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Safe Platooning in Automated Highway Systems

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Safe Platooning in Automated Highway Systems¹

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Abstract

This report addresses the problem of designing safe controllers for the hybrid system composed by the interaction of the regulation and coordination layers in the hierarchical PATH AHS architecture. Traffic is assumed to be organized into platoons of closely spaced vehicles. Conditions to achieve safe platooning under normal mode of operation are investigated. The notion of safety is related with the absence of collisions that exceed a given relative velocity threshold. State dependent safety regions for the platoons are designed in such a way that, whenever the state of a platoon is inside these safety regions, it is guaranteed that platoon maneuvering will be safe and follow that the behavior prescribed by the finite state machines that compose the coordination layer. It is shown that it is possible to design control laws that keep the state of the platoons inside these safety regions. Velocity profiles inside these safety regions are derived for all the single lane maneuvers and a nonlinear velocity tracking controller is designed to track these profiles. This controller attempts to complete the maneuvers with comfort in minimum time, whenever safety is not compromised. The results obtained allow one to decouple the design and verification of the regulation and coordination layers in the PATH AHS architecture. The overall complexity of the design and verification of the AHS as an hybrid system is therefore greatly reduced. The regulation layer control schemes presented in this report were implemented and tested using SmartPath AHS simulation software. Simulation results were in complete agreement with theoretical predictions.

Keywords

Automated Highway Systems, safe platooning, Safety regions, feedback based maneuvers, velocity tracking control, hybrid systems.

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Executive Summary

The increment in highway capacity in the PATH AHS architecture proposed in (Varaiya and Shladover, 1991) is achieved by organizing traffic into platoons of closely spaced vehicles. The tight spacing between vehicles within a platoon prevents intraplatoon collisions at high relative velocities, while the large gaps between platoons prevent interplatoon collisions. Platoons are formed or broken up by two basic maneuvers (Varaiya, 1993; Hsu *et al.*, 1991): join and split. The relative motion between platoons during the join and split maneuvers increases the risk of high relative velocity collisions and therefore compromise a safe operation.

The PATH AHS, currently under design, is supposed to be fully automated, and therefore a safe operation is of primary concern. All the tasks that directly involve the control of the motion of vehicles are executed by the coordination and regulation layers. For this reason, when the safe operation of the PATH AHS hierarchical architecture has to be guaranteed it is necessary to study the hybrid system composed by the interaction between these two layers.

The main contribution in this report is related to safe platooning in the regulation layer level. The notion of safety is that no platoon is allowed to collide with the platoon ahead of it at a relative velocity greater than a prescribed limit. The results show that for a safe normal mode operation of AHS, it is necessary to establish bounds on the parameters that determine the vehicle's behavior during the execution of the regulation layer maneuvers. It is shown that, under the set of safety related constraints introduced in this report, the optimal safe strategy for the vehicles joining or splitting consists in applying full brakes when the vehicle ahead applies and holds maximum braking, as originally presented in (Frankel *et al.*, 1994; Puri and Varaiya, 1995; Li *et al.*, 1997a). Collision propagation in the highway is analyzed, although under restrictive assumptions. It is concluded that, with a similar approach that the one used for the join and split control laws, this collision propagation can be avoided by constraining the behavior of platoons executing the leader control law. It is also shown that it is possible to design feedback control laws for the regulation layer such that the overall safety of the AHS can be guaranteed, under the given notion of safety.

Most importantly, the results for safe platooning are also analyzed for the case in which no collisions are desired to occur during the execution of AHS maneuvers. It is found that it is possible to avoid collisions when platoons are maneuvering if vehicles' braking deceleration is controlled so as to make the braking capability of any vehicle larger than the braking capability of the vehicle ahead in the same platoon. Combining the results in this report with the ideas of (Swaroop, 1994), expressions to guarantee this braking capability requirements are presented.

These results allow to decouple the design and verification of the regulation and coordination layers. The behavior of the regulation layer is guaranteed to conform with the required behavior by the finite state machines that compose the coordination layer. The overall complexity of the design and verification of the AHS as an hybrid system is therefore greatly reduced.

Based on the safe platooning analysis, velocity profiles are derived for all the single lane maneuvers. These profiles are described in the state space of the platoons' relative

motion and are therefore suited for feedback control implementation. A non-linear feedback velocity tracking controller is presented. This controller allows the maneuvers to be completed in minimum time and with comfort values of jerk and acceleration, whenever safety is not compromised. The simulation results presented illustrate the effectiveness of the designed control laws. The approach here presented to design the control laws for the maneuvers in the normal mode of operation of the regulation layer is also being applied to the degraded mode maneuvers. Simulation results in SmartPath (Eskafi *et al.*, 1992) are presented.

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Chapter 1

Introduction

The concept of Automated Highway Systems (AHS) has been proposed to increase capacity and safety in current surface transportation systems (Varaiya, 1993). The increment in highway capacity in the AHS architecture presented in (Varaiya and Shladover, 1991) is achieved by organizing traffic into platoons of closely spaced vehicles. The tight spacing between vehicles within a platoon prevents intraplatoon collisions at high relative velocities, while the large gaps between platoons prevent interplatoon collisions. The relative motion between platoons during the join and split maneuvers, that allow to form and brake platoons, increases the risk of high relative velocity collisions and therefore compromise a safe operation.

The AHS architectures in (Varaiya and Shladover, 1991) consists of five hierarchical layers (Varaiya and Shladover, 1991): network, link, coordination regulation and physical layer (see figure 1.1). There are different abstractions for each layer. In the physical and regulation layers, the abstraction is a continuous time model of the closed loop controlled vehicle dynamics. In the coordination layer, the execution of maneuvers is modeled through finite state machines that incorporate the structured communication between vehicles. The link layer uses a flow model to abstract macroscopic highway vehicular density and traffic flow. The proper abstraction for the network layer is yet to be determined. For examples of these different abstractions the reader is advised to consult, for example, (Hsu *et al.*, 1991; Swaroop *et al.*, 1994; Godbole and Lygeros, 1994; Frankel *et al.*, 1996; Li *et al.*, 1997b; Rao and Varaiya, 1994; Papageorgiou, 1990; Papageorgiou *et al.*, 1990).

This report addresses one important control problem in the AHS hierarchical architecture of the the California PATH program in (Varaiya and Shladover, 1991): the design of safe controllers for the regulation layer maneuvers.

The PATH AHS design in (Varaiya and Shladover, 1991) envisions fully automated lanes. A safe operation of these lanes is of primary concern. All the tasks that directly involve the control of the motion of vehicles are executed by the coordination and regulation layers. For this reason, when the safe operation of the PATH AHS hierarchical architecture has to be guaranteed it is necessary to study the interaction between these two layers. Originally, the coordination and regulation layer design and verification for safety were performed independently (Hsu *et al.*, 1991; Godbole and Lygeros, 1993). The underlying assumption was that, once the performance of each layer was verified, the properties achieved would

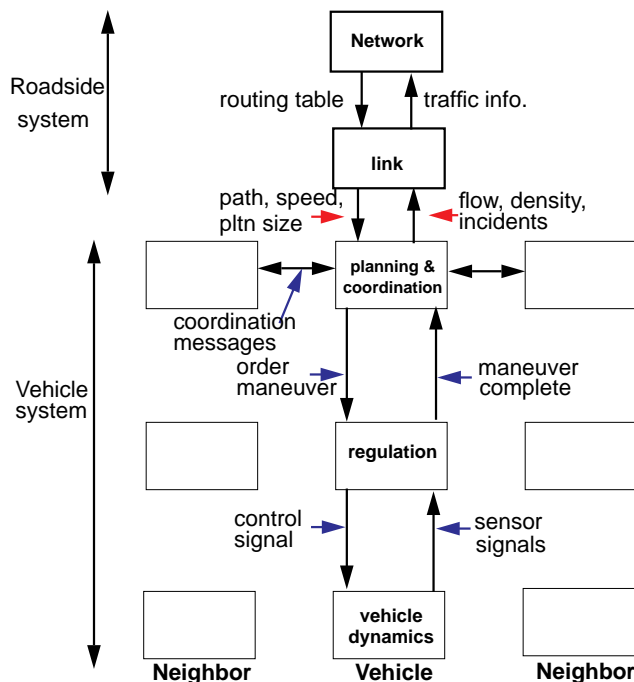


Figure 1.1: Hierarchical architecture of AVHS in the PATH program

survive after interconnection. Early simulations in SmartPath (Eskafi *et al.*, 1992) indicated that this was not the case. High speed collisions between platoons were reported in (Lygeros and Godbole, 1993). As a consequence, safe operation could not be guaranteed. In an attempt to properly model the interaction between the coordination and regulation layers, the use of hybrid systems tools was proposed in (Puri and Varaiya, 1994; Godbole *et al.*, 1994; Godbole, 1994).

Two complementary approaches have been taken to guarantee a safe operation the AHS architecture in (Varaiya, 1993). One approach is to design the regulation layer control laws in such a way that a safe operation of the AHS is guaranteed in the normal mode of operation (Frankel *et al.*, 1994; Li *et al.*, 1997a). The other approach, in (Lygeros *et al.*, 1995), is to extend the original set of normal mode maneuvers to handle degraded conditions of operation that could imply a safety risk. The design and implementation of the coordination layer finite state machines and the regulation layer control laws for these degraded modes of operation is in progress.

This report presents results that are based on the first approach mentioned above. As in (Frankel *et al.*, 1994; Li *et al.*, 1997a), the notion of safety is related with the absence of collisions between platoons that exceed a given relative velocity threshold. The safety of platoons in an AHS under this notion is analyzed. The aim is to effectively allow the decoupling of the design and verification analysis of the regulation and coordination layers. A set of regulation layer feedback based control laws is designed which guarantees that the state in the discrete event system that constitute the coordination layer evolves according with design specifications. In essence, the control laws of the continuous dynamical system that constitute the regulation layer are designed so as to make the discrete event system

that constitute the coordination layer verifiable and predictable. In this way, the safety of the hybrid system composed by the coordination and regulation layers can be guaranteed.

As the behavior of a platoon is determined by the control law that is applied to its leader, the results presented focus on the control laws that are applied to platoon leaders. Under normal operation, there are five control laws for the leader of a platoon: leader law, join law, split law, decelerate to change lane law and change lane law. The leader law is used to keep a platoon traveling at a target velocity and at a safe distance from the platoon ahead. Join and split laws are used to perform these longitudinal maneuvers. The decelerate to change lane law is used to execute an additional longitudinal maneuver that creates a safe distance from platoons in different lanes before a change lane can occur. The change lane law controls the lateral motion of a vehicle when it goes from one lane to another. The safety analysis of the follower law, that applies to non-leader vehicles in a platoon was carried out in (Swaroop, 1994; Swaroop and Hedrick, 1996).

Relevant prior work is analyzed below. The controller designed in (Godbole and Lygeros, 1993) relied on the use of nominal open-loop trajectories that the platoon executing the control law attempted to track. Control laws were safe and comfortable for passengers under normal circumstances. However, when the platoon that is ahead of the one performing the maneuver undergoes large accelerations or decelerations, comfort and safety can be compromised. If the acceleration capabilities of the platoon tracking the trajectory are lower than expected, the maneuvers may not complete at all.

In (Frankel *et al.*, 1994) the design of feedback based control laws that allow a safe operation of the regulation layer in the normal mode of operation was presented. The controllers used a finite-state machine that switched among feedback laws, in order to keep the velocity of the platoon within a safety limit. The controllers also kept the jerk and acceleration within comfort boundaries, except when safety was not compromised. Completion of the maneuvers in this design did not depend on meeting a desired open-loop acceleration trajectory. The control laws prevented even low-speed collisions in all but the most extreme cases of lead platoon deceleration. If the platoon ahead applied and hold maximum braking, a collision could still occur, but the relative velocity at impact would be within a specified acceptable limit. The control laws were designed under the assumptions that all platoons had the same acceleration capabilities and that the linearized platoon dynamics can be represented using a second order model with a pure delay.

In (Li *et al.*, 1997a) the safety of the regulation layer control laws designed in (Frankel *et al.*, 1994) was rigorously proved, under the same assumptions described above. A unified control strategy for the single lane control laws: leader, merge, split and decelerate to change lane was presented. The controller design was realized in two stages. In the first stage, for each control law, a desired velocity profile for the platoon leader was derived. This profile guarantees that high speed collisions will be avoided under single lane disturbances. Whenever safety is not compromised, the platoon will attempt to achieve a target velocity and separation from the platoon ahead in minimum time and by using acceleration and jerk within comfort limits. In the second stage, a nonlinear velocity tracking controller was designed. This controller allows the platoon to track the desired velocity within a given error bound. As in (Frankel *et al.*, 1994), the cost of improved safety and comfort is in the increased time that a maneuver takes to be completed. The results obtained in (Li *et al.*, 1997a) were

independently confirmed in (Puri and Varaiya, 1995) under similar assumptions.

In (Lygeros, 1996a) the decoupling approach for the design of the regulation and coordination layers is presented in a more general setting for hybrid systems. A game-theoretic approach is used to formulate the problem of AHS safe operation. Controllers for the regulation layer are assumed to be derived from the optimal solution of a two players, zero sum game. However, in some of the maneuvers, such as the join or split, no explicit feedback controller design is presented that guarantees safety performance, when the parameters that determine the behavior of platoon leaders during the maneuvers are fixed. These parameters include, for example, the acceleration capabilities, the braking delays, the relative impact velocity threshold, etc.

In this report the results in (Li *et al.*, 1997a) are extended. The case when the two platoons involved in a maneuver have different braking capability is now considered. This situation was first discussed in (Lygeros, 1996a), where some simulation examples showed the difference in acceleration capability may require a different control strategy under extreme conditions. The results here presented show that, in order for the operation of AHS to be safe under all normal mode operating conditions, it is necessary to establish bounds on the parameters that determine the vehicle's behavior during the execution of the regulation layer maneuvers. The effect of these bounds is to rule out the cases reported in (Lygeros, 1996a) in which safety can be compromised because of the platoons' different braking capabilities. It is also shown that, if joins and splits are to be executed, it is not possible to guarantee the existence of feedback based solutions for the optimal control in (Lygeros, 1996a) unless the maneuvers satisfy a set of safety related constraints presented in this report. Moreover, under these safety related constraints, the optimal safe strategy for the vehicles joining or splitting consists in applying full brakes when the vehicle ahead applies and holds maximum braking, as originally presented in (Frankel *et al.*, 1994; Puri and Varaiya, 1995; Li *et al.*, 1997a).

Collision propagation in the highway is analyzed. It is concluded that, with a similar approach to the one used for the join and split control laws, this collision propagation can be avoided by constraining the behavior of platoons executing the leader control law.

The results for safe platooning are also analyzed for the case where no collisions are to occur during the execution of AHS maneuvers. It is found that it is possible to avoid collisions when platoons are maneuvering if vehicles' braking deceleration is controlled. The goal is to make the braking capability of any vehicle larger than the braking capability of the vehicle ahead in the same platoon. Interplatoon distance is designed so as to keep safety between platoons when this controlled braking is applied. Following the ideas of (Swaroop, 1994) and the results derived for the safety of the leader law, expressions to constrain the braking capability of vehicles involved in a maneuver are presented.

Desired velocity profiles for each one of the single lane maneuvers are derived. These profiles allow maneuvers to be completed in minimum time, while guaranteeing that the state during the maneuver will always be safe. Comfort accelerations and jerks are used, whenever safety is not compromised. The non-linear velocity tracking controller that is used to track these desired velocity profiles uses now the full state of the platoons relative motion, as opposed to the design in (Li *et al.*, 1997a) in which only a part of the state was used. The results obtained with the use of this controller are illustrated with simulation examples.

If the requirement of safe operation of AHS in normal mode of operation is to have

meaning, it can not be separated from the requirement of high performance that these systems are expected to deliver. An empty AHS is always safe. While the approach presented in (Lygeros, 1996*a*) also addresses the problem of safe operation of AHS, the results presented do not allow to draw conclusions on the performance of the system. This occurs because of the hierarchical structure of cost functions that was adopted in (Lygeros, 1996*a*). In this structure safety has, as expected, the top priority. Whenever safety is compromised, the other lower priority cost functions, normally related to system performance, are ignored. This situation, in which safety is the only cost function that is considered, can happen even in absence of disturbances. The procedure that is suggested in (Lygeros, 1996*a*) to overcome this problem is to allow multiple interactions between the regulation and coordination layers. There is not, however, any indication of the implications of this interaction on the overall safety or performance.

In the approach presented in this report it is proposed to constrain the behavior of the PATH AHS regulation layer in the normal mode of operation in such a way that both the safety and performance requirements are considered simultaneously. In this framework, safety and performance are always guaranteed in the absence of single lane disturbances. Safety is always guaranteed, even in the presence of this kind of disturbances.

This report is divided in four chapters and one appendix. Chapter 2 contains the safe platooning analysis and chapter 3 the design of the velocity tracking controller that executes the safe maneuvers. In chapter 4 the simulation results for the regulation layer control laws are presented. Chapter 5 contains the conclusions of the report. Appendix A includes detailed calculations for derivation in chapter 3.

Chapter 2

Safe Platooning

2.1 Two platoons at a time with a second order model

2.1.1 Safe control

In this section conditions to guarantee the safety of two adjacent platoons in the same lane are derived. Results will be extended to the general case in the following section. The notion of safety is similar to the one used in (Frankel *et al.*, 1994; Li *et al.*, 1997a; Lygeros, 1996a): the platoon performing the control law will not collide with the platoon ahead at a relative velocity greater than a prescribed limit, v_{allow} .

Assumption 2.1 *Safe control laws analysis is accomplished under the following assumptions:*

1. *Single lane maneuvers.*
2. *Bounded acceleration of vehicles in the highway.*
3. *Positive velocity of vehicles, i.e. reverse motions will never occur.*
4. *Maximum braking acceleration can be achieved d seconds after a full braking command is issued.*
5. *Bounded maximum velocity of vehicles in the highway.*

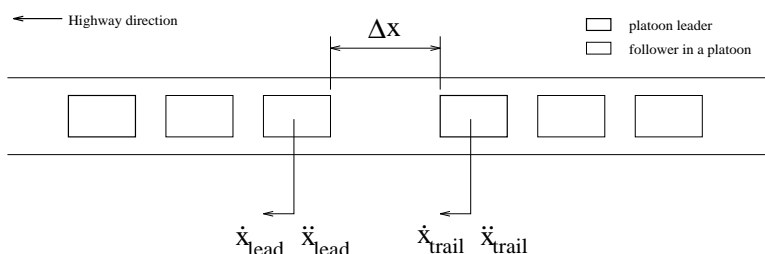


Figure 2.1: Notation for two platoons on the highway.

Consider two platoons traveling on the automated highway as shown in Fig. 2.1, the trail platoon is moving behind the lead platoon in the same lane. Let $x_{lead}(t)$ be the position

at time t of the lead platoon's back and $x_{trail}(t)$ be the position at time t of the trail platoon's front. Let $\dot{x}_{lead}(t)$, $\dot{x}_{trail}(t)$, $\ddot{x}_{lead}(t)$ and $\ddot{x}_{trail}(t)$ denote the first and second time derivatives of these positions at time t . $\dot{x}_{lead}(t)$ and $\dot{x}_{trail}(t)$ will also be denoted by $v_{lead}(t)$ and $v_{trail}(t)$ respectively. Let the accelerations of the lead platoon be $w(t)$ and that of the trail platoon be $u(t)$.

If an input/output linearization procedure is applied to a dynamic model of the vehicles, as in (Swaroop, 1994), the dynamics of the platoons' motion become

$$\ddot{x}_{lead}(t) = w(t) , \quad (2.1)$$

$$\ddot{x}_{trail}(t) = u(t) , \quad (2.2)$$

where $w(t) \in [-a_{min}^{lead}, a_{max}^{lead}]$ and $u(t) \in [-a_{min}^{trail}, a_{max}^{trail}]$ for all time t , $a_{min}^{lead}, a_{max}^{lead}, a_{min}^{trail}, a_{max}^{trail} > 0$; $w(t)$ and $u(t)$ are such that $\dot{x}_{lead}(t)$ and $\dot{x}_{trail}(t)$ remain positive for all t .

Define the relative distance between the platoons to be

$$\Delta x(t) = x_{lead}(t) - x_{trail}(t) . \quad (2.3)$$

For the analysis of platoon collisions, the relevant dynamics are independent of the absolute positions, x_{lead} or x_{trail} . Hence, the dynamics of the relative motion between the lead and trail platoons is given by

$$\Delta \dot{x}(t) = \dot{x}_{lead}(t) - \dot{x}_{trail}(t) , \quad (2.4)$$

$$\Delta \ddot{x}(t) = \ddot{x}_{lead}(t) - \ddot{x}_{trail}(t) = w(t) - u(t) , \quad (2.5)$$

$$\dot{v}_{lead}(t) = w(t) , \quad (2.6)$$

where $\Delta \dot{x}(t)$ and $\Delta \ddot{x}(t)$ denote the relative velocity and the relative acceleration between the platoons. Eq. (2.6) is necessary to account for the independence of $w(t)$ and $u(t)$.

Definition 2.1 (Unsafe impact) *An unsafe impact is said to happen at time t if*

$$\Delta x(t) \leq 0 \quad \text{and} \quad -\Delta \dot{x}(t) \geq v_{allow} , \quad (2.7)$$

with $v_{allow} \geq 0$ being the maximum allowable impact velocity.

The set $X_{MS} \subset \mathfrak{R}^3$ denotes the set of all triples $(\Delta x, \Delta \dot{x}, v_{lead})$ such that (2.7) is not satisfied, $\Delta x > 0$ and $0 \leq v_{lead} \leq v_{max}$, where v_{max} is the maximum highway velocity for the lead platoon.

Definition 2.2 (Safe control) *A control law for the acceleration of the trail platoon, $u(t)$, is said to be safe for an initial condition $(\Delta x(0), \Delta \dot{x}(0), v_{lead}(0))$ if the following is true: For any arbitrary lead platoon acceleration $w(\tau)$; $\tau \geq 0$ such that $w(\tau) \in [-a_{min}^{lead}, a_{max}^{lead}]$ and $0 \leq v_{lead}(\tau) \leq v_{max}$, $(\Delta x(t), \Delta \dot{x}(t), v_{lead}(t)) \in X_{MS}$ for all $t \geq 0$.*

The notion of safety is therefore given by the condition that the trail platoon will not collide with the lead platoon at a relative speed greater than the prescribed $v_{allow} \geq 0$, regardless of the behavior of the lead platoon. The choice of v_{allow} depends on the particular maneuver, the braking capabilities of vehicles and the maximum velocity of vehicles in the highway. The selection of v_{allow} is also determined by the tradeoffs between the time the maneuver takes to complete and the risk of injuries. For example, for a join law v_{allow} is set to be a positive number for the maneuver to be completed in a finite reasonable time; whereas in a leader law, v_{allow} is set to 0, since no impacts are expected to happen for platoons in this case.

2.1.2 Safety feasible region

Define the regions

$$\mathbf{R}_1(\Delta x,) = \left\{ (\Delta x, \Delta \dot{x}) : 0 \leq -\Delta \dot{x} \leq -(a_{min}^{trail} + a_{max}^{trail})d + \sqrt{v_{allow}^2 - (a_{min}^{lead} - a_{min}^{trail})(2\Delta x + (a_{max}^{trail} + a_{min}^{trail})d^2)} \right\} \quad (2.8)$$

and

$$\mathbf{R}_2(\Delta x, v_{max}) = \left\{ (\Delta x, \Delta \dot{x}) : 0 \leq -\Delta \dot{x} \leq -(a_{max}^{trail} + a_{min}^{trail})d - v_{max} + \sqrt{2a_{min}^{trail}\Delta x + \alpha v_{max}^2 + v_{allow}^2 + a_{min}^{trail}(a_{max}^{trail} + a_{min}^{trail})d^2} \right\}, \quad (2.9)$$

with

$$\alpha = a_{min}^{trail}/a_{min}^{lead} > 0, \quad (2.10)$$

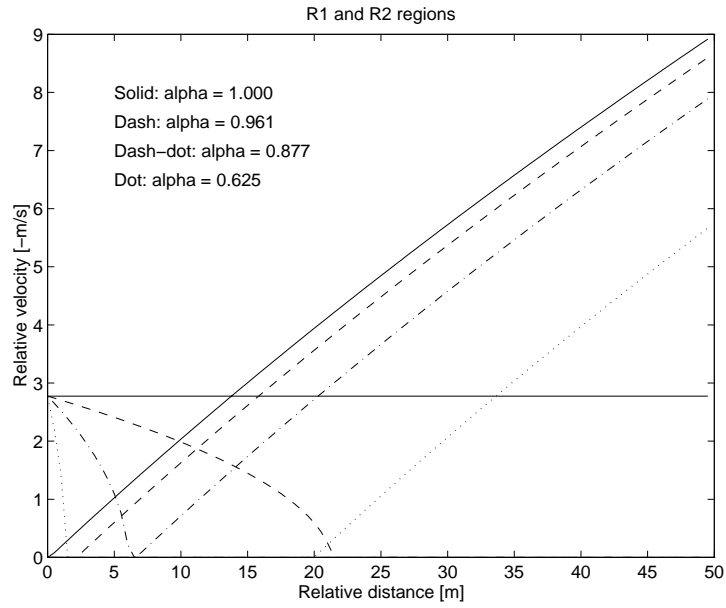
and $(\Delta x, \Delta \dot{x}, v_{max}) \in X_{MS}$.

Figure 2.1.2 shows some examples of these regions for different selections of α and a given value of v_{max} . As will be shown subsequently, the role of these regions is crucial to determine the existence of safe control laws. To understand the rationale behind the definition of these regions, assume the lead platoon is traveling at maximum speed v_{max} and suddenly it applies full braking. The trail platoon is also assumed to apply full braking after the delay d has passed. The lower boundary of $\mathbf{R}_2(\Delta x, v_{max})$ corresponds to the case in which the trail platoon is far enough from the lead platoon so that this maximum deceleration will stop the lead platoon before the trail platoon hits it at v_{allow} . The lower boundary of $R(\Delta x)$ represents the case in which the lead platoon is still in motion and is hit by the trail platoon at v_{allow} .

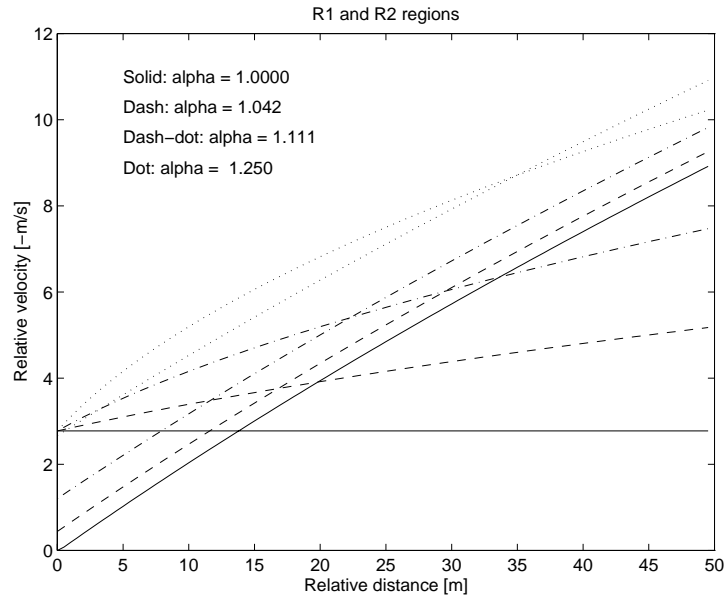
Assume $(\Delta x_F, \Delta \dot{x}_F, v_{lead_F})$ is the final state for a join or split maneuver. The following definition establishes a link between the this final state and the regions $\mathbf{R}_1(\Delta x,)$ and $\mathbf{R}_2(\Delta x, v_{max})$.

Definition 2.3 (Safety feasible region) *A safety feasible region is said to exist for a final state $(\Delta x_F, \Delta \dot{x}_F, v_{lead_F})$ if $v_{lead_F} \leq v_{max}$, $\mathbf{R}_1(\Delta x,)$ and $\mathbf{R}_2(\Delta x, v_{max})$ are connected and $(\Delta x_F, \Delta \dot{x}_F) \in \mathbf{R}_1(\Delta x,) \cup \mathbf{R}_2(\Delta x, v_{max})$*

The safety feasible region has some important properties that will be introduced in the following lemmas. This region is defined to constrain the behavior of platoons performing single lane maneuvers. For the moment it is enough to say that, when a safety feasible region exists, the initial and final states for all the trajectories generated by all safe control laws are such that, when plotted in the phase plane $(\Delta x, \Delta \dot{x})$, these states will necessarily be above the lower boundary of the safety feasible region, for any choice of velocity of the lead platoon.



(a)



(b)

Figure 2.2: Regions $\mathbf{R}_1(\Delta x,)$ and $\mathbf{R}_2(\Delta x, v_{max})$. Plots obtained with $v_{allow} = 3 \text{ m/s}$, $a_{min}^{trail} = 5 \text{ m/s}^2$, $d = 0.03 \text{ s}$, $a_{max}^{trail} = 2.5 \text{ m/s}^2$ and $v_{max} = 23 \text{ m/s}$. a) $\alpha \leq 1$. b) $\alpha \geq 1$.

Lemma 2.1 Assume $\mathbf{R}_1(\Delta x)$ and $\mathbf{R}_2(\Delta x, v_{max})$ are connected then $\mathbf{R}_1(\Delta x)(\Delta x)$ and $\mathbf{R}_2(\Delta x, v_{lead})$ are also connected for all $v_{lead} \leq v_{max}$.

