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ABSTRACT

The Feynman-Dyson formulation of a perturbation expansion for quantum field theory allows one to give a general combinatorial treatment to the Feynman diagrams involved. A very simple analysis for the total number of such diagrams, $T(n, \epsilon, \rho)$, in quantum electrodynamics, leads to:

$$T(n, \epsilon, \rho) = \frac{(n!)^2}{(\epsilon!)^2 (n-\epsilon)!} \cdot \frac{n!}{\rho! \left[\frac{1}{2}(n-\rho)\right]! \, 2^{\frac{1}{2}(n-\rho)}},$$

in which n is the order of the perturbation and ϵ, ρ are the number of external electron and photon lines, respectively. The first factor is the number of different diagrams using only the electron lines and the second is that for the photon lines. In this total set of diagrams are many undesired ones; these are removed by means of generating functions. Relations which these functions satisfy are obtained, and from them one may readily find the exact numbers of diagrams desired, for not too large n. The generating functions are also used to find the asymptotic dependence on n, and it is found that this dependence is essentially unaffected by removing any specific type of graph. The sign alternations of the matrix elements in quantum electrodynamics are also considered in terms of similar generating functions. The generalization of the analysis to other types of interactions is also discussed.

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I. Introduction

The Feynman-Dyson formulation of quantum field theory in terms of the S-matrix yields a straightforward method for obtaining the matrix elements corresponding to a particular physical process by means of a perturbation expansion in powers of the coupling constant. In this theory it has been shown that each matrix element may be uniquely represented by a linear graph called a Feynman diagram. The problem of performing a calculation to a given order then separates into two parts: (a) the combinatorial problem of obtaining the number of Feynman diagrams of a particular type which there are; and (b), the actual evaluation of the associated matrix elements. It is the purpose of this paper to investigate problem (a) in some detail. In addition to finding the exact number of diagrams, one may also ask for the asymptotic behavior as the order, n , of the perturbation becomes large. Evidently it is the asymptotic behavior (together with the asymptotic properties of the associated matrix elements) which will be of interest in determining the ultimate convergence or divergence of the perturbation expansion. The question of the asymptotic behavior has been investigated by Hurst[1] using a different

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[1] C. A. Hurst, Proc. Roy. Soc. A, 214, 44 (1952).

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method from the one employed here; however, the author feels that the present method is somewhat simpler and more conventional, and also that it provides more insight into the structure of the graphs than does the former.

Due to the fact that certain diagrams may either have matrix elements which are equal to zero, or which identically cancel with others, these diagrams should be removed from consideration in the number of contributing diagrams. In addition, other graphs have infinite parts which must be removed by the renormalization technique, and so these, too, should not be counted. In the present study generating functions will be derived from which the desired number of the remaining reduced diagrams for any process can readily be obtained.

II. The Feynman-Dyson Method

It has been shown by Dyson[2] that the S-matrix, which represents the transition probability amplitude from a state at $t = -\infty$ to one at $t = \infty$, can be written as:

$$S[\infty,-\infty] = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-i}{\hbar c}\right)^n \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} d^4x_1 \cdots d^4x_n \; P\left[H^I(x_1)\cdots H^I(x_n)\right]$$

in which $H^I(x)$ is the interaction hamiltonian for the fields involved, and P is Dyson's time ordering operator. In general the interaction hamiltonian will be a sum of products of field operators, which in turn are sums of creation and annihilation operators. In order to obtain the S-matrix for a specific process one may convert these time ordered operators into ones in which all of the annihilation operators appear to the right of the creation operators by multiple permutations[3]. When this is done, one finds that in addition to the

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[2] F. J. Dyson, Phys. Rev. 75, 486 (1949).

[3] G. C. Wick, Phys. Rev. 80, 268 (1950).

terms corresponding to the operators present initially in the time ordered arrangement, there are also terms with fewer operators than before which are produced by the commutation relation of the fields. The latter simply represent the vacuum fluctuations. Now, in order to obtain the desired portion of the S-matrix, we simply choose all those terms which have the proper number of remaining operators to destroy the particles in the initial state and to create those in the final states. All other parts will have vanishing matrix elements between the states of interest.

