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CREEP BUCKLING OF A NONLINEARLY VISCOELASTIC BEAM-COLUMN

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ABSTRACT

The stability of an initially crooked, simply-supported, H-section beam-column, subjected to an axial compressive load, is investigated. The material of the column is taken to behave as a general nonlinear viscoelastic solid. It is assumed that the constitutive relation of the material can be represented by a Volterra-Fréchet functional polynomial.

Conditions sufficient to assure instantaneous, short term and long term (i.e., asymptotic) stability are established. It is shown that complete knowledge of the material creep functions (i.e., the kernels appearing in the functional polynomial representation of the stress-strain relation) is not required in order to determine stability conditions. A program of experiments to characterize the material for stability studies is presented.

A formal analogy for instantaneous and short term stability conditions is established between the column under consideration and an initially straight, concentrically loaded column fabricated from an imaginary, nonlinearly viscoelastic material. The relationship between the actual viscoelastic material under consideration and the imaginary viscoelastic material is explicitly given. A further analogy - this one between the asymptotic stability of the column under consideration and the stability of a fictitious, nonlinearly elastic column - is also established.

I. Introduction

Investigations of the stability of nonlinear hereditary structural systems give rise to the consideration of two fundamental problems. Firstly, there is the need for the development of more potent analytical procedures for the better representation of the mechanical behavior of structural materials under a variety of service conditions. (This demands a deeper study of constitutive equations.) Then, there is the requirement for the application of more refined methods of analysis to investigate the stability of the solutions of nonlinear differential, integral and integro-differential equations. This follows since the problem of the stability of nonlinear hereditary systems may be reduced to the stability of an equation in terms of such nonlinear operators.

With regard to the first problem, it may be noted that although the construction of unique, comprehensive and fully detailed constitutive equations for structural materials of significance would seem to be an important goal, at present it does not appear that such an objective is practicable. It is perhaps for this reason that most investigations concerned with the stability of particular nonlinear viscoelastic structures -- such as the creep buckling of viscoelastic columns, snap-through of viscoelastic arches, etc. -- are based on the use of quite specific nonlinear differential or quasilinear integral operators. These operators -- whose coefficients have, in some cases, been experimentally obtained -- hopefully represent the behavior of the material under conditions similar to those expected during the life of the structure. This approach may prove to be useful for the solution of specific problems, but it certainly lacks generality. Instead of attempting to obtain closed, or nearly closed, solutions to this type of problem (based on particular stress-strain relationships), it would seem at least as useful to seek pertinent properties of the solution based on a quite general statement of the material behavior.

In this regard, the use of a general nonlinear stress-strain relationship represented by means of a Volterra-Fréchet functional expansion has proved to be an expedient instrument of analysis. Such a representation allows for a general and comprehensive treatment of problems of stability. (See references [1] and [2].) Although increased efforts are being made to experimentally evaluate the kernel functions appearing in the functional expansion for a given material [3], it has been shown [1], [2] that a complete knowledge of the kernel functions is not necessary to establish the conditions of stability. Indeed, only a relatively limited amount of information is required. This paper is an example of the application of previously developed theory to the problem of creep buckling of an initially crooked column -- or equivalently, a beam-column. As a matter of convenience, the column cross-section is taken to be an H-section.

Short term stability is investigated under very general assumptions of material behavior. A program of experiments to determine the essential features of material behavior required in an analysis of short term stability is discussed. Some particular cases of interest are briefly considered. The last part of the paper deals with the determination of conditions under which the beam-column is asymptotically stable. A formal analogy between the conditions for asymptotic stability and instantaneous buckling is stated. The case of nonaging materials (and the procedure for evaluating asymptotic deflections in that instance) is briefly discussed.

Finally, it is perhaps useful to clarify the way in which the term "stability" is employed in this paper: it is used to signify stability in the sense of Lagrange [4]. Hereafter, "short term stability" will be used to denote stability of the mechanical system under consideration for all finite values of the time t. "Asymptotic stability" will signify that in addition to being stable for all finite values of t, the system is also stable for $t \to \infty$. It is noted that for the problem under consideration, "stability" may be used synonymously for "boundedness of deflection." This is so, because for the crooked column no neutral state of equilibrium is possible. Such a state could exist only for an initially straight column.

II. Constitutive Equations

An initially crooked, H-section, simply-supported column is subjected to a compressive load, $P(t)$, varying arbitrarily with time, as indicated in Figure 1. It is assumed that the load possesses a finite positive limit

$$\lim_{t \to \infty} P(t) = P_\infty < \infty \tag{1}$$

Dynamic effects will be disregarded in the present analysis. Furthermore, $P(t)$ will be taken to be zero for $t < t_o$.

The material is considered to be a general nonlinear viscoelastic one, for which the constitutive equation relating the stress σ and the strain ϵ in a uniaxial test is given by

$$\epsilon(t) = \mathop{\mathcal{F}}_{\tau=-\infty}^{\tau=t} [\sigma(\tau)], \tag{2}$$

where $\mathcal{F}$ represents a continuous nonlinear functional. (Herein, for convenience, stresses and strains will be considered positive if they are compressive.) Utilizing the generalized Weierstrass polynomial theorem for continuous functionals(due to Fréchet [5]) equation (2) may be represented to any desired degree of accuracy by a functional polynomial of the form

$$\epsilon(t) = \sum_{n=1}^{m} \frac{1}{n!} \int_{t_o}^{t^+}\int_{t_o}^{t^+}\cdots\int_{t_o}^{t^+} \sigma(\tau_1)\sigma(\tau_2)\ldots\sigma(\tau_n) f_n(t;\tau_1,\tau_2,\ldots,\tau_n)\, d\tau_1 d\tau_2\ldots d\tau_n, \tag{3}$$

where f_n are the "material creep functions". The material is assumed to be in a quiescent state for all $t < t_o$. The kernel functions f_n, which include products of delta functions of argument $t-\tau_i$ (with $i=1,2,...,n$) to account for immediate nonlinear elastic behavior, are identically zero whenever any of the arguments τ_i has a value less than the value of the argument t. It should be noted that in previous analyses of the problem [6], different explicit expressions were assumed for the material in tension and in compression. Here it is not necessary to do so, because the general constitutive law (2) already includes the possibility of a different behavior in tension than in compression.