Proof: For $\alpha \leq 1$ it suffices to show that $\mathbf{R}_2(\Delta x, v_{max}) \subset \mathbf{R}_2(\Delta x, v_{lead})$. Take an arbitrary Δx . Let $\Delta \dot{x}(\Delta x, v_{max})$ and $\Delta \dot{x}(\Delta x, v_{lead})$ denote the minimum values of $\Delta \dot{x}$ that satisfy Eq. (2.9) for v_{max} and v_{lead} , respectively, and a given Δx . Then

$$\begin{aligned} -(\Delta \dot{x}(\Delta x, v_{lead}) - \Delta \dot{x}(\Delta x, v_{max})) &= v_{max} - v_{lead} + \sqrt{c_1 + \alpha v_{lead}^2} - \sqrt{c_1 + \alpha v_{max}^2} \\ &\geq v_{max} - v_{lead} + \sqrt{c_1 + v_{lead}^2} - \sqrt{c_1 + v_{max}^2} > 0, \end{aligned}$$

with $c_1 = 2a_{min}^{trail}\Delta x + v_{allow}^2 + a_{min}^{trail}(a_{max}^{trail} + a_{min}^{trail})d^2$. Thus

$$\Delta \dot{x}(\Delta x, v_{lead}) < \Delta \dot{x}(\Delta x, v_{max}) \Rightarrow \mathbf{R}_2(\Delta x, v_{max}) \subset \mathbf{R}_2(\Delta x, v_{lead}).$$

For $\alpha > 1$ the result follows from the fact that the lower bounds of $\mathbf{R}_1(\Delta x)$ and $\mathbf{R}_2(\Delta x, \cdot)$ are monotonically decreasing with Δx for any v_{lead} . ■

Remark: Lemma 2.1 assures that the existence of a safety feasible region is a property that depends on v_{max} .

Lemma 2.2 Assume $v_{allow} < (a_{max}^{trail} + a_{min}^{trail})d$ and $\alpha < 1$, then there is no safety feasible region.

Proof: First notice that $\alpha < 1 \Rightarrow (a_{min}^{lead} - a_{min}^{trail}) > 0$. In this case there are two possibilities for the expression inside the square root in Eq. (2.8). If the argument is negative, $\mathbf{R}_1(\Delta x)$ is not defined and therefore can not be connected to $\mathbf{R}_2(\Delta x, v_{max})$. When the argument is positive

$$(a_{max}^{trail} + a_{min}^{trail})d > v_{allow} > \sqrt{v_{allow}^2 - (a_{min}^{lead} - a_{min}^{trail})(2\Delta x + (a_{max}^{trail} + a_{min}^{trail})d^2)},$$

thus $\Delta \dot{x}$ is negative. Hence $\mathbf{R}_1(\Delta x) = \emptyset$ and therefore $\mathbf{R}_1(\Delta x)$ and $\mathbf{R}_2(\Delta x, v_{max})$ can not be connected. ■

Remarks:

1. To perform a join maneuver, the velocity of the trail platoon has to be greater than that of the lead platoon. This allows the distance between platoons to decrease with time. Lemma 2.2 states that if no collisions are allowed during a join maneuver ($v_{allow} = 0 < (a_{max}^{trail} + a_{min}^{trail})d$), then it is not possible to design a safe control law for the trail platoon with a safety feasible region, when $\alpha < 1$, i.e., when the trail platoon has less braking capability than the lead platoon. This conclusion can be extended to the split maneuver, as disturbances from the lead platoon can make the distance between platoons to decrease instead of increasing, as it is normally the case in the split maneuver.

2. Assume that the trail platoon is performing a join maneuver, i.e., the relative distance is decreasing, and that $\alpha < 1$. If no collision is to take place in a join, then during the approaching process the minimum distance between platoons Δx_{min} should satisfy

$$\Delta x_{min} \geq \frac{\left((a_{max}^{trail} + a_{min}^{trail})d + v_{lead}\right)^2 - \alpha v_{lead}^2 - a_{min}^{trail}(a_{max}^{trail} + a_{min}^{trail})d^2}{2a_{min}^{trail}} ; \alpha < 1 . \quad (2.11)$$

Lemma 2.3 Assume $\alpha \geq 1$ and $v_{allow} \geq (a_{max}^{trail} + a_{min}^{trail})d$, then there is always a safety feasible region.

Proof: $\alpha \geq 1 \Rightarrow (a_{min}^{lead} - a_{min}^{trail}) \leq 0$, therefore the argument in the square root of Eq. (2.8) is always positive. The choice of v_{allow} guarantees that $-\Delta \dot{x} > 0$. Hence $\mathbf{R}_1(\Delta x)$ always exists and its lower bound is monotonically decreasing with Δx . From here it follows that $\mathbf{R}_1(\Delta x)$ and $\mathbf{R}_2(\Delta x, v_{max})$ are always connected. ■

Lemma 2.4 Given v_{max} , if the value of α is such that

$$\alpha < \alpha_{min}(v_{max}) = \frac{(a_{min}^{trail} + a_{max}^{trail})d - v_{allow}}{v_{max}} + 1 ,$$

then there is not a safety feasibility region.

Proof: The intersection of the lower boundaries of $\mathbf{R}_1(\Delta x)$ and $\mathbf{R}_2(\Delta x, v_{max})$ is given by the curve

$$R_3(v_{max}) = (\alpha - 1)v_{max} - c_2 + v_{allow} . \quad (2.12)$$

with

$$c_2 = (a_{min}^{trail} + a_{max}^{trail})d .$$

Set $R_3(v_{max}) = 0$ and solve Eq. (2.12) for α . This is the value of α_{min} . For any value of $\alpha < \alpha_{min}$ the intersection of the lower boundaries of $\mathbf{R}_1(\Delta x)$ and $\mathbf{R}_2(\Delta x, v_{max})$ occurs below the $\Delta \dot{x}$ axis; from here it follows that $R_3(v_{max}) < 0$ and $\mathbf{R}_1(\Delta x) = \emptyset$ and therefore $\mathbf{R}_1(\Delta x)$ and $\mathbf{R}_2(\Delta x, v_{max})$ are not connected. ■

Remarks:

1. If $\alpha_{min} < 1$ then the minimum distance during an approaching process is

$$\Delta x_{min} = 0 .$$

2. Otherwise, if $\alpha_{min} \geq 1$, this minimum distance is

$$\Delta x_{min} = \frac{c_2 - v_{allow} + v_{max}}{2a_{min}^{trail}} (c_2 + v_{allow}) - \frac{c_2 d}{2} ,$$

2.1.3 Safety theorem

Assumption 2.2

1. For the given set of parameters and final state there exists a safety feasible region.
2. The relative motion of the lead and trail platoons is given by Eqs (2.4)-(2.6).
3. The full state is observable.

The following theorem establishes a subset of X_{MS} such that, when the given set of parameters and final state has a safety feasible region, a control law exists which is safe for any initial conditions $(\Delta x(0), \Delta \dot{x}(0), v_{lead}(0))$ that lies in this subset. It is necessary first to define the auxiliary curve

$$S(\Delta x, v_{lead}) = \max(R_1(\Delta x), R_3(v_{lead})) \quad (2.13)$$

Theorem 2.1 *Let $X_{safe} \subset X_{MS} \subset \mathfrak{R}^3$ be the set of $(\Delta x, \Delta \dot{x}, v_{lead}) \in X_{MS}$ that satisfy:*

$$-\Delta \dot{x} < \begin{cases} R_2(\Delta x, v_{lead}); & R_2(\Delta x, v_{lead}) > S(\Delta x, v_{lead}), \\ R_1(\Delta x); & R_2(\Delta x, v_{lead}) \leq S(\Delta x, v_{lead}), \end{cases} \quad (2.14)$$

where

$$\begin{aligned} R_1(\Delta x) &= -c_2 + \sqrt{v_{allow}^2 + \frac{\alpha - 1}{\alpha} a_{min}^{trail} (2\Delta x + c_2 d)}, \\ R_2(\Delta x, v_{lead}) &= -c_2 - v_{lead} + \sqrt{2a_{min}^{trail} \Delta x + \alpha v_{lead}^2 + v_{allow}^2 + a_{min}^{trail} c_2 d}, \\ R_3(v_{lead}) &= (\alpha - 1)v_{max} - c_2 + v_{allow}, \\ S(\Delta x, v_{lead}) &= \max(R_1(\Delta x), R_3(v_{lead})), \\ c_2 &= (a_{max}^{trail} + a_{min}^{trail})d. \end{aligned}$$

Under assumptions 2.1 and 2.2, there exists a control law that is safe for any initial condition $(\Delta x(0), \Delta \dot{x}(0), v_{lead}(0)) \in X_{safe}$, in the sense of Definition 2.2.

Moreover, any control law that applies maximum braking whenever $(\Delta x(t), \Delta \dot{x}(t), v_{lead}(t)) \notin X_{safe}$ is safe for any initial condition $(\Delta x(0), \Delta \dot{x}(0), v_{lead}(0)) \in X_{safe}$. Under such control law, $(\Delta x(t), \Delta \dot{x}(t), v_{lead}(t)) \in X_{bound} \subset X_{MS} \subset \mathfrak{R}^3$. The elements of X_{bound} satisfy

$$-\Delta \dot{x} < \begin{cases} B_2(\Delta x, v_{lead}); & B_2(\Delta x, v_{lead}) > T(\Delta x, v_{lead}), \\ B_1(\Delta x); & B_2(\Delta x, v_{lead}) \leq T(\Delta x, v_{lead}), \end{cases} \quad (2.15)$$

where

$$\begin{aligned} B_1(\Delta x) &= \sqrt{v_{allow}^2 + \frac{\alpha - 1}{\alpha} 2a_{min}^{trail} \Delta x} = \sqrt{v_{allow}^2 - (a_{min}^{lead} - a_{min}^{trail}) 2\Delta x}, \\ B_2(\Delta x, v_{lead}) &= -v_{lead} + \sqrt{2a_{min}^{trail} \Delta x + \alpha v_{lead}^2 + v_{allow}^2}, \\ B_3(v_{lead}) &= (\alpha - 1)v_{max} + v_{allow}, \\ T(\Delta x, v_{lead}) &= \max(B_1(\Delta x), B_3(v_{lead})). \end{aligned}$$

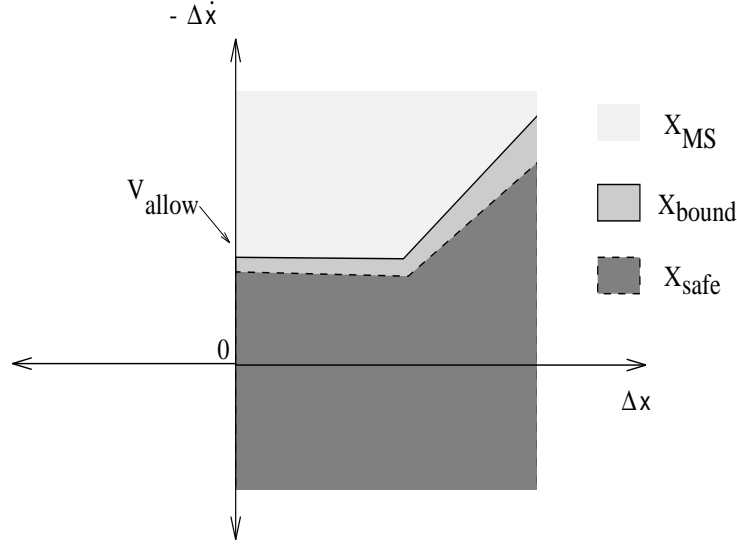


Figure 2.3: Relationships between X_{MS} , X_{bound} and X_{safe} . Notice that the vertical axis is “-” relative velocity, i.e. $-\Delta\dot{x}$

Notice that $X_{safe} \subset X_{bound} \subset X_{MS}$. The relations between X_{MS} , X_{safe} and X_{bound} are illustrated in figure 2.3, when v_{lead} is constant.

Proof: Consider a control law that would apply maximum braking whenever $(\Delta x, \Delta\dot{x}, v_{lead}) \notin X_{safe}$. By assumption, the trail platoon will decelerate, at $-a_{min}^{trail}$, d seconds after the maximum braking is applied. If maximum braking is applied at time t , the acceleration of the trail platoon at time $\tau \in [t, t + d]$ can take values in $[-a_{min}^{lead}, a_{max}^{lead}]$.

Suppose that $(\Delta x(0), \Delta\dot{x}(0), v_{lead}(0)) \in X_{safe}$. It will be shown that under the proposed control law, $(\Delta x(t), \Delta\dot{x}(t), v_{lead}(t)) \in X_{bound}$ for all $t \geq 0$. Firstly, notice that since $X_{safe} \in X_{bound}$, this is true if $(\Delta x(t), \Delta\dot{x}(t), v_{lead}(t)) \in X_{safe}$ for all $t \geq 0$. Because $(\Delta x(t), \Delta\dot{x}(t), v_{lead}(t))$ is continuous in t , if $(\Delta x(t), \Delta\dot{x}(t), v_{lead}(t)) \notin X_{safe}$ for some time $t > 0$, then there exists T_1 , $t \geq T_1 > 0$ when $(\Delta x(T_1), \Delta\dot{x}(T_1), v_{lead}(T_1))$ lies on the boundary of X_{safe} . For $t \in [T_1, T_1 + d]$,

$$v_{lead}(t) = \int_{T_1}^t w(\sigma) d\sigma + v_{lead}(T_1) ,$$

$$v_{trail}(t) = \int_{T_1}^t u(\sigma) d\sigma + v_{trail}(T_1) ,$$

$$\Delta x(t) = \int_{T_1}^t \int_{T_1}^{\tau} w(\sigma) d\sigma d\tau - \int_{T_1}^t \int_{T_1}^{\tau} u(\sigma) d\sigma d\tau + \Delta\dot{x}(T_1)(t - T_1) + \Delta x(T_1) .$$

Consider the following function that is the separation of $\Delta\dot{x}$ from the velocity boundary of X_{bound}

$$g(\Delta x, v_{lead}, v_{trail}) = \Delta\dot{x} - b_1(\Delta x, \Delta\dot{x}, v_{lead}) ,$$

where

$$b_1(\Delta x, \Delta \dot{x}, v_{lead}) = \begin{cases} v_{lead} - \sqrt{2a_{min}^{trail} \Delta x + \alpha v_{lead}^2 + v_{allow}^2}; & B_2(\Delta x, v_{lead}) > T(v_{lead}), \\ -\sqrt{v_{allow}^2 - (a_{min}^{lead} - a_{min}^{trail})2\Delta x}; & B_2(\Delta x, v_{lead}) \leq T(v_{lead}). \end{cases}$$

is the relative velocity boundary of the set X_{bound} . Hence, for $\Delta x \geq 0$, the triple $(\Delta x, \Delta \dot{x}, v_{lead}) \in X_{bound}$ if and only if $g(\Delta x, v_{lead}, v_{trail}) \geq 0$ and $\Delta x \geq 0$. The time derivative of $g(\Delta x, v_{lead}, v_{trail})$ is

$$\begin{aligned} \dot{g}(\Delta x, v_{lead}, v_{trail}) &= \frac{\partial g(\Delta x, v_{lead}, v_{trail})}{\partial \Delta x} \Delta \dot{x} + \frac{\partial g(\Delta x, v_{lead}, v_{lead})}{\partial v_{lead}} \dot{v}_{lead} + \\ &\quad \frac{\partial g(\Delta x, v_{lead}, v_{trail})}{\partial v_{trail}} \dot{v}_{trail} = \frac{\partial g(\Delta x, v_{lead}, v_{trail})}{\partial \Delta x} \Delta \dot{x} \\ &\quad + \frac{\partial g(\Delta x, v_{lead}, v_{trail})}{\partial v_{lead}} w(t) + \frac{\partial g(\Delta x, v_{lead}, v_{trail})}{\partial v_{trail}} u(t), \end{aligned} \quad (2.16)$$

after substitution of (2.1) and (2.2). Notice that

$$\frac{\partial g(\Delta x, v_{lead}, v_{trail})}{\partial v_{lead}} \geq 0, \quad (2.17)$$

$$\frac{\partial g(\Delta x, v_{lead}, v_{trail})}{\partial v_{trail}} \leq 0. \quad (2.18)$$

From the existence of a safety feasible region it follows that

$$(\Delta x, \Delta \dot{x}, v_{lead}) \in X_{bound} - X_{safe} \Rightarrow \Delta \dot{x} \leq 0. \quad (2.19)$$

From relationships (2.17)-(2.19) into (2.16) and since $w(t) \in [-a_{min}^{lead}, a_{max}^{lead}]$ and $u(t) \in [-a_{min}^{trail}, a_{max}^{trail}]$, for any $t \in [T_1, T_1 + d]$ it follows that

$$\begin{aligned} &\min\{\dot{g}(\Delta x(t), v_{lead}(t), v_{trail}(t))\} = \\ &\min_{w(t), u(t)} \left\{ \frac{\partial g(\Delta x, v_{lead}, v_{trail})}{\partial \Delta x} \Delta \dot{x} + \frac{\partial g(\Delta x, v_{lead}, v_{trail})}{\partial v_{lead}} w(t) + \frac{\partial g(\Delta x, v_{lead}, v_{trail})}{\partial v_{trail}} u(t) \right\} = \\ &\quad \frac{\partial g(\Delta x, v_{lead}, v_{trail})}{\partial \Delta x} \Delta \dot{x} - \frac{\partial g(\Delta x, v_{lead}, v_{trail})}{\partial v_{lead}} a_{min}^{lead} + \frac{\partial g(\Delta x, v_{lead}, v_{trail})}{\partial v_{trail}} a_{max}^{trail} \end{aligned}$$

Notice that the combination of (2.17)-(2.19) implies uniqueness in the choice of $w(t)$ and $u(t)$.

Define for $t \in [T_1, T_1 + d]$,

$$\bar{v}_{lead}(t) = v_{lead}(T_1) - (t - T_1)a_{min}^{lead},$$

$$\bar{v}_{trail}(t) = v_{trail}(T_1) + (t - T_1)a_{max}^{trail},$$

$$\bar{\Delta x}(t) = -\frac{(t - T_1)^2}{2}(a_{min}^{lead} + a_{max}^{trail}) + \Delta \dot{x}(T_1)(t - T_1) + \Delta x(T_1).$$

Thus, for $t \in [T_1, T_1 + d]$,

$$g(\Delta x(t), v_{lead}(t), v_{trail}(t)) \geq g(\overline{\Delta x}(t), \bar{v}_{lead}(t), \bar{v}_{trail}(t)) \geq g(\overline{\Delta x}(d), \bar{v}_{lead}(d), \bar{v}_{trail}(d)).$$

It will be shown that $\bar{g} = g(\overline{\Delta x}(T_1 + d), \bar{v}_{lead}(T_1 + d), \bar{v}_{trail}(T_1 + d)) \geq 0$.

At $t = T_1$, since $(\Delta x(T_1), \Delta \dot{x}(T_1), v_{lead}(T_1))$ is on ∂X_{safe} , then either

$$v_{trail}(T_1) = -c_2 + \sqrt{2a_{min}^{trail}\Delta x(T_1) + \alpha v_{lead}^2(T_1) + v_{allow}^2 + a_{min}^{trail}c_2d}, \quad (2.20)$$

or

$$v_{trail}(T_1) = v_{lead}(T_1) - c_2 + \sqrt{v_{allow}^2 - (a_{min}^{lead} - a_{min}^{trail})(2\Delta x(T_1) + c_2d)} \quad \text{or} \quad (2.21)$$

Suppose (2.20) is true, then the bound $\bar{g} = g(\overline{\Delta x}(T_1 + d), \bar{v}_{lead}(T_1 + d), \bar{v}_{trail}(T_1 + d))$ is given by:

$$\begin{aligned} \bar{g} &= v_{lead}(T_1) - v_{trail}(T_1) - c_2 - v_{lead}(T_1) + a_{min}^{lead}d \\ &\quad + \sqrt{2a_{min}^{trail}\overline{\Delta x}(T_1 + d) + \alpha \bar{v}_{lead}^2(T_1 + d) + v_{allow}^2} \\ &= -\sqrt{2a_{min}^{trail}\Delta x(T_1) + \alpha v_{lead}^2(T_1) + v_{allow}^2 + a_{min}^{trail}c_2d} + a_{min}^{trail}d \\ &\quad + \sqrt{2a_{min}^{trail}\overline{\Delta x}(T_1 + d) + \alpha \bar{v}_{lead}^2(T_1 + d) + v_{allow}^2} \\ &= -\sqrt{2a_{min}^{trail}\Delta x(T_1) + \alpha v_{lead}^2(T_1) + v_{allow}^2 + a_{min}^{trail}c_2d} + a_{min}^{trail}d \\ &\quad + \sqrt{-2a_{min}^{trail}\left(\frac{d}{2}c_2 - \Delta \dot{x}(T_1)d - \Delta x(T_1)\right) + \alpha (v_{lead}(T_1) - a_{min}^{lead}d)^2 + v_{allow}^2} \\ &= -\sqrt{2a_{min}^{trail}\Delta x(T_1) + \alpha v_{lead}^2(T_1) + v_{allow}^2 + a_{min}^{trail}c_2d} + a_{min}^{trail}d \\ &\quad - \sqrt{\left(\sqrt{2a_{min}^{trail}\Delta x(T_1) + \alpha v_{lead}^2(T_1) + v_{allow}^2 + a_{min}^{trail}c_2d} - a_{min}^{trail}d\right)^2} = 0. \end{aligned}$$

If, on the other hand (2.21) is true, then defining

$$c_3(t) = \sqrt{v_{allow}^2 - (a_{min}^{lead} - a_{min}^{trail})(2\Delta x(t) + (a_{max}^{trail} + a_{min}^{trail})d^2)},$$

$$\begin{aligned} \bar{g} &= v_{lead}(T_1) - v_{trail}(T_1) - c_2 + \sqrt{v_{allow}^2 - 2(a_{min}^{lead} - a_{min}^{trail})\overline{\Delta x}(T_1 + d)} \\ &= v_{lead}(T_1) - v_{lead}(T_1) + c_2 - c_3(T_1) - c_2 \\ &\quad + \sqrt{v_{allow}^2 - 2(a_{min}^{lead} - a_{min}^{trail})\left(-c_2\frac{d}{2} + c_2d - c_3(T_1)d + \Delta x(T_1)\right)} \\ &= -(a_{min}^{lead} - a_{min}^{trail})d - c_3(T_1) + \sqrt{(c_3(T_1) + (a_{min}^{lead} - a_{min}^{trail})d)^2} = 0. \end{aligned}$$

Thus, if either (2.20) or (2.21) is true, then for any $t \in [T_1, T_1 + d]$,

$$g(\Delta(t), v_{lead}(t), v_{trail}(t)) \geq \bar{g} \geq 0 .$$

For $t \geq T_1 + d$, full braking is achieved i.e. $u(t) = -a_{min}^{trail}$.

It is now shown that if $g((\Delta(T_1 + d), v_{lead}(T_1 + d), v_{trail}(T_1 + d))) \leq 0$, then $g(\Delta(t), v_{lead}(t), v_{trail}(t)) \leq 0$ for all $t \geq T_1 + d$. From relationships (2.17) and (2.19), $g(\Delta(t), v_{lead}(t), v_{trail}(t))$ is minimized if $v_{lead}(t)$ is minimized. This is achieved if $w(t) = -a_{min}^{lead}$ for $t \in [T_1 + d, \infty)$ or until $v_{lead}(t) = 0$.

Under this worst case scenario, for the first choice in the argument of $g(\Delta(t), v_{lead}(t), v_{trail}(t))$

$$\begin{aligned} g(\Delta(t), v_{lead}(t), v_{trail}(t)) &= \Delta\dot{x}(T_1 + d) - v_{lead}(T_1 + d) + a_{min}^{trail}(t - T_1 - d) + \\ &\quad \sqrt{2a_{min}^{trail}\Delta x(t) + \alpha v_{lead}^2(t) + v_{allow}^2} \\ &= \Delta\dot{x}(T_1 + d) - v_{lead}(T_1 + d) + a_{min}^{trail}(t - T_1 - d) + \\ &\quad \sqrt{2a_{min}^{trail}\Delta x(T_1 + d) + 2a_{min}^{trail}\Delta\dot{x}(T_1 + d)(t - T_1 - d) + \alpha v_{lead}^2(T_1 + d)} \\ &\quad \frac{-2a_{min}^{trail}v_{lead}(T_1 + d)(t - T_1 - d) + (a_{min}^{trail})^2(t - T_1 - d)^2 + v_{allow}^2}{-2a_{min}^{trail}v_{lead}(T_1 + d)(t - T_1 - d) + (a_{min}^{trail})^2(t - T_1 - d)^2 + v_{allow}^2} \\ &= \Delta\dot{x}(T_1 + d) - v_{lead}(T_1 + d) + a_{min}^{trail}(t - T_1 - d) + \\ &\quad \sqrt{2a_{min}\Delta x(T_1 + d) + \alpha v_{lead}^2(T_1 + d) + v_{allow}^2} \\ &\quad \frac{-2a_{min}^{trail}v_{trail}(T_1 + d)(t - T_1 - d) + (a_{min}^{trail})^2(t - T_1 - d)^2}{-2a_{min}^{trail}v_{trail}(T_1 + d)(t - T_1 - d) + (a_{min}^{trail})^2(t - T_1 - d)^2} . \end{aligned}$$

When $(\Delta x, \Delta\dot{x}, v_{lead}) \notin X_{bound}$, $v_{trail} \geq \sqrt{2a_{min}^{trail}\Delta x(T_1 + d) + \alpha v_{lead}^2(T_1 + d) + v_{allow}^2} = c_4$, then

$$\begin{aligned} g(\Delta(t), v_{lead}(t), v_{trail}(t)) &\leq \Delta\dot{x}(T_1 + d) - v_{lead}(T_1 + d) + a_{min}^{trail}(t - T_1 - d) \\ &\quad + \sqrt{c_4^2 - 2a_{min}^{trail}c_4(t - T_1 - d) + (a_{min}^{trail})^2(t - T_1 - d)^2} \\ &= \Delta\dot{x}(T_1 + d) - v_{lead}(T_1 + d) + a_{min}^{trail}(t - T_1 - d) + \sqrt{(c_4 - a_{min}^{trail}(t - T_1 - d))^2} \\ &= \Delta\dot{x}(T_1 + d) - v_{lead}(T_1 + d) + c_4 = g(T_1 + d) \leq 0 . \end{aligned}$$

or, for the second choice in the argument of $g(\Delta(t), v_{lead}(t), v_{trail}(t))$

$$\begin{aligned} g(\Delta(t), v_{lead}(t), v_{trail}(t)) &= \Delta\dot{x}(T_1 + d) - (a_{min}^{lead} - a_{min}^{trail})(t - T_1 - d) \\ &\quad + \sqrt{v_{allow}^2 - 2(a_{min}^{lead} - a_{min}^{trail})\Delta x(t - T_1 - d)} \\ &= \Delta\dot{x}(T_1 + d) - (a_{min}^{lead} - a_{min}^{trail})(t - T_1 - d) + \sqrt{v_{allow}^2 - 2(a_{min}^{lead} - a_{min}^{trail})} \\ &\quad \left(\Delta x(T_1 + d) - \frac{1}{2}(a_{min}^{lead} - a_{min}^{trail})(t - T_1 - d)^2 + \Delta x(T_1 + d)(t - T_1 - d) \right) . \end{aligned}$$

When $(\Delta x, \Delta \dot{x}, v_{lead}) \notin X_{bound}$ it follows that

$$-\Delta \dot{x}(T_1 + d) \geq \sqrt{v_{allow}^2 - 2(a_{min}^{lead} - a_{min}^{trail})\Delta x(T_1 + d)} = c_5 .$$

Hence

$$\begin{aligned} g(\Delta(t), v_{lead}(t), v_{trail}(t)) &\leq \Delta \dot{x}(T_1 + d) - (a_{min}^{lead} - a_{min}^{trail})(t - T_1 - d) \\ &+ \sqrt{(c_5 + (a_{min}^{lead} - a_{min}^{trail})(t - T_1 - d))^2} = \Delta \dot{x}(T_1 + d) + c_5 = g(T_1 + d) \leq 0 . \end{aligned}$$

■

Remarks:

1. Theorem 2.1 will be used to guarantee that a control law for a maneuver is safe. In the control laws that are proposed in this report, whenever $(\Delta x(t), \Delta \dot{x}(t), v_{lead}(t)) \notin X_{safe}$, maximum braking is applied. Hence, by Theorem 2.1, if $(\Delta x(0), \Delta \dot{x}(0), v_{lead}(0)) \in X_{safe}$, the safe control laws maintain the relationship, $(\Delta x(\tau), \Delta \dot{x}(\tau), v_{lead}(\tau)) \in X_{bound} \subset X_{MS}$ for all $\tau \geq 0$. Thus, an unsafe impact will not occur.
2. The fact that the final state $(\Delta x_F, \Delta \dot{x}_F, v_{lead_F})$ is required to be inside X_{safe} for all maneuvers guarantees that, as long as the trajectory generated by a safe control law remains inside X_{safe} , it will not be necessary to apply full brakes in the absence of disturbances, i.e., full braking of the lead platoon.
3. Notice that when the delay $d = 0$, i.e. maximum braking can be achieved instantaneously, the sets X_{safe} and X_{bound} are the same. Thus, when $d = 0$, $\overline{X_{bound}}$, the closure of X_{bound} is invariant if the control law consists of applying maximum braking whenever $(\Delta x, \Delta \dot{x}, v_{lead})$ lies outside X_{bound} . Since $\overline{X_{bound}} \subset X_{MS}$, an unsafe impact will not occur. However, since maximum braking cannot be achieved until after a delay of d seconds, the condition to apply maximum braking is more stringent (outside X_{safe}). Indeed, the relationship between the boundaries ∂X_{safe} and ∂X_{bound} of X_{safe} and X_{bound} , respectively, is such that if maximum braking is applied at time t when $(\Delta x(t), \Delta \dot{x}(t), v_{lead}(t)) \in \partial X_{safe}$, and for $\tau \in [0, d]$, the worst case scenario which is $w(\tau) = -a_{min}^{lead}$ and $u(\tau) = a_{max}^{trail}$ takes place, then $(\Delta x(t + d), \Delta \dot{x}(t + d), v_{lead}(t + d)) \in \partial X_{bound}$.