To relate this procedure to the Feynman diagrams, we now put n points on a piece of paper, and then for those operators $\phi(x_i)$ which annihilate the initial particles or create the final ones we draw lines from the associated points, i , off the paper (different lines for each type of particle), and for pairs of operators for which we have employed the commutation relations we draw lines between the corresponding points. The total S-matrix is represented by all possible graphs drawn in this manner with the given number of each type of external line. For charged fields, as a result of the two degrees of freedom, one finds that the lines must be considered to be directed, with pairs of lines at each point, one arriving and one leaving to preserve charge conservation. Finally, if one has a non-linear coupling, for example, $\phi^3(x)$, then the permutations of the fields may occur between any of the three fields in the product at each point, and for the purposes of enumeration of graphs each of these must be counted. Thus in drawing the graphs one might consider that for such a field each point would have several different places to which a line could be drawn.

III. The Total Number of Diagrams

As an example, let us now consider quantum electrodynamics, for which

$$H^I(x) = -e/c\ \bar{\psi}\ \gamma_\mu\ \psi\ A_\mu$$

It will be simple to generalize to other interactions from the results of this case. For the total number of diagrams, the electron and photon lines may connect the n points independently of each other, so that each may be considered separately and the total number is the product of the two. Thus we consider:

(A) The Number of Electron Diagrams. From the interaction hamiltonian we see that each point of the diagram has one electron line coming to it, and one leaving it. From this it is evident that for each line coming onto the paper, there is also one which must leave it. Let the number of such free electron lines (at infinity) be ϵ . In order to find the total number of diagrams, given ϵ , we may consider that a line is drawn from each point. It is then necessary to make sure that each line then ends on any other point such that only one line goes to a given point. To begin the formation of an electron diagram, we simply choose ϵ points from the total of n points and let lines proceed to infinity from them. Each remaining point must then have a line drawn from it to some point on the paper (only one line is allowed to go to each point, of course). Then, to complete the diagram each of the ϵ remaining points which has no line coming to it must be joined to a line from off the paper. Thus we find that the total number of ways, $N^e(n, \epsilon)$

to connect lines to the electrons is:

$$N^{e}(n,\epsilon) = \frac{n!}{\epsilon!\,(n-\epsilon)!} \cdot n(n-1)\cdots(\epsilon+1)$$

$$= \frac{(n!)^2}{(\epsilon!)^2\,(n-\epsilon)!}\,, \qquad (1)$$

in which the first factor is the number of ways to choose the ϵ points for the external lines, and the second is the number of ways to connect the $(n-\epsilon)$ points internally to the n points to which they may go.

This result may also be obtained in a somewhat more illuminating fashion by considering the precise ways in which the points are joined by electron lines. Evidently the electron lines will form ϵ open polygons, and in addition there may be any number of closed loops. Let ϵ_i be the number of open polygons of i points, and λ_j the number of closed loops of j points. With this specification of a diagram, we may form the electron diagrams in

$$\frac{n!}{\prod_i \epsilon_i!\,(i!)^{\epsilon_i} \prod_j \lambda_j!\,(j!)^{\lambda_j}} \prod_i (i!)^{\epsilon_i} \prod_j \left[(j-1)!\right]^{\lambda_j}$$

different ways. The first factor is the number of ways that we can choose the points to be placed on the various lines, while the second is the number of ways that these points can then be arranged on the lines or loops. Since we have to do with directed lines, any permutation of points on the open polygons is different, while for the closed loops cyclic permutations of a configuration give nothing new whereas any other permutation gives a new diagram.