For the sake of generality, only very mild restrictions will be imposed on the material functions. For example,

$$\epsilon^{(n)}(t^+;t_1,t_2,...,t_n) = \int_{t_1}^{t^+}\int_{t_2}^{t^+}...\int_{t_n}^{t^+} f_n(t^+;\tau_1,\tau_2,...,\tau_n)\, d\tau_1 d\tau_2 ... d\tau_n,$$

$$n=1,2,...,m, \qquad (4)$$

will be assumed to be piecewise continuous positive functions exhibiting, at most, step discontinuities at $t_i=t$.

It is evident that bounded asymptotic creep is a necessary - although not a sufficient - condition for asymptotic stability of viscoelastic structures. Therefore, when discussing asymptotic stability the following further restriction will be imposed on the functions $\epsilon^{(n)}$:

$$\lim_{t \to \infty} \epsilon^{(n)}(t;t_1,t_2,...,t_n) < \infty, \quad n=1,2,...,m, \qquad (5)$$

for all values of t_i $(i=1,2,\ldots,n)$, including $t_i \to \infty$.

Moreover it will be assumed that the material ages asymptotically. By this is meant that after a long period of time the material properties will be time invariant. This implies that for large values of the variables t and τ_i (i.e., for $t, \tau_i \to \infty$), the function $\epsilon^{(n)}$ will tend asymptotically to a limit function of the form

$$\epsilon^{(n)}(t;\tau_1,\tau_2,\ldots,\tau_n) \to \epsilon_\infty^{(n)}(t-\tau_1,t-\tau_2,\ldots t-\tau_n),\ n=1,2,\ldots m, \quad \text{\underline{for large values of t and τ_i.}} \tag{6}$$

III. Functional Equation Governing the Deflection

Referring to Figure 1, it follows from geometrical considerations (where attention has been restricted to a geometrically linearized analysis) that

$$\epsilon_2 - \epsilon_1 = -h \frac{\partial^2 w}{\partial x^2} \quad . \tag{7}$$

In the above, ϵ_1 and ϵ_2 are, respectively, the strains at the convex and concave side of the beam-column, h is the depth of the section, w_o is the initial crookedness, and w is the deflection measured from w_o. From equilibrium it follows that

$$\begin{aligned} \sigma_1 &= \frac{P}{A} - \frac{2M}{Ah} \\ \sigma_2 &= \frac{P}{A} + \frac{2M}{Ah} \ , \end{aligned} \tag{8}$$

where σ_1 and σ_2 are the stresses in the convex and concave flanges, respectively. The bending moment, M, is given by

$$M = P(w+w_o) \quad . \tag{9}$$

Eliminating $\sigma_1, \sigma_2, \epsilon_1, \epsilon_2$ and M between equations (2), (7), (8) and (9), the following functional equation is obtained

$$\mathcal{F}\left\{ \bar{\sigma}\left[1 + \frac{2}{h}(w+w_o)\right]\right\}_{\tau=t_o}^{\tau=t} - \mathcal{F}\left\{ \bar{\sigma}\left[1 - \frac{2}{h}(w+w_o)\right]\right\}_{\tau=t_o}^{\tau=t} = -h \frac{\partial^2 w}{\partial x^2} \quad , \tag{10}$$

where $\bar{\sigma}$ is the average stress, given by

$$\bar{\sigma}(t) = \frac{P(t)}{A} \quad . \tag{11}$$

Assume the initially crooked shape to be of the form

$$w_o = a \sin \alpha x \ , \ \alpha = \frac{\pi}{\ell} \ , \tag{12}$$

where ℓ is the length of the column, and further assume that

$$w = b(t) \sin \alpha x \ . \tag{13}$$

In what follows we shall refer to "a" as the "initial crookedness."

Substituting equations (12) and (13) into equation (10), and, in the usual manner for problems of this type [7], collocating the solution at the center of the bar, the following functional equation is obtained

$$\mathcal{F}_{\tau=t_o}^{\tau=t} [\bar{\sigma}\ (1+W)] - \mathcal{F}_{\tau=t_o}^{\tau=t} [\bar{\sigma}\ (1-W)] - h\ \alpha^2 b = 0 \ , \tag{14}$$

where

$$W(t) = \frac{2}{h} [a + b(t)]. \tag{15}$$

In order to investigate the asymptotic stability of the deflection function b(t), it is convenient to recast equation (14) in the form of a nonlinear functional expansion. To accomplish this, expand the functional $\mathcal{F}$ appearing in equation (14) as in equation (3), utilizing equation (15)

$$\sum_{n=1}^{m} \frac{1}{n!} \sum_{i=1}^{n} \left(\frac{2}{h}\right)^{i} [1-(-1)^{i}] \binom{n}{i} \sum_{j=o}^{i} \binom{i}{j} a^{i-j} \underbrace{\int_{t_o}^{t^+}\int_{t_o}^{t^+}\cdots\int_{t_o}^{t^+}}_{n} b(\tau_1)b(\tau_2)\ldots b(\tau_j)$$

$$\bar{\sigma}(\tau_1)\bar{\sigma}(\tau_2)\ldots\bar{\sigma}(\tau_n)\ f_n(t;\tau_1,\tau_2,\ldots,\tau_n)\ d\tau_1 d\tau_2\ldots d\tau_n - h\,\alpha^2 b(t) = 0\ . \tag{16}$$