2.1.4 Lack of a safety feasible region

The existence of a safety feasible region on assumption 2.2 rules out the cases reported in (Lygeros, 1996a; Lygeros, 1996b) in which the lead platoon is assumed to have more acceleration or deceleration capability than the trail platoon. The existence of a safety feasible region guarantees that the sign of $\Delta \dot{x}$ in ∂X_{safe} is always negative. The solution to the minimization of $g(\Delta x(t), \Delta \dot{x}(t), v_{lead}(t))$ is then unique.

To discuss with more details the implication of the lack of a safety feasible region, consider the example discussed on page 9 of (Lygeros, 1996b) in which a split maneuver is

attempted. Figure 2.4 shows the regions $\mathbf{R}_1(\Delta x)$ and $\mathbf{R}_2(\Delta x, v_{lead})$ that corresponds to the given parameters: $a_{max}^{lead} = a_{max}^{trail} = 3 m/s^2$, $a_{min}^{lead} = 8 m/s^2$, $a_{min}^{trail} = 5 m/s^2$, $v_{lead} = 23 m/s$, $\Delta x = 0.01 m$, $\Delta \dot{x} = 3 m/s$, $a_{lead}(0) = 3 m/s^2$, $a_{trail}(0) = 2 m/s^2$. A jerk of $+/- 15 m/s^3$ is used as control. It can be noticed that the regions $\mathbf{R}_1(\Delta x)$ and $\mathbf{R}_2(\Delta x, v_{lead})$ are not connected.

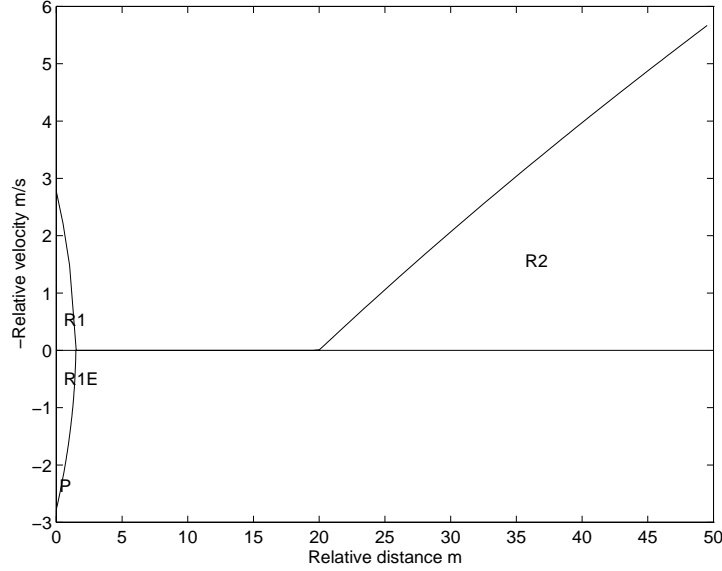


Figure 2.4: Effect of the lack of a safety feasible region

To illustrate the situation depicted in this example, define the extended region $\mathbf{R}_{1E}(\Delta x)$. This region is obtained by considering the symmetry of the original $\mathbf{R}_1(\Delta x)$ region about the the Δx axis. The initial position of the trail platoon at the beginning of the maneuver, marked as P in figure 2.4, is inside $\mathbf{R}_{1E}(\Delta x)$. If the state of the trail platoon can be kept inside this region, no collision above v_{allow} will occur with the lead platoon. Two remarks are necessary:

1. Notice that if the trail platoon is to complete the split maneuver, the trajectory of the split will have to eventually leave the region $\mathbf{R}_{1E}(\Delta x)$. Once this happens, if the lead platoon applies and holds full brakes an unsafe impact will occur. Therefore, it is not possible to safely perform the split maneuver under this kind of disturbance.
2. In the absence of a safety feasible region, there are multiple possibilities for the trail platoon to leave the region $\mathbf{R}_{1E}(\Delta x)$. One possibility corresponds to the case reported in the example under analysis, in which the lead platoon initially applies full acceleration. This acceleration increases the relative velocity $\Delta \dot{x}$ and drives the point P out of the region $\mathbf{R}_{1E}(\Delta x)$. A similar situation would happen if the trail platoon applies initially full brakes.

The only way in which the trail platoon can remain inside $\mathbf{R}_{1E}(\Delta x)$ is to keep a short separation from the lead platoon. In the example considered, the lead car is initially applying full acceleration. Even when the trail platoon is applying full positive jerk to try

to remain inside $\mathbf{R}_{1E}(\Delta x)$, the given parameters are such that there is no possibility for the trail platoon to catch up with the lead platoon and to avoid being driven out of $R_{1E}(\Delta x)$. Once this happens, if the lead car applies and hold full brakes an unsafe impact will occur. This is exactly the situation in the example of (Lygeros, 1996b).

2.2 Two platoons at a time with a third order model

2.2.1 Safe control

In this section the jerks are now considered as the control inputs in the dynamics of the relative motion of platoons, as opposed to the acceleration with time delay controls that was considered in section 2.1. The notion of safety is still the same: the platoon performing the control law will not collide with the platoon ahead at a relative velocity greater than a prescribed limit, v_{allow} .

Assumption 2.3 *For the analysis of the safety of control laws assume:*

1. *Single lane maneuvers.*
2. *Bounded acceleration of vehicles in the highway.*
3. *Bounded jerks of vehicles in the highway.*
4. *Positive velocity of vehicles, i.e. reverse motions will never occur.*
5. *Bounded maximum velocity of vehicles in the highway.*

If an input/output linearization procedure is applied to a dynamic model of the vehicles, as in (Sheikholeslam and Desoer, 1990), the dynamics of the platoons' motion become, using the same notation as in chapter 2

$$\ddot{x}_{lead}(t) = w(t) , \quad (2.22)$$

$$\ddot{x}_{trail}(t) = u(t) , \quad (2.23)$$

where $w(t) \in [-j_{min}^{lead}, j_{max}^{lead}]$ and $u(t) \in [-j_{min}^{trail}, j_{max}^{trail}]$ for all time t and $j_{min}^{lead}, j_{max}^{lead}, j_{min}^{trail}, j_{max}^{trail} > 0$.

The controls $w(t)$ and $u(t)$ are such that $\dot{x}_{lead}(t)$ and $\dot{x}_{trail}(t)$ remain positive for all t and $a_{lead}(t) \in [-a_{min}^{lead}, a_{max}^{lead}]$, $a_{trail}(t) \in [-a_{min}^{trail}, a_{max}^{trail}]$, with $a_{min}^{lead}, a_{max}^{lead}, a_{min}^{trail}, a_{max}^{trail} > 0$.

The bounds on accelerations imply that

$$a_{lead}(t) \leq -a_{min}^{lead} \Rightarrow w(t) = 0$$

$$a_{lead}(t) \geq a_{max}^{lead} \Rightarrow w(t) = 0$$

$$a_{trail}(t) \leq -a_{min}^{trail} \Rightarrow u(t) = 0$$

$$a_{trail}(t) \geq a_{max}^{trail} \Rightarrow u(t) = 0 .$$

The dynamics of the relative motion between the lead and trail platoons is given by

$$\Delta \dot{x}(t) = v_{lead}(t) - v_{trail}(t) , \quad (2.24)$$

$$\dot{v}_{lead}(t) = a_{lead}(t) , \quad (2.25)$$

$$\dot{v}_{trail}(t) = a_{trail}(t) , \quad (2.26)$$

$$\dot{a}_{lead}(t) = w(t) , \quad (2.27)$$

$$\dot{a}_{trail}(t) = u(t) . \quad (2.28)$$

The states $(\Delta x, v_{lead}, v_{trail}, a_{lead}, a_{trail})$ and $(\Delta x, \Delta \dot{x}, v_{lead}, a_{lead}, a_{trail})$ will be used indistinctively by considering $\Delta \dot{x} = v_{lead} - v_{trail}$.

Definition 2.4 (Unsafe impact) *An unsafe impact is said to happen at time t if*

$$\Delta x(t) \leq 0 \quad \text{and} \quad -\Delta \dot{x}(t) \geq v_{allow} , \quad (2.29)$$

with $v_{allow} \geq 0$ being the maximum allowable impact velocity.

The set $X_{MS} \subset \mathfrak{R}^5$ denotes the set of quintuples $(\Delta x, v_{lead}, v_{trail}, a_{lead}, a_{trail})$ such that (2.7) is not satisfied, $\Delta x > 0$, $v_{lead} \in [0, v_{max}]$, where v_{max} is the maximum highway velocity for the lead platoon, $v_{trail} \geq 0$ and the bounds on accelerations $a_{lead} \in [-a_{min}^{lead}, a_{max}^{lead}]$, $a_{trail} \in [-a_{min}^{trail}, a_{max}^{trail}]$, are satisfied.

Definition 2.5 (Safe control) *A control law for the jerk of the trail platoon, $u(t)$, is said to be safe for an initial condition $(\Delta x(0), v_{lead}(0), v_{trail}(0), a_{lead}(0), a_{trail}(0))$ if the following is true: For any arbitrary lead platoon jerk $w(\tau)$; $\tau \geq 0$ such that $w(\tau) \in [-j_{min}^{lead}, j_{max}^{lead}]$, $(\Delta x(t), v_{lead}(t), v_{trail}(t), a_{lead}(t), a_{trail}(t)) \in X_{MS}$ for all $t \geq 0$.*

2.2.2 Safety feasible region

Define the regions

$$\mathbf{R}_1(\Delta x) = \left\{ (\Delta x, \Delta \dot{x}) : 0 \leq -\Delta \dot{x} \leq -d_1 + \sqrt{v_{allow}^2 - (a_{min}^{lead} - a_{min}^{trail})(2\Delta x + d_2)} \right\} \quad (2.30)$$

and

$$\mathbf{R}_2(\Delta x, v_{max}) = \left\{ (\Delta x, \Delta \dot{x}) : 0 \leq -\Delta \dot{x} \leq -d_1 - v_{max} + \sqrt{2a_{min}^{trail} \Delta x + \alpha v_{max}^2 + v_{allow}^2 + d_3} \right\} , \quad (2.31)$$

where $(\Delta x, v_{max}, v_{trail}, a_{lead}, a_{trail}) \in X_{MS}$ and

$$\alpha = a_{min}^{trail} / a_{min}^{lead} > 0 ,$$

$$d_1 = \frac{(a_{trail}^{max} + a_{trail}^{min})^2}{2j_{trail}^{min}} > 0 ,$$

$$d_2 = \frac{(a_{trail}^{max} + a_{trail}^{min})^3}{3(j_{trail}^{min})^2} > 0 ,$$

$$d_3 = \frac{a_{trail}^{min}}{3(j_{trail}^{min})^2} (a_{trail}^{max} + a_{trail}^{min})^3 > 0 .$$

Assume $(\Delta x_F, \Delta \dot{x}_F, v_{lead_F}, a_{lead_F}, a_{trail_F})$ is the final state for a join or split maneuver. The following definition establishes a link between the this final state and the regions $\mathbf{R}_1(\Delta x,)$ and $\mathbf{R}_2(\Delta x, v_{max})$.

Definition 2.6 (Safety feasible region) *A safety feasible region is said to exist for a final state $(\Delta x_F, \Delta \dot{x}_F, v_{lead_F}, a_{lead_F}, a_{trail_F})$ if $v_{lead_F} \leq v_{max}$, $a_{lead_F} \in [-a_{min}^{lead}, a_{max}^{lead}]$, $a_{trail_F} \in [-a_{min}^{trail}, a_{max}^{trail}]$, $\mathbf{R}_1(\Delta x,)$ and $\mathbf{R}_2(\Delta x, v_{max})$ are connected and $(\Delta x_F, \Delta \dot{x}_F) \in \mathbf{R}_1(\Delta x,) \cup \mathbf{R}_2(\Delta x, v_{max})$*

The safety feasible region has the same properties that were introduced in the previous section in Lemmas 2.1-2.3. The intention is again to constrain the behavior of platoons performing single lane maneuvers. The projection of the initial and final state in the phase plane $(\Delta x, \Delta \dot{x})$ for all the trajectories generated by a safe control law will be always above the lower boundary of the safety feasible region, for any choice of velocity of the lead platoon and bounded accelerations.

2.2.3 Safety theorem

Assumption 2.4

1. For the given set of parameters and final state, there exists a safety feasible region.
2. The relative motion of the lead and trail platoons is given by Eqs (2.24)-(2.28).
3. Only the states Δx , v_{lead} and v_{trail} are observed.

The following theorem establishes a subset of X_{MS} such that, when the given set of parameters has a safety feasible region, a control law exists which is safe for any initial conditions $(\Delta x(0), v_{lead}(0), v_{trail}(0), a_{lead}(0), a_{trail}(0))$ that lies in this subset.

Theorem 2.2 *Let $X_{safe} \subset X_{MS} \subset \mathfrak{R}^3$ be the set of $(\Delta x, \Delta \dot{x}, v_{lead}) \in X_{MS}$ that satisfy:*

$$-\Delta \dot{x} < \begin{cases} R_2(\Delta x, v_{lead}); & R_2(\Delta x, v_{lead}) > S(\Delta x, v_{lead}), \\ R_1(\Delta x); & R_2(\Delta x, v_{lead}) \leq S(\Delta x, v_{lead}), \end{cases} \quad (2.32)$$

where

$$\begin{aligned} R_1(\Delta x) &= -d_1 + \sqrt{v_{allow}^2 + \frac{\alpha - 1}{\alpha} a_{min}^{trail} (2\Delta x + d_2)}, \\ R_2(\Delta x, v_{lead}) &= -d_1 - v_{lead} + \sqrt{2a_{min}^{trail} \Delta x + \alpha v_{lead}^2 + v_{allow}^2 + a_{min}^{trail} d_3}, \\ R_3(v_{lead}) &= (\alpha - 1)v_{max} - d_1 + v_{allow}, \\ S(\Delta x, v_{lead}) &= \max(R_1(\Delta x), R_3(v_{lead})). \end{aligned}$$

Under assumptions 2.3 and 2.4, there exists a control law that is safe for any initial condition $(\Delta x(0), v_{lead}(0), v_{trail}(0), a_{lead}(0), a_{trail}(0)) \in X_{safe}$, in the sense of Definition 2.2.

Moreover, any control law that applies maximum braking whenever $(\Delta x(t), v_{lead}(t), v_{trail}(t), a_{lead}(t), a_{trail}(t)) \notin X_{safe}$ is safe for any initial condition $(\Delta x(0), v_{lead}(0), v_{trail}(0), a_{lead}(0), a_{trail}(0)) \in X_{safe}$. Under such control law,

$(\Delta x(t), v_{lead}(t), v_{trail}(t), a_{lead}(t), a_{trail}(t)) \in X_{bound} \subset X_{MS} \subset \mathfrak{R}^5$. The elements of X_{bound} satisfy

$$-\Delta \dot{x} < \begin{cases} B_2(\Delta x, v_{lead}); & B_2(\Delta x, v_{lead}) > T(\Delta x, v_{lead}), \\ B_1(\Delta x); & B_2(\Delta x, v_{lead}) \leq T(\Delta x, v_{lead}), \end{cases} \quad (2.33)$$

where

$$\begin{aligned} B_1(\Delta x) &= \sqrt{v_{allow}^2 + \frac{\alpha - 1}{\alpha} 2a_{min}^{trail} \Delta x} = \sqrt{v_{allow}^2 - (a_{min}^{lead} - a_{min}^{trail}) 2\Delta x}, \\ B_2(\Delta x, v_{lead}) &= -v_{lead} + \sqrt{2a_{min}^{trail} \Delta x + \alpha v_{lead}^2 + v_{allow}^2}, \\ B_3(v_{lead}) &= (\alpha - 1)v_{max} + v_{allow}, \\ T(\Delta x, v_{lead}) &= \max(B_1(\Delta x), B_3(v_{lead})). \end{aligned}$$

Proof: Define the state vector $\mathbf{z} = (\Delta x, v_{lead}, v_{trail}, a_{lead}, a_{trail})$ and the observed state vector $\mathbf{y} = (\Delta x, v_{lead}, v_{trail})$. Notice that the bounds on a_{lead} and a_{trail} are already considered in X_{MS} . Therefore, for the analysis it is enough to take into account the constraints in \mathbf{y} .

Consider a control law that would apply maximum braking whenever the state $\mathbf{z} \notin X_{safe}$. Suppose the initial state $\mathbf{z}(0) \in X_{safe}$, It will be shown that under the proposed control law, $\mathbf{z}(t) \in \overline{X_{bound}}$ for all $t \geq 0$. Firstly, notice that since $X_{safe} \in \overline{X_{bound}}$, this is true if $\mathbf{z}(t) \in X_{safe}$ for all $t \geq 0$. Because $\mathbf{z}(t)$ is continuous in t , if $\mathbf{z}(t) \notin X_{safe}$ for some time $t > 0$, then there exists $T_1, t \geq T_1 > 0$ when $\mathbf{z}(T_1)$ lies on the boundary of X_{safe} . For $t \in [T_1, T_1 + d]$,

$$a_{lead}(t) = \min \left\{ \max \left\{ -a_{lead}^{min}, \int_{T_1}^t w(\sigma) d\sigma + a_{lead}(T_1) \right\}, a_{lead}^{max} \right\}, \quad (2.34)$$

$$a_{trail}(t) = \min \left\{ \max \left\{ -a_{trail}^{min}, \int_{T_1}^t u(\sigma) d\sigma + a_{trail}(T_1) \right\}, a_{trail}^{max} \right\}, \quad (2.35)$$

$$v_{lead}(t) = \int_{T_1}^t a_{lead}(\sigma) d\sigma + v_{lead}(T_1),$$

$$v_{trail}(t) = \int_{T_1}^t a_{trail}(\sigma) d\sigma + v_{trail}(T_1),$$

$$\Delta x(t) = \int_{T_1}^t \int_{T_1}^{\tau} a_{lead}(\sigma) d\sigma d\tau - \int_{T_1}^t \int_{T_1}^{\tau} a_{trail}(\sigma) d\sigma d\tau + \Delta \dot{x}(T_1)(t - T_1) + \Delta x(T_1),$$

where the possible saturation on accelerations is accounted for. Consider the following function which is the separation of $\Delta \dot{x}$ from the velocity boundary of X_{bound}

$$g(\mathbf{y}) = \Delta \dot{x} - b_1(\Delta x, v_{lead}),$$

where

$$b_1(\Delta x, v_{lead}) = \begin{cases} -\sqrt{v_{allow}^2 - (a_{min}^{lead} - a_{min}^{trail})(2\Delta x)}; & B_2(\Delta x, v_{lead}) > T(\Delta x, v_{lead}) \\ +v_{lead} - \sqrt{2a_{min}^{trail} \Delta x + \alpha v_{lead}^2 + v_{allow}^2}; & B_2(\Delta x, v_{lead}) \leq T(\Delta x, v_{lead}), \end{cases} \quad (2.36)$$

is the relative velocity boundary of the set X_{bound} . Hence, for $\Delta x \geq 0$, $\mathbf{z} \in X_{bound}$ if and only if $g(\mathbf{y}) \geq 0$ and $\Delta x \geq 0$. The time derivative of $g(\mathbf{y})$ is

$$\begin{aligned} \dot{g}(\mathbf{y}) &= \frac{\partial g}{\partial \Delta x} \Delta \dot{x} + \frac{\partial g}{\partial v_{trail}} \dot{v}_{lead} + \frac{\partial g}{\partial v_{trail}} \dot{v}_{trail} \\ &= \frac{\partial g}{\partial \Delta x} \Delta \dot{x} + \frac{\partial g}{\partial v_{lead}} a_{lead} + \frac{\partial g}{\partial v_{trail}} a_{trail} . \end{aligned} \quad (2.37)$$

Notice that

$$\frac{\partial g(\mathbf{y})}{\partial v_{lead}} \geq 0 , \quad (2.38)$$

$$\frac{\partial g(\mathbf{y})}{\partial v_{trail}} \leq 0 , \quad (2.39)$$

for the two choices in (2.36).

The sign of the other partial derivative depends on α :

$$\frac{\partial g(\mathbf{y})}{\partial \Delta x} > 0 ; \quad \forall \alpha > 1 , \quad (2.40)$$

$$\frac{\partial g(\mathbf{y})}{\partial \Delta x} < 0 ; \quad \forall \alpha < 1 . \quad (2.41)$$

From the assumption 2.4, the existence of a safety feasible region implies

$$\mathbf{z} \in X_{bound} - X_{safe} \Rightarrow \Delta \dot{x} \leq 0 . \quad (2.42)$$

From inequalities (2.38)-(2.42) and in (2.37) it follows that the value of $g(\mathbf{y})$ is minimized when a_{lead} is minimized and a_{trail} is maximized. From (2.34) and (2.35) it follows that

$$\min_{w(t)} \{a_{lead}\} = -a_{min}^{lead} \quad (2.43)$$

$$\max_{u(t)} \{a_{trail}\} = -a_{max}^{trail} \quad (2.44)$$

Thus, from (2.43) and (2.44) it follows that if $\mathbf{z} \in \partial X_{safe}$, the worst possible case, in terms of the approaching velocity to the boundary of X_{bound} , occurs when the lead platoon is applying full brakes and the trail platoon is accelerating at maximum capability. This is the same result that was obtained in Theorem 2.1.

Assume at $t = T_1$ the trail platoon applies full brakes, i.e., $j_{trail} = -j_{min}^{trail}$. For $T_1 \leq t \leq T_1 + T_2$, define $\bar{\mathbf{z}}(t) = (\bar{\Delta x}(t), \bar{v}_{lead}(t), \bar{v}_{trail}(t), \bar{a}_{lead}(t), \bar{a}_{trail}(t))$, where

$$\bar{a}_{lead}(t) = -a_{min}^{lead} ,$$

$$\bar{a}_{trail}(t) = a_{max}^{trail} - j_{min}^{trail}(t - T_1) ,$$

$$\bar{v}_{lead}(t) = v_{lead}(T_1) - (t - T_1)a_{min}^{lead} ,$$

$$\bar{v}_{trail}(t) = v_{trail}(T_1) + (t - T_1)a_{max}^{trail} - j_{min}^{trail} \frac{(t - T_1)^2}{2} ,$$

$$\bar{\Delta x}(t) = \Delta x(T_1) + \Delta \dot{x}(T_1)(t - T_1) - \frac{(t - T_1)^2}{2} (a_{min}^{lead} + a_{max}^{trail}) - j_{min}^{trail} \frac{(t - T_1)^3}{6} ,$$

with T_2 the amount of time necessary for the trail platoon to achieve full deceleration, when starting at maximum acceleration. T_2 is given by

$$T_2 = \frac{a_{max}^{trail} + a_{min}^{trail}}{j_{min}^{trail}}$$

Thus, for $T_1 \leq t \leq T_1 + T_2$,

$$g(\mathbf{y}(t)) \geq g(\bar{\mathbf{y}}(t)) ,$$

where $\bar{\mathbf{y}}(t)$ is defined similarly to $\bar{\mathbf{z}}(t)$.

It will be shown that $\bar{g} = g(\bar{\mathbf{y}}(T_1 + T_2)) \geq 0$. At $t = T_1$, since $(\mathbf{z}(T_1))$ is on ∂X_{safe} , then either

$$v_{trail}(T_1) = -d_1 + \sqrt{2a_{min}^{trail}\Delta x(T_1) + \alpha v_{lead}^2(T_1) + v_{allow}^2 + d_3} \quad (2.45)$$

or

$$v_{trail}(T_1) = v_{lead}(T_1) - d_1 + \sqrt{v_{allow}^2 - (a_{min}^{lead} - a_{min}^{trail})(2\Delta x(T_1) + d_2)} . \quad (2.46)$$

Suppose (2.45) is true, then

$$\begin{aligned} \bar{g} &= v_{lead}(T_1) - v_{trail}(T_1) - (a_{max}^{trail} + a_{min}^{lead})T_2 - v_{lead}(T_1) + a_{min}^{lead}T_2 + j_{min}^{trail}\frac{T_2^2}{2} \\ &\quad + \sqrt{2a_{min}^{trail}\Delta x(T_1 + T_2) + \alpha v_{lead}^2(T_1 + T_2) + v_{allow}^2} \\ &= -\sqrt{2a_{min}^{trail}\Delta x(T_1) + \alpha v_{lead}^2(T_1) + v_{allow}^2 + d_3} + a_{min}^{trail}T_2 \\ &\quad + \sqrt{2a_{min}^{trail}\Delta x(T_1 + T_2) + \alpha v_{lead}^2(T_1 + T_2) + v_{allow}^2} \\ v_{allow}^2 + &= -\sqrt{2a_{min}^{trail}\Delta x(T_1) + \alpha v_{lead}^2(T_1) + v_{allow}^2 + d_3} + a_{min}^{trail}T_2 \\ &\quad - \sqrt{\left(\sqrt{2a_{min}^{trail}\Delta x(T_1) + \alpha v_{lead}^2(T_1) + v_{allow}^2 + d_3} - a_{min}^{trail}T_2\right)^2} = 0 . \end{aligned}$$

If, on the other hand (2.46) is true, then defining

$$d_4(t) = \sqrt{v_{allow}^2 - (a_{min}^{lead} - a_{min}^{trail})(2\Delta x(t) + d_2)} ,$$

$$\begin{aligned} \bar{g} &= v_{lead}(T_1) - v_{trail}(T_1) - (a_{min}^{lead} + a_{max}^{trail})T_2 + j_{min}^{trail}\frac{T_2^2}{2} \\ &\quad + \sqrt{v_{allow}^2 - 2(a_{min}^{lead} - a_{min}^{trail})\Delta x(T_1 + T_2)} \\ &= v_{lead}(T_1) - v_{lead}(T_1) + d_1 - d_4(T_1) - (a_{min}^{lead} + a_{max}^{trail})T_2 + j_{min}^{trail}\frac{T_2^2}{2} \\ &\quad + \sqrt{v_{allow}^2 - 2(a_{min}^{lead} - a_{min}^{trail})\left(j_{min}^{trail}\frac{T_2^3}{6} - (a_{min}^{lead} + a_{max}^{trail})\frac{T_2^2}{2} + d_1T_2 - d_4(T_1)T_2 + \Delta x(T_1)\right)} \\ &= -(a_{min}^{lead} + a_{max}^{trail})T_2 - d_4(T_1) + \sqrt{(d_4(T_1) + (a_{min}^{lead} + a_{max}^{trail})T_2)^2} = 0 . \end{aligned}$$

Thus, if either (2.45) or (2.46) is true, then for any $t \in [T_1, T_1 + T_2]$,

$$g(\bar{\mathbf{y}}(t)) \geq \bar{g} \geq 0 .$$

For $t \geq T_1 + T_2$, full braking is achieved i.e. $u(t) = -a_{min}^{trail}$.

It is now shown that if $g(\mathbf{y}(T_1 + T_2)) \leq 0$, then $g(\mathbf{y}(t)) \leq 0$ for all $t \geq T_1 + T_2$. From relationships (2.38)-(2.42), $g(\mathbf{y}(t))$ is minimized if $v_{lead}(t)$ is minimized. This is achieved if $w(t) = -a_{min}^{lead}$ for $t \in [T_1 + T_2, \inf)$ or until $v_{lead}(t) = 0$.

