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Idempotent Ideals in Integral Group Rings II

By

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Let π be a finite group and let $\mathbb{Z}\pi$ be its integral group ring. In a recent paper [1], the author investigated the following problem. Determine which finite groups π have the property that $\mathbb{Z}\pi$ be without non-trivial idempotent ideals (two-sided). It was proved that, if π is nilpotent then $\mathbb{Z}\pi$ has no non-trivial idempotent ideals. Also, if π is not solvable then there is a non-trivial idempotent ideal (contained in the augmentation ideal) of $\mathbb{Z}\pi$. The remaining case where π is solvable but not nilpotent is still unresolved. We conjectured that the complete solution to the problem was precisely the solvable groups.

In the present paper we prove the following result.

Theorem 1. *If π is a Frobenius group of order $p^\alpha q^\beta$ where p and q are distinct primes, then $\mathbb{Z}\pi$ has no non-trivial idempotent ideals.*

Note that in this case, by [5, Ch. 2, Theorem 7.6], the Frobenius kernel π' is a sylow subgroup of π and π is isomorphic to the semidirect product $\pi' \cdot \pi/\pi'$. Also, it is known [5, Ch. 10, Theorem 3.1] that π/π' is cyclic or generalized quaternion.

In the process of establishing Theorem 1 we verify the above conjecture for idempotent ideals contained in the augmentation ideal of $\mathbb{Z}\pi$ (Theorem 2). As in [1], the main step is to show that an idempotent ideal is $\mathbb{Z}\pi$ -projective. By means of an example we show why the argument used in [1, Theorem 3.1] to prove this step fails for groups which are solvable but not nilpotent.

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Theorem 2. *Let π be a finite group and I be the augmentation ideal of $\mathbb{Z}\pi$. Then $\mathbb{Z}\pi$ has no non-trivial idempotent ideals contained in I if and only if π is solvable.*

Proof. The proof of the necessity was given in [1, Theorem 2.1]. We shall prove the sufficiency by induction on the order of π . For this purpose let π' be a minimal normal subgroup of π . It is known and not difficult to verify that π' is a p -group for some prime p . Let I' and I'' denote the augmentation ideals of $\mathbb{Z}\pi'$ and $\mathbb{Z}\pi/\pi'$ respectively. The sequence of ideals

$$0 \rightarrow \mathbb{Z}\pi \cdot I' \rightarrow I \rightarrow I'' \rightarrow 0$$

is exact. Now let an ideal $A = A^2$ be in I . Since the image of A in I'' is idempotent, it follows by induction that $A \subset \mathbb{Z}\pi \cdot I'$. But as in the proof of [1, Theorem 3.1],

$\mathbf{Z}\pi \cdot I'^m \subset p\mathbf{Z}\pi$ for some integer m . Since π' is normal in π , $\mathbf{Z}\pi \cdot I' = I' \cdot \mathbf{Z}\pi$ and hence

$$A \subset \bigcap (\mathbf{Z}\pi \cdot I')^n \subset \bigcap p^n \mathbf{Z}\pi = (0).$$

This completes the proof of the theorem.

Note that in the proof of Theorem 2 A need only be a subring of $\mathbf{Z}\pi$.

By combining the above Theorem with the fact [4, Theorem 7] that the Jacobson radical of $\mathbf{Z}\pi$ is zero, we get the following improvement of [1, Corollary 4.2].

Corollary. *Let $\pi \neq \{e\}$ be a finite solvable group and I be the augmentation of $\mathbf{Z}\pi$. Then I^{n+1} is a proper ideal of I^n for all $n \geq 1$.*

As in [1] we point out that the intersection $\bigcap I^n = (0)$ if and only if π is a p -group by a result of Gruenberg [6, Theorem B]. For a more general version see [8].

We now turn to the proof of Theorem 1.