Grouping terms of equal order, the following mth order nonlinear integral equation is obtained

$$-G_o(t) = \int_{t_o}^{t^+} b(\tau_1)\left[\bar{\sigma}(\tau_1)G_1(t;\tau_1) - h\,\alpha^2\delta(t-\tau_1)\right]d\tau_1 \ +$$

$$+\frac{1}{2!}\int_{t_o}^{t^+}\int_{t_o}^{t^+} b(\tau_1)b(\tau_2)\ \bar{\sigma}(\tau_1)\bar{\sigma}(\tau_2)\ G_2(t;\tau_1,\tau_2)\ d\tau_1 d\tau_2 + \ldots\ldots$$

$$\ldots\ldots+\frac{1}{m!}\int_{t_o}^{t^+}\int_{t_o}^{t^+}\cdots\int_{t_o}^{t^+} b(\tau_1)b(\tau_2)\ldots b(\tau_m)\ \bar{\sigma}(\tau_1)\bar{\sigma}(\tau_2)\ldots\bar{\sigma}(\tau_m)\ G_m(t;\tau_1,\tau_2,\ldots,\tau_m)$$

$$d\tau_1 d\tau_2\ldots d\tau_m, \tag{17}$$

where

$$G_r(t;\tau_1,\tau_2,\ldots,\tau_r) = r!\sum_{n=1}^{m}\frac{1}{n!}\sum_{i=1}^{n}\left(\frac{2}{h}\right)^{i}\left[1-(-1)^{i}\right]\binom{n}{i}\binom{i}{r}a^{i-r}\underbrace{\int_{t_o}^{t^+}\int_{t_o}^{t^+}\cdots\int_{t_o}^{t^+}}_{n-r}$$

$$\bar{\sigma}(\tau_{r+1})\bar{\sigma}(\tau_{r+2})\ldots\bar{\sigma}(\tau_n)\ f_n(t;\tau_1,\tau_2,\ldots,\tau_n)\ d\tau_{r+1}d\tau_{r+2}\ldots d\tau_n,\ r=0,1,2,\ldots,n, \tag{18}$$

and δ is the Dirac delta function.

IV. Conditions for Stability

Conditions for the stability of the structure will follow from the investigation of the stability of the solution of the mth order nonlinear integral equation (17). In order to carry out this investigation, it is convenient to invert equation (17) so as to obtain b(t) as an explicit functional. Equation (17) may be inverted by using an algorithm developed by Volterra [8], giving

$$b(t) = F_1(t) + \frac{1}{2!} F_2(t) + \frac{1}{3!} F_3(t) + \ldots . \tag{19}$$

where the functions $F_i(t)$ are obtained from the inversion of the following infinite triangular system of linear integral equations

$$\int_{t_o}^{t^+} F_i(\tau_1) K_1(t;\tau_1) d\tau_1 = - S_i(t), \ i = 1, 2, \ldots, \tag{20}$$

with

$$S_1(t) = G_o(t)$$

$$S_2(t) = \int_{t_o}^{t^+}\int_{t_o}^{t^+} F_1(\tau_1)F_1(\tau_2)\tilde{G}_2(t;\tau_1,\tau_2)d\tau_1 d\tau_2$$

$$S_3(t) = \int_{t_o}^{t^+}\int_{t_o}^{t^+}\int_{t_o}^{t^+} F_1(\tau_1)F_1(\tau_2)F_1(\tau_3)\ \tilde{G}_3(t;\tau_1,\tau_2,\tau_3)\ d\tau_1 d\tau_2 d\tau_3 +$$

$$+ 3 \int_{t_o}^{t^+}\int_{t_o}^{t^+} F_1(\tau_1)F_2(\tau_2)\ \tilde{G}_2(t;\tau_1,\tau_2)d\tau_1 d\tau_2 \tag{21}$$

. . .

and

$$K_1(t;\tau_1) = \bar{\sigma}(\tau_1)G_1(t;\tau_1) - h\,\alpha^2\,\delta\,(t-\tau_1) \tag{22}$$

$$\tilde{G}_r(t;\tau_1,\tau_2,\ldots,\tau_r) = \bar{\sigma}(\tau_1)\bar{\sigma}(\tau_2)\ldots\bar{\sigma}(\tau_r)G_r(t;\tau_1,\tau_2,\ldots,\tau_r),$$

$$r = 2, 3, \ldots, m. \tag{23}$$

The problem is now reduced to finding conditions for the boundedness of the functions $F_i(t)$. In the problem under consideration, it is desired to investigate short term and long term (i.e., asymptotic) stability of the structure. This requires the investigation of the boundedness of the functions $F_i(t)$ for finite values of t (short term stability) and for $t \to \infty$ (asymptotic stability).