Under this worst case scenario, for the first choice in the argument of $g(\mathbf{y}(t))$

$$\begin{aligned} g(\mathbf{y}(t)) &= \Delta \dot{x}(T_1 + T_2) - v_{lead}(T_1 + T_2) + a_{min}^{trail}(t - T_1 - T_2) + \\ &\quad \sqrt{2a_{min}^{trail} \Delta x(t) + \alpha v_{lead}^2(t) + v_{allow}^2} \\ &= \Delta \dot{x}(T_1 + T_2) - v_{lead}(T_1 + T_2) + a_{min}^{trail}(t - T_1 - T_2) + \\ &\quad \frac{\sqrt{2a_{min}^{trail} \Delta x(T_1 + T_2) + 2a_{min}^{trail} \Delta \dot{x}(T_1 - T_2)(t - T_1 - T_2) + \alpha v_{lead}^2(T_1 + T_2)} \\ &\quad - 2a_{min}^{trail} v_{lead}(T_1 + T_2)(t - T_1 - T_2) + (a_{min}^{trail})^2(t - T_1 - T_2)^2 + v_{allow}^2}}{2} \\ &= \Delta \dot{x}(T_1 + T_1) - v_{lead}(T_1 + T_2) + a_{min}^{trail}(t - T_1 - T_2) + \\ &\quad \frac{\sqrt{2a_{min} \Delta x(T_1 + T_2) + \alpha v_{lead}^2(T_1 + T_2) + v_{allow}^2}}{2} \\ &\quad - \frac{2a_{min}^{trail} v_{trail}(T_1 + T_2)(t - T_1 - T_2) + (a_{min}^{trail})^2(t - T_1 - T_2)^2}{2} . \end{aligned}$$

When $\mathbf{z} \notin X_{bound}$, $v_{trail}(T_1 + T_2) \geq \sqrt{2a_{min}^{trail} \Delta x(T_1 + T_2) + \alpha v_{lead}^2(T_1 + T_2) + v_{allow}^2} = d_5$.

$$\begin{aligned} g(\mathbf{y}(t)) &\leq \Delta \dot{x}(T_1 + T_2) - v_{lead}(T_1 + T_2) + a_{min}^{trail}(t - T_1 - T_2) \\ &\quad + \sqrt{d_5^2 - 2a_{min}^{trail} d_5(t - T_1 - T_2) + (a_{min}^{trail})^2(t - T_1 - T_2)^2} \\ &= \Delta \dot{x}(T_1 + T_2) - v_{lead}(T_1 + T_2) + a_{min}^{trail}(t - T_1 - T_2) + \sqrt{(d_5 - a_{min}^{trail}(t - T_1 - T_2))^2} \\ &= \Delta \dot{x}(T_1 + T_2) - v_{lead}(T_1 + T_2) + d_5 = g(\mathbf{y}(T_1 + T_2)) \leq 0 . \end{aligned}$$

For the second choice in the argument of $g(\mathbf{y}(t))$

$$\begin{aligned} g(\mathbf{y}(t)) &= \Delta \dot{x}(T_1 + T_2) - (a_{min}^{lead} - a_{min}^{trail})(t - T_1 - T_2) \\ &\quad + \sqrt{v_{allow}^2 - 2(a_{min}^{lead} - a_{min}^{trail}) \Delta x(t - T_1 - T_2)} \\ &= \Delta \dot{x}(T_1 + T_2) - (a_{min}^{lead} - a_{min}^{trail})(t - T_1 - T_2) + \sqrt{v_{allow}^2 - 2(a_{min}^{lead} - a_{min}^{trail})} \\ &\quad \left(\Delta x(T_1 + T_2) - \frac{1}{2}(a_{min}^{lead} - a_{min}^{trail})(t - T_1 - T_2)^2 + \Delta x(T_1 + T_2)(t - T_1 - T_2) \right) . \end{aligned}$$

When $\mathbf{y} \notin X_{bound}$ it follows that

$$-\Delta \dot{x}(T_1 + T_2) \geq \sqrt{v_{allow}^2 - 2(a_{min}^{lead} - a_{min}^{trail}) \Delta x(T_1 + T_2)} = d_6 .$$

Hence

$$\begin{aligned}
g(\mathbf{y}(t)) &\leq \Delta \dot{x}(T_1 + T_2) - (a_{min}^{lead} - a_{min}^{trail})(t - T_1 - T_2) \\
&\quad + \sqrt{(d_6 + (a_{min}^{lead} - a_{min}^{trail})(t - T_1 - T_2))^2} \\
&= \Delta \dot{x}(T_1 + T_2) + d_6 = g(T_1 + T_2) \leq 0 .
\end{aligned}$$

■

Remark:

1. Theorem 2.2 is similar to Theorem 2.1 if

$$d = \frac{a_{max}^{trail} + a_{min}^{trail}}{j_{min}^{trail}} ,$$

however these theorems are not equivalent. The width of the layer between X_{safe} and X_{bound} in Theorem 2.2 is smaller than the one in Theorem 2.1.

2. It can be shown that, when the acceleration of the trail platoon is assumed to be known, Theorem 2.2 still holds if the value of a_{max}^{trail} is substituted by $a_{trail}(t)$ in the proof. This conclusion follows from the fact that $y(\mathbf{y}(t))$ is not a function of $a_{trail}(t)$. The minimization of $\dot{g}(\mathbf{y}(t))$ is still achieved when the lead platoon applies full brakes and the trail platoon tries to reach also maximum acceleration. The width of the layer between X_{safe} and X_{bound} is dynamic in this case, as it depends on the current acceleration of the trail platoon. However, if the trail platoon is accelerating at a_{max}^{trail} when the lead platoon applies full braking, the width of the layer remains the same.
3. The knowledge of the acceleration of the lead platoon a_{lead} leads to results that further reduce the width of the layer between X_{safe} and X_{bound} as for Theorem 2.2 it is assumed that the lead platoon is braking at maximum capability.

2.3 An arbitrary number of platoons at a time

Collisions in a highway produce instantaneous changes in the velocity of the vehicles involved in them. If, for example, the interplatoon distance in the highway is too small, these changes in velocity can, in turn, produce other collisions. If collisions in an AHS can be confined to the two platoons involved in a join or split maneuver, then the overall safety of the AHS can be guaranteed. In this section, conditions on the state of platoons executing the leader law that avoid the propagation of impacts are derived.

Consider a one lane highway in which platoons are distributed as is shown in figure 2.5. Assume that platoon #1 is colliding with platoon #2 and that, at the same time, platoon number #4 is also colliding with platoon #3. An important design problem is to set the state of the leader law of platoon #3 in such a way that these two collisions do not cause a collision between platoons #2 and #3. The following theorem establishes a new region $X_{leader} \subset X_{safe}$, such that when the initial state of a platoon executing the leader law belongs to this region, no impact propagation occurs in the highway.

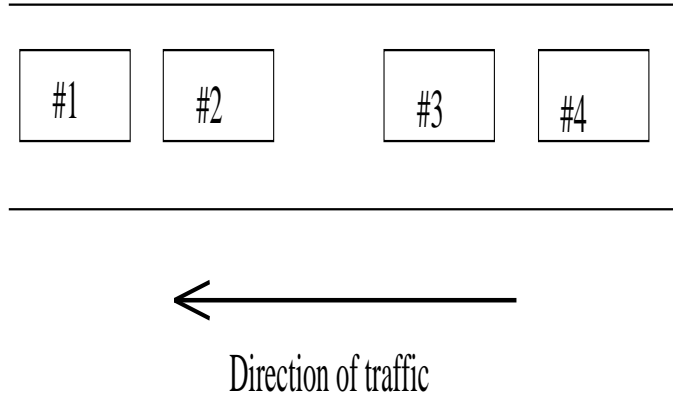


Figure 2.5: Distribution of platoons in a highway

Assumption 2.5 *Platoons organization in an AHS satisfies:*

1. *No front impacts are allowed for vehicles executing the leader control law.*
2. *Vehicles executing the join or split control laws are allowed to have collisions equal or below v_{allow} .*
3. *No adjacent platoons can perform simultaneously a join or split maneuver.*
4. *Collisions at low relative speed behave like perfectly elastic impacts (Meriam and Kraige, 1992; Lygeros, 1996a).*
5. *The braking capability of the vehicles involved in a low relative velocity impact is preserved after the impact.*
6. *The masses of vehicles involved in a collision are equal.*

Theorem 2.3 *Let $X_{leader} \subset \mathfrak{R}^3$ be the set of $(\Delta x, \Delta \dot{x}, v_{lead})$ that satisfy:*

$$-\Delta \dot{x} < - (a_{max}^{trail} + a_{min}^{trail})d - v_{lead} - v_{allow} + \sqrt{2a_{min}^{trail} \Delta x + \alpha(v_{lead} - v_{allow})^2 + a_{min}^{trail} (a_{max}^{trail} + a_{min}^{trail})d^2} ,$$

and $0 \leq v_{lead} \leq v_{max}$.

Under assumptions 2.1-2.5,

- i. *There exists a leader control law that is safe for any initial condition $(\Delta x(0), \Delta \dot{x}(0), v_{lead}(0)) \in X_{leader}$, in the sense of Definition 2.2.*
- ii. *There is no impact propagation in the highway, in the sense that no front collisions occur in platoons performing the leader control law. Safe front collisions in the highway occur only between platoons performing the join and split control laws and their respective lead platoons .*

Moreover, any control law that applies maximum braking whenever $(\Delta x(t), \Delta \dot{x}(t), v_{lead}(t)) \notin X_{leader}$ is safe for any initial condition $(\Delta x(0), \Delta \dot{x}(0), v_{lead}(0)) \in$

X_{leader} . Under such control law, $(\Delta x(t), \Delta \dot{x}(t), v_{lead}(t)) \in X_{bound_1} \subset \mathfrak{R}^3$. The elements of X_{bound_1} satisfy

$$-\Delta \dot{x} < -v_{lead} + \sqrt{2a_{min}^{trail} \Delta x + \alpha v_{lead}^2},$$

with $0 \leq v_{lead} \leq v_{max}$.

Proof: Take an arbitrary platoon in the highway whose leader is executing a leader law. Several cases of collisions have to be considered.

Case a: Consider first the case in which, at time $t = 0$ there are two collisions. The lead platoon collides with the platoon in front of it at a relative speed of v_{allow} and the trail platoon has also a collision with the platoon in back at relative speed of v_{allow} . If $t = 0^-$ and $t = 0^+$ denote the times just before the collisions and after them, respectively, the discontinuities in the velocities can be expressed as:

$$v_{lead}(0^+) = v_{lead}(0^-) - v_{allow}, \quad (2.47)$$

$$v_{trail}(0^+) = v_{trail}(0^-) + v_{allow}. \quad (2.48)$$

If $(\Delta x(0^-), \Delta \dot{x}(0^-), v_{lead}(0^-)) \in X_{leader}$ then

$$\begin{aligned} -\Delta \dot{x}(0^-) &< -(a_{max}^{trail} + a_{min}^{trail})d - v_{lead}(0^-) - v_{allow} \\ &+ \sqrt{2a_{min}^{trail} \Delta x + \alpha (v_{lead}(0^-) - v_{allow})^2 + a_{min}^{trail} (a_{max}^{trail} + a_{min}^{trail})d^2}. \end{aligned} \quad (2.49)$$

From Eqs. (2.47) and (2.48) in inequality (2.49) it follows than

$$\begin{aligned} -\Delta \dot{x}(0^+) &< -(a_{max}^{trail} + a_{min}^{trail})d - v_{lead}(0^+) \\ &+ \sqrt{2a_{min}^{trail} \Delta x + \alpha v_{lead}(0^+)^2 + a_{min}^{trail} (a_{max}^{trail} + a_{min}^{trail})d^2}. \end{aligned} \quad (2.50)$$

Define, at $t = 0^-$, the auxiliary set $X_{bound_2} \supset X_{leader}$ whose elements satisfy

$$-\Delta \dot{x}(0^-) < -v_{lead}(0^-) - v_{allow} + \sqrt{2a_{min}^{trail} \Delta x + \alpha (v_{lead}(0^-) - v_{allow})^2}. \quad (2.51)$$

Using again Eqs. (2.47) and (2.48) in inequality (2.51)

$$-\Delta \dot{x}(0^+) < -v_{lead}(0^+) + \sqrt{2a_{min}^{trail} \Delta x + \alpha v_{lead}(0^+)^2}. \quad (2.52)$$

From inequalities (2.50) and (2.52) it follows that

$$(\Delta x(0^-), \Delta \dot{x}(0^-), v_{lead}(0^-)) \in \partial X_{leader} \Rightarrow (\Delta x(0^+), \Delta \dot{x}(0^+), v_{lead}(0^+)) \in \partial X_{safe}$$

and

$$(\Delta x(0^-), \Delta \dot{x}(0^-), v_{lead}(0^-)) \in X_{bound_2} \Rightarrow (\Delta x(0^+), \Delta \dot{x}(0^+), v_{lead}(0^+)) \in X_{bound_1}.$$

The safety of the leader law follows from theorem 2.1 with $v_{allow} = 0$.

Case b: Consider now the case in which the lead platoon collides with the platoon in front of it at time $t = 0$ with a relative speed of v_{allow} . The discontinuities in the velocities can be expressed as:

$$v_{lead}(0^+) = v_{lead}(0^-) - v_{allow} , \quad (2.53)$$

$$v_{trail}(0^+) = v_{trail}(0^-) . \quad (2.54)$$

From Eqs. (2.53) and (2.54) in Eq. (2.49) it follows than

$$\begin{aligned} -\Delta\dot{x}(0^+) &< -(a_{max}^{trail} + a_{min}^{trail})d - v_{lead}(0^+) - v_{allow} \\ &+ \sqrt{2a_{min}^{trail}\Delta x + \alpha v_{lead}(0^+)^2 + a_{min}^{trail}(a_{max}^{trail} + a_{min}^{trail})d^2} . \end{aligned} \quad (2.55)$$

It should be noticed that $(\Delta x(0^+), \Delta\dot{x}(0^+), v_{lead}(0^+)) \notin X_{leader}$ and therefore at $t = 0^+$ full brakes are applied.

Two possibilities are analyzed. The first one corresponds with no collision happening between the the platoon executing the leader law and the platoon in back after full brakes are applied. Substituting Eqs. (2.53) and (2.54) in inequality (2.51) and using similar arguments as in theorem 2.1, it can be concluded that for $t \geq 0^+ + d$ the state $(\Delta x, \Delta\dot{x}, v_{lead}) \in X_{bound_3}$ whose elements satisfy

$$-\Delta\dot{x} < -v_{lead} - v_{allow} + \sqrt{2a_{min}^{trail}\Delta x + \alpha v_{lead}^2} ; \quad \forall t \geq 0^+ + d . \quad (2.56)$$

It is clear that $X_{bound_3} \subset X_{bound_1}$ and therefore the safety of the leader law follows from theorem 2.1.

The second possibility is that at some time $t = T_1 \geq d > 0$ the platoon in back collides with the trail platoon at relative speed of v_{allow} while applying full brakes. Then if T_1^- and T_1^+ denote the time just before and after this second collision, respectively, the discontinuities in velocities are given by

$$v_{lead}(T_1^+) = v_{lead}(T_1^-) , \quad (2.57)$$

$$v_{trail}(T_1^+) = v_{trail}(T_1^-) + v_{allow} . \quad (2.58)$$

From Eqs. (2.57) and(2.58) in inequality (2.56) it follows than

$$-\Delta\dot{x}(T_1^+) < -v_{lead}(T_1^+) + \sqrt{2a_{min}^{trail}\Delta x + \alpha v_{lead}(T_1^+)^2} . \quad (2.59)$$

From inequality (2.59) it follows that

$$(\Delta x(T_1^-), \Delta\dot{x}(T_1^-), v_{lead}(T_1^-)) \in X_{bound_3} \Rightarrow (\Delta x(T_1^+), \Delta\dot{x}(T_1^+), v_{lead}(T_1^+)) \in X_{bound} .$$

The safety of the leader law follows again from theorem 2.1 with $v_{allow} = 0$.

The other case in which the lead platoon does not collide, the argument is straightforward. It can be concluded that $(\Delta x(0^+), \Delta\dot{x}(0^+), v_{lead}(0^+)) \in X_{leader} \subset X_{safe}$ and $(\Delta x(0^+), \Delta\dot{x}(0^+), v_{lead}(0^+)) \in X_{bound_1} \subset X_{bound}$.

From assumption 2.5 and considering that the selection of the platoon executing the leader law is arbitrary, it follows that no platoon executing the leader law can have a collision with the platoon in front of it, provided $(\Delta x(0^-), \Delta \dot{x}(0^-), v_{lead}(0^-)) \in X_{leader}$, and therefore no impact propagation occurs. ■

Remark:

1. The structure of the region X_{leader} in theorem 2.3 is different from the one presented in (Lygeros, 1996a) in which the safety regions for the leader law would be obtained by a translation in the direction of the relative distance axis, Δx , of the safety region X_{safe} in theorem 2.1. The translation of the safety region X_{safe} in theorem 2.1 along the Δx axis produces a region equal to the region X_{leader} only for the case in which the acceleration capabilities of vehicles are the same.
2. The region X_{leader} in theorem 2.3 is a conservative estimate when the two possible neighbor platoons are also executing the leader law. It is possible to derive another regions for these other cases $X_{leader_1} \supset X_{leader}$, $X_{leader_2} \supset X_{leader}$ and $X_{leader_3} \supset X_{leader}$ under the assumptions of one collision in front for the case in which the platoon in front is involved in a joint/split, on collision in back, for the case when the rear platoon is attempting a join/split, or zero collisions when both the front and rear platoons are not attempting joins/splits. However, the implication of having three different safe regions for platoons executing in leader law will be that before any join can be attempted by the neighbor platoons, the lead platoon would have to increase its relative distance with the previous platoon in order to prevent for the possibility of collisions. This increment of relative distance could be propagated downstream producing a generalized decrement in the velocity of the platoons, when spacing is tight. The case considered in theorem 2.3 guarantees that this propagation will not happen and therefore a more steady behavior of platoons' velocity in the highway can be achieved.
3. It is possible to think on extending the results in theorem 2.3 to the case in which the masses of the vehicles involved in a collision are not equal. If ϕ is the maximum ratio of the masses of any two vehicles in a given lane, then the region X_{leader} has to be modified to

$$-\Delta \dot{x} < - (a_{max}^{trail} + a_{min}^{trail})d - v_{lead} - \phi v_{allow} + \sqrt{2a_{min}^{trail} \Delta x + \alpha(v_{lead} - \phi v_{allow})^2 + a_{min}^{trail}(a_{max}^{trail} + a_{min}^{trail})d^2} \quad ,$$

and $0 \leq v_{lead} \leq v_{max}$. The notion of safety, however, has also to be modified because the difference in the mass of vehicles can produce a velocity increase after a collision. v_{allow} would have to be lowered to prevent for such increases. A more careful analysis of this case is still needed.

2.4 Join and split with no collisions

In the previous two sections, conditions for safe platooning in AHS are established. The state of the relative motion of platoons is $(\Delta x, \Delta \dot{x}, v_{lead})$ and the set of parameters is arbitrary.

It is concluded that the join and split maneuvers implied risk of low speed collisions, when the ratio of braking capabilities satisfies

$$\alpha = \frac{a_{min}^{trail}}{a_{min}^{lead}} < 1 .$$

On the other hand, according with Lemma 2.3, when $\alpha \geq 1$ and $v_{allow} \geq (a_{max}^{trail} + a_{min}^{trail})d$, there is always a safety feasibility region for Theorem 2.1. This result can be refined to establish a bound on the value of α such that the value of v_{allow} can be decreased to zero. The following lemma defines a value $\alpha_n > 1$ such that whenever $\alpha \geq \alpha_n$ joins and splits can be completed with no collisions, i.e., with $v_{allow} = 0$.

Lemma 2.5 *If the final state for a join/split maneuver is given by $(\Delta x_F, \Delta \dot{x}_F, v_{lead_F}) \in X_{safe}$ with $v_{lead_F} \leq v_{max}$, $v_{allow} = 0$ and α satisfies*

$$\alpha \geq \alpha_m = \frac{a_{min}^{trail} (2\Delta x_F + (a_{min}^{trail} + a_{max}^{trail})d^2)}{a_{min}^{trail} (2\Delta x_F + (a_{min}^{trail} + a_{max}^{trail})d^2) - ((a_{min}^{trail} + a_{max}^{trail})d - \Delta \dot{x}_F)^2} > 1 , \quad (2.60)$$

then there is always a safety feasibility region.

Proof: Eq. (2.60) follows directly from the intersection of the curve $R_1(\Delta x)$ with the point $(\Delta x_F, \Delta \dot{x}_F)$ when $v_{allow} = 0$. ■

In this section, conditions to perform maneuvers with no collision are investigated. The problem is to establish additional assumptions under which it can be guaranteed that α satisfies (2.60). This problem is related with the problem of intraplatoon behavior that was extensively studied in (Swaroop, 1994), where the string stability for different follower control laws was analyzed¹. There are two conclusions in (Swaroop, 1994) of relevance for the problem of join and split maneuvers without collisions.

1. To guarantee robust string stability performance it is necessary to broadcast the lead vehicle acceleration and velocity to all the vehicles that conform a platoon. When lead vehicle relative position information is also broadcasted, performance is enhanced.
2. Given an acceleration profile for the leader of a platoon, the magnitude of the acceleration increases with the distance to the leader of the platoon, although the rate of increment decreases with the same distance.

Two approaches are presented that allow join and split maneuvers with no collisions. For the first approach it is considered that the measurements available to the trail platoon are the same that those in Assumption 2.2 and that the acceleration in a platoon propagates according with the results reported in (Swaroop, 1994). In the second approach it is considered that during the join and split maneuvers the acceleration of the leader of the lead platoon can be broadcasted to all the vehicles in the trail platoon. If this is the case, then the results on robust string stability performance that can be obtained with the use of this acceleration can be applied to the lead and trail platoon simultaneously.

For the first approach consider the following additional assumption.

¹Roughly speaking, by string stability it is meant that the tracking errors within a platoon are bounded and are not amplified.

Assumption 2.6

1. There exists a finite ratio $\mu \geq 1$ such that if a_{min}^{leader} is the magnitude of the minimum acceleration of the leader of a platoon, then the magnitude of the minimum acceleration of the last vehicle in the platoon, a_{min}^{last} , satisfies

$$a_{min}^{last} \leq \mu a_{min}^{leader} . \quad (2.61)$$

2. The magnitude of the maximum deceleration for all the vehicles in the highway has an overall maximum A_{MIN} .

Lemma 2.6 *If the magnitude of the maximum deceleration of the leader of the lead platoon in a join or split maneuver satisfies*

$$a_{min}^{leader} \leq \frac{A_{MIN}}{\mu^2 \alpha_m} , \quad (2.62)$$

then, under assumption 2.6, it is possible to perform the join and split maneuvers without collisions.

Proof: Apply (2.61) to the lead platoon, then (2.62) becomes

$$\frac{a_{min}^{lead}}{\mu} \leq a_{min}^{leader} \leq \frac{A_{MIN}}{\mu^2 \alpha_m} \quad \Rightarrow \quad a_{min}^{lead} \leq \frac{A_{MIN}}{\mu \alpha_m} . \quad (2.63)$$

Applying now (2.61) to the trail platoon it follows that, if A_{MIN} is the magnitude of the maximum deceleration of the last vehicle of the trail platoon, then

$$A_{MIN} \leq \mu a_{min}^{trail} . \quad (2.64)$$

From (2.64) in (2.63)

$$a_{min}^{lead} \leq \frac{A_{MIN}}{\mu \alpha_m} \leq \frac{a_{min}^{trail}}{\alpha_m} . \quad (2.65)$$

Hence

$$\frac{a_{min}^{trail}}{a_{min}^{lead}} = \alpha \geq \alpha_m . \quad (2.66)$$

The lemma follows directly from Lemma 2.5 and Theorem 2.1. ■

Remark: Lemma 2.6 sets the minimum acceleration for the leader of the lead platoon involved in a join/split in such a way that whenever the maneuver is executed, the last vehicle in the trail platoon is able to accommodate for the braking requirements in terms of string stability and safety of the maneuver.

The second approach considers the following additional assumption regarding the acceleration of the leader of the lead platoon.

Assumption 2.7

1. The acceleration and velocity of the leader of the lead platoon involved in a join or split maneuver is broadcasted to all vehicles in the trail platoon.
2. The number of vehicles, N_{lead} of the lead platoon is broadcasted to all vehicles in the trail platoon.
3. Given the position of a vehicle inside a platoon, n , there is a positive non-decreasing function, $\rho(n)$, such that the magnitude of the maximum deceleration of the n -th vehicle in the platoon, a_{min}^n satisfies

$$a_{min}^n \leq \rho(n) a_{min}^{leader} , \quad (2.67)$$

where a_{min}^{leader} is the magnitude of the maximum deceleration of the leader of the platoon.

Lemma 2.7 If a_{min}^{trail} satisfies

$$a_{min}^{trail} \geq \alpha_m \rho(N_{lead}) a_{min}^{leader} \quad (2.68)$$

and the magnitude of the maximum deceleration of the leader of the lead platoon in a join or split maneuver satisfies

$$a_{min}^{leader} \leq \frac{A_{MIN}}{\mu \alpha_m} , \quad (2.69)$$

then, under assumptions 2.6 and 2.7, it is possible to perform the join or split maneuvers without collisions.

Proof: If a_{min}^{trail} satisfies (2.68) then, from assumption 2.7, Lemma 2.5 and Theorem 2.1, it can be concluded that the join or split maneuvers can be performed without collisions. Eq. (2.69) and assumption 2.7 guarantee that the maximum deceleration for the leader of the lead platoon is set in such a way that the last vehicle in the trail platoon will decelerate within its maximum deceleration capabilities. ■

Remark: Lemma 2.7 states that if the velocity and acceleration of the leader of the lead platoon are known to all vehicles of the platoons involved in a join or split maneuver, then all the vehicles performing the follower law in the trail platoon will use this information in their followers law. String stability for the lead and trail platoons combined follows from the results in (Swaroop, 1994).

2.5 Platooning with no collisions

This section presents the calculations to determine a steady state headway that must be kept by a platoon leader executing the leader law, when no collision are desired in the AHS. The intention of these calculations is to illustrate the use of the safety results contained in this chapter in other safety and capacity analysis tools. This headway, or interplatoon

distance, is an important factor in determining the AHS capacity, when vehicles are traveling organized in platoons (Broucke and Varaiya, 1996).

The underlying idea in the calculations is to guarantee that all vehicles in the AHS are capable of achieving the levels of acceleration or deceleration that an operation with no possibility of collisions requires. Consider first the following assumption, that is complementary to assumption 2.6.

Assumption 2.8

1. There exists a finite ratio $\mu \geq 1$ such that if a_{max}^{leader} is the magnitude of the maximum acceleration of the leader of a platoon, then the magnitude of the maximum acceleration of the last vehicle in the platoon, a_{max}^{last} , satisfies

$$a_{max}^{last} \leq \mu a_{max}^{leader} . \tag{2.70}$$

2. The magnitude of the maximum acceleration for all the vehicles in the highway has an overall maximum A_{MAX} .

Assume that two platoons are involved in a join or split maneuver, as depicted in figure 2.6 and that the last vehicle of the trail platoon has the maximum acceleration and deceleration capabilities, A_{MAX} and A_{MIN} , respectively, where A_{MIN} , as defined in assumption 2.6, is the maximum deceleration for all the vehicles in the highway.

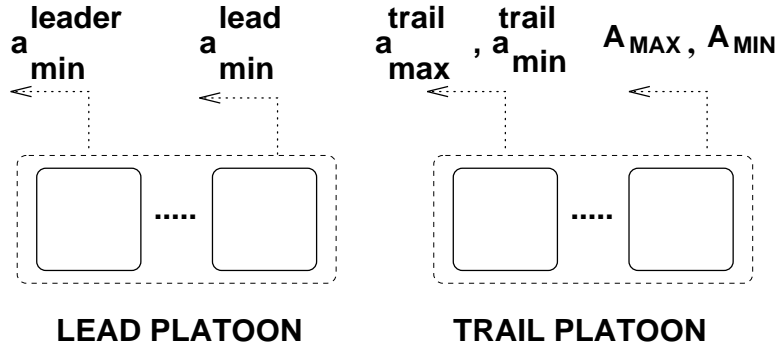


Figure 2.6: Accelerations in two adjacent platoons involved in a join or split maneuver

According to assumptions 2.6 and 2.8, the acceleration and deceleration of the leader of the trail platoon should respectively satisfy

$$a_{min}^{trail} = \frac{1}{\mu} A_{MIN} \tag{2.71}$$

$$a_{max}^{trail} = \frac{1}{\mu} A_{MAX} . \tag{2.72}$$

After the substitution of (2.71) and (2.72) into (2.60) the value of the minimum ratio of braking capabilities, α_m is

$$\alpha_m = \frac{A_{MIN} (2\mu\Delta x_F + (A_{MIN} + A_{MAX})d^2)}{A_{MIN} (2\mu\Delta x_F + (A_{MIN} + A_{MAX})d^2) - ((A_{MIN} + A_{MAX})d^2 - \mu\Delta \dot{x}_F)^2} > 1 , \tag{2.73}$$

By Lemma 2.6, the acceleration of the leader of the lead platoon should satisfy

$$a_{min}^{leader} \leq \frac{A_{MIN}}{\mu^2 \alpha_m} < A_{MIN} , \quad (2.74)$$

$$a_{max}^{leader} \leq \frac{A_{MAX}}{\mu} , \quad (2.75)$$

where a_{max}^{leader} is the maximum acceleration of the leader of the lead platoon.

Now assume the situation depicted in figure 2.7. In this case another platoon, denoted as the front platoon in the figure, is ahead of the lead platoon. The last car of this front platoon is assumed to have a deceleration capability of A_{MIN} and to travel at v_{max} , the maximum possible velocity in the highway. Notice that, by (2.74), the braking capability of the leader of the lead platoon a_{min}^{leader} is less than the assumed braking capability for the last car of the front platoon, A_{MIN} . If the approach established in Theorem 2.3 is used to analyze the safe behavior of the lead platoon with respect to the front platoon behavior, a value for the platoon headway Δx_{leader} between the lead and front platoons can be determined such that this platoons are safe with respect to each other. This value corresponds to the point in which the boundary of X_{leader} intersects with the axis $\Delta \dot{x} = 0$. This value is given by

$$\Delta x_{leader} \geq \frac{\alpha_m \left(\mu v_{max} + \left(\frac{1}{\mu \alpha_m} A_{MIN} + A_{MAX} \right) d \right)^2 - v_{max}^2 - \frac{A_{MIN}}{\mu} \left(\frac{1}{\mu \alpha_m} A_{MIN} + A_{MAX} \right) d^2}{2A_{MIN}} . \quad (2.76)$$

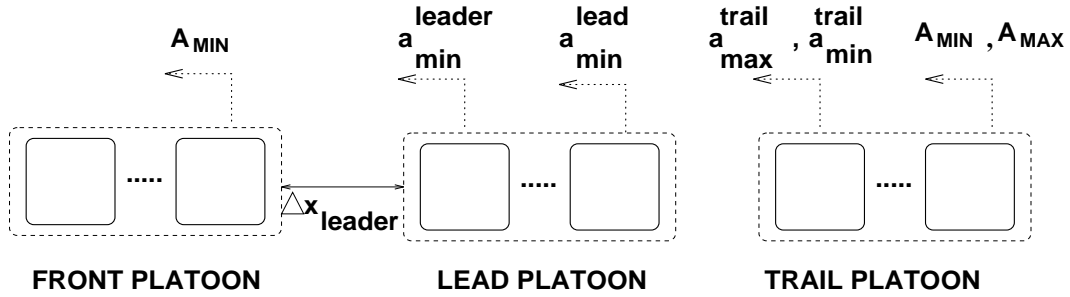


Figure 2.7: Accelerations in two adjacent platoons. The trail platoon is executing the leader law

When nominal values for the parameters in (2.73) are used to evaluate α_m its value can be very close to one. From the point of view of time for maneuver completion it is desirable to pick a value of $\alpha > \alpha_m$; the larger the value of α the less time to complete a join or split. From (2.74), however, it is clear that increasing α arbitrarily will also increase arbitrarily the value of Δx_{leader} . At a certain point Δx_{leader} can exceed the capacity of the relative distance sensor, Δx_{range} . If the value of α is chosen such that $\Delta x_{leader} > \Delta x_{range}$ the leader of the lead platoon is in risk of colliding with the last vehicle of front platoon, if this front platoon suddenly appears in the relative distance sensor range. According to this

there is a maximum value of α given by

$$\alpha \leq \alpha_M = \frac{2A_{MIN} (\Delta x_{range} - v_{max}d)}{\mu^2 v_{max}^2}. \quad (2.77)$$

Eq. (2.77) is calculated by assuming that the leader of the lead platoon is traveling at v_{max} and that maximum braking is attained after a time delay d .