To find the total number of diagrams, we must now sum over all possible choices of the ϵ_i, λ_j with the restrictions that the number of external lines is ϵ and the total number of points is n. Thus we have

$$N^e(n,\epsilon) = \sum''_{(\epsilon_i,\lambda_j)} \frac{n!}{\prod_i \epsilon_i! \prod_j (j)^{\lambda_j} \lambda_j!},$$

in which the sum is to be taken over all ϵ_i, λ_j with the restrictions (indicated by the double prime):

$$\sum_i \epsilon_i = \epsilon, \quad \sum_i i\epsilon_i + \sum_j j\lambda_j = n.$$

To remove the restrictions and allow easier summations we may multiply the equation by $x^{\sum_i i\epsilon_i + \sum_j j\lambda_j} \times y^{\sum_i \epsilon_i}$, after which we can sum over all ϵ_i, λ_j independently, choosing only the coefficient of $x^n y^\epsilon$ in the final result. Thus:

$$N^e(n,\epsilon) = \text{Coeff. } x^n y^\epsilon \text{ in: } n! \sum \prod_i \frac{(x^i y)^{\epsilon_i}}{\epsilon_i!} \prod_j \frac{(x^j/j)^{\lambda_j}}{\lambda_j!} \quad (2)$$

$$= \text{Coeff. } x^n y^\epsilon \text{ in: } n!\, e^{\sum_i x^i y}\, e^{\sum_j x^j/j} \quad (3)$$

$$= \text{Coeff. } x^n y^\epsilon \text{ in: } n!\,(1-x)^{-1}\, e^{xy/(1-x)} \quad (4)$$

One readily finds that the coefficient of $x^n y^\epsilon$ from this generating function agrees with Eq. (1).

(B) The Photon Lines. From the interaction hamiltonian we see that each point should have one undirected photon line connected to it. To construct a diagram for these, we choose ρ points for the external lines while the remaining points, of which there are $n-\rho$, which must be an even number, are joined in pairs. Thus for the total of the photon diagrams we find

$$N^P(n,\rho) = \frac{n!}{\rho!\,(n-\rho)!} \frac{\left[\frac{(n-\rho)(n-\rho-1)}{2}\right] \cdots \left[\frac{2\cdot 1}{2}\right]}{\left[\frac{(n-\rho)}{2}\right]!}$$

where the first factor is again the number of ways to choose the ρ points for the external lines, while the second is the number of ways that $n-\rho$ points may be joined in pairs. Thus we have

$$N^P(n,\rho) = \frac{n!}{\rho!\left(\frac{n-\rho}{2}\right)!\,2^{\frac{n-\rho}{2}}} \tag{5}$$

The total number of diagrams, $T(n, \epsilon, \rho)$, is the product of Eqs. (1) and (5):

$$T(n,\epsilon,\rho) = \frac{(n!)^3}{(\epsilon!)^2(n-\epsilon)!\,\rho!\left(\frac{n-\rho}{2}\right)!\,2^{\frac{n-\rho}{2}}}.$$

If one introduces the Sterling approximation into this expression, it is found that $T(n, \epsilon, \rho)/n! \sim n^{n/2}$ as $n \to \infty$.

It is perhaps worth mentioning that one can form simple generating functions for $N^e(n, \epsilon)$ and $N^p(n, \rho)$ as:

$$\sum_{\epsilon} \frac{N^e(n, \epsilon)}{n!} y^{\epsilon} = L_n(-y),$$

$$\sum_{\rho} \frac{N^p(n, \rho)}{n!} z^{\rho} = \frac{1}{(i\sqrt{2})^n} H_n(-i\sqrt{2}\, z),$$

in which $L_n(x)$ and $H_n(x)$ are the n'th order Laguerre and Hermite polynomials, respectively[4]. Using the well known generating function for the Laguerre polynomials one can easily form the "grand generating function" for the electron lines, Eq. (4).

[4] Margenau and Murphy. The Mathematics of Physics and Chemistry. New York, D. Van Nostrand Co., 1943.