A. Instantaneous and Short Term Stability

It is apparent that the existence of a bounded solution of equations (20) for finite values of t will depend on the behavior of the singularities associated with the kernel function $K_1(t;\tau_1)$. To explicitly separate these singularities, recall equations (18) and write the function $G_1(t;\tau_1)$ in the form

$$G_1(t^+;\tau_1) = \frac{4}{h}\sum_{n=1}^{m} \alpha_n \frac{1}{n!} \underbrace{\int_{t_o}^{t^+}\int_{t_o}^{t^+}\ldots\int_{t_o}^{t^+}}_{n-1} \bar{\sigma}(\tau_2)\ldots\bar{\sigma}(\tau_n)f_n(t^+;\tau_1,\tau_2,\ldots,\tau_n)$$

$$d\tau_2 d\tau_3\ldots d\tau_n, \tag{24}$$

where

$$\alpha_n = \frac{1}{2}\sum_{i=1}^{n}\left(\frac{2}{h}\right)^{i-1}\left[1-(-1)^i\right]\binom{i}{1}\binom{n}{i}a^{i-1} = \frac{1}{2}\,n\left[\left(1+\frac{2a}{h}\right)^{n-1}+\left(1-\frac{2a}{h}\right)^{n-1}\right]. \quad (25)$$

Introducing the function

$$B(t^+;\nu) = \frac{h}{4}\int_{\nu}^{t^+} G_1(t^+;\tau_1)\,\frac{\bar{\sigma}(\tau_1)}{\bar{\sigma}(t)}\,d\tau_1 =$$

$$= \frac{1}{\bar{\sigma}(t)}\sum_{n=1}^{m}\alpha_n\frac{1}{n!}\int_{\nu}^{t^+}\int_{t_o}^{t^+}\cdots\int_{t_o}^{t^+}\bar{\sigma}(\tau_1)\bar{\sigma}(\tau_2)\cdots\bar{\sigma}(\tau_n)f_n(t^+;\tau_1,\tau_2,\ldots,\tau_n)$$

$$d\tau_1 d\tau_2\ldots d\tau_n, \quad (26)$$

the following relation holds

$$\frac{h}{4}\,\frac{\bar{\sigma}(\tau_1)}{\bar{\sigma}(t)}\,G_1(t^+;\tau_1) = -\frac{\partial}{\partial\tau_1}B(t^+;\tau_1) . \quad (27)$$

Substituting $K_1(t;\tau_1)$ given by equation (22) in equations (20), and taking into account equation (27), equations (20) may be rewritten as follows

$$-\frac{h^2\alpha^2}{4}F_i(t) - \bar{\sigma}(t)\int_{t_o}^{t^+}F_i(\tau_1)\,\frac{\partial}{\partial\tau_1}B(t^+;\tau_1)\,d\tau_1 = -\frac{h}{4}S_i(t^+),$$

$$i = 1,\ 2,\ \ldots \quad (28)$$

Recalling equation (26), and taking into account the delta function behavior of f_n for $\tau_1 = t$, equations (28) now take the form

$$\left[\bar{\sigma}(t)B(t^+;t) - \frac{h^2\alpha^2}{4}\right]F_i(t) + \bar{\sigma}(t)\int_{t_o}^{t^+} F_i(\tau_1)\frac{\partial}{\partial\tau_1}B(t^+;\tau_1)d\tau_1 = -\frac{h}{4}S_i(t),$$

$$i = 1, 2, \ldots, \qquad (29)$$

where the delta function contribution associated with the term $\frac{\partial}{\partial\tau}B(t;\tau)$ has been separated out and explicitly included in the first term on the left hand side of equations (29). Thus the term $\frac{\partial B}{\partial\tau_1}$ appearing in these equations exhibits at most a step discontinuity at $\tau_1 = t$.

The analysis of the solution of equations (29) around the zeros of the function $\left[\bar{\sigma}(t)\ B(t^+;t) - \frac{1}{4}h^2\alpha^2\right]$ establishes that if

$$\bar{\sigma}(t)\ B(t^+;t) < \frac{1}{4}h^2\alpha^2\ , \qquad (30)$$

then the functions $F_i(t)$, and consequently the deflections, remain bounded. (See reference [1].) If a real, finite, positive value of time, t_1, exists such that

$$\bar{\sigma}(t_1)\ B(t_1^+;t_1) = \frac{1}{4}h^2\alpha^2\ , \qquad (31)$$

then buckling occurs at $t = t_1$. This time will be called the "critical time."

If, for a given material, a complete knowledge of the material creep functions, f_n, is available, then, from equation (26), the function $B(t^+;t)$

may be evaluated for a given average stress history $\bar{\sigma}(t)$. The problem of determining the critical time then reduces to the evaluation of the lowest real, finite, positive root, t_1, of equation (31). However, this is not the case which is most interesting for practical applications. In fact, although the analysis given above entirely solves the short term stability problem under very general conditions, the complete experimental determination of the kernel functions f_n for a given material is, in general, an extremely difficult and time-consuming undertaking. (See references [1] and [3].) The principal aim of this paper is to show how, from a very general point of view, it is possible to solve specific problems by means of a limited, well-planned sequence of experiments.

In order to develop this idea, recall equation (26) and write the condition for buckling given by equation (31) in the form

$$\bar{\sigma}(t)B(t^+;t) = \sum_{n=1}^{m} \alpha_n \frac{1}{n!} \int_t^{t^+} \int_{t_o}^{t^+} \cdots \int_{t_o}^{t^+} \bar{\sigma}(\tau_1)\bar{\sigma}(\tau_2)\ldots\bar{\sigma}(\tau_n)\; f_n(t^+;\tau_1,\tau_2,\ldots,\tau_n)\, d\tau_1 d\tau_2 \ldots d\tau_n = \frac{1}{4}\, h^2\, \alpha^2 . \tag{32}$$

Consider now an element of the material submitted to a certain uniaxial stress history $\bar{\sigma}(t)$. At a certain time t_i a small increment of the axial load is applied and the corresponding instantaneous increment of strain occurring at t_i is recorded. This operation may be repeated a number of times, utilizing a sequence of different small stress increments $\delta\sigma_j$, so as to generate a corresponding sequence of small instantaneous strain increments $\delta\epsilon_j$ at the time t_i. The limit

$$E_T(t_i) = \lim_{\delta\sigma_j \to 0} \frac{\delta\sigma_j}{\delta\epsilon_j}$$

will be defined as the tangent modulus of the material at time t_i. Note that in general the tangent modulus depends on the stress history $\bar{\sigma}(t)$.