Once a value of $\alpha = \alpha_c$ such that $\alpha_m \leq \alpha_c \leq \alpha_M$ has been chosen², the distance Δx_{leader} between the leader of the lead platoon and the rear vehicle of the front platoon can be determined considering two possibilities:

1. The front platoon is within the relative distance sensor range, Δx_{range} and the velocity of the front platoon can be measured.
2. The front platoon is within the relative distance sensor range, Δx_{range} and the velocity of the front platoon can not be measured.

For the first case the maximum inter-platoon distance satisfies

$$\Delta x_{leader}^{max} = \frac{\alpha_c \left(\mu v_{max} + \left(\frac{1}{\mu} A_{MIN} + A_{MAX} \right) d \right)^2 - v_{max}^2 - A_{MIN} \left(\frac{1}{\mu^2} A_{MIN} + A_{MAX} \right) d^2}{2A_{MIN}} \quad (2.78)$$

and for the second possibility this spacing is

$$\Delta x_{leader}^{max} = \frac{\alpha_c \left(\mu v_{max} + \left(\frac{1}{\mu} A_{MIN} + A_{MAX} \right) d \right)^2 - v_{max}^2}{2A_{MIN}} \quad (2.79)$$

It should be noted that the lower bound of Δx_{leader}^{max} , as given by Eqs. (2.78) and (2.79), is a necessary and sufficient condition to avoid collisions in the worst possible scenarios. These scenario occur when both the front platoon and the lead platoon are traveling at maximum speed v_{max} and the front platoon applies maximum braking and when the lead platoon senses a stopped front platoon within sensor range while traveling at maximum speed.

For AHS capacity analysis purposes the value of Δx_{leader}^{max} is used, as in this situation the highway is supposed to be at its maximum level of occupancy and therefore it is expected for platoons to be within relative sensor distance range.³

During the normal operation of the AHS the lower bound of the headway given by (2.78) can be reduced if the velocity of the last vehicle of the front platoon, v_{front} , is known. In this case the platoon headway satisfies

$$\Delta x_{leader}^{max} \geq \frac{\alpha_c \left(\mu v_{front} + \left(\frac{1}{\mu} A_{MIN} + A_{MAX} \right) d \right)^2 - v_{front}^2 - A_{MIN} \left(\frac{1}{\mu^2} A_{MIN} + A_{MAX} \right) d^2}{2A_{MIN}}. \quad (2.80)$$

²If $\alpha_M < \alpha_m$ there is no possibility of achieving platooning with no collisions for the given relative distance sensor range.

³If the following set of values is used: $\alpha_c = 1.15$, $v_{max} = 25$ m/s, $\mu = 1.12$, $A_{MIN} = 5$ m/s², $A_{MAX} = 2.5$ m/s², $d = 0.03$ s, $\Delta x_F = 1$ m and $\dot{\Delta x}_F = 0$ m/s then $\Delta x_{leader}^{max} \approx 30$ m.

Another value for Δx_{leader}^{max} can be found if a similar result is derived using of Lemma 2.7 instead of Lemma 2.6. As previously noted, Lemma 2.7 requires more information to be available in the AHS.

Chapter 3

Velocity Tracking Control

3.1 Velocity Profiles

In this section the velocity for the relative motion of platoons during a maneuver is expressed as desired velocity profiles in the state space $(\Delta x, \Delta \dot{x}, v_{lead})$. These profiles must allow the platoons to complete the maneuvers in minimum time while guaranteeing that the trajectories remain inside the safety region X_{bound} defined in the previous chapter. The desired velocity profiles are established considering the following assumption.

Assumption 3.1

1. *Whenever safety is not compromised, platoons should keep the acceleration and jerk within comfort bounds.*
2. *Maneuvers are executed one at a time. No maneuver can begin before the previous one is completed.*

3.1.1 Join Law

The goal of the control law in a join maneuver is to decrease the initial relative displacement between the lead platoon and the trail platoon, $\Delta x(0)$, to a desired intraplatoon spacing Δx_{join} . The relative velocity, $\Delta \dot{x}$, should be null at the end of the join maneuver. The resulting trajectory of the state $(\Delta x, \Delta \dot{x}, v_{lead})$ during the join maneuver, according to Theorem 2.1, must be within the safety set X_{bound} .

In order to decrease the time the join maneuver takes to complete, the relative velocity between the trail and lead platoons, $\Delta \dot{x}$, should be minimized while observing the safety limits. This suggests that the state $(\Delta x, \Delta \dot{x}, v_{lead})$ of the join maneuver should be kept, as much as possible, in the boundary ∂X_{safe} of the safety set X_{safe} in Theorem 2.1. This boundary consists of two smooth portions:

1. In the first portion, the trail platoon is far enough from the lead platoon so that maximum deceleration will stop the lead platoon before the trail platoon hits it at v_{allow} , if a collision occurs.

2. The other portion of the maximum safe velocity curve represents the case when full braking does not stop the lead platoon before the trail platoon hits it at v_{allow} , if a collision occurs.

According with Eq. (2.14), the maximum safe velocity v_{trail} of the trail platoon, for a given Δx and v_{lead} is

$$v_{trail}(v_{lead}, \Delta x) = \begin{cases} -c_2 + \sqrt{2a_{min}^{trail} \Delta x + \alpha v_{lead}^2 + v_{allow}^2 + c_2 d}; \\ \quad R_2(\Delta x, v_{lead}) > S(\Delta x, v_{lead}), \\ -c_2 + v_{lead} + \sqrt{v_{allow}^2 + \frac{\alpha-1}{\alpha} a_{min}^{trail} (2\Delta x + c_2 d)}; \\ \quad R_2(\Delta x, v_{lead}) \leq S(\Delta x, v_{lead}), \end{cases} \quad (3.1)$$

where

$$\begin{aligned} R_1(\Delta x) &= -c_2 + \sqrt{v_{allow}^2 + \frac{\alpha-1}{\alpha} a_{min}^{trail} (2\Delta x + c_2 d)}, \\ R_2(\Delta x, v_{lead}) &= -c_2 - v_{lead} + \sqrt{2a_{min}^{trail} \Delta x + \alpha v_{lead}^2 + v_{allow}^2 + a_{min}^{trail} c_2 d}, \\ R_3(v_{lead}) &= (\alpha - 1)v_{max} - c_2 + v_{allow}, \\ S(\Delta x, v_{lead}) &= \max(R_1(\Delta x), R_3(v_{lead})), \\ c_2 &= (a_{max}^{trail} + a_{min}^{trail})d. \end{aligned}$$

Ideally, the maximum desired velocity for the trail platoon while performing a join maneuver should be the one indicated by (3.1). It should be noticed, however, that in (3.1) it is assumed that the velocity of the trail platoon, v_{trail} , is on the boundary of X_{safe} . Therefore, when the state $(\Delta x, \Delta \dot{x}, v_{lead})$ is not in this boundary, the actual value of the other state, $\Delta \dot{x}$, affects the trajectory of the state as time passes. Consider, for example, the situation depicted in Fig. 3.1. The flow departing from point p in the figure will have an initial direction given by the resultant of $\Delta \dot{x}$ and the effective acceleration a at that particular instant of time. For this reason it is suggested to set the maximum desired velocity for the trail platoon, v_{safe} as a function of the full state $(\Delta x, \Delta \dot{x}, v_{lead})$ as follows

$$v_{safe}(\Delta x, \Delta \dot{x}, v_{lead}) = \begin{cases} -c_2 + \sqrt{2a_{min}^{trail} (\Delta x + \eta \Delta \dot{x}) + \alpha v_{lead}^2 + v_{allow}^2 + a_{min}^{trail} c_2 d}; \\ \quad R_2(\Delta x, v_{lead}) > S(\Delta x, v_{lead}), \\ -c_2 + v_{lead} + \sqrt{v_{allow}^2 + \frac{\alpha-1}{\alpha} a_{min}^{trail} (2(\Delta x + \eta \Delta \dot{x}) + c_2 d)}; \\ \quad R_2(\Delta x, v_{lead}) \leq S(\Delta x, v_{lead}), \end{cases} \quad (3.2)$$

where $\eta > 0$ is a gain.

To finish the join maneuver in minimum time, it is necessary to slow the trail platoon to v_{lead} at the end of the join. According with assumption 3.1 the trail platoon should

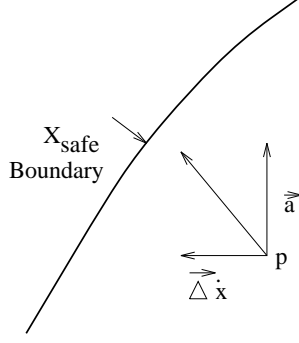


Figure 3.1: Effect of the relative velocity $\Delta\dot{x}$ on the desired velocity for the trail platoon.

decelerate at the maximum comfortable level. The velocity in the deceleration curve, v_{min} , written as a function of $(\Delta x, \Delta\dot{x}, v_{lead})$ is

$$v_{min}(\Delta x, \Delta\dot{x}, v_{lead}) = \min \left\{ \begin{array}{l} v_{lead} + \sqrt{2a_{com}(\Delta x + \eta\Delta\dot{x} - \Delta x_{join})} \\ v_{fast} \end{array} \right. . \quad (3.3)$$

where a_{com} is the magnitude of the comfort acceleration and deceleration for vehicles in a highway, Δx_{join} is the desired intraplatoon distance and v_{fast} is the maximum recommend velocity for a platoon to travel on the highway.

In order for the join control law to be safe and to allow the maneuver to be completed in minimum time, the velocity of the trail platoon should satisfy

$$v_d(\Delta x, \Delta\dot{x}, v_{lead}) = \min(v_{min}, v_{safe}) .$$

Eqs. (3.2)-(3.3) define a desired velocity profile for the trail platoon during a safe join law. Figure 3.2 shows an example of this desired velocity profile in the $\Delta x(\cdot)$ vs. $\Delta\dot{x}(\cdot)$ phase plane. For the profile in figure 3.2 it is assumed that the lead platoon is traveling at constant velocity and that the braking capabilities of the lead and trail platoons are equal. The acceleration portion in Figure 3.2 will be produced by the velocity tracking controller to be described in the next section.

The desired phase-plane trajectory for the trail platoon velocity includes abrupt changes in acceleration at the points where curve different sections intersect. It is convenient to smooth these transitions so as not to violate jerk comfort constraints. Cubic splines are used for this purpose (see Appendix A).

3.1.2 Split Law

In the split maneuver the goal is to increase the distance between the lead and trail platoon, Δx , to a desired value Δx_{split} . To accomplish this increment in relative distance, the relative speed between platoons, $\Delta\dot{x}$, must necessarily be positive. For this reason, in most cases, the velocity of the trail platoon will be lower than the velocity of the lead platoon, and thus the threat of high-speed collisions during a split maneuver will be inherently reduced.

To design the desired velocity profile for a split maneuver, a similar approach to the one used for the join maneuver can also be used. Two boundary curves are established for

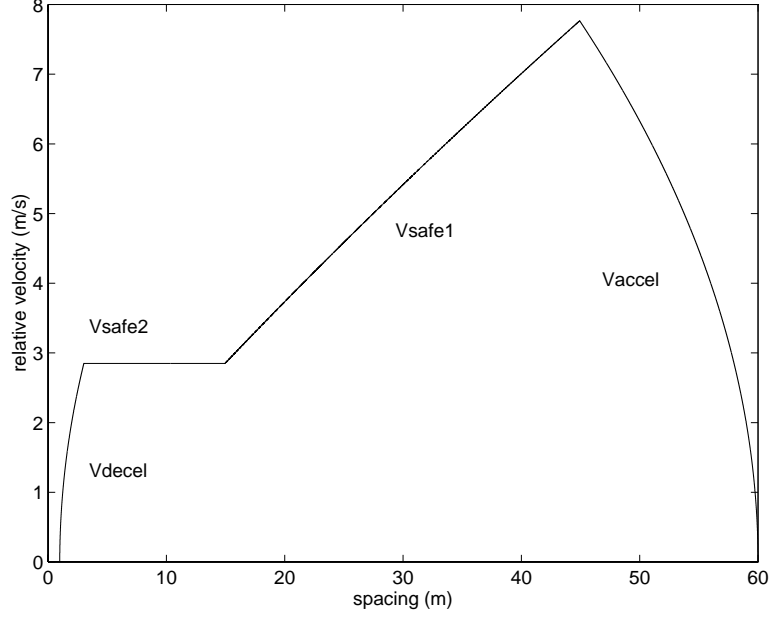


Figure 3.2: Basic velocity profile for 60 m initial spacing. The lead platoon is moving at a constant velocity of 25 m/s.

the velocity of the trail platoon, v_{trail} . The first one, related to safety, is derived from Eq. (2.14) by assuming $v_{allow} = 0$. Thus, for a given state $(\Delta x, \Delta \dot{x}, v_{lead})$, the maximum velocity of the trail platoon for the split law to be safe is

$$v_{safe}(\Delta x, \Delta \dot{x}, v_{lead}) = \begin{cases} -c_2 + \sqrt{2a_{min}^{trail}(\Delta x + \eta\Delta \dot{x}) + \alpha v_{lead}^2 + a_{min}^{trail}c_2d}; & R_2(\Delta x, v_{lead}) > S(\Delta x, v_{lead}), \\ -c_2 + v_{lead} + \sqrt{\frac{\alpha-1}{\alpha}a_{min}^{trail}(2(\Delta x + \eta\Delta \dot{x}) + c_2d)}; & R_2(\Delta x, v_{lead}) \leq S(\Delta x, v_{lead}), \end{cases} \quad (3.4)$$

where

$$\begin{aligned} R_1(\Delta x) &= -c_2 + \sqrt{\frac{\alpha-1}{\alpha}a_{min}^{trail}(2\Delta x + c_2d)}, \\ R_2(\Delta x, v_{lead}) &= -c_2 - v_{lead} + \sqrt{2a_{min}^{trail}\Delta x + \alpha v_{lead}^2 + a_{min}^{trail}c_2d}, \\ R_3(v_{lead}) &= (\alpha-1)v_{max} - c_2, \\ S(\Delta x, v_{lead}) &= \max(R_1(\Delta x), R_3(v_{lead})), \\ c_2 &= (a_{max}^{trail} + a_{min}^{trail})d. \end{aligned}$$

The other boundary curve is related to time-optimality. This curve establishes a lower bound on the velocity of the trail platoon. To determine this lower bound, it is assumed that, for a given state $(\Delta x, \Delta \dot{x}, v_{lead})$, if the trail platoon is traveling at this minimum velocity,

then it will reach the desired intraplatoon distance Δx_{split} with null relative velocity by applying maximum comfort acceleration. It is also assumed that there exists a minimum velocity v_{slow} below which it is not recommended to travel on the highway under normal circumstances. The minimum velocity of the trail platoon is therefore given by

$$v_{min}(\Delta x, \Delta \dot{x}, v_{lead}) = \max \left\{ \begin{array}{l} v_{lead} - \sqrt{2a_{com}(\Delta x_{split} - \Delta x - \eta \Delta \dot{x})}, \\ v_{slow} . \end{array} \right. \quad (3.5)$$

At any particular state $(\Delta x, \Delta \dot{x}, v_{lead})$ of a split maneuver, the velocity of the trail platoon should satisfy the safety requirements, therefore from Eqs. (3.4) and (3.5)

$$v_d(\Delta x, \Delta \dot{x}, v_{lead}) = \min(v_{min}, v_{safe}) .$$

3.1.3 Decelerate to Change Lane Law

The decelerate to change lane control law attempts to create a safe distance between platoons in different lanes before any actual change lane maneuver can take place. The decelerate to change lane law can be treated similarly to the split law. The only distinction is that, while safety is considered in terms of the lead platoon in the same lane, the time optimal part of the trajectory has to be calculated in terms of the lead platoon in the lane where the trail platoon is changing into. Notice that, according with assumption 3.1, it is enough to calculate for safety only for the platoon in the same lane, because the change lane maneuver will not occur until the decelerate to change lane maneuver is completed. The maximum safe velocity for the trail platoon is therefore the same as in the split law in Eq. (3.4).

The minimum velocity of the trail platoon is established in the same way as in the split control law, but considering the target velocity and distance with respect to the platoon in the adjacent lane. Thus

$$v_{min}(\Delta x_{next}, \Delta \dot{x}_{next}, v_{next}) = \max \left\{ \begin{array}{l} v_{next} - \sqrt{2a_{com}(\Delta x_{change} - \Delta x_{next} - \eta \Delta \dot{x}_{next})}, \\ v_{slow} . \end{array} \right. \quad (3.6)$$

where $(\Delta x_{next}, \Delta \dot{x}_{next}, v_{next})$ is the state of the platoon performing the change lane maneuver, relative to the lead platoon that is in the lane where the trail platoon is changing into and Δx_{change} is the required spacing after the decelerate to change lane maneuver is completed. At any particular stage of a decelerate to change lane maneuver, the velocity of the trail platoon should satisfy the safety requirements, therefore from Eqs. (3.4) and (3.6)

$$v_d(\Delta x, \Delta \dot{x}, v_{lead}, \Delta x_{next}, \Delta \dot{x}_{next}, v_{next}) = \min(v_{min}, v_{safe}) .$$

3.1.4 Leader Law

The leader law is intended to keep a platoon traveling on a highway at a target velocity and at a safe distance from the platoon ahead. When no propagation of collisions is desired in the highway, the state $(\Delta x, \Delta \dot{x}, v_{lead})$ of a platoon executing the leader law has to remain within

the set X_{leader} defined in Theorem 2.3. When safety is not critical, the target velocity for a platoon leader executing the leader law is no longer the velocity of the platoon ahead, but some desired velocity v_{link} . This velocity is given by a highway link layer traffic controller according to the section of the highway where the leader of the platoon is currently located (Li *et al.*, 1995).

The maximum safe velocity curve v_{safe} for a platoon in leader law, given $(\Delta x, \Delta \dot{x}, v_{lead})$, is according with Theorem 2.3

$$v_{safe}(\Delta x, \Delta \dot{x}, v_{lead}) = -c_2 - v_{allow} + \sqrt{2a_{min}^{trail} (\Delta x + \eta \Delta \dot{x}) + \alpha (v_{lead} - v_{allow})^2 + a_{min}^{trail} c_2 d} . \quad (3.7)$$

The desired velocity for a platoon under the leader law is therefore

$$v_d(\Delta x, \Delta \dot{x}, v_{lead}) = \min(v_{link}, v_{safe}) .$$

It is also important to remark that whenever $v_{link} > v_{safe}$ then the relative spacing Δx will decrease until it reaches

$$\Delta x_{leader} = \frac{(v_{lead} + v_{allow} + c_2)^2 - \alpha (v_{lead} - v_{allow})^2 - a_{min}^{trail} c_2 d}{2a_{min}^{trail}} . \quad (3.8)$$

Substituting the value of Δx_{leader} in (3.8) into (3.7), $v_d(\Delta x_{leader}, 0, v_{lead}) = v_{lead}$ and therefore the desired velocity of the trail platoon will be the velocity of the lead platoon .

3.2 Velocity profile tracking controller

In this section a velocity tracking controller is introduced. This controller commands the actual velocity of a platoon to follow the desired velocity profile derived in the previous section. The design of this controller is based on the following assumptions.

Assumption 3.2

1. *Positions and velocities of both the lead and the trail platoons are measured quantities.*
2. *The acceleration of the trail platoon is known.*
3. *The acceleration of the lead platoon is estimated.*
4. *The jerk of the lead platoon is modeled as noise.*

The velocity tracking controller is designed using the backstepping procedure (Krstic *et al.*, 1995). This controller design combines an observer for the lead platoon state with a nonlinear controller for the jerk of the trail platoon.

3.2.1 Backstepping Design

Let $v_d(\Delta x, \Delta \dot{x}, v_{lead})$ be the value of the desired velocity flow field for the trail platoon. Introduce, for convenience, the change of variables

$$(\Delta x, \Delta \dot{x}, v_{lead}) \iff (\Delta x, v_{lead}, v_{trail}) ,$$

that follows directly from $v_{trail} = v_{lead} - \Delta \dot{x}$. Define the velocity error by

$$e = v_{trail} - v_d(\Delta x, v_{lead}, v_{trail}).$$

The velocity error dynamics is given by

$$\dot{e} = a_{trail} - \begin{pmatrix} \frac{\partial v_d}{\partial \Delta x} & \frac{\partial v_d}{\partial v_{lead}} & \frac{\partial v_d}{\partial v_{trail}} \end{pmatrix} \begin{pmatrix} v_{lead} - v_{trail} \\ a_{lead} \\ a_{trail} \end{pmatrix}, \quad (3.9)$$

where a_{lead} and a_{trail} denote the second time derivative of x_{lead} and x_{trail} , respectively. According to assumption 3.2, let \hat{a}_{lead} be the estimated acceleration of the lead car. Define the lead platoon acceleration estimation error, \tilde{a}_{lead} as

$$\tilde{a}_{lead} = \dot{v}_{lead} - \hat{a}_{lead} = a_{lead} - \hat{a}_{lead}. \quad (3.10)$$

From (3.10) into (3.9)

$$\dot{e} = a_{trail} - \begin{pmatrix} \frac{\partial v_d}{\partial \Delta x} & \frac{\partial v_d}{\partial v_{lead}} & \frac{\partial v_d}{\partial v_{trail}} \end{pmatrix} \begin{pmatrix} v_{lead} - v_{trail} \\ \hat{a}_{lead} \\ a_{trail} \end{pmatrix} - \frac{\partial v_d}{\partial v_{lead}} \tilde{a}_{lead}. \quad (3.11)$$

Assume, for the moment, that there is not error in the estimation of the acceleration, i.e., $\tilde{a}_{lead} = 0$, then, if the dynamics of e is desired to be stable, it is possible to define a fictitious control for the acceleration of the trail platoon as

$$, (\Delta x, v_{lead}, v_{trail}, \hat{a}_{lead}, a_{trail}) = -\lambda_1 e + \begin{pmatrix} \frac{\partial v_d}{\partial \Delta x} & \frac{\partial v_d}{\partial v_{lead}} & \frac{\partial v_d}{\partial v_{trail}} \end{pmatrix} \begin{pmatrix} v_{lead} - v_{trail} \\ \hat{a}_{lead} \\ a_{trail} \end{pmatrix}, \quad (3.12)$$

Using (3.12) into (3.11)

$$\dot{e} = -\lambda_1 e + a_{trail} - , - \frac{\partial v_d}{\partial v_{lead}} \tilde{a}_{lead}. \quad (3.13)$$

Define $\tilde{,}$ to be the difference between a_{trail} and $, ,$ i.e.,

$$\tilde{,} (t) = a_{trail}(t) - , (\Delta x, v_{lead}, v_{trail}, \hat{a}_{lead}, a_{trail}). \quad (3.14)$$

From (3.14) in (3.13), the velocity error dynamics is

$$\dot{e} = -\lambda_1 e + \tilde{,} - \frac{\partial v_d}{\partial v_{lead}} \tilde{a}_{lead}. \quad (3.15)$$

Consider now the dynamics of $\tilde{\cdot}$,

$$\dot{\tilde{\cdot}} = j_{trail} - \left(\frac{\partial,}{\partial \Delta x} \quad \frac{\partial,}{\partial v_{lead}} \quad \frac{\partial,}{\partial v_{trail}} \quad \frac{\partial,}{\partial \hat{a}_{lead}} \quad \frac{\partial,}{\partial a_{trail}} \right) \begin{pmatrix} v_{lead} - v_{trail} \\ a_{lead} \\ a_{trail} \\ \hat{a}_{lead} \\ j_{trail} \end{pmatrix}, \quad (3.16)$$

where $j_{trail} = d^3 x_{trail}/dt^3$ is the control jerk of the trail platoon and \hat{a}_{lead} is the time derivative of the estimate of the lead platoon's acceleration. The expression for the latter depends on the implementation of the lead platoon state observer and will be defined later when this observer is presented.

The following control for j_{trail} is proposed

$$\left(1 - \frac{\partial,}{\partial a_{trail}} \right) j_{trail} = -\lambda_2 \tilde{\cdot} - \beta e + \left(\frac{\partial,}{\partial \Delta x} \quad \frac{\partial,}{\partial v_{lead}} \quad \frac{\partial,}{\partial \dot{x}_{trail}} \quad \frac{\partial,}{\partial \hat{a}_{lead}} \right) \begin{pmatrix} v_{lead} - \dot{x}_{trail} \\ \hat{a}_{lead} \\ \ddot{x}_{trail} \\ \hat{\tilde{a}}_{lead} \end{pmatrix}, \quad (3.17)$$

where $\hat{\tilde{a}}_{lead}$ is an estimate of the time derivative of the lead platoon acceleration. When \hat{a}_{lead} is estimated using a full order observer, $\hat{\tilde{a}}_{lead} = \hat{a}_{lead}$; when \hat{a}_{lead} is estimated using a reduced order observer, their difference is proportional to the error in the estimate of the lead platoon acceleration. Thus,

$$\hat{a}_{lead} - \hat{\tilde{a}}_{lead} = d_1 \hat{a}_{lead}, \quad (3.18)$$

where $d_1 = 0$, when a full order observer is used, and is a known constant, when a reduced order observer is used.

The dynamics of $\tilde{\cdot}$ under (3.17) becomes

$$\dot{\tilde{\cdot}} = -\beta e - \lambda_2 \tilde{\cdot} - \left(\frac{\partial,}{\partial v_{lead}} + \frac{\partial,}{\partial \hat{a}_{lead}} d_1 \right) \tilde{a}_{lead}.$$

Define

$$\mathbf{g} = \begin{pmatrix} -\frac{\partial v_d}{\partial v_{lead}} \\ -\frac{\partial,}{\partial v_{lead}} - d_1 \frac{\partial,}{\partial \hat{a}_{lead}} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\partial v_d}{\partial v_{lead}} \\ -(\lambda_1 + d_1) \frac{\partial v_d}{\partial v_{lead}} - \frac{\partial v_d}{\partial \Delta x} - \frac{\partial^2 v_d}{\partial \Delta x \partial v_{lead}} (v_{lead} - v_{trail}) - \frac{\partial^2 v_d}{\partial v_{lead}^2} \hat{a}_{lead} - \frac{\partial^2 v_d}{\partial v_{trail} \partial v_{lead}} a_{trail} \end{pmatrix},$$

the combined dynamics of e and $\tilde{\cdot}$ are given by

$$\frac{d}{dt} \begin{pmatrix} e \\ \tilde{\cdot} \end{pmatrix} = \begin{pmatrix} -\lambda_1 & 1 \\ -\beta & -\lambda_2 \end{pmatrix} \begin{pmatrix} e \\ \tilde{\cdot} \end{pmatrix} + \mathbf{g} \tilde{a}_{lead}. \quad (3.19)$$

Notice that the state evolution matrix in (3.19) is stable when λ_1 , λ_2 and β are positive. The design values of these parameters can be obtained by minimizing the effect of \tilde{a}_{lead} on e using linear methods and assuming constant values of \mathbf{g} (see appendix A for more details).

3.2.2 Lead Platoon State Observers

The velocity profile tracking controller makes use of the estimate of the acceleration of the lead platoon that, by assumption 3.2, is not measured. Two observers to estimate this lead platoon acceleration are presented. The first one is a full order observer that estimates the position, velocity and acceleration of the lead platoon. The second is a reduced order observer that estimates only the acceleration of the lead platoon.

According with assumption 3.2, the dynamics of the lead platoon is given by

$$\frac{d^3}{dt^3} x_{lead} = j_{lead}, \quad (3.20)$$

where j_{lead} is the jerk input to the lead platoon. Defining

$$\mathbf{x}_{lead} = \begin{pmatrix} x_{lead} \\ v_{lead} \\ a_{lead} \end{pmatrix}, \quad \mathbf{y}_{lead} = \begin{pmatrix} x_{lead} \\ v_{lead} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

(3.20) can be rewritten as

$$\dot{\mathbf{x}}_{lead} = \mathbf{A} \mathbf{x}_{lead} + \mathbf{B} j_{lead}, \quad \mathbf{y}_{lead} = \mathbf{C} \mathbf{x}_{lead}.$$

It is straightforward to check that (\mathbf{A}, \mathbf{B}) is controllable and (\mathbf{A}, \mathbf{C}) is observable.

Full order observer

A full order state observer for the lead platoon acceleration is

$$\begin{aligned} \dot{\hat{\mathbf{x}}}_{lead} &= \mathbf{A} \hat{\mathbf{x}}_{lead} - \mathbf{L} (\mathbf{y}_{lead} - \mathbf{C} \hat{\mathbf{x}}_{lead}) + \mathbf{q}, \\ \hat{a}_{lead} &= (0 \ 0 \ 1) \hat{\mathbf{x}}_{lead}, \end{aligned} \quad (3.21)$$

where $\hat{\mathbf{x}}_{lead} = (\hat{x}_{lead}, \hat{v}_{lead}, \hat{a}_{lead})^T \in \mathcal{R}^3$ is the state estimate, the observer gain $\mathbf{L} \in \mathcal{R}^{3 \times 2}$ is such that $\mathbf{A} - \mathbf{L}\mathbf{C}$ is asymptotically stable, and $\mathbf{q} = \mathbf{q}(t)$ is a tuning function to be determined. For the full order observer, \hat{a}_{lead} can be expressed without error using known quantities from (3.21), i.e. $d_1 = 0$ in (3.18).