If one wishes to generalize these results to cases in which the interaction hamiltonian involves factors of the type ϕ^k, it is only necessary to note that, as pointed out at the end of Section II, one may consider each point as consisting of a set of k subpoints. Thus we need only consider the number of photon-like diagrams of $k \cdot n$ instead of n points in such a case. Evidently as k increases from one with fixed large n, the number of diagrams increases very rapidly.

IV. Removal of Unwanted Diagrams

The counting scheme of Section III for the total number of diagrams must now be modified in order to remove certain graphs for which the matrix elements either give no contribution, or must be removed by renormalization. In these categories we have graphs involving:

(a) Odd electron loops. For quantum electrodynamics, Furry's theorem states that such matrix elements will cancel in pairs.

(b) Closed parts. Dyson[2] has shown that these only multiply the S-matrix by a constant factor.

(c) Self energy parts for electron or photon lines. The former must be renormalized, the latter give no contribution.

(d) Vertex parts. These lead to the self charge divergence and must also be renormalized. The removal of (d), (e) leaves one with the number of "irreducible" graphs[5].

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[5] F. J. Dyson, Phys. Rev. 75, 1736 (1949).

(e) Separated parts with one external electron and one external photon line. These matrix elements vanish since energy and momentum cannot be conserved.

We will remove these various diagrams in order; however, it will be evident that any step may be omitted in case one wishes to retain such elements, as, for instance, in a case in which Furry's theorem does not apply so that one would like to retain the odd loops.

In order to remove the odd loops, it is only necessary to sum over the even j's in Eq. (2), with the result that

$$N_1^e(n,\epsilon) = \text{Coeff. } x^n y^\epsilon \text{ in: } n!\,(1-x^2)^{-1/2} e^{x(y+1/y)}.$$

and so

$$N_1^e(n,\epsilon) = \text{Coeff. } x^{n-\epsilon} \text{ in: } \frac{n!}{\epsilon!}(1-x)^{-(\epsilon+1/2)}(1+x)^{-1/2}.$$

For large n this may be estimated as follows. Using the binomial expansion for the two factors, we have

$$N_1^e(n,\epsilon) = \frac{n!}{\epsilon!}\sum_{k=0}^{\infty}\binom{\epsilon+k-1/2}{k}\binom{n-\epsilon-k-1/2}{n-\epsilon-k}(-1)^{n-\epsilon-k},$$

and using the expressions for the binomial coefficients, we find

$$N_1^e(n,\epsilon) = (-1)^{n-\epsilon}\frac{n!}{\epsilon!}\sum_k \frac{\Gamma(\epsilon+k+1/2)}{k!\,\Gamma(\epsilon+1/2)}\frac{\Gamma(n-\epsilon-k+1/2)}{(n-\epsilon-k)!\,\Gamma(1/2)}(-1)^k.$$

If the integral representation for the gamma functions are introduced and a change of integration variables is made one obtains

$$N_1^e(n,\epsilon) = \frac{(-1)^{n-\epsilon}\binom{n}{\epsilon}}{\Gamma(\epsilon+1/2)\Gamma(1/2)2^{\epsilon-1}} \int_0^\infty u^n e^{-u}\,du \times \times \int_{-1}^{1} dx\,(1-x)^{\epsilon-1/2} \times^{n-\epsilon} (1+x)^{-1/2}.$$

The first integral gives n! . Evidently, as $n \to \infty$ the major contribution to the second integral comes near $x = \pm 1$, and for $\epsilon > 0$, we need only consider the region about $x = -1$. If we replace $(1 - x)$ by 2 in this region, the remaining integral from 0 to 1 is a beta function so that we find

$$N_1^e(n,\epsilon) \cong \binom{n}{\epsilon} \frac{\sqrt{2}\; n!}{\Gamma(\epsilon+1/2)} \frac{(n-\epsilon)!}{\Gamma(n-\epsilon+3/2)} \qquad (6)$$

Using the Stirling approximation, one sees that the order of electron graphs is thus reduced by $n^{\frac{1}{2}}$ compared to Eq. (1).