It is not difficult to prove that the tangent modulus $E_T(t)$ of a material submitted to a uniaxial stress history $\bar{\sigma}(t)$ is given by

$$\frac{1}{E_T(t)} = \frac{1}{\bar{\sigma}(t)} \sum_{n=1}^{m} n \frac{1}{n!} \int_t^{t^+} \int_{t_o}^{t^+} \cdots \int_{t_o}^{t^+} \bar{\sigma}(\tau_1) \cdots \bar{\sigma}(\tau_n)\, f_n(t^+; \tau_1, \ldots, \tau_n)\, d\tau_1 \cdots d\tau_n \, . \qquad (33)$$

In order to simplify the notation, write equation (32) in the form

$$B(t^+; t) = \frac{1}{\bar{\sigma}(t)} \sum_{n=1}^{m} \alpha_n \frac{1}{n!} A_n(t) \, , \qquad (34)$$

where

$$A_n(t) = \int_t^{t^+} \int_{t_o}^{t^+} \cdots \int_{t_o}^{t^+} \bar{\sigma}(\tau_1) \cdots \bar{\sigma}(\tau_n)\, f_n(t^+; \tau_1, \ldots, \tau_n)\, d\tau_1 \cdots d\tau_n \, . \qquad (35)$$

Let $\beta_i (i=1,2,\ldots,m)$ be m different real numbers, and let $E_{T_i}(t)$ be the tangent modulus of the material submitted to a stress history $\beta_i \bar{\sigma}(t)$. The tangent modulus $E_{T_i}(t)$ is experimentally determined according to the procedure previously outlined. But now, recalling equations (33) and (35), it is seen that $E_{T_i}(t)$ is given by the expression

$$\frac{1}{E_{T_i}(t)} = \frac{1}{\bar{\sigma}(t)} \sum_{n=1}^{m} n\, \beta_i^{n-1} \frac{1}{n!} A_n(t), \quad i = 1, 2, \ldots, m . \tag{36}$$

This system of m linearly independent equations permits the evaluation of the m coefficients $A_n(t)$ at any desired time, t. This in turn allows for the determination of $B(t^+;t)$ by substitution into equation (34).

In what follows some special cases will be presented and briefly discussed.

<u>Instantaneous Buckling</u>

Instantaneous buckling will occur provided equation (32) is satisfied for $t = t_o$. Recalling equation (4), the condition for instantaneous buckling may be written

$$\sum_{n=1}^{m} \alpha_n \frac{1}{n!} \bar{\sigma}^{-n}(t_o)\, \varepsilon^{(n)}(t_o^+;t_o,t_o,\ldots,t_o) = \frac{1}{4} h^2 \alpha^2 . \tag{37}$$

The coefficients $\varepsilon^{(n)}(t_o^+;t_o,\ldots,t_o) = A_n(t_o)$ may be obtained by performing the sequence of experiments already discussed. However, in this specific case it is perhaps more convenient to evaluate the coefficients A_n

by observing that the instantaneous response of the material under a constant uniaxial stress σ_{oi} is given by

$$\epsilon_i = \sum_{n=1}^{m} \frac{1}{n!} \sigma_{oi}^{n} \, \epsilon^{(n)}(t_o^+; t_o, \ldots, t_o) \; .$$

By performing m experiments at m different stress levels $\sigma_{oi}(i=1,2,\ldots m)$ it is therefore possible to directly evaluate $\epsilon^{(n)}(t_o^+; t_o, \ldots, t_o)$.

Small Initial Crookedness

If the crookedness parameter $\frac{2a}{h}$ is small compared with unity, then it is seen from equation (25) that $\alpha_n \to n$. Hence the condition for buckling in this case will be, if due account is also taken of equations (31), (33), (34) and (35),

$$\frac{\bar{\sigma}(t)}{E_T(t)} = \frac{1}{4} h^2 \alpha^2 \; . \tag{38}$$

This result is essentially equivalent to that obtained in reference [1], if account is taken that the notations are different, and that, in this case, the problem is restricted to the investigation of an H-section.

It is worth noting that comparison of equations (32) and (33) yields that $1/B(t^+;t)$ is the tangent modulus of a virtual material, submitted to a uniaxial stress history $\bar{\sigma}(t)$ whose material creep functions are $\frac{\alpha_n}{n} f_n(t;\tau_1,\tau_2,\ldots,\tau_n)$. Since the coefficients $\frac{\alpha_n}{n} > 1$, for $n > 2$, and they increase with the crookedness parameter $\frac{2a}{h}$, then the apparent tangent modulus, $1/B(t^+;t)$, will be smaller than the actual tangent modulus of the material, $E_T(t)$, and it will decrease as the initial crookedness increases. The coefficients α_n play the role, in the problem of the crooked column, of increasing the contribution of the higher order creep functions f_n.

The Linear and the Quadratic Case

When m = 1 the condition for short term buckling is

$$\bar{\sigma}(t) = \frac{1}{4} h^2 \alpha^2 E_T(t) \quad , \tag{39}$$

where the tangent modulus

$$E_T(t) = \frac{1}{\varepsilon^{(1)}(t^+;t)} \tag{40}$$

is independent of the load history and the initial crookedness.