The dynamics of the acceleration estimation error \tilde{a}_{lead} is given by

$$\begin{aligned} \dot{\tilde{\mathbf{x}}}_{lead} &= (\mathbf{A} - \mathbf{L}\mathbf{C}) \tilde{\mathbf{x}}_{lead} + \mathbf{B} j_{lead} - \mathbf{q}, \\ \tilde{a}_{lead} &= (0 \ 0 \ 1) \tilde{\mathbf{x}}_{lead}, \end{aligned} \quad (3.22)$$

where $\tilde{\mathbf{x}}_{lead} = \mathbf{x}_{lead} - \hat{\mathbf{x}}_{lead}$.

Reducer order observer

A reduced order observer that estimates only the acceleration of the lead platoon, \hat{a}_{lead} , can be designed as follows

$$\begin{aligned} \dot{r} &= -L_2 r - (L_1 L_2 \quad L_2^2 + L_1) \mathbf{y}_{lead} + q, \\ \hat{a}_{lead} &= r + L_1 x_{lead} + L_2 v_{lead}, \end{aligned} \quad (3.23)$$

where, for this case, L_1 and L_2 are the two components of the matrix \mathbf{L} with $L_2 > 0$ for the observer to be stable and $q = q(t) \in \mathcal{R}$ is a tuning function to be determined.

It can be shown that the acceleration estimation error $\tilde{a}_{lead} = a_{lead} - \hat{a}_{lead}$ is given by

$$\dot{\tilde{a}}_{lead} = -L_2 \tilde{a}_{lead} + \dot{j}_{lead} - q(t). \quad (3.24)$$

Notice that, because of the structure in (3.23), the reduced order observer does not allow $\dot{\hat{a}}_{lead}$ to be computed using known quantities. $\dot{\hat{a}}_{lead}$ can be estimated by

$$\dot{\hat{a}}_{lead}^{\sim} = \dot{r} + L_1 v_{lead} + L_2 \hat{a}_{lead}.$$

Thus $\dot{\hat{a}}_{lead} - \dot{\hat{a}}_{lead}^{\sim} = L_2 \tilde{a}_{lead}$, i.e. $d_1 = L_2$ in (3.18).

3.2.3 Stability Analysis

In the stability analysis for the velocity tracking controller, the designs for the trail platoon jerk control and the lead platoon acceleration observer are combined. The value of the observer tuning function, \mathbf{q} and q for the full order observer and the reducer order observer, respectively, is related with the value of the nonlinear term in the jerk control, \mathbf{g} , in such a way that a stable behavior can be obtained for both the error dynamics of $(e, \tilde{\cdot})^T$ in (3.19) and the lead platoon acceleration estimation error, \tilde{a}_{lead} , in (3.22) or (3.23).

Assumption 3.3

1. The jerk of the lead platoon is bounded, i.e., $\|j_{lead}(\cdot)\|_{\infty} \leq j_{max}$.

Full order observer

The following theorem establishes a bound on the velocity tracking error $e(t) = v_{trail}(t) - v_d(\Delta x(t), v_{lead}(t), v_{trail}(t))$, when the full order observer is used to estimate the state of the lead platoon.

Theorem 3.1 *Let the dynamics of $\mathbf{s} = (e, \tilde{\cdot})^T$ be given by (3.19) with $\lambda_1, \lambda_2, \beta > 0$ and the dynamics of lead platoon acceleration estimation error, \tilde{a}_{lead} , be given by (3.22). Choose the control law for the jerk of the trail platoon according with (3.12) and (3.17). Then, under assumptions 3.2 and 3.3, there is time t_1 such that for any $t \geq t_1$ and any $\epsilon > 0$ the velocity tracking error of the trail platoon $e(t) = v_{trail}(t) - v_d(\Delta x(t), v_{lead}(t), v_{trail}(t))$ satisfies*

$$|e(t)| \leq \nu_f (1 + \epsilon); \quad \nu_f > 0. \quad (3.25)$$

Proof: First notice that $\lambda_1, \lambda_2, \beta > 0$ implies that the matrix

$$\mathbf{F} = \begin{pmatrix} -\lambda_1 & 1 \\ -\lambda_2 & \beta \end{pmatrix},$$

is stable.

Define $\mathbf{A}_F = \mathbf{A} - \mathbf{L}\mathbf{C}$ to be the stable evolution matrix of the full order observer in (3.22). Let $\mathbf{Q} \in \mathcal{R}^{2 \times 2}$, and $\mathbf{P} \in \mathcal{R}^{3 \times 3}$ be positive definite symmetric matrices that satisfy the Lyapunov equations

$$\mathbf{Q}\mathbf{F} + \mathbf{F}^T\mathbf{Q} = -2\mathbf{C}_1; \quad \mathbf{P}\mathbf{A}_F + \mathbf{A}_F^T\mathbf{P} = -2\mathbf{C}_2,$$

where $\mathbf{C}_1 \in \mathcal{R}^{3 \times 3}$ and $\mathbf{C}_2 \in \mathcal{R}^{2 \times 2}$ are positive definite matrices.

Consider the Lyapunov function

$$V(e, \tilde{s}, \tilde{x}_{lead}) = \frac{1}{2}\mathbf{s}^T\mathbf{Q}\mathbf{s} + \gamma\frac{1}{2}\tilde{\mathbf{x}}_{lead}^T\mathbf{P}\tilde{\mathbf{x}}_{lead}, \quad (3.26)$$

where $\gamma > 0$. The time derivative of (3.26) is

$$\dot{V} = -\mathbf{s}^T\mathbf{C}_1\mathbf{s} - \gamma\tilde{\mathbf{x}}_{lead}^T\mathbf{C}_2\tilde{\mathbf{x}}_{lead} + \mathbf{s}^T\mathbf{Q}\mathbf{g}(0 \ 0 \ 1)\tilde{\mathbf{x}}_{lead} - \gamma\tilde{\mathbf{x}}_{lead}^T\mathbf{P}\mathbf{q} + \gamma j_{lead}\mathbf{B}^T\mathbf{P}\tilde{\mathbf{x}}_{lead}, \quad (3.27)$$

after substitution of (3.19) and (3.22) into (3.27).

Choose the tuning function \mathbf{q} in (3.22) to be

$$\mathbf{q} = \frac{(\mathbf{g}^T\mathbf{Q}\mathbf{s})}{\gamma}\mathbf{P}^{-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (3.28)$$

Then

$$\dot{V} = -\mathbf{s}^T\mathbf{C}_1\mathbf{s} - \gamma\tilde{\mathbf{x}}_{lead}^T\mathbf{C}_2\tilde{\mathbf{x}}_{lead} + \gamma j_{lead}\mathbf{B}^T\mathbf{P}\tilde{\mathbf{x}}_{lead}. \quad (3.29)$$

Let $\mathbf{F} = \mathbf{T}_1\mathbf{\Lambda}_1\mathbf{T}_1^{-1}$ and $\mathbf{A}_F = \mathbf{T}_2\mathbf{\Lambda}_2\mathbf{T}_2^{-1}$ be the real Schur decomposition (Golub and Loan, 1989) of \mathbf{F} and \mathbf{A}_F , respectively. Pick

$$-2\mathbf{C}_1 = \mathbf{T}_1^{-T}(\mathbf{\Lambda}_1 + \mathbf{\Lambda}_1^{-T})\mathbf{T}_1^{-1}, \quad -2\mathbf{C}_2 = \mathbf{T}_2^{-T}(\mathbf{\Lambda}_2 + \mathbf{\Lambda}_2^T)\mathbf{T}_2^{-1}. \quad (3.30)$$

It can be shown that

$$\mathbf{Q} = \mathbf{T}_1^{-T}\mathbf{T}_1^{-1}, \quad \mathbf{P} = \mathbf{T}_2^{-T}\mathbf{T}_2^{-1}. \quad (3.31)$$

Using (3.30) and (3.31) in (3.29)

$$\dot{V} = -\mathbf{s}^T\mathbf{Q}\mathbf{F}\mathbf{s} - \gamma\tilde{\mathbf{x}}_{lead}^T\mathbf{P}\mathbf{A}_F\tilde{\mathbf{x}}_{lead} + \gamma j_{lead}\mathbf{B}^T\mathbf{P}\tilde{\mathbf{x}}_{lead}. \quad (3.32)$$

Hence, it follows that

$$\dot{V} \leq -2\zeta V + \tilde{\mathbf{x}}_{lead}^T\mathbf{P} \cdot_3 j_{lead}, \quad (3.33)$$

where -2ζ is minimum real part of the eigenvalues of \mathbf{F} and \mathbf{A}_F and $\mathbf{P}_{\cdot 3}$ is the third column of \mathbf{P} .

From (3.26) it follows that

$$e^2 \leq \rho V, \quad \text{where} \quad \rho = \frac{2Q_{22}}{Q_{11}Q_{22} - Q_{12}^2} \quad (3.34)$$

and

$$(\mathbf{P}_{\cdot 3}^T \tilde{\mathbf{x}}_{lead})^2 \leq \frac{\delta}{\gamma} V, \quad \text{where} \quad \delta = 2\mathbf{P}_{\cdot 3}^T \mathbf{P}^{-1} \mathbf{P}_{\cdot 3} \quad (3.35)$$

and Q_{ij} are the (i, j) th element of the matrix \mathbf{Q} .

The time derivative of the square root of (3.26) is

$$\frac{d}{dt} \left(V^{\frac{1}{2}} \right) = \frac{1}{2} V^{-\frac{1}{2}} \dot{V} \quad (3.36)$$

Using (3.33), (3.34) and (3.35) in (3.36) and by assumption 3.3

$$\frac{d}{dt} \left(V^{\frac{1}{2}} \right) \leq -\zeta V^{\frac{1}{2}} + \sqrt{\frac{\delta}{\gamma}} j_{max} . \quad (3.37)$$

Hence, from (3.34) and (3.37), it follows that for any initial conditions $(\mathbf{s}(0), \tilde{\mathbf{x}}_{lead}(0))$, and for any $\epsilon > 0$, there is a time t_1 s.t. if $t \geq t_1$,

$$\sqrt{\frac{1}{\rho}} |e(t)| \leq V^{\frac{1}{2}}(t) \leq \frac{j_{max}}{\zeta} \sqrt{\frac{\delta}{\gamma}} (1 + \epsilon) ,$$

Therefore, after a long enough time

$$|e(t)| \leq \nu_f (1 + \epsilon) ,$$

where

$$\nu_f = \frac{j_{max}}{\zeta} \sqrt{\frac{\delta \rho}{\gamma}} .$$

■

Reduced order observer

The following theorem establishes a bound on the velocity tracking error $e(t) = v_{trail}(t) - v_d(\Delta x(t), v_{lead}(t), v_{trail}(t))$, when the reducer order observer is used to estimate the acceleration of the lead platoon.

Theorem 3.2 *Let the dynamics of $(e, \tilde{\cdot})^T$ be given by (3.19) and the dynamics of lead platoon acceleration estimation error, \tilde{a}_{lead} , be given by (3.23). Choose the control law for the jerk of the trail platoon according with (3.12) and (3.17). Then, under assumptions 3.2 and 3.3, there is time t_1 such that for any $t \geq t_1$ and any $\epsilon > 0$ the velocity tracking error of the trail platoon $e(t) = v_{trail}(t) - v_d(\Delta x(t), v_{lead}(t), v_{trail}(t))$ satisfies*

$$|e(t)| \leq \nu_r(1 + \epsilon) ; \quad \nu_r > 0 . \quad (3.38)$$

Proof: Consider the Lyapunov function

$$V(e, \tilde{\cdot}, \tilde{a}_{lead}) = \frac{1}{2}\beta e^2 + \frac{1}{2}\tilde{\cdot}^2 + \frac{1}{2}\gamma \tilde{a}_{lead}^2 . \quad (3.39)$$

Using (3.19) and (3.24), the time derivative of (3.39) is

$$\dot{V}(e, \tilde{\cdot}, \tilde{a}_{lead}) = -\beta\lambda_1 e^2 - \lambda_2 \tilde{\cdot}^2 + (\beta e \quad \tilde{\cdot}) \mathbf{g} \tilde{a}_{lead} - \gamma \tilde{a}_{lead} q + \gamma j_{lead} \tilde{a}_{lead} - \gamma L_2 \tilde{a}_{lead}^2 . \quad (3.40)$$

If the tuning function q in (3.23) is set to be

$$q = \frac{1}{\gamma}(\beta e \quad \tilde{\cdot}) \mathbf{g} , \quad (3.41)$$

then, from (3.41) into (3.40)

$$\dot{V} = -\beta\lambda_1 e^2 - \lambda_2 \tilde{\cdot}^2 - \gamma L_2 \tilde{a}_{lead}^2 + \gamma \tilde{a}_{lead} j_{lead} .$$

This shows that if $\|j_{lead}(\cdot)\|_\infty \leq j_{max}$, then

$$\dot{V} \leq -2\zeta V + \sqrt{2\gamma} V^{\frac{1}{2}} j_{max} , \quad (3.42)$$

where $\zeta = \min(\lambda_1, \lambda_2, L_2)$. Similarly, it can be shown that

$$\frac{d}{dt} \left(V^{\frac{1}{2}} \right) \leq -\zeta V^{\frac{1}{2}} + \sqrt{\frac{\gamma}{2}} j_{max} .$$

Hence, for any initial conditions $(e(0), \tilde{\cdot}(0), \tilde{a}_{lead}(0))$, and for any $\epsilon > 0$, there is a time t_1 such that if $t \geq t_1$,

$$\sqrt{\frac{\beta}{2}} |e(t)| \leq V^{\frac{1}{2}}(t) \leq \frac{j_{max}}{\zeta} \sqrt{\frac{\gamma}{2}} (1 + \epsilon) ,$$

Therefore, after a long enough time

$$|e(t)| \leq \nu_r(1 + \epsilon) ,$$

where

$$\nu_r = \frac{j_{max}}{\zeta} \sqrt{\frac{\gamma}{\beta}} .$$

■

3.2.4 Lead platoon jerk saturation effect on stability

Theorems 3.1 and 3.2 establish a bound on the velocity tracking error $e(t)$, when the jerk of the lead car satisfies assumption 3.3. When maneuver trajectories are inside the safety sets X_{safe} and X_{leader} defined in Theorems 2.1 and 2.3, safety is not compromised and the jerk of the lead platoon is expected to satisfy

$$|j_{trail}(\cdot)| \leq j_{com} ,$$

where j_{com} is the comfort jerk. When safety is compromised, and trajectories are inside X_{bound} , it was shown that the worst behavior of the lead platoon in terms of safety, was to apply and hold full brakes. If a jerk control for the lead platoon is assumed, this implies that the maximum jerk can be applied for at most d seconds. In this section two corollaries to Theorems 3.1 and 3.2 are presented. They establish that, whenever the lead platoon applies and holds full brakes, the velocity tracking error will eventually go to zero.

Assumption 3.4

1. *The maximum braking jerk of the lead platoon can be sustained for at most d seconds.*

Full order observer

Corollary 3.1 *Let the dynamics of $\mathbf{s} = (e, \tilde{\cdot})^T$ be given by (3.19) with $\lambda_1, \lambda_2, \beta > 0$ and the dynamics of lead platoon acceleration estimation error, \tilde{a}_{lead} , be given by (3.22). Choose the control law for the jerk of the trail platoon according with (3.12) and (3.17). Then, under assumptions 3.2 and 3.4,*

$$\lim_{t \rightarrow \infty} |e(t)| = 0 .$$

Proof: Proceede as in Theorem 3.1, then

$$\dot{V} \leq -2\zeta V + \tilde{\mathbf{x}}_{lead}^T \mathbf{P} \cdot j_{lead} .$$

By assumption 3.4, $j_{lead}(t) = 0 ; \forall t \geq d$. Therefore

$$\dot{V}(t) \leq -2\zeta V(t) \leq 0 ; \quad \forall t \geq d .$$

■

Reducer order observer

Corollary 3.2 *Let the dynamics of $(e, \tilde{\cdot})^T$ be given by (3.19) and the dynamics of lead platoon acceleration estimation error, \tilde{a}_{lead} , be given by (3.23). Choose the control law for the jerk of the trail platoon according with (3.12) and (3.17). Then, under assumptions 3.2 and 3.4,*

$$\lim_{t \rightarrow \infty} |e(t)| = 0 .$$

Proof: Following the same procedure as in Theorem 3.2,

$$\dot{V}(t) = -\beta\lambda_1 e(t)^2 - \lambda_2 \tilde{\omega}(t)^2 - \gamma L_2 \tilde{a}_{lead}(t)^2 + \gamma \tilde{a}_{lead}(t) j_{lead}(t).$$

By assumption 3.4, $j_{lead}(t) = 0; \forall t \geq d$. Therefore

$$\dot{V}(t) = -\beta\lambda_1 e(t)^2 - \lambda_2 \tilde{\omega}(t)^2 - \gamma L_2 \tilde{a}_{lead}(t)^2 \leq -2\zeta V(t) \leq 0; \quad \forall t \geq d,$$

where $\zeta = \min(\lambda_1, \lambda_2, L_2)$. ■

Appendix A contains issues related to the implementation of the velocity tracking controller.

Chapter 4

Regulation Layer Simulation Results

The control laws simulation results shown here are from a Matlab program that simulates just two adjacent platoons involved in a maneuver. The program was written to test the control laws for different behaviors of the platoon ahead. The control used was the velocity tracking controller presented in chapter 3. Most of the control laws were also implemented in SmartPath (Eskafi *et al.*, 1992). The results of both Matlab and SmartPath simulations were the same concerning vehicle safety and performance.

Some of the parameter values for the simulations are shown in Table 4.1.

Five plots are included for each simulation:

1. Relative distance Δx vs. time.
2. Relative velocity $-\Delta \dot{x}$ vs. time.
3. Acceleration of the trail platoon vs. time.
4. Jerk of the trail platoon vs. time.
5. Phase portrait in the Δx - $\Delta \dot{x}$ plane. The plot includes the two safety boundaries defined in chapter 2, ∂X_{safe} and ∂X_{bound} . The controller reference is obtained by reducing ∂X_{safe} by a constant factor to account for discrete time and controller tuning effects.

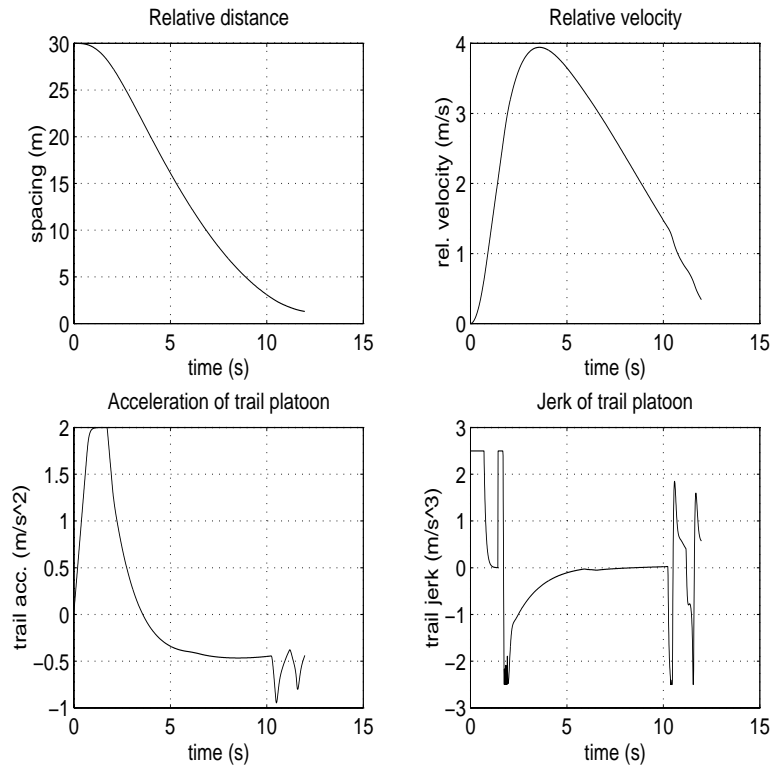
4.1 Simulations with no collisions allowed ($v_{allow} = 0$)

The following set of results uses the approach presented in this report to produce regulation layer maneuvers in which not even low speed collisions will occur. For these simulations the remaining parameters are shown in Table 4.2

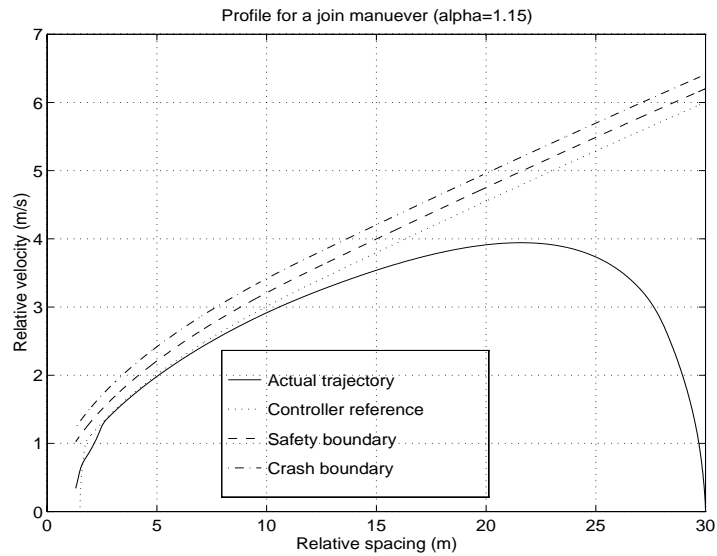
Figure 4.1 shows results for a merge from 30 *m* initial spacing. The velocity of the platoon ahead was constant at 25 *m/s*. The maneuver was completed in 11.9 *s*. Jerk and acceleration comfort limits were not exceeded. The final relative velocity is not zero as the simulation only ran to the point where the follower law takes effect.

Figure 4.2 shows results from a merge with an initial spacing of 60 *m*. The lead platoon maintained a constant velocity. The merge took 16.5 *s* in this case.

Figure 4.3 show the case in which the lead platoon applies maximum braking when the trail platoon has maximum relative velocity. Note that the simulation shows no collisions, as expected. This figure includes a large spike in jerk. The controller is designed so that

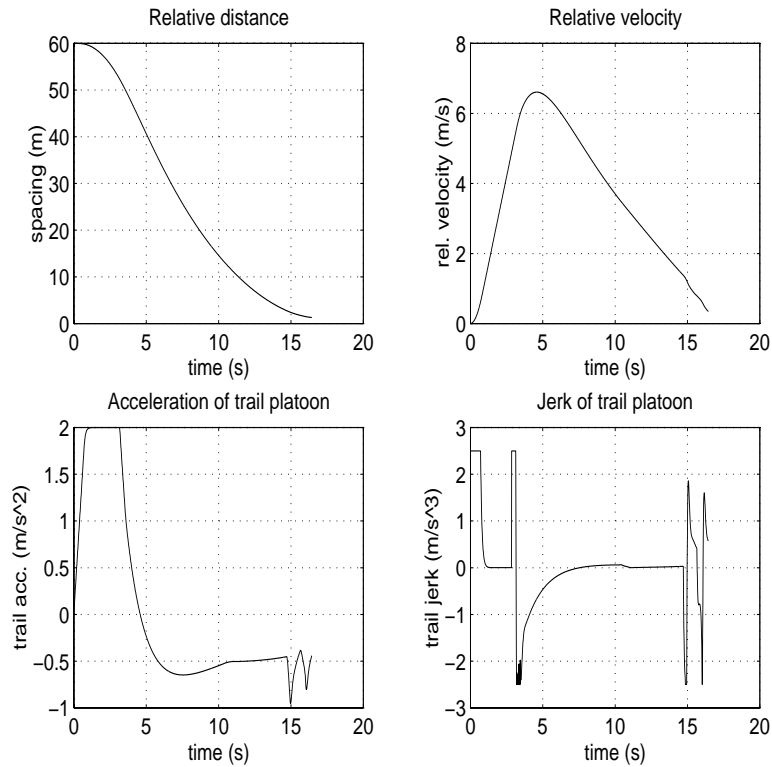


(a)

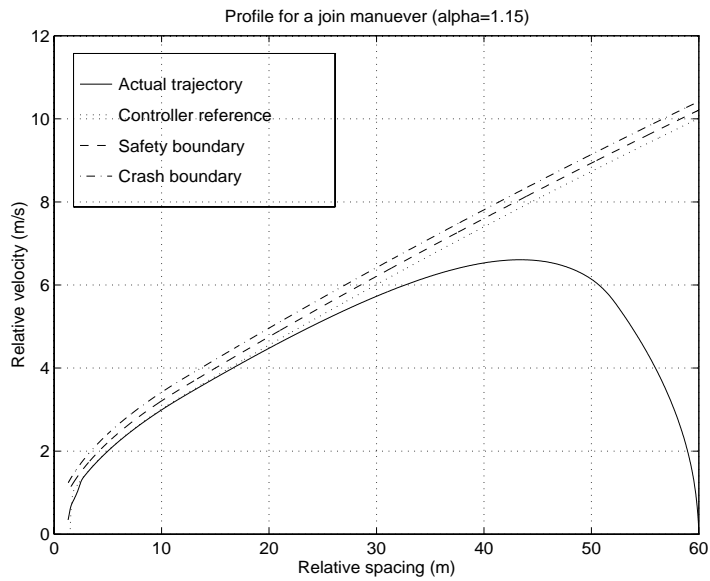


(b)

Figure 4.1: Simulation results of merge from 30 m initial spacing: The initial velocity of both lead and trail platoons was 25 m/s . a) Results vs. time. b) Results in the phase plane Δx vs. $\Delta \dot{x}$.



(a)



(b)

Figure 4.2: Simulation results of merge from 60 m initial spacing: The initial velocity of both lead and trail platoons was 25 m/s. a) Results vs. time. b) Results in the phase plane Δx vs. $\Delta \dot{x}$.

$a_{com} = \pm 2 \text{ m/s}^2$	This is the value used in the current merge (Godbole and Lygeros, 1993). It is commonly accepted in the literature. See (Hitchcock, 1993a; Chiu <i>et al.</i> , 1977).
$\mu = 1.12$	This value was derived graphically from the results for the follower law in (Swaroop, 1994).
$A_{MIN} = 5 \text{ m/s}^2$	This is the absolute value of the maximum deceleration. This value is used in the current merge.
$A_{MAX} = -2.5 \text{ m/s}^2$	This is a rough approximation based on data presented in (Gillespie, 1992). The road is assumed to be flat. The vehicles are assumed to have automatic transmissions in third gear.
$j_{com} = \pm 2.5 \text{ m/s}^3$	Lygeros and Godbole (Godbole and Lygeros, 1993) set the comfortable jerk limit at 5m/s^3 in the current merge. Most examples in the literature suggest the limit is between 2m/s^3 and 2.5m/s^3 . See (Hitchcock, 1993a; Sklar <i>et al.</i> , 1979; Chiu <i>et al.</i> , 1977).
$j_{max} = -50 \text{ m/s}^3$	This value was selected as a physical limit on jerk. It is less than the one given in (Fenton, 1979).
$\Delta x_{join} = 1 \text{ m}$	This is the current intraplatoon spacing.
$\Delta x_{split} = 60 \text{ m}$	This is the current interplatoon distance.
$\Delta x_{range} = 91 \text{ m}$	This value corresponds to the maximum range of the sensor currently used in PATH.
$d = 30 \text{ ms}$	Simple brake models often include pure time delays of about 50 ms . It is shown in (Gerdes <i>et al.</i> , 1993), however, that delays in the current braking system for PATH are greater than 150 ms . By redesigning the brake system, delays near 20 ms could be achieved (Gerdes and Hedrick, 1995). Delays from sensing, filtering and differentiating are also possible, but they could be small at a high sample rate.

Table 4.1: Parameters used for the simulations

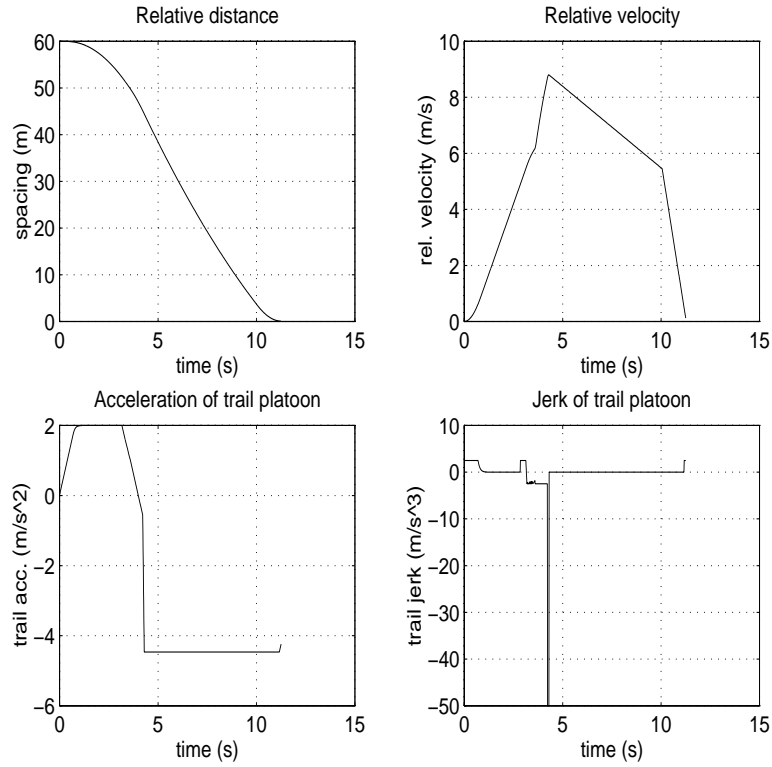
comfort limits are disregarded when safety becomes critical. In these cases, the comfort jerk was overridden once the large lead platoon deceleration was detected.