Proof. Let $A \neq (0)$ be an idempotent ideal of $\mathbf{Z}\pi$. In view of [1, Corollary 1.4 and Proposition 1.1], it is sufficient to prove that A is $\mathbf{Z}\pi$ -projective. For this purpose we assume that the Frobenius kernel π' is the p -syllow subgroup of π (see paragraph following Theorem 1). We have an exact sequence

$$0 \rightarrow \mathbf{Z}\pi \cdot I' \rightarrow \mathbf{Z}\pi \rightarrow \mathbf{Z}\pi/\pi' \rightarrow 0.$$

According to [1, Theorem 3.1], the image of A in $\mathbf{Z}\pi/\pi'$ is (0) or $\mathbf{Z}\pi/\pi'$. Therefore, Theorem 2 implies that $\mathbf{Z}\pi \cdot I' + A = \mathbf{Z}\pi$. Hence by the proof of [1, Theorem 3.1], it follows that the index $n = |\mathbf{Z}\pi : A|$ is finite and $(n, p) = 1$. Let $M = \mathbf{Z}\pi/A$. According to [3, Ch. XII, Corollary 2.7], the cohomology groups $\hat{H}(\pi', M) = (0)$. Since $I + A = \mathbf{Z}\pi$, it is easy to show that $\hat{H}^{-1}(\pi, M) = (0)$. Let N' denote the norm element in $\mathbf{Z}\pi'$. Since $N'M = (0)$ and $\hat{H}^0(\pi', M) = (0)$, the invariant elements in M by the action of π' , $M^{\pi'} = (0)$. But $M^{\pi} \subset M^{\pi'}$ and so $\hat{H}^0(\pi, M) = (0)$. Let S be a q -syllow subgroup of π . Since π is a Frobenius group, a result of Babakhanian [2, 1.1] implies that the restriction, $\text{res}: \hat{H}(\pi, M) \rightarrow \hat{H}(S, M)$ is an isomorphism. In particular, both $\hat{H}^{-1}(S, M)$ and $\hat{H}^0(S, M)$ are trivial. Therefore by the Tate-Nakayama-Rim theorem [7, Theorem 4.12], M has homological dimension ≤ 1 . But $0 \rightarrow A \rightarrow \mathbf{Z}\pi \rightarrow M \rightarrow 0$ is a resolution of M . Therefore A is projective and the proof of the Theorem is complete.

To conclude this paper we show why the argument used in [1] fails for idempotent ideals not contained in the augmentation ideal. In this case the key fact proved in [1, Theorem 3.1] for π nilpotent was the following. Let $L \neq (0)$ be an idempotent left ideal of $\mathbf{Z}\pi$. Let n be the index $|\mathbf{Z}\pi : L|$. Then it was shown that $(n, |\pi|) = 1$. This key fact implied that L was $\mathbf{Z}\pi$ -projective [9, Proposition 7.1]. However, for solvable groups which are not nilpotent, n and $|\pi|$ need not be relatively prime and yet the ideal could still be $\mathbf{Z}\pi$ -projective. Here is an example [1, Section 5]. Let $\pi = S_3$, the symmetric group on three letters, $\pi = \langle x, y : x^3 = 1, y^2 = 1, yxy = x^{-1} \rangle$. Let L be the left ideal of $\mathbf{Z}\pi$ generated by $2, y + 1, x^2 + x + 1$. Then $|\mathbf{Z}\pi : L| = 2$ and $L = L^2$ via the equations

$$\begin{aligned}x^2 + x + 1 &= (x^2 + x + 1)^2 - 2(x^2 + x + 1), \\y + 1 &= (y + 1)(x^2 + x + 1) - (y + 1)x(y + 1), \\y - 1 &= (y - 1)(x^2 + x + 1) + (y - 1)x(y - 1).\end{aligned}$$

But the proof of Theorem 1 implies that L is $\mathbf{Z}\pi$ -projective. More directly, let $t = 2y + xy(y + 1) - xy(x^2 + x + 1) = x + y - yx$. Then $(1 + x + y - yx)t = 2$, $(y + 1)t = y + 1$ and $(x^2 + x + 1)t = x^2 + x + 1$. Since t has zero annihilator, $L = (\mathbf{Z}\pi)t$ is isomorphic to $\mathbf{Z}\pi$ under the module mapping $\lambda \rightarrow \lambda t (\lambda \in \mathbf{Z}\pi)$. That L is $\mathbf{Z}\pi$ -free is to be expected since it is known that projective modules over $\mathbf{Z}S_3$ are free [10, p. 289].

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