When m = 2, (i.e., when the material may be represented by a second order functional polynomial) $\alpha_1 = 1$ and $\alpha_2 = 2$. Then the condition for buckling is

$$\bar{\sigma}(t)\ \varepsilon^{(1)}(t^+;t) + \int_t^{t^+}\int_{t_o}^{t^+} \bar{\sigma}(\tau_1)\bar{\sigma}(\tau_2)\cdot f_2(t^+;\tau_1,\tau_2)d\tau_1 d\tau_2 =$$

$$= \frac{\bar{\sigma}(t)}{E_T(t,\bar{\sigma})} = \frac{1}{4} h^2 \alpha^2 \quad , \tag{41}$$

where $E_T(t,\bar{\sigma})$ symbolizes the tangent modulus of the material submitted to a stress history equal to exactly the true average stress history $\bar{\sigma}$ acting on the actual column. This interesting result shows that, in this case also, the apparent tangent modulus $1/B(t^+;t)$ does not depend on the initial crookedness -- although it does depend on the load history.

B. Asymptotic Stability

Let $\bar{\sigma}(t)$ be an average stress history acting on a column with an initial crookedness of amount "a," and assume that equation (30) remains satisfied (i.e., that no buckling occurs for any finite value of time). Suppose -- as was assumed in equation (1) -- that $P(t)$, and consequently $\bar{\sigma}(t)$, possesses a finite limit as $t \to \infty$. The problem is now to investigate under what conditions the deflection will remain bounded as $t \to \infty$. Hitherto, very mild restrictions were imposed on the functions f_n [or on the functions $\epsilon^{(n)}$, related to f_n through equations (4)]. To investigate asymptotic stability, equations (5) and (6) are assumed to be satisfied. The asymptotic stability of the structure will follow from the investigation of the asymptotic stability of the mth order nonlinear integral equation (17). A similar type of investigation has already been performed in a previous work [2]. It may be shown from that work that if equation (5) is satisfied, and if a function $\bar{K}(t-\tau_1)$ exists such that it approximates the function $K_1(t;\tau_1)$ [given by equation (22)] in the sense that

$$\lim_{t\to\infty} \int_{t_o}^{\infty} \left| \bar{K}(t-\tau_1) - K_1(t;\tau_1) \right| e^{-\eta(t-\tau_1)} d\tau_1 = 0, \qquad \mathrm{Re}\ \eta > 0 , \tag{42}$$

then if

$$\int_{t_o}^{\infty} \bar{K}(\tau)\, e^{-\nu\tau} d\tau \neq 0 \;, \; \text{Re}\ \nu \geqslant 0 \;, \tag{43}$$

b(t) is bounded as $t \to \infty$.

To construct a function $\bar{K}_1(t-\tau_1)$ which approximates $K_1(t;\tau_1)$ in the sense of equation (42), it is natural to utilize the asymptotic form of the "imperfect" (i.e., time-varying) kernel $K_1(t;\tau_1)$. Taking into account equations (4), (6) and (24), it is not difficult to establish that

$$\bar{K}_1(t-\tau_1) = \bar{\sigma}_\infty\, \bar{G}_1(t-\tau_1) - h\, \alpha^2\, \delta\,(t-\tau_1), \tag{44}$$

where

$$\bar{\sigma}_\infty = \lim_{t\to\infty} \frac{P(t)}{A} = \frac{P\infty}{A} \tag{45}$$

and

$$\bar{G}_1(t-\tau_1) = -\frac{4}{h}\frac{\partial}{\partial\tau_1} \sum_{n=1}^{m} \frac{\alpha_n}{n} \cdot \frac{\bar{\sigma}_\infty^{\,n-1}}{(n-1)!}\, \epsilon_\infty^{(n)}(t-\tau_1,\ \infty,\ \infty,\dots,\ \infty) \;, \tag{46}$$

approximates $K_1(t;\tau_1)$ in the sense of equation (42).

Substitution of equation (44) into equation (43) yields

$$\int_{t_o}^{\infty} g(\tau)\, e^{-\nu\tau} d\tau \neq 1 \;, \; \text{Re}\ \nu \geqslant 0 \;, \tag{47}$$

where

$$g(\tau) = \frac{4\,\bar{\sigma}_\infty}{h^2\alpha^2} \cdot \frac{\partial}{\partial\tau} \sum_{n=1}^{m} \frac{\alpha_n}{n} \cdot \frac{\bar{\sigma}_\infty^{\,n-1}}{(n-1)!}\, \epsilon_\infty^{(n)}(\tau, \infty, \ldots, \infty) \,. \tag{48}$$

Consider now that a completely aged sample of the material under consideration is subjected to a uniaxial creep test under the stress $\bar{\sigma}_\infty H(t)$ where H is the heaviside unit step function, and $\bar{\sigma}_\infty$ is <u>any</u> stress level. Let the strain response in this creep test be denoted by ϵ_∞. Now imagine another completely aged sample of this material to be subjected to a stress history $\bar{\sigma}_\infty H(t) + \delta\bar{\sigma}_\infty H(t-t_o)$, where $\delta\bar{\sigma}_\infty$ is a positive infinitesimal increment of stress. Denote the strain response in this test by $\epsilon_\infty + \delta\epsilon_\infty$. Let τ be the time increment $t - t_o$, and consider the case when $t_o \to \infty$. On physical grounds it is to be expected that the increment of strain, $\delta\epsilon_\infty(\tau)$, due to the increment of stress, $\delta\bar{\sigma}_\infty$, will be a positive, monotonically increasing function of τ. In what follows, attention will be restricted to materials which behave in such a manner.