To remove the graphs with closed parts, we note that every diagram will consist of a subdiagram (which may be the diagram itself) which has no closed parts, and a remainder which is made up of one or more closed parts with no external lines.

Thus we have

$$T(n,\epsilon,\rho) = \sum_{n_1,n_2}{}' \frac{n!}{n_1!\,n_2!}\, T(n_1,0,0)\, F(n_2,\epsilon,\rho)$$

in which $F(n, \epsilon , \rho)$ is the number of diagrams with no closed parts, and

the sum is over all n_1, n_2 with the restriction that $n_1 + n_2 = n$. The summand simply represents the number of ways that one can make up diagrams from their constituent parts, given n_1, n_2. If we again use the generating functions, we find:

$$\mathcal{F}(x, \epsilon, \rho) = \frac{J(x, \epsilon, \rho)}{J(x, 0, 0)}, \tag{7}$$

where:

$$J(x, \epsilon, \rho) = \sum_n \frac{T(n, \epsilon, \rho)}{n!} x^n,$$

$$\mathcal{F}(x, \epsilon, \rho) = \sum_n \frac{F(n, \epsilon, \rho)}{n!} x^n,$$

and in which $T(0, 0, 0)$ is defined to be $= 1$ for convenience.

The next problem to be considered is that of the number of irreducible diagrams. Before doing this, however, it should be remarked that if one had the exact propagation functions for electrons and photons it would only be necessary to consider the irreducible diagrams. Nevertheless, for the purposes of calculation the renormalized functions are not known, and the only way in which they can be obtained is by obtaining the matrix elements related to the reducible diagrams, removing the divergent parts order by order. For this purpose it is useful to have those graphs from which only the self energies in the external lines have been removed. To calculate this number, we see that a diagram may have a central part of n' points together with n_i external electron lines of i points in self energy graphs, and n_j external photon lines of j points in self energy parts. The total number of graphs is then:

$$F(n,\epsilon,\rho) = \sum''' \frac{n!}{n'! \prod_i (i!)^{n_i} n_i! \prod_j (j!)^{n_j} n_j!} \times$$

$$\times (2\epsilon)(2\epsilon-1)\cdots(2\epsilon-n''+1)\cdot\rho(\rho-1)\cdots(\rho-n'''+1) \times$$

$$\times \left[F(i,1,0)\right]^{n_i} \left[F(j,0,2)\right]^{n_j} 2^{n'''} G(n',\epsilon,\rho).$$

The sum is to be carried over all arrangements of points in groups, with the restrictions that $n = \sum_i i\, n_i + \sum_j j\, n_j + n'$, $n'' = \sum_i n_i$, and $n''' = \sum_j n_j$. $G(n', \epsilon, \rho)$ is the number of diagrams with no self energy parts in the external lines. The factors involving $(2\epsilon) \ldots (2\epsilon - n'' + 1)$ and $\rho \ldots (\rho - n''' + 1)$ represent the number of ways in which the electron and photon self energies may be inserted in the external portions of the lines, and $2^{n'''}$ is due to the fact that each photon self energy may be inserted in two different ways. Evidently the number of electron self energy diagrams of n points is $F(n, 1, 0)$, and the number for photons in $F(n, 0, 2)$. If we introduce x to keep track of the first restriction, ξ for the second, and η for the third, we find

$$\mathcal{F}(x,\epsilon,\rho) = \sum_{n'',n'''} \frac{(2\epsilon)!}{(2\epsilon-n'')!} \frac{\rho!}{(\rho-n''')!} \mathcal{G}(x,\epsilon,\rho) \times$$

$$\times \text{Coeff. of } \xi^{n''}\eta^{n'''} \text{ in: } \exp\left[\xi\, \mathcal{F}(x,1,0) + 2\eta\, \mathcal{F}(x,0,2)\right]$$

so that:

$$\mathcal{F}(x,\epsilon,\rho) = \mathcal{G}(x,\epsilon,\rho) \sum_{m'',\,m'''} \binom{2\epsilon}{m''} \left[\mathcal{F}(x,1,0)\right]^{m''} \times \binom{\rho}{m'''} \left[2\,\mathcal{F}(x,0,2)\right]^{m'''}.$$

