$	0.25	$-\pi/2$
1	7	0	2	0
2	0.5	$\pi/2$	1.15	$\pi/2$
3	2	0	1.25	π
4	1.15	$-\pi/2$	1.25	$\pi/2$
5	1.25	π	0	0
6	1.25	$-\pi/2$	1	$-\pi/2$
7	0	0	0	0
8	1	$\pi/2$	0.2	$-\pi/2$
9	0	0	0	0
10	0.2	$\pi/2$	0	0

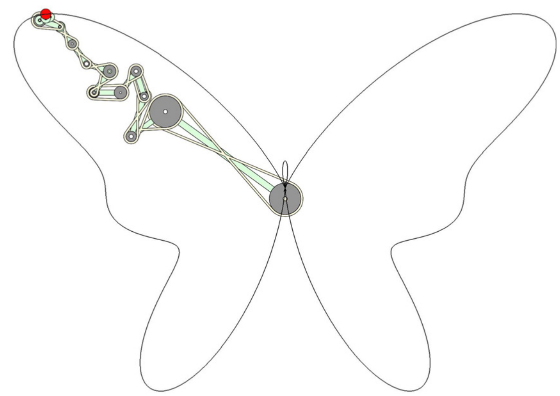

Fig. 11 The constraint coupled serial chain to draw this butterfly curve consists of 14 terms

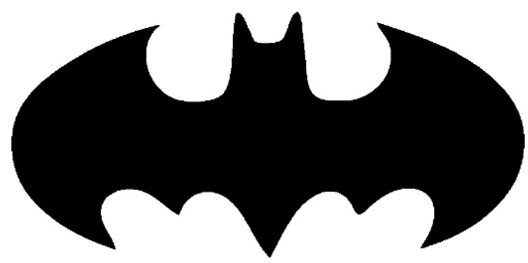

Fig. 12 The shape of the Batman logo

4.2 Butterfly Using a Single-Coupled Serial Chain. In order to define the single-coupled serial chain that draws the butterfly curve, we use the coefficients of the trigonometric form of the equation, Table 4, and compute the dimensions, L_k and M_k, and initial angles, ψ_k and η_k, using Eqs. (12) and (13). These dimensions are listed in Table 6. The constant terms $k=0$ define the location of the first ground pivot $(-0.5, 0)$. There are 14 links in this coupled serial chain driven by a single input. The single-coupled serial chain that draws the butterfly curve is shown in Fig. 11.

5 Mechanisms That Draw the Batman Logo

In this example, we use the software MATHEMATICA to extract 3235 points to define the boundary curve of the Batman logo shown in Fig. 12. Twenty terms of the discrete Fourier transform of these points yield the curve $\boldsymbol{P} = (x(\theta), y(\theta))$ in the trigonometric form that approximates this Batman logo curve, see Table 7. The formulas obtained above yield mechanisms that draw our approximation to the Batman logo. Figure 13 shows the Batman logo curve achieved by 20 terms of Fourier series.

Table 7 Coefficients of the component trigonometric functions for the Batman logo

k	a_k	b_k	c_k	d_k
0	480.22	0	317.57	0
1	11.36	−175.47	−80.64	−14.59
2	2.08	−18.25	−2.05	−0.60
3	−8.80	19.26	21.64	12.16
4	−2.80	2.14	5.38	3.28
5	5.49	−5.70	−22.26	−10.47
6	1.48	0.92	3.54	−1.62
7	−6.68	5.58	−11.81	−1.03
8	−2.53	0.84	−0.64	1.85
9	1.00	0.59	4.17	1.90
10	−1.04	2.64	−3.47	−1.05
11	3.59	−5.21	0.99	1.19
12	−1.65	0.79	−5.35	−3.42
13	−0.19	−0.20	−2.64	−1.41
14	−1.93	0.18	−3.03	−2.42
15	−2.44	1.36	−0.41	0.35
16	0.35	−0.17	−1.95	−0.42
17	0.44	−0.30	1.77	2.25
18	1.03	0.30	−1.05	0.59
19	1.58	−0.28	0.75	1.25

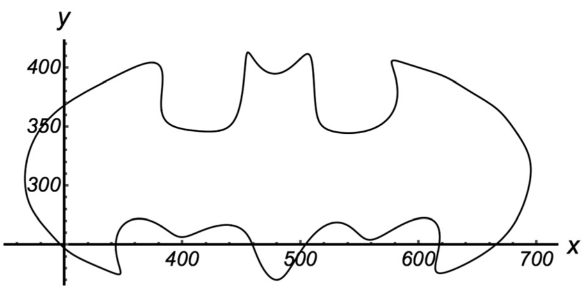

Fig. 13 Batman logo curve obtained using 20 terms of the discrete Fourier transform of boundary points