In the final merge simulation, the lead platoon braked at comfortable deceleration. No collision occurred. The results are shown in figure 4.4. It should be noticed that in the last part of the maneuver the acceleration of the trail platoon exceeded the comfort limit. The controller was designed to allow this behavior in order to avoid collisions.

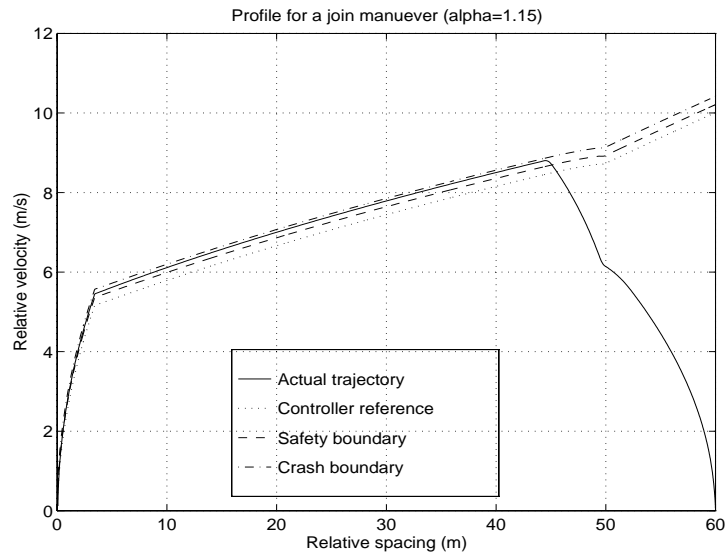
The split law was also simulated. Figures 4.5 and 4.6 show the results of split from 1 and 30 m to 60 m spacing, respectively. The cases when the lead platoon applies comfort and full braking while the trail platoon is attempting a split are shown in Figures 4.7 and 4.8, respectively.

The last simulation results in Fig. 4.9 correspond to the leader law. An extreme case was simulated. The lead platoon detects a stopped platoon in front of it, while traveling at maximum speed. Notice that in this case the controller reference is calculated assuming comfort braking levels that are not possible in this situation.

Table 4.3 shows the effect of the delay d in the time for completion of a join maneuver and table 4.4 the effect of the range of the relative distance sensor in the value of α_M .

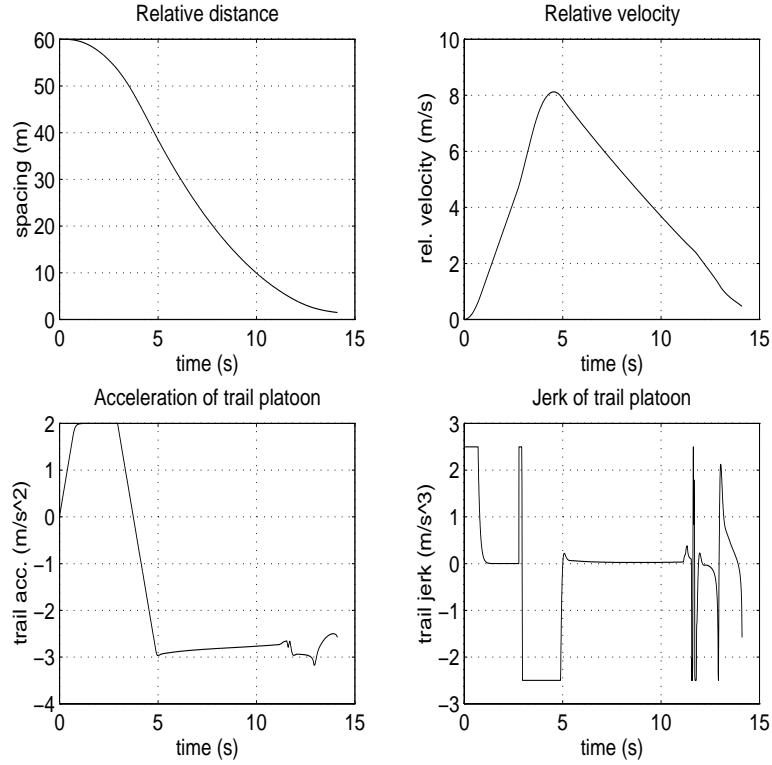


(a)

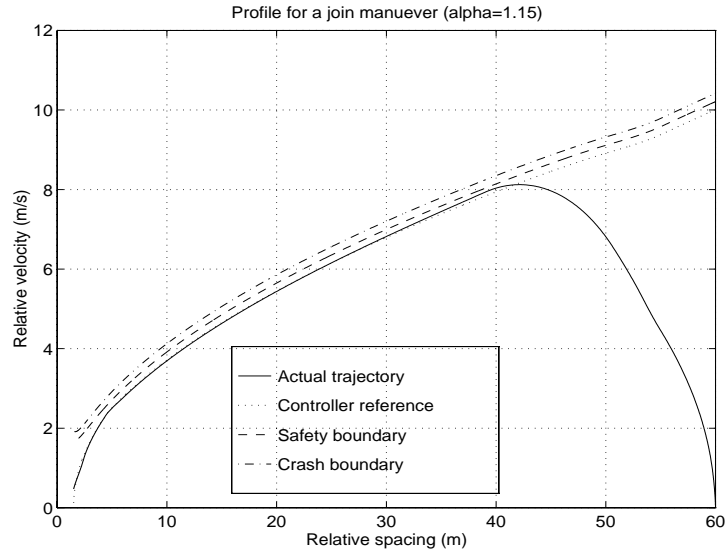


(b)

Figure 4.3: Simulation results of merge from 60 m initial spacing: The initial velocity of both lead and trail platoons was 25 m/s. The lead platoon applied maximum braking at 3.5 s. a) Results vs. time. b) Results in the phase plane Δx vs. $\Delta \dot{x}$.

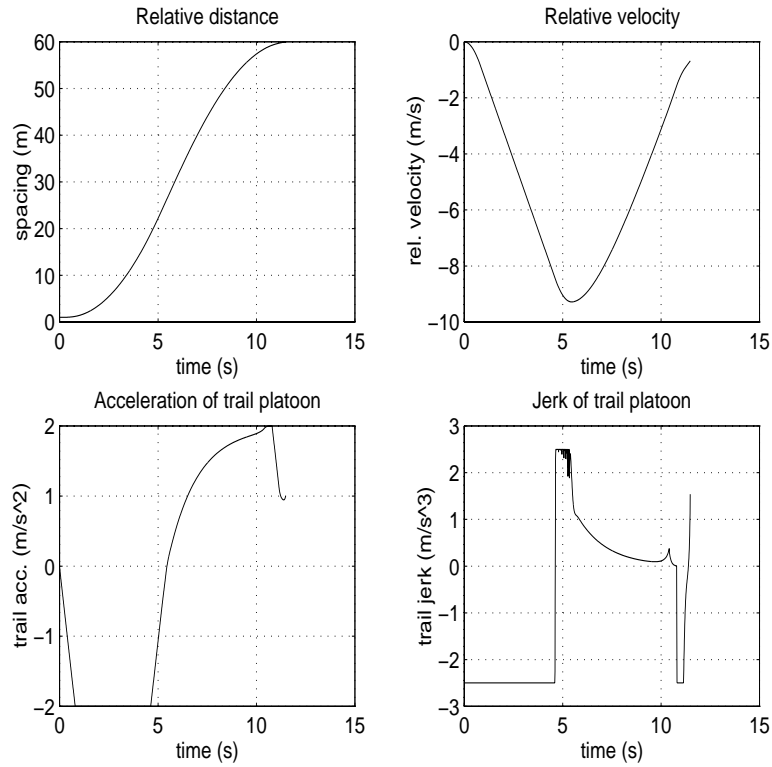


(a)

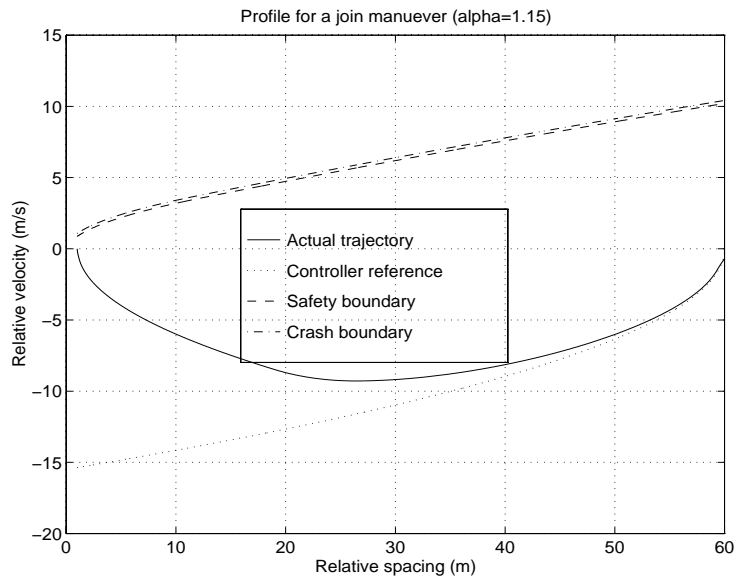


(b)

Figure 4.4: Simulation results of merge from 60 m initial spacing: The initial velocity of both lead and trail platoons was 25 m/s. The lead platoon applied comfort braking at 4.1 s. a) Results vs. time. b) Results in the phase plane Δx vs. $\Delta \dot{x}$.

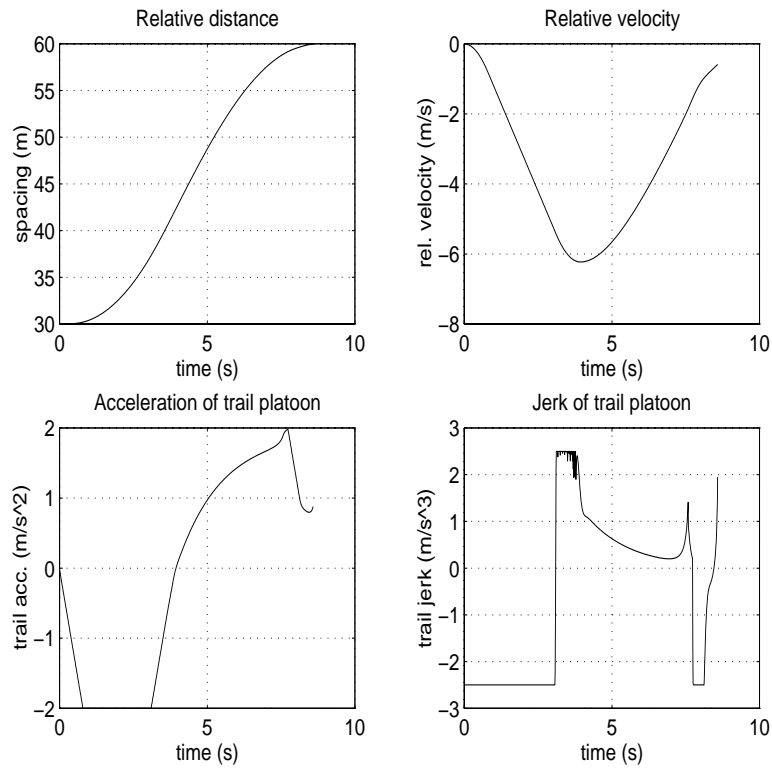


(a)

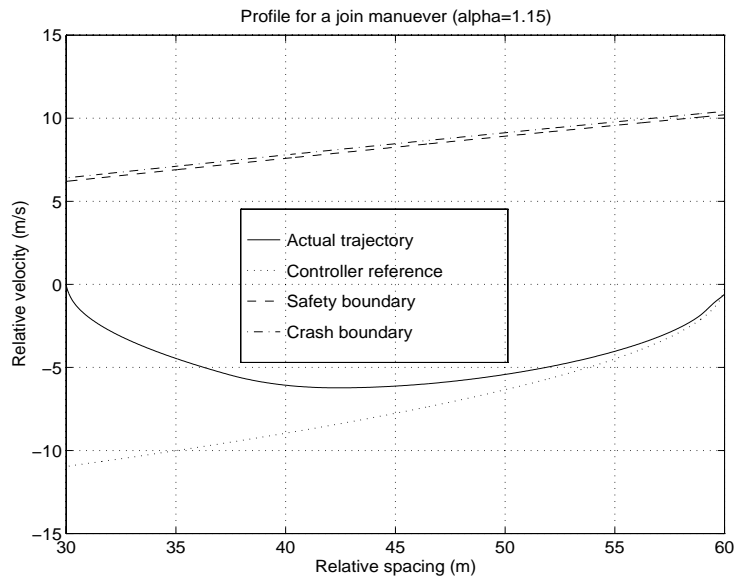


(b)

Figure 4.5: Simulation results of split from 1 to 60 m spacing: The initial velocity of both platoons was 25 m/s. a) Results vs. time. b) Results in the phase plane Δx vs. $\Delta \dot{x}$.

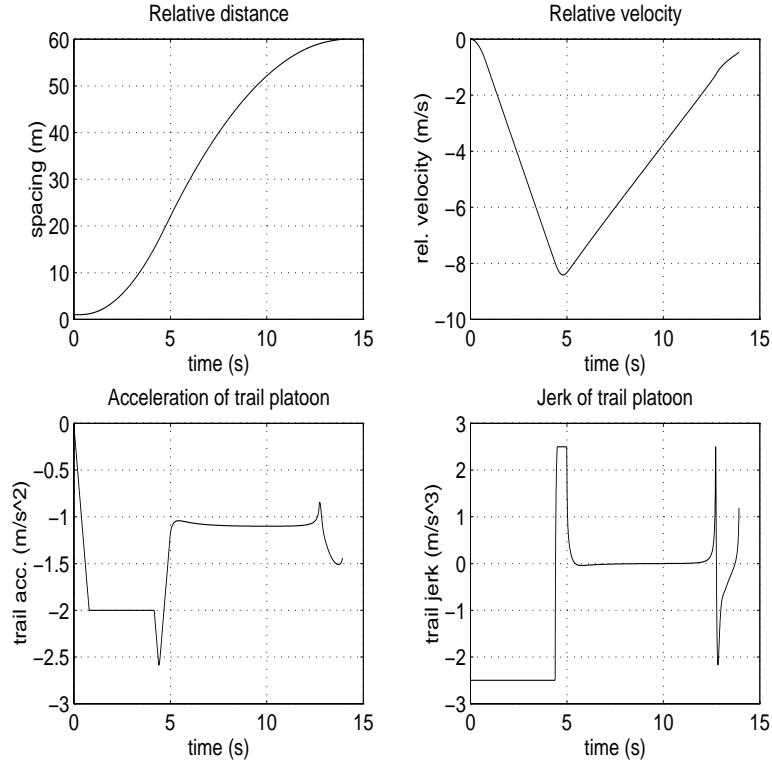


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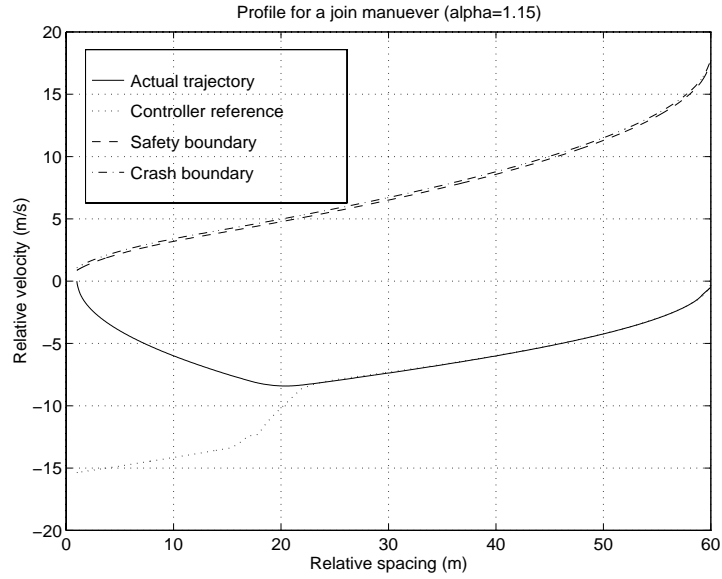


(b)

Figure 4.6: Simulation results of split from 30 to 60 m spacing: The initial velocity of both platoons was 25 m/s. a) Results vs. time. b) Results in the phase plane Δx vs. $\Delta \dot{x}$.

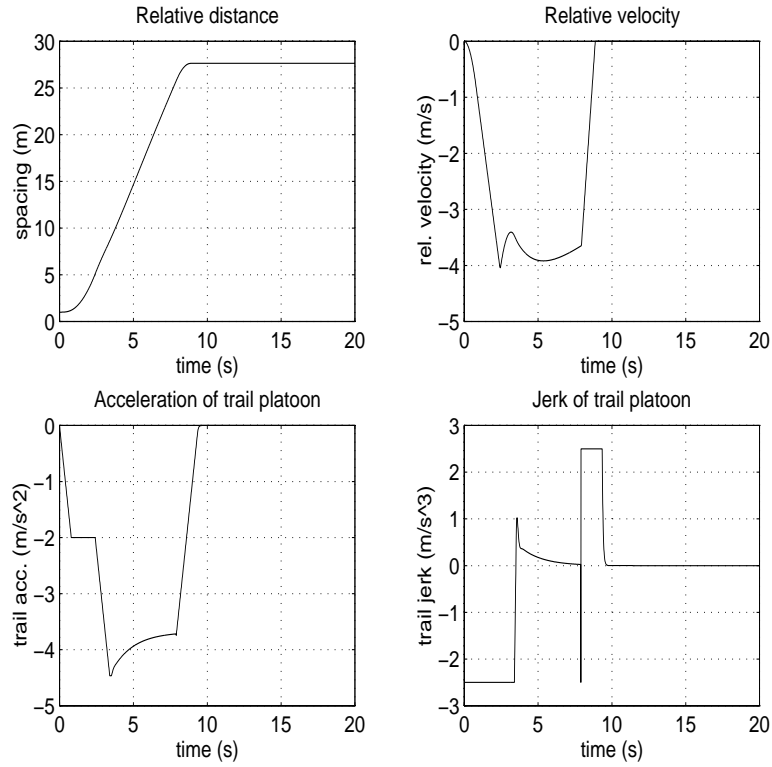


(a)

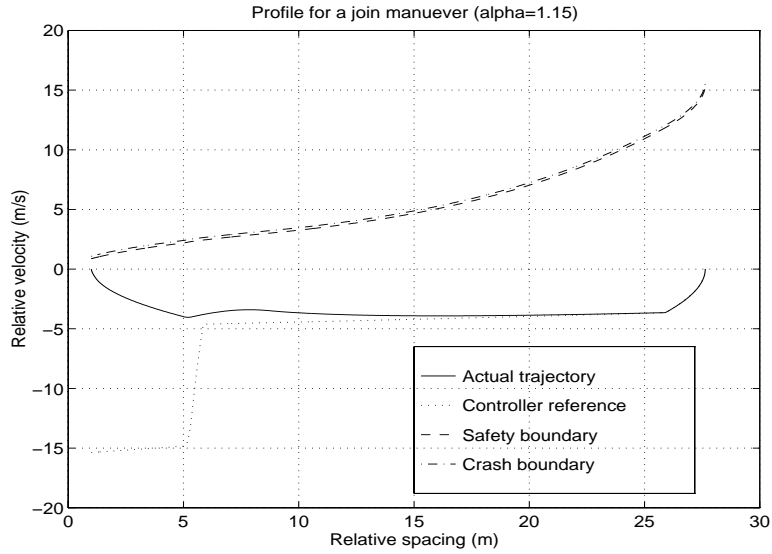


(b)

Figure 4.7: Simulation results of split from 1 to 60 m spacing: The initial velocity of both platoons was 25 m/s. The lead platoon applies comfort braking at $\Delta x = 15$ m. a) Results vs. time. b) Results in the phase plane Δx vs. $\Delta \dot{x}$.

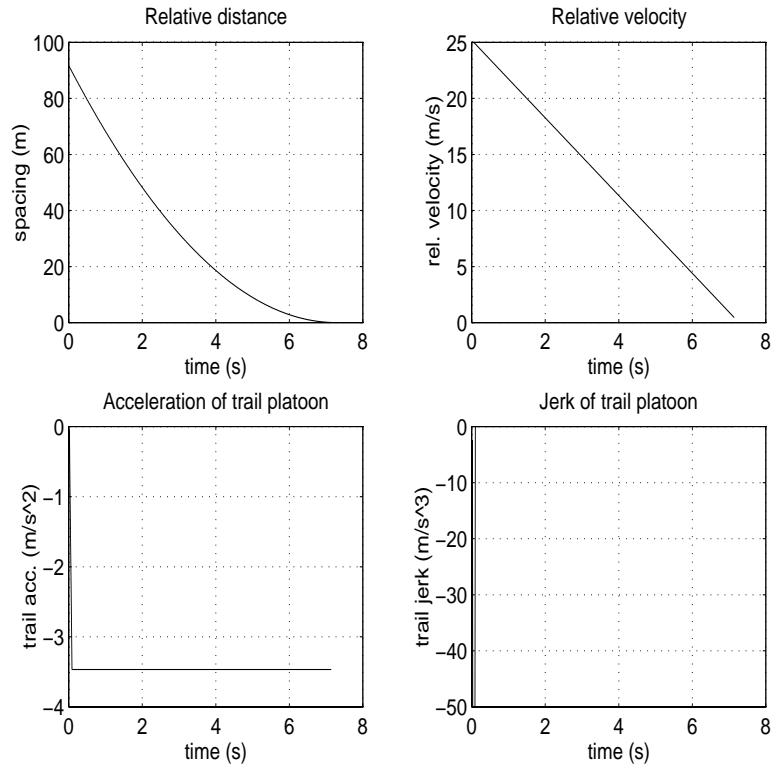


(a)

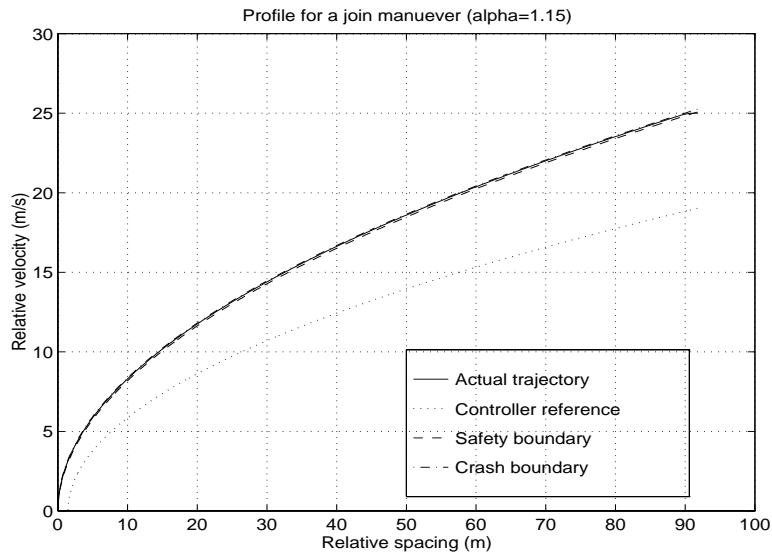


(b)

Figure 4.8: Simulation results of split from 1 to 60 m spacing: The initial velocity of both platoons was 25 m/s. The lead platoon applies maximum braking at $\Delta x = 5$ m. a) Results vs. time. b) Results in the phase plane Δx vs. $\Delta \dot{x}$.



(a)



(b)

Figure 4.9: Simulation results of the leader law. A platoon is traveling at 25 m/s and detects an stopped platoon 90 m in front of it. a) Results vs. time. b) Results in the phase plane Δx vs. $\Delta \dot{x}$.

$v_{allow} = 0 \text{ m/s}$	Desired condition for normal mode of operation in AHS.
$\alpha_c = 1.15$	This value was obtained from Eq. (2.77).
$a_{min}^{trail} = A_{MIN}/\mu = 4.46 \text{ m/s}^2$	For the join and split laws.
$a_{min}^{trail} = A_{MIN}/\mu^2\alpha_c = 3.4661$	For the leader law.
$a_{min}^{lead} = A_{MIN}/\mu\alpha_c = 3.88 \text{ m/s}^2$	For the join and split laws.
$a_{min}^{lead} = A_{MIN}$	For the leader law.
$a_{max}^{trail} = 2.0 \text{ m/s}^2 < A_{MAX}$	
$v_{max} = 25 \text{ m/s} \approx 55 \text{ mi/hr}$	

Table 4.2: Additional parameters used for the simulations in the no collision case.

4.2 Low speed collisions allowed, $v_{allow} > 0$

In this section similar results to those presented in the previous sections are shown. The intention is to remark the advantages in the use of controlled braking during platoon maneuvering as proposed in this report.

The parameters used for the simulations for the case in which low speed collisions are acceptable are shown in Table 4.5.

Figure 4.10 shows results for a merge from 30 m initial spacing. The velocity of the platoon ahead was constant at 25 m/s . The maneuver was completed in 12.3 s . Jerk and acceleration comfort limits were not exceeded. The final relative velocity is not zero as the simulation only ran to the point where the follower law takes effect.

Figure 4.11 shows results from a merge with an initial spacing of 60 m . The lead platoon maintained a constant velocity. The merge took 17.1 s in this case.

Figure 4.12 show the case in which the lead platoon applies maximum braking when the trail platoon has maximum relative velocity. The simulations were allowed to run until the trail platoon either stopped or collided. Note that the simulation shows a collision that, as expected, has an impact speed lower than v_{allow} . This figure includes a large spike in jerk. The controller is designed so that comfort limits are disregarded when safety becomes critical. In these cases, the comfort jerk was overridden once the large lead platoon deceleration was detected.

In the final merge simulation, the lead platoon braked at comfortable deceleration. No collision occurred. The results are shown in figure 4.13.

The split law was also simulated. Figures 4.14 and 4.15 show the results of split from 1 and 30 m to 60 m spacing, respectively. The cases when the lead platoon applies comfort and full braking while the trail platoon is attempting a split are shown in Figures 4.16 and 4.17, respectively.

The last simulation results in Fig.4.18 correspond to the leader law. An extreme case was simulated. The lead platoon detects a stopped platoon in front of it, while traveling at maximum speed. As expected, there is a collision at the end of the maneuver with relative velocity equal to v_{allow} . Notice also that in this case the controller reference is calculated assuming comfort braking levels that are not possible in this situation.

Maneuver	Delay (s)			
	0.03	0.05	0.20	0.15
Join from 30 m	11.9	12.5	14.7	17.1
Join from 60 m	16.4	17.1	19.4	22.1

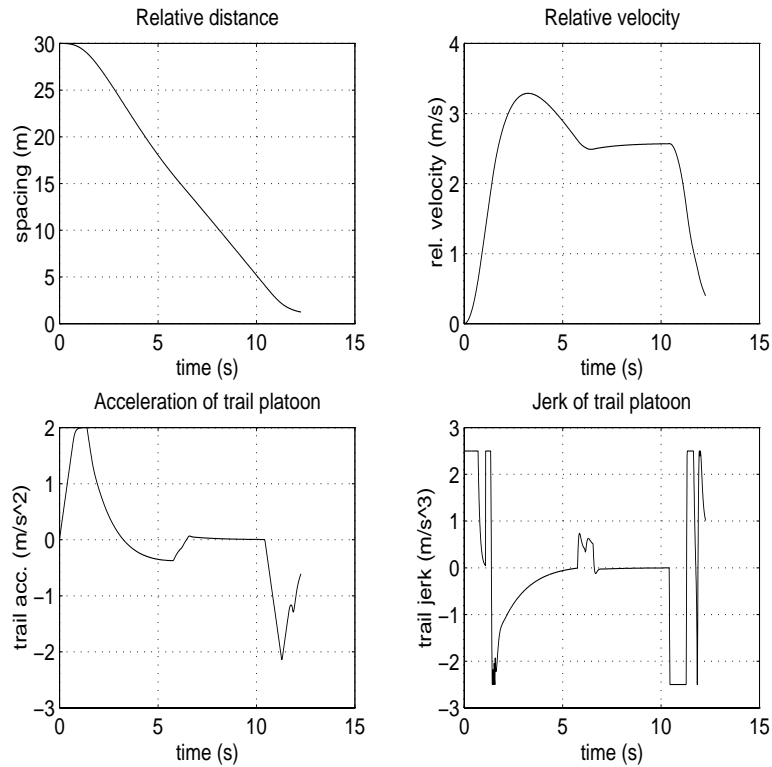
Table 4.3: Effect of the time delay in the time for maneuvering completion

Range	α_M
40	0.50
50	0.63
60	0.76
70	0.88
80	1.01
90	1.14
100	1.27
110	1.40

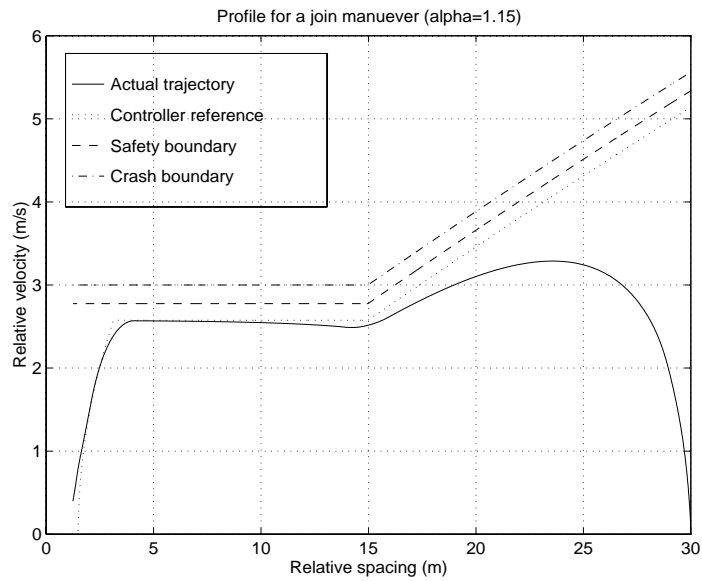
Table 4.4: α_M vs. relative distance sensor range

$v_{allow} = 3 \text{ m/s}$	The severity of injuries in automobile accidents is measured on the Abbreviated Injury Scale (AIS). Injuries rated from 3 to 6 on this scale are considered serious. Injuries of AIS = 2 are moderate and not life threatening. Using actual crash data, Hitchcock related AIS values to relative velocity at impact (Hitchcock, 1993b). For crashes at or below 3.3 m/s, he found no probability of fatalities or injuries rated AIS ≥ 3 . The probability of injuries rated AIS = 2 at that speed or slower is low.
$\alpha_c = 1.0$	This choice implies same braking capabilities, i.e., $a_{min}^{trail} = a_{min}^{lead}$
$\mu = 1$	All vehicles are assumed to have same braking capability.
$a_{min}^{trail} = A_{MIN}/\mu = 5.0 \text{ m/s}^2$	For the join and split laws.
$a_{min}^{trail} = A_{MIN}/\mu^2\alpha_c = 5.0$	For the leader law.
$a_{min}^{lead} = A_{MIN}/\mu\alpha_c = 5.0 \text{ m/s}^2$	For the join and split laws.
$a_{min}^{lead} = A_{MIN} = 5.0 \text{ m/s}^2$	For the leader law.
$a_{max}^{trail} = 2.0 \text{ m/s}^2 < A_{MAX}$	
$v_{max} = 25 \text{ m/s} \approx 55 \text{ mi/hr}$	

Table 4.5: Additional parameters used for the simulations in the low speed collision case.