Thus:

$$\mathcal{G}(x,\epsilon,\rho) = \frac{\mathcal{F}(x,\epsilon,\rho)}{\left[1+\mathcal{F}(x,1,0)\right]^{2\epsilon}\left[1+2\,\mathcal{F}(x,0,2)\right]^{\rho}} \tag{8}$$

To find the number of irreducible diagrams, $I(n, \epsilon, \rho)$, we proceed similarly, removing self energy parts in any of the electron or photon lines and in addition removing all vertex parts. In this case we construct the reducible diagrams by choosing an irreducible diagram of n_0 points and then putting in the various possible additional parts at the vertices or in the lines. The number of vertex parts of n points is $F(n, 1, 1)$. Thus, we have

$$F(n,\epsilon,\rho) = \sum_{(n_v,n_e,n_p)}{}'''' \frac{n!}{\prod_v (v!)^{n_v} n_v! \prod_e (e!)^{n_e} (n_e)! \prod_p (p!)^{n_p} n_p!} \times I(n_0,\epsilon,\rho) \prod_v \left[G(v,1,1)\right]^{n_v} \prod_e \left[F(e,1,0)\right]^{n_e} \prod_p \left[F(p,0,2)\right]^{n_p} \times (n_0+\epsilon)(n_0+\epsilon-1)\cdots(n_0+\epsilon-n'+1)\, 2^{n''} \left(\frac{n_0+\rho}{2}\right)\left(\frac{n_0+\rho-1}{2}\right)\cdots\left(\frac{n_0+\rho}{2}-n''+1\right) \tag{9}$$

in which the first two factors represent the number of ways to choose the points and form the subgraphs among the set of n_v vertex parts of v points, n_e electron self energy parts of e points, and n_p photon self energy parts of p points. The remaining factors represent the number of ways that the self energy parts can be inserted in the $n_o + \epsilon$ electron lines and the $\frac{n_o+\rho}{2}$ photon lines. The sum is carried over all n_i's, with the restrictions that $n = \sum v\, n_v + \sum e n_e + \sum p n_p$, $n_o = \sum n_v$, $n' = \sum n_e$, and $n'' = \sum n_p$. Note that all points in the diagram are in one of the subgraphs, and the number of ways to put the vertex parts on the irreducible graph is just the number that we are seeking. G(v, 1, 1) is chosen for the vertex parts rather than F(v, 1, 1) in order that we only count a self energy subgraph in a line once. Using the generating function method again, we find that Eq. (9) becomes

$$\mathcal{F}(x,\epsilon,\rho) = \sum_{n_o,n',n''} \frac{I(n_o,\epsilon,\rho)\,(n_o+\epsilon)!\,\left(\frac{n_o+\rho}{2}\right)!}{(n_o+\epsilon-n')!\left(\frac{n_o+\rho}{2}-n''\right)!} \times 2^{n''} \times$$

$$\times \text{Coeff. of } \xi^{n_o}\eta^{n'}\zeta^{n''} \text{ in } \exp\left[\xi\, G(x,1,1) + \eta\, \mathcal{F}(x,1,0) + \zeta \mathcal{F}(x,0,2)\right]$$

and so

$$\mathcal{F}(x,\epsilon,\rho) = \sum_{n_o,n_1,n_2} \frac{I(n_o,\epsilon,\rho)}{n_o!} \binom{n_o+\epsilon}{n'} \binom{\frac{n_o+\rho}{2}}{n''} \times$$

$$\times \left[G(x,1,1)\right]^{n_o} \left[\mathcal{F}(x,1,0)\right]^{n'} \left[2\,\mathcal{F}(x,0,2)\right]^{n''}$$

$$= \sum_{n_o} \frac{I(n_o,\epsilon,\rho)}{n_o!} \left\{ G(x,1,1)\left[1+\mathcal{F}(x,1,0)\right]\left[1+2\,\mathcal{F}(x,0,2)\right]^{1/2} \right\}^{n_o}$$

$$\times \left[1+\mathcal{F}(x,1,0)\right]^{\epsilon} \left[1+2\,\mathcal{F}(x,0,2)\right]^{\rho/2}.$$