It is easy to establish that, to within higher order terms in the infinitesimal stress increment $\delta\bar{\sigma}_\infty$,

$$\delta\epsilon_\infty(\tau) = \delta\bar{\sigma}_\infty \sum_{n=1}^{m} \frac{\bar{\sigma}_\infty^{\,n-1}}{(n-1)!}\, \epsilon_\infty^{(n)}(\tau, \infty, \infty, \ldots, \infty). \tag{49}$$

From the above discussion, it then follows that

$$\sum_{n=1}^{m} \frac{\bar{\sigma}_\infty^{\,n-1}}{(n-1)!}\, \epsilon_\infty^{(n)}(\tau, \infty, \infty, \ldots, \infty) > 0, \ \underline{\text{for any } \bar{\sigma}_\infty}\,, \tag{50}$$

and

$$\frac{\partial}{\partial\tau} \sum_{n=1}^{m} \frac{\bar{\sigma}_\infty^{\,n-1}}{(n-1)!} \; \epsilon_\infty^{(n)} (\tau, \infty, \infty, \ldots, \infty) > 0, \; \underline{\text{for any } \bar{\sigma}_\infty} \; . \tag{51}$$

If equation (48) is rewritten, taking into account equation (25), it takes the form

$$g(\tau) = \frac{2\,\bar{\sigma}_\infty}{h^2 \alpha^2} \left\{ \frac{\partial}{\partial\tau} \sum_{n=1}^{m} \frac{[\bar{\sigma}_\infty(1 + 2a/h)]^{n-1}}{(n-1)!} \; \epsilon_\infty^{(n)} (\tau, \infty, \infty, \ldots, \infty) \right.$$

$$\left. + \frac{\partial}{\partial\tau} \sum_{n=1}^{\infty} \frac{[\bar{\sigma}_\infty(1 - 2a/h)]^{n-1}}{(n-1)!} \; \epsilon_\infty^{(n)} (\tau, \infty, \infty, \ldots, \infty) \right\} . \tag{52}$$

It then follows from equation (51), recalling that $\bar{\sigma}_\infty > 0$, that $g(\tau) > 0$. Consequently equation (47) will be satisfied if, and only if,

$$\int_{t_o}^{\infty} g(\tau) \, d\tau < 1 \; , \tag{53}$$

or, equivalently,

$$\sum_{n=1}^{m} \alpha_n \frac{\bar{\sigma}_\infty^{\,n}}{n!} \; \epsilon_\infty^{(n)} (\infty, \infty, \ldots, \infty) < \frac{1}{4} h^2 \alpha^2 \; . \tag{54}$$

This equation provides a sufficient condition for asymptotic boundedness of the deflection. (It may be proved to be a necessary condition provided some further very weak restrictions on the kernels $\varepsilon^{(n)}$ are assumed.) The critical value of the asymptotic average stress, $\bar{\sigma}_\infty^*$, may be obtained by replacing the inequality sign in equation (54) by an equality sign, and solving the resulting nonlinear algebraic equation in $\bar{\sigma}_\infty^*$. It is important to note -- as is clearly indicated in equation (54) -- that the critical load does not depend on a complete knowledge of the material functions $\varepsilon^{(n)}$ appearing in the constitutive equation (3), but only on their asymptotic values.

In order to directly determine the values of $\varepsilon_\infty^{(n)}(\infty, \infty, \ldots, \infty)$ experimentally, consider a specimen of the material which has been completely aged, and submit it to a constant stress σ_i. From equation (3), and utilizing the asymptotic form of $\varepsilon^{(n)}$ given by equation (6), the expression for the asymptotic strain $\varepsilon_i(\infty)$ of the completely aged material under the constant stress σ_i is given by

$$\varepsilon_i(\infty) = \sum_{n=1}^{m} \frac{\sigma_i^n}{n!} \varepsilon_\infty^{(n)}(\infty, \infty, \ldots, \infty) . \tag{55}$$

By performing m such experiments at m different stress levels σ_i $(i = 1, 2, \ldots, m)$, the m outputs $\varepsilon_i(\infty)$ will be produced. Then the solution of the system of m linear algebraic equations given by equation (55), with $i = 1, 2, \ldots, m$, uniquely determines the values of the m coefficients $\varepsilon_\infty^{(n)}(\infty, \ldots, \infty)$, $n = 1, 2, \ldots, m$.

A significant physical interpretation of equation (54), which gives the condition for asymptotic boundedness of the deflection, follows from comparison of this equation with the condition for boundedness of the deflection under instantaneous loading. The condition for instantaneous buckling was already given by equation (37), and it is immediately recognized that the condition for boundedness under instantaneous loading is given by

$$\sum_{n=1}^{m} \alpha_n \frac{1}{n!} \bar{\sigma}^n(t_o)\ \epsilon^{(n)}(t_o^+;\ t_o, \ldots, t_o) < \frac{1}{4} h^2 \alpha^2 . \tag{56}$$

Comparison of equations (54) and (56) shows that the condition for asymptotic boundedness given by equation (54) is nothing but the condition for boundedness of the deflection of a nonlinear elastic column for which the nonlinear elastic stress-strain relationship is

$$\epsilon = \sum_{n=1}^{m} \frac{\sigma^n}{n!}\ \epsilon_\infty^{(n)}(\infty, \infty, \ldots, \infty) . \tag{57}$$