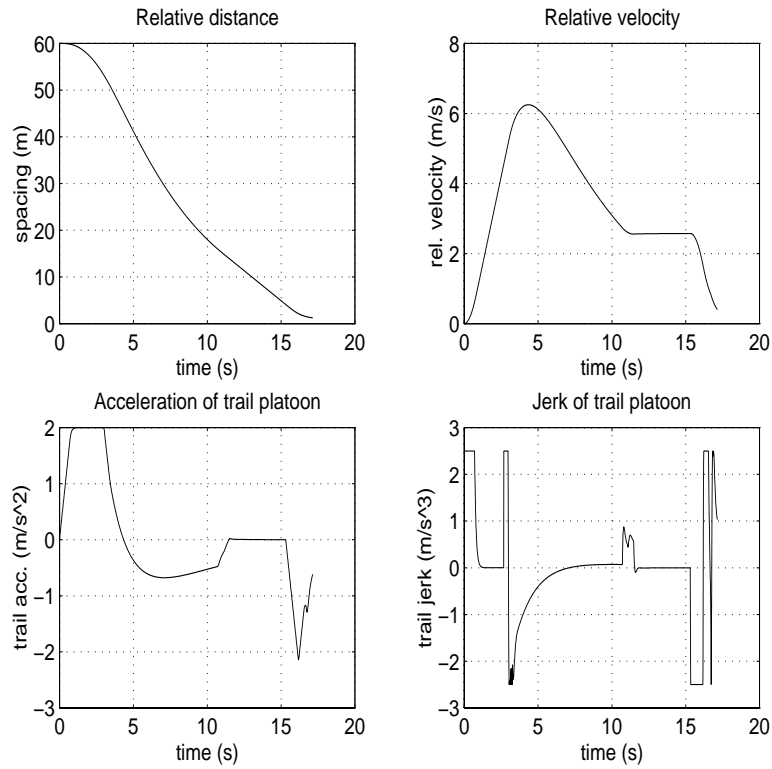


(a)

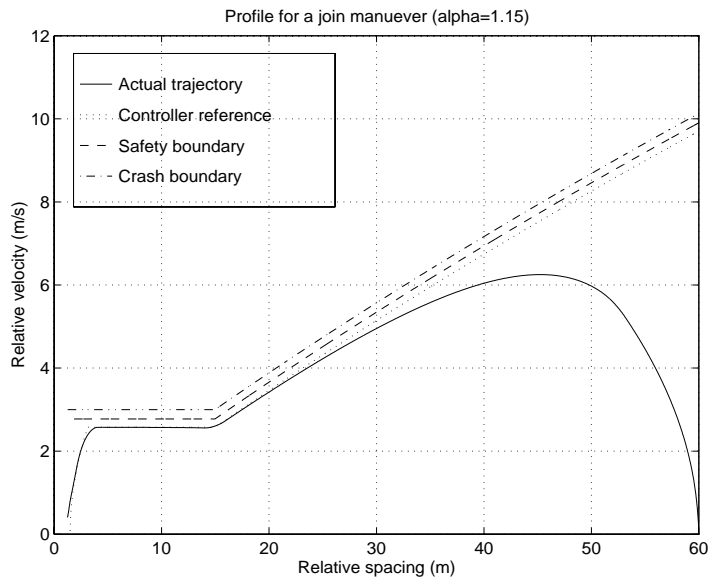


(b)

Figure 4.10: Simulation results of merge from 30 m initial spacing: The initial velocity of both lead and trail platoons was 25 m/s . a) Results vs. time. b) Results in the phase plane Δx vs. $\Delta \dot{x}$.

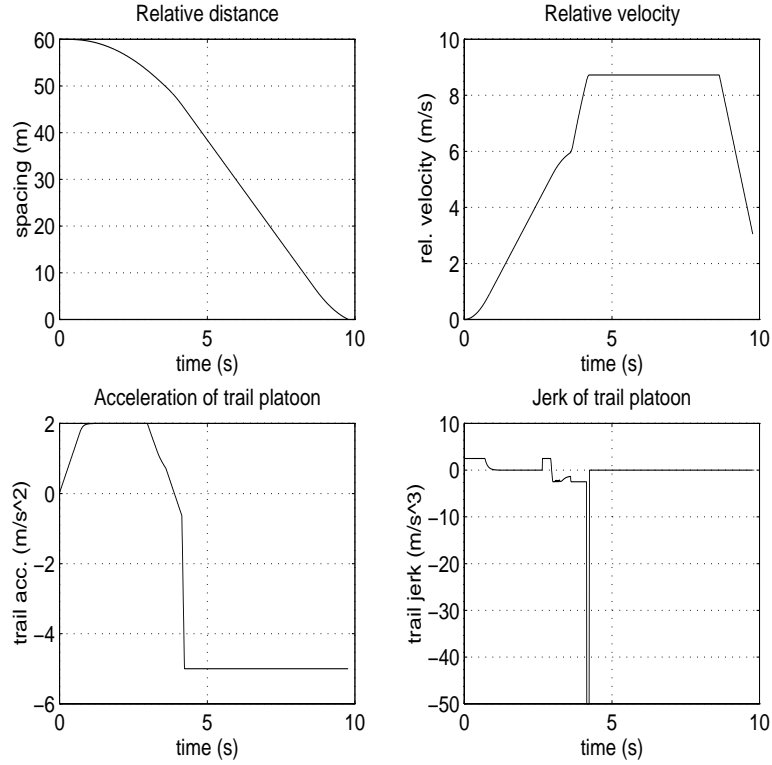


(a)

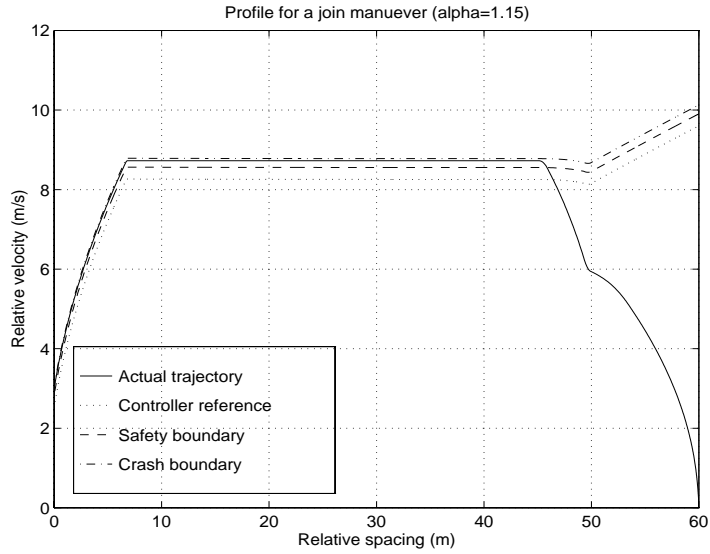


(b)

Figure 4.11: Simulation results of merge from 60 m initial spacing: The initial velocity of both lead and trail platoons was 25 m/s. a) Results vs. time. b) Results in the phase plane Δx vs. $\Delta \dot{x}$.

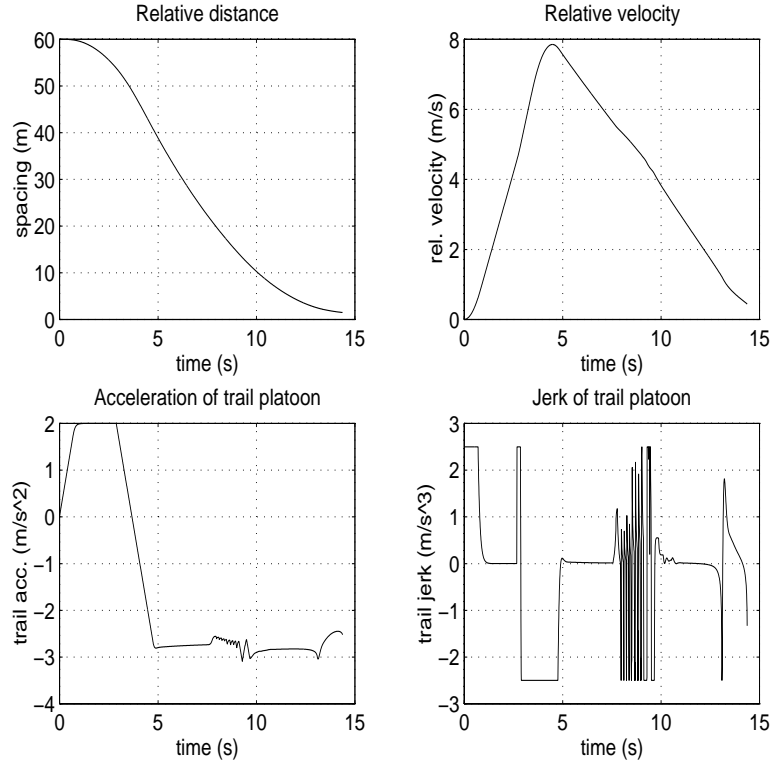


(a)

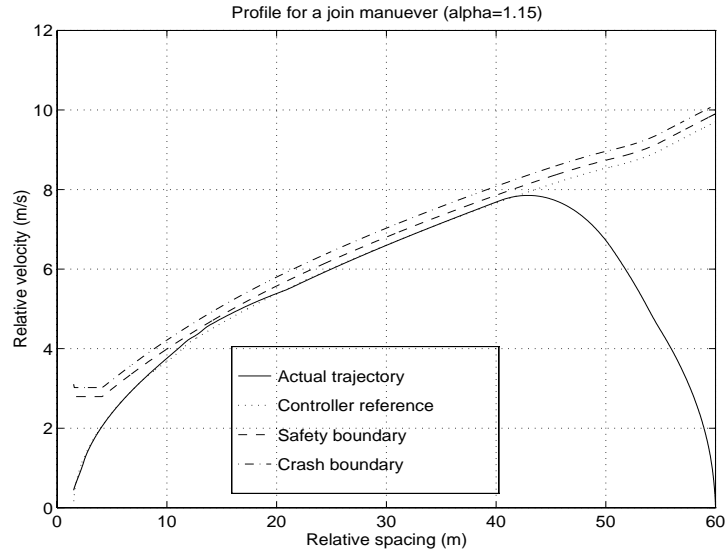


(b)

Figure 4.12: Simulation results of merge from 60 m initial spacing: The initial velocity of both lead and trail platoons was 25 m/s. The lead platoon applied maximum braking at 3.5 s. a) Results vs. time. b) Results in the phase plane Δx vs. $\Delta \dot{x}$.

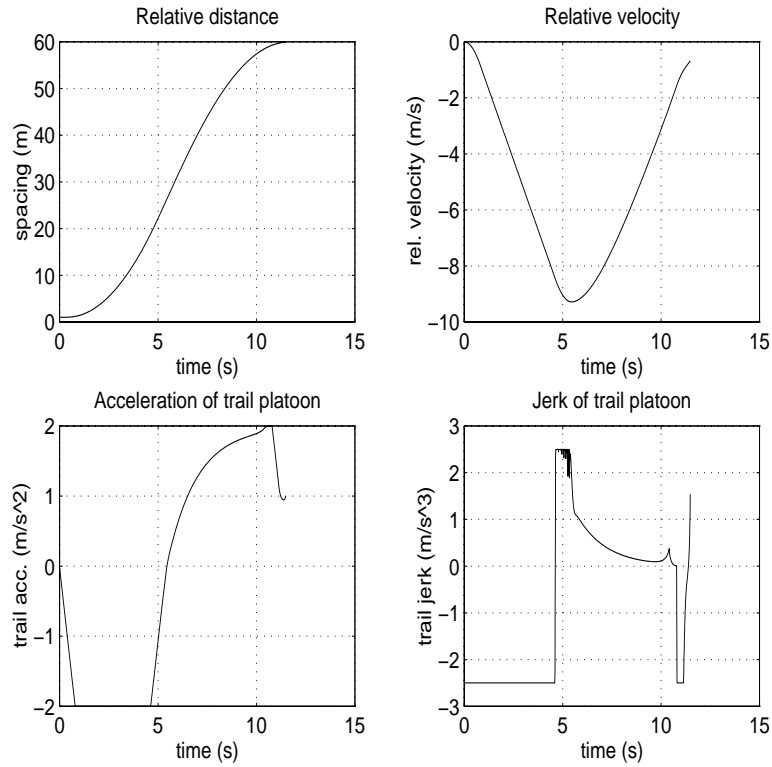


(a)

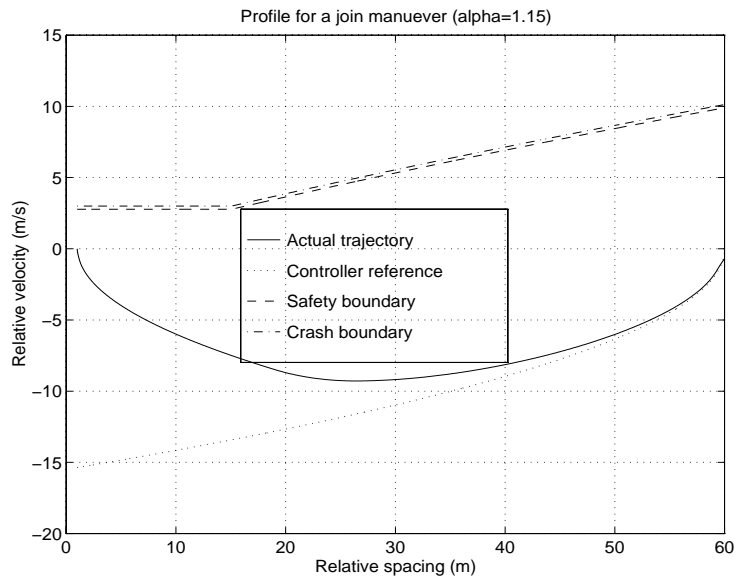


(b)

Figure 4.13: Simulation results of merge from 60 m initial spacing: The initial velocity of both lead and trail platoons was 25 m/s. The lead platoon applied comfort braking at 4.1 s. a) Results vs. time. b) Results in the phase plane Δx vs. $\Delta \dot{x}$.

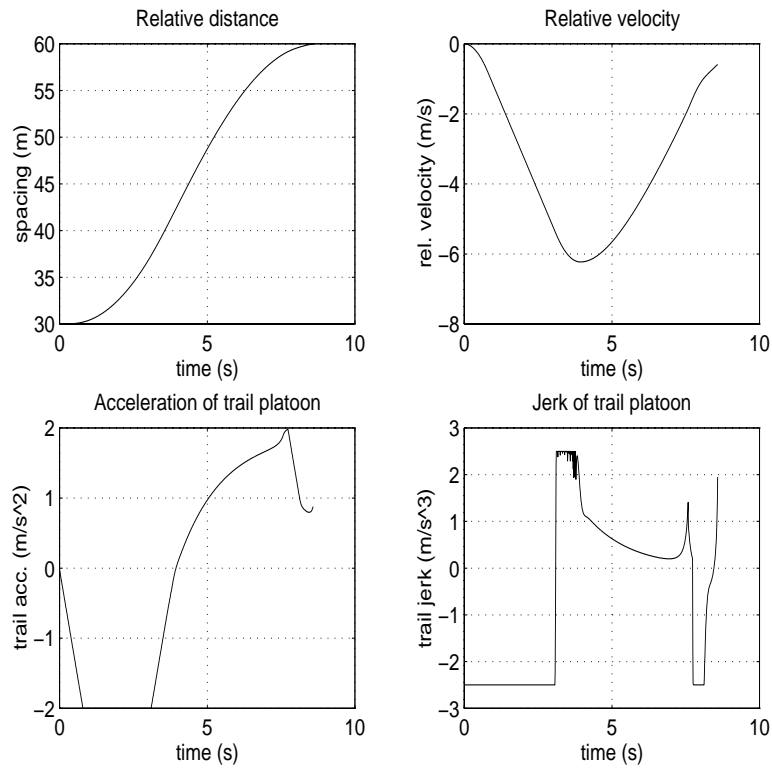


(a)

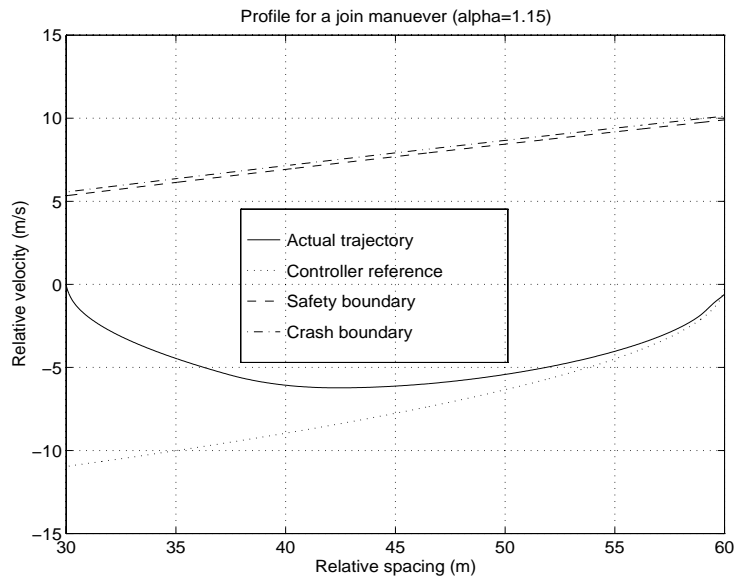


(b)

Figure 4.14: Simulation results of split from 1 to 60 m spacing: The initial velocity of both platoons was 25 m/s. a) Results vs. time. b) Results in the phase plane Δx vs. $\Delta \dot{x}$.

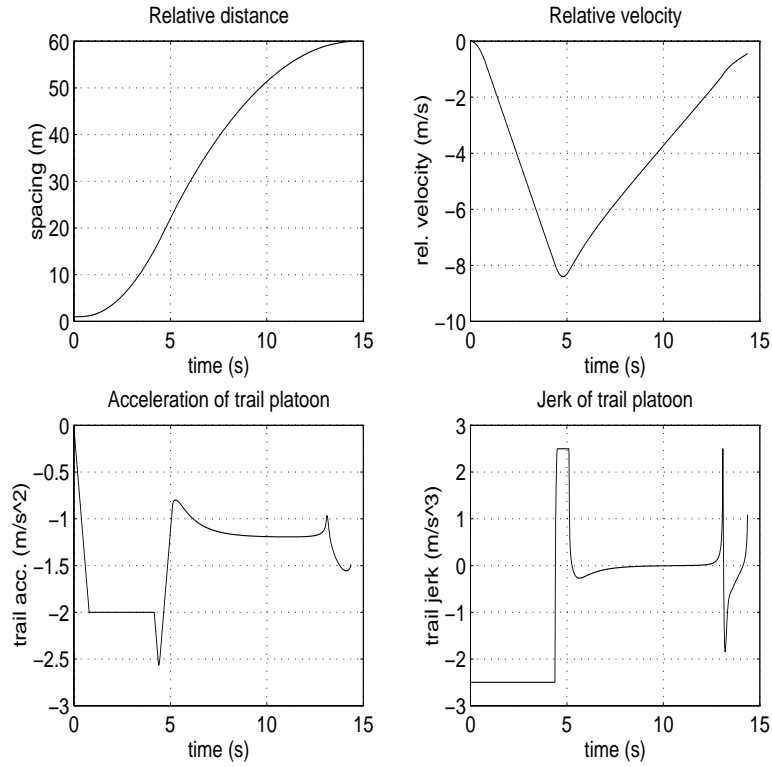


(a)

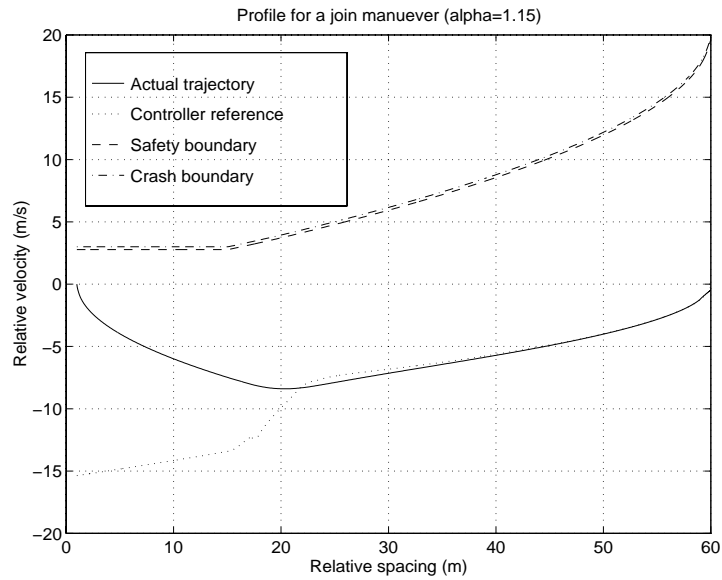


(b)

Figure 4.15: Simulation results of split from 30 to 60 m spacing: The initial velocity of both platoons was 25 m/s. a) Results vs. time. b) Results in the phase plane Δx vs. $\Delta \dot{x}$.

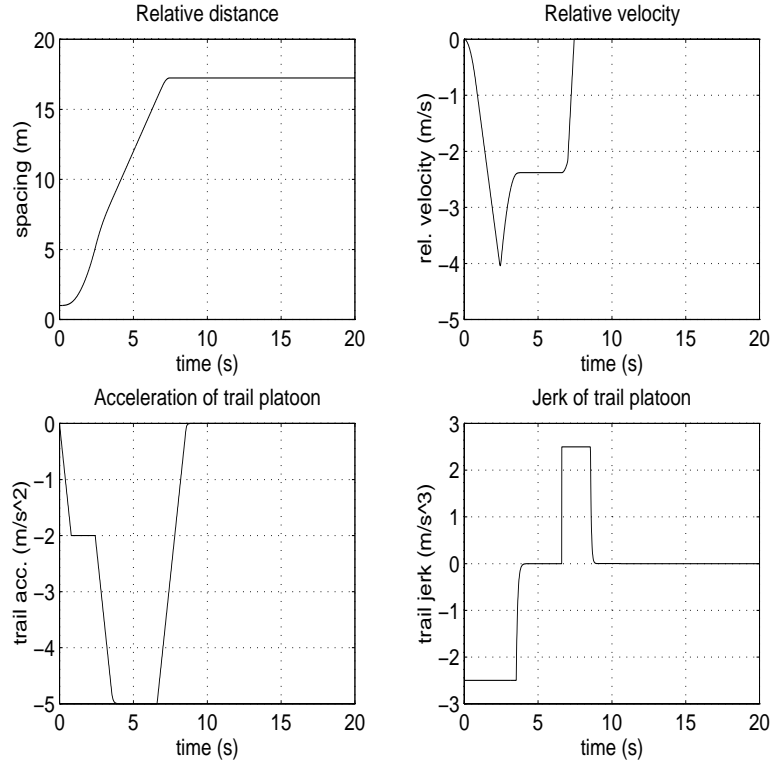


(a)

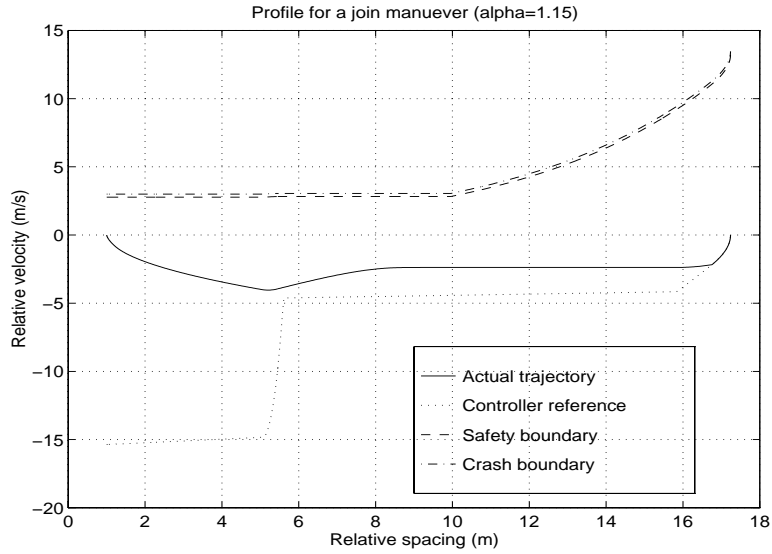


(b)

Figure 4.16: Simulation results of split from 1 to 60 m spacing: The initial velocity of both platoons was 25 m/s. The lead platoon applies comfort braking at $\Delta x = 15$ m. a) Results vs. time. b) Results in the phase plane Δx vs. $\Delta \dot{x}$.

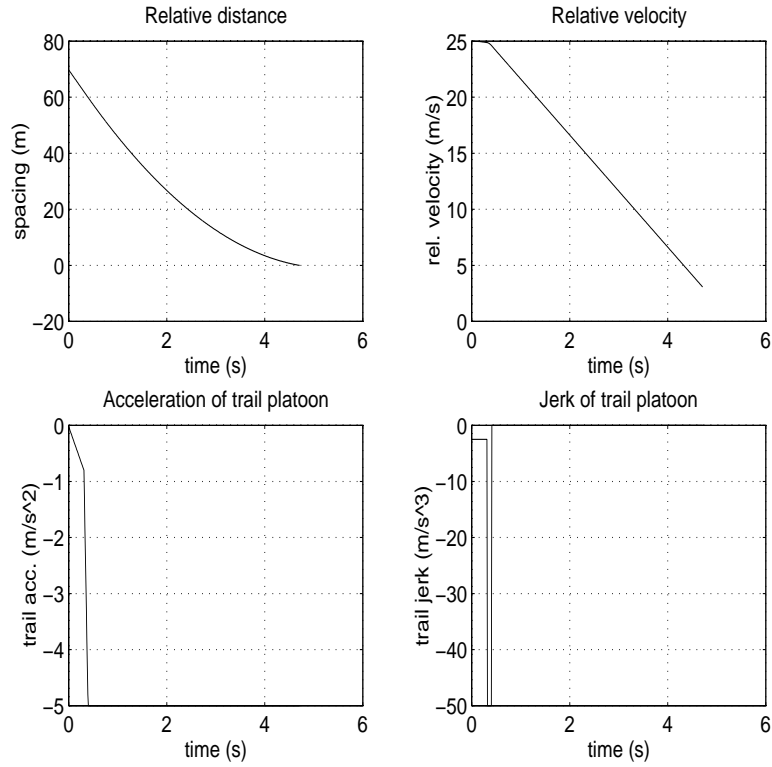


(a)

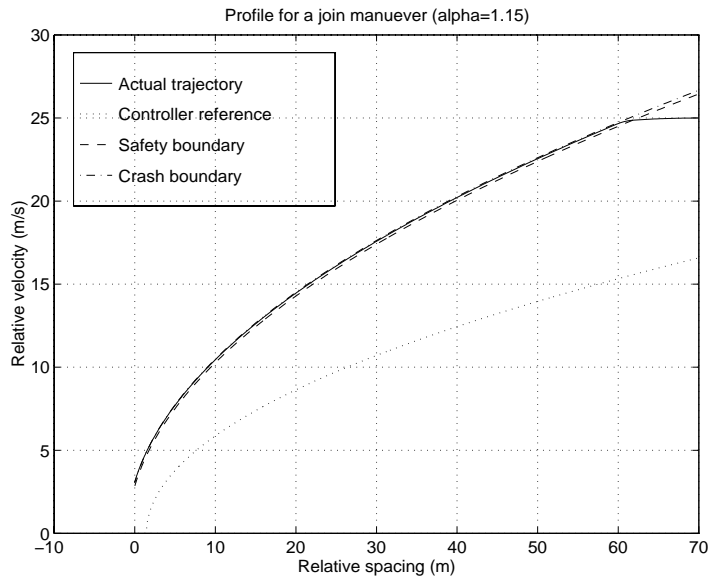