Thus we find for the generating function:

$$\mathcal{J}(\phi(x), \epsilon, \rho) = \frac{\mathcal{F}(x, \epsilon, \rho)}{[1 + \mathcal{F}(x, 1, 0)]^{\epsilon} [1 + 2\mathcal{F}(x, 0, 2)]^{\rho/2}} , \quad (10)$$

in which

$$\phi(x) = \mathcal{G}(x, 1, 1) [1 + \mathcal{F}(x, 1, 0)] [1 + 2\mathcal{F}(x, 0, 2)]^{1/2} .$$

In order to find $I(n, \epsilon, \rho)$, we must invert the series $\phi = \phi(x)$ and find $x = x(\phi)$. This is then substituted into the right hand side of Eq. (10), and a comparison of equal powers in ϕ gives the desired result.

To remove the graphs with separated parts of only one external electron and one photon line, we again write the equation satisfied by such diagrams:

$$F(n, \epsilon, \rho) = \sum_{(n_\nu)}''' \frac{n!}{(n')! \prod_\nu (\nu!)^{n_\nu} n_\nu!} H(n', \epsilon, \rho) \prod_\nu [F(\nu, 1, 1)]^{n_\nu}$$

in which the restrictions on the sum are $n = n' + \sum_\nu \nu n_\nu$, $\epsilon = \epsilon' + \sum_\nu n_\nu$, and $\rho = \rho' + \sum_\nu n_\nu$. This equation is treated as before, and one finds

$$\mathcal{H}(x, y, z) = \mathcal{F}(x, y, z) \exp[-yz\mathcal{F}(x, 1, 1)] ,$$

in which $\mathcal{F}(x, y, z)$ is the "grand generating function" for the number of diagrams:

$$\mathcal{F}(x, y, z) = \sum_{\epsilon, \rho} \mathcal{F}(x, \epsilon, \rho)\, y^{\epsilon} z^{\rho}$$

and

$$\mathcal{H}(x, y, z) = \sum_{n, \epsilon, \rho} \frac{H(n, \epsilon, \rho)}{n!}\, x^{n} y^{\epsilon} z^{\rho}.$$

V. Conclusion

The generating functions which were derived in Section IV have a twofold use. In the first place, they may be used in conjunction with an exact calculation of the matrix elements for a particular process, as a check to ensure that all of the elements have been considered. Since the number of diagrams increases extremely rapidly with n, such a check is quite useful for all but the very low orders. Secondly, an investigation of the asymptotic behavior may yield valuable evidence regarding the overall convergence of perturbation expansions in field theories. From Eqs. (7), (8) it is not difficult to show that the removal of closed parts and self energy parts in external lines does not affect the asymptotic behavior of the number of matrix elements. The treatment of Eq. (10) is more difficult, but Hurst[1] has shown that the number of irreducible diagrams of a particularly simple subset is asymptotically also $\sim n^{3n/2}$, so that the asymptotic behavior is not effectively changed.

If one now considers the sum of all the matrix elements, he finds that in order that the perturbation expansion converge, the average contribution per element to the S-matrix must decrease with n at least as rapidly as

$n^{-n/2}$, since $(n!)^{-1}$ multiplies the n'th term in the expansion. This extremely rapid decrease seems to put a severe requirement on any field theory*. One may

* It has been shown by Hurst, Proc. Cambridge Phil. Soc. 48, 625 (1952), that for the simple case of a ϕ^3 contact coupling of a boson field to itself the perturbation expansion is indeed divergent.

also remark that the magnitude of the coupling constant would seem to have rather little to do with the convergence of the theory. Of course if quantum electrodynamics does not converge, the excellent agreement (after renormalization) with experiment would lead one to believe that somehow he is dealing with an asymptotic expansion. If this is the case, the size of the coupling constant will be essential, since for small values the series carried out to a minimum term may be an excellent approximation, while for a larger constant there may not even be such a minimum and there may be no good approximation.