Table 8 Dimensions for the component Scotch yoke mechanisms to draw the Batman logo

k	L_k	ψ_k	M_k	η_k
0	480.22	0	317.57	0
1	175.84	−1.50	81.95	−2.96
2	18.37	−1.45	2.14	−2.85
3	21.18	1.99	24.83	0.51
4	3.52	2.48	6.30	0.54
5	7.92	−0.80	24.60	−2.70
6	1.75	0.55	3.90	−0.42
7	8.71	2.44	11.85	−3.05
8	2.66	2.82	1.95	1.90
9	1.16	0.53	4.58	0.42
10	2.84	1.94	3.63	−2.84
11	6.33	−0.96	1.55	0.87
12	1.83	2.69	6.35	−2.57
13	0.28	−2.34	3.00	−2.65
14	1.94	3.04	3.88	−2.46
15	2.80	2.63	0.54	2.42
16	0.40	−0.46	2.00	−2.92
17	0.53	−0.61	2.87	0.90
18	1.08	0.28	1.21	2.62
19	1.61	−0.17	1.46	1.03

5.1 Batman Logo Using Scotch Yoke Mechanisms. The coefficients of the component trigonometric equations listed in Table 7 can be used to determine the dimensions of a set of component Scotch yoke mechanisms, using Eqs. (3) and (6).

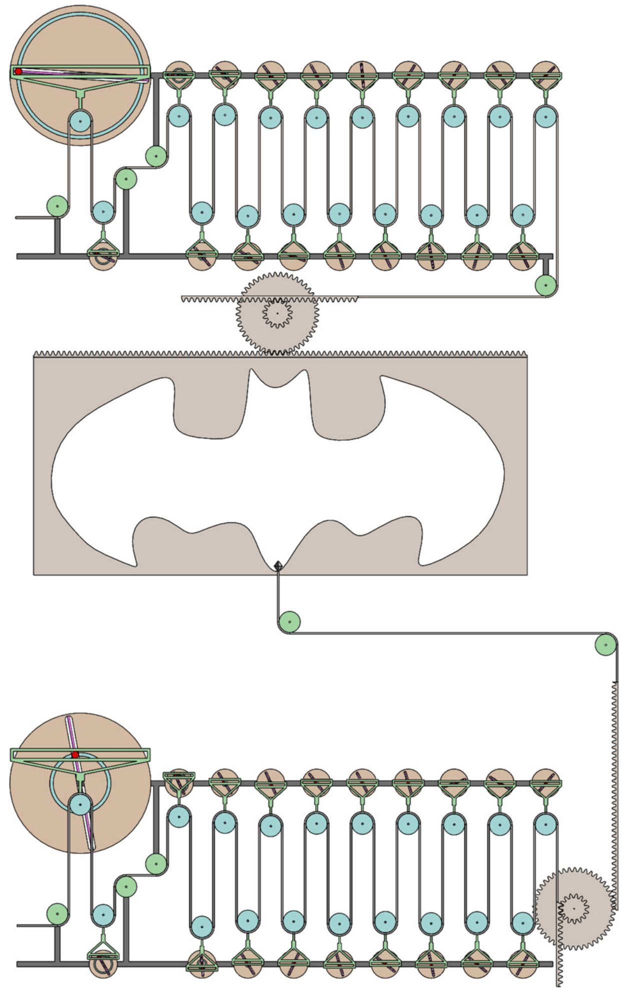

Fig. 14 The system of component Scotch yoke mechanisms that draw the Batman logo consists of two sets of 19 mechanisms

These dimensions are listed in Table 8. Figure 14 shows the system of Scotch yoke mechanisms that draw this curve.

5.2 Batman Logo Using a Single-Coupled Serial Chain. By using the coefficients of the component trigonometric equations listed in Table 7, Eqs. (12) and (13) yield the dimensions, L_k and M_k, and the initial angles, ψ_k and η_k, of the serial chain that draws the Batman logo. These dimensions are listed in Table 9. The $k=0$ terms identify the ground pivot coordinates $(480.22, 317.57)$. This system is a serial chain consisting of 38 links with one input that draws our approximation of the Batman logo. Figure 15 shows the single-coupled serial chain that can draw this Batman logo.

6 Physical Prototype for the Trifolium

We used the Stratasys Fortus additive manufacturing system to build components for the trifolium linkage using acrylonitrile butadiene styrene plastic. Figure 16 shows the individual components. Figure 17 is the assembled trifolium drawing mechanism.

In order to show the curve generated by the mechanism, we attached a blue light emitting diode (LED) to the drawing tip and used a DC motor to drive the base at 100 rpm. A photograph of the trace of the LED is shown in Fig. 18.