Nonaging Materials - Asymptotic Deflections

If the material is nonaging, the creep functions appearing in equation (3) will be functions of the differences of the arguments -- i.e., they will have the form $f_n(t-\tau_1, t-\tau_2, \ldots, t-\tau_n)$. Then the material functions will reduce to the form $\epsilon^{(n)}(t-\tau_1, t-\tau_2, \ldots, t-\tau_n)$. It is now easy to show that the condition for asymptotic boundedness reduces to

$$\sum_{n=1}^{m} \alpha_n \frac{\bar{\sigma}_\infty^n}{n!}\ \epsilon^{(n)}(\infty, \infty, \ldots, \infty) < \frac{1}{4} h^2 \alpha^2 . \tag{58}$$

This is essentially equivalent to equation (54), except that the asymptotic form (for large values of t and τ_i) $\epsilon_{\infty}^{(n)}(\infty, \infty, \ldots, \infty)$ of the material functions $\epsilon^{(n)}(t;\tau_1, \tau_2, \ldots, \tau_n)$ has been replaced by the actual limit

$$\epsilon^{(n)}(\infty, \infty, \ldots, \infty) = \lim_{t\to\infty} \epsilon^{(n)}(t-\tau_1, t-\tau_2, \ldots, t-\tau_n) ,$$

which is independent of the values of τ_i .

An advantage of dealing with nonaging materials is that if the input of the system is asymptotically bounded then the output of the system will not depend on the input history. This property, and its consequences, was studied in detail in reference [2]. If the theory developed in that reference is applied to the problem under consideration, it may be shown that the asymptotic deflection of the column may be computed as the deflection of a nonlinear elastic column submitted to a load $P(\infty)$, and for which the mechanical behavior of the material is governed by the nonlinear elastic stress-strain relationship

$$\epsilon = \sum_{n=1}^{m} \frac{\sigma^n}{n!} \epsilon^{(n)}(\infty, \infty, \ldots, \infty) .$$

V. Conclusions

An investigation was conducted of the conditions under which a simply-supported, H-section, nonlinear viscoelastic column is stable. The material was assumed to be a quite general nonlinear viscoelastic one whose stress-strain relationship is given by a Volterra-Fréchet functional expansion. The investigation has shown that if at a certain finite time the axial load P(t) approaches the value

$$P(t) = \frac{\pi^2 [1/B(t^+;t)]\, I}{\ell^2} \qquad , \qquad I = \frac{h^2}{4} A \ ,$$

then buckling occurs. If the function $1/B(t^+;t)$, given by equation (26), is interpreted as an apparent (or virtual) tangent modulus, then a formal analogy exists between creep buckling of a column with arbitrary initial crookedness and inelastic buckling, in the sense of Shanley [9], of an initially straight column under a concentric load [1]. Further -- and tightening the analogy -- it has been shown that $1/B(t^+;t)$ may be considered as the actual tangent modulus of a virtual material which has been submitted to the given average stress history. For the virtual material, the creep functions appearing in the functional expansion given by equation (3) are $\frac{\alpha_n}{n} f_n$ instead of f_n. Then -- as was to be expected -- the apparent tangent modulus depends not only on the stress history but also on the initial crookedness of the column. It has been demonstrated that if the stress-strain relationship is not explicitly known for a certain material, then the measurement of the actual tangent modulus of the real material at m different levels of stress history allows for the evaluation of the apparent tangent modulus.

The investigation of the conditions for which the bar is asymptotically stable was conducted under the further assumptions that the material exhibits bounded creep, ages asymptotically and the axial load possesses a finite limit as $t \to \infty$. Application of a previously developed theory [2] allowed for the determination of the condition for asymptotic stability. A formal analogy is also obtained in this case, between the condition for asymptotic stability of the real column and the condition for buckling of a corresponding nonlinearly elastic column.

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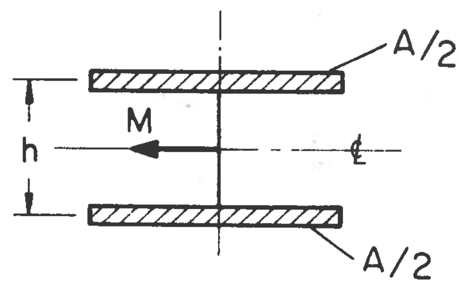

FIG. 1a CROSS SECTION OF IDEALIZED **H**-SECTION BEAM

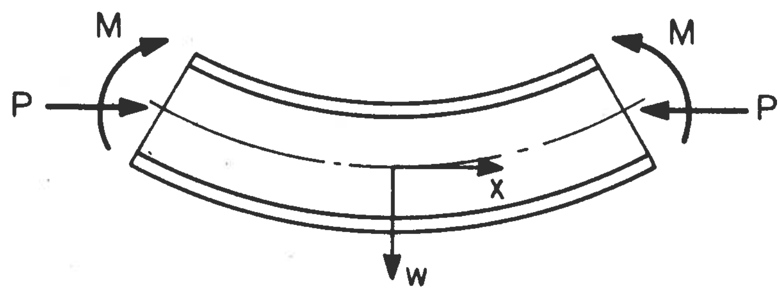

FIG. 1b PORTION OF BENT BEAM-COLUMN

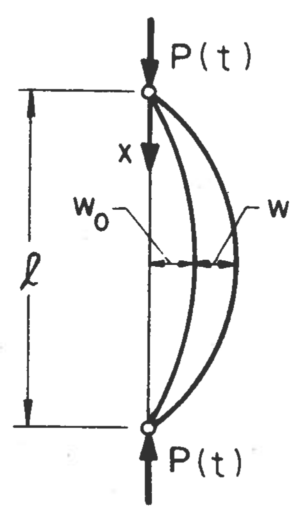

FIG. 1c DEFLECTIONS OF SIMPLY SUPPORTED COLUMN