With regard to the convergence of the expansion we must still notice that the signs of the various matrix elements are not necessarily alike, and there is still the possibility that judicious cancellations could produce a sufficiently rapid decrease in the average of the elements, even though the average magnitude remains large. For quantum electrodynamics such a sign alternation is associated with the various electron diagrams due to the fact that the electron field satisfies anti-commutation relations. It has been shown by Dyson[2] that each matrix element carries an intrinsic sign which is given by $(-1)^p$, in which p is the number of free electron lines of an even number of points plus the number of even loops. The sum of the numbers of matrix elements with this sign included can readily be obtained, either with all loops allowed or with the odd ones removed. In the first case, we multiply the summand in Eq. (2) by the sign

factor and then sum as before. We may let

$$p = \sum (i-1)\,\epsilon_i + \sum (j-1)\,\lambda_j ,$$

since the odd parts in i, j will not contribute to the sign, and due to the restrictions on ϵ_i, λ_j we find

$$p = n - \epsilon + \sum \lambda_j$$

If we multiply the terms in Eq. (2) by this factor and then sum, we find:

$$S^e(n,\epsilon) = \text{Coeff. of } x^n y^\epsilon \text{ in: } (-1)^{n-\epsilon}\, n!\,(1-x)\exp\left(\frac{xy}{1-x}\right),$$

$$= (-1)^{n-\epsilon}\,\frac{n!}{\epsilon!}\binom{n-2}{n-\epsilon},$$

where $S^e(n, \epsilon)$ is the sum over all the electron diagrams with the above choice of signs. Eq. (11) differs from Eq. (1) only by the factor $\epsilon(\epsilon - 1)/n(n-1)$. If one removes the odd loops, then he finds:

$$S_1^e(n,\epsilon) = \text{Coeff. of } x^n y^\epsilon \text{ in: } n!\,(-1)^{n-\epsilon}(1-x^2)^{1/2}\exp\left(\frac{xy}{1-x}\right).$$

This may be estimated as was $N'^{(e)}(n, \epsilon)$ in Section IV, with the result that

$$S_1^e(n,\epsilon) \cong (-1)^{n-\epsilon}\,\frac{n!}{\epsilon!}\,\frac{(n-4)!}{\Gamma(\epsilon-\tfrac{1}{2})\,\Gamma(n-\epsilon-\tfrac{1}{2})\,2^{5/2}},$$

which is of order n^{-2} times Eq. (6). These numbers are still exceedingly large, and the convergence would seem doubtful. There is still a difficulty in the signs since even with the above choice of sgin, the remaining part of the matrix elements is not positive definite, and so one cannot be sure that this will not produce additional cancellation. As an argument in favor of convergence, it is to be noted that since all but a fraction n^{-2} of the graphs are cancelled by the alternation in sign considered, it would only be necessary that the matrix elements be changed slightly (though exceedingly judiciously) from a constant value over the entire set in order to produce convergence. That is, it is only necessary to assume that the ratio of the average contribution to the matrix elements of one sign compared to those of the other is $1 + O(n^{-2})$ in order to produce complete cancellation. Finally, it may be remarked that the Dyson formulation of the S-matrix represents an infinite series in which each term is also a series. Procedures for obtaining the S-matrix which are based on a pattern-type of approximation essentially are based on a rearrangement of that double series. Thus, even though the perturbation theory were to converge due to fortuitous cancellations it is not at all necessary that such rearranged series would converge or that they would yield the same results as perturbation theory even if convergent. The equivalence of the two expansions would of necessity be based on the absolute convergence of the double series, rather than on the far weaker condition of ordinary convergence of the original single series.

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