Table 9 Dimensions of the single-coupled serial chain that draws the Batman logo

k	L_k	ψ_k	M_k	η_k
0	287.86	0.58	287.86	0.58
1	47.44	1.60	128.71	−1.46
2	8.13	1.47	10.24	−1.43
3	2.06	0.61	22.98	2.04
4	1.63	1.42	4.84	2.25
5	8.64	−1.86	16.10	−1.05
6	1.31	1.62	2.72	0.96
7	9.51	−1.98	4.20	−2.30
8	0.81	−2.00	2.19	3.09
9	2.30	0.88	2.42	1.75
10	3.23	−1.90	0.41	−1.54
11	3.92	0.91	2.42	−1.05
12	3.98	−2.26	2.44	−1.20
13	1.46	−2.15	1.54	−1.16
14	2.71	−2.50	1.44	−1.40
15	1.37	−2.43	1.47	2.81
16	0.88	−1.60	1.13	−1.22
17	1.70	0.65	1.16	2.46
18	1.06	−0.69	0.43	−1.04
19	1.51	0.35	0.28	0.95

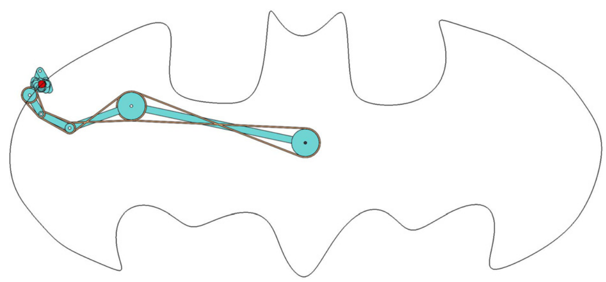

Fig. 15 The single-coupled serial chain that draws the Batman logo consists of 38 links

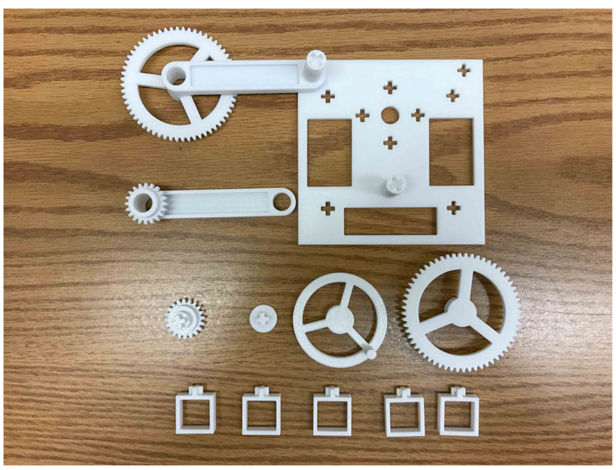

Fig. 16 A photograph of the component parts of the trifolium drawing mechanism that were manufactured from acrylonitrile butadiene styrene using additive manufacture

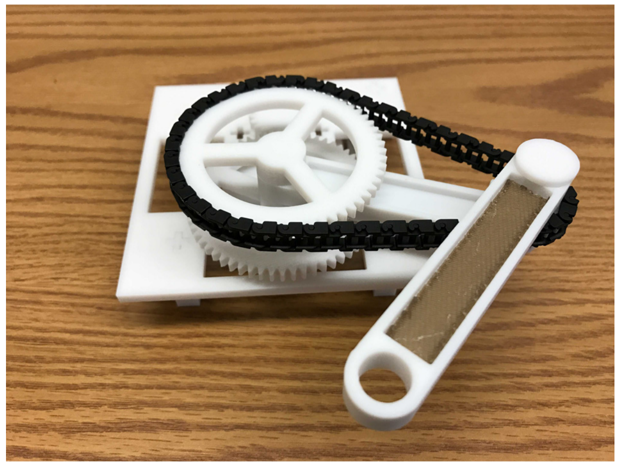

Fig. 17 A photograph of the assembled mechanism that draws the trifolium

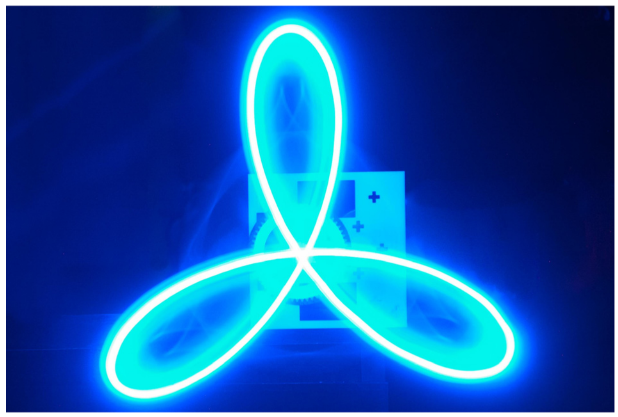

Fig. 18 The photograph shows the trifolium traced by the trajectory of a LED positioned at the drawing point of the mechanism

7 Conclusion

This paper shows how to determine the dimensions of mechanical systems that draw plane curves which have coordinate functions given by finite Fourier series, called trigonometric curves. We show how to use the coefficients of these components functions to determine the dimensions of individual Scotch yoke mechanisms that act on a belt or cable drive to move a cursor to draw the curve and how those dimensions yield two-coupled serial chains that draw the curve. Finally, we show how to determine the dimensions of a single-coupled serial chain that draws the curve with its end-point. In each case, the mechanical system has one degree-of-freedom and draws the curve exactly.

This methodology is demonstrated on three increasingly complicated curves, the trifolium, the butterfly curve, and the Batman logo curve. The single-coupled serial chain for these cases requires 2, 14, and 38 links, respectively. We present a physical prototype for the trifolium. These results contribute to our understanding of the connection between plane curves and mechanical systems illustrated by Kempe's universality theorem.

References

[1] Coros, S., Thomaszewski, B., Noris, G., Sueda, S., Forberg, M., Sumner, R. W., Matusik, W., and Bickel, B., 2013, "Computational Design of Mechanical Characters," ACM Trans. Graphics, **32**(4), p. 83.

[2] Thomaszewski, B., Coros, S., Gauge, D., Megaro, V., Grinspun, E., and Gross, M., 2014, "Computational Design of Linkage-Based Characters," ACM Trans. Graphics, **33**(4), p. 64.

[3] Nolle, H., 1974, "Linkage Coupler Curve Synthesis: A Historical Review—II: Developments After 1875," Mech. Mach. Theory, **9**(3–4), pp. 325–348.

[4] Nolle, H., 1974, "Linkage Coupler Curve Synthesis: A Historical Review—I: Developments Up To 1875," Mech. Mach. Theory, **9**(2), pp. 147–168.

[5] Koetsier, T., 1983, "A Contribution to the History of Kinematics—I: Watt's Straight-Line Linkages and the Early French Contributions to the Theory of the Planar 4-Bar Coupler Curve," Mech. Mach. Theory, **18**(1), pp. 37–42.
[6] Koetsier, T., 1983, "A Contribution to the History of Kinematics—II: The Work of English Mathematicians on Linkages During the Period 1869-78," Mech. Mach. Theory, **18**(1), pp. 43–48.
[7] Kempe, A. B., 1876, "On a General Method of Describing Plane Curves of the nth Degree by Linkwork," Proc. London Math. Soc., **VII**(102), pp. 213–216.
[8] Kempe, A. B., 1877, *How to Draw a Straight Line*, Macmillan, London.
[9] Jordan, D., and Steiner, M., 1999, "Configuration Spaces of Mechanical Linkages," Discrete Comput. Geom., **22**(2), pp. 297–315.
[10] Kapovich, M., and Millson, J. J., 2002, "Universality Theorems for Configuration Spaces of Planar Linkages," Topology, **41**(6), pp. 1051–1107.
[11] Kobel, A., 2008, "Automated Generation of Kempe Linkages for Algebraic Curves in a Dynamic Geometry System," BCS thesis, Saarland University, Saarbrucken, Germany.
[12] Saxena, A., 2011, "Kempe's Linkages and the Universality Theorem," Resonance, **16**(220), pp. 220–237.
[13] Liu, Y., and McCarthy, J. M., 2017, "Synthesis of a Linkage to Draw a Plane Algebraic Curve," Mech. Mach. Theory, **111**, pp. 10–20.
[14] Hong, H., and Schicho, J., 1998, "Algorithms for Trigonometric Curves (Simplification, Implicitization, Parameterization)," J. Symbolic Comput., **26**(3), pp. 279–300.
[15] Artobolevskii, I. I., 1964, *Mechanisms for the Generation of Plane Curves*, Pergamon Press, London.
[16] Miller, D. C., 1916, "A 32-Element Harmonic Synthesizer," J. Franklin Inst., **181**(1), pp. 51–81.
[17] Nie, X., and Krovi, V., 2005, "Fourier Methods for Kinematic Synthesis of Coupled Serial Chain Mechanisms," ASME J. Mech. Des., **127**(2), pp. 232–241.
[18] Oppenheim, A. V., and Schafer, R. W., 2010, *Discrete-Time Signal Processing*, Pearson Higher Education, Upper Saddle River, NJ.
[19] Kendig, K., 2011, *A Guide to Plane Algebraic Curves*, MAA, Washington, DC.
[20] Shikin, E. V., 1995, *Handbook and Atlas of Curves*, CRC Press, Boca Raton, FL.
[21] Fay, T. H., 1989, "The Butterfly Curve," Am. Math. Mon., **96**(5), pp. 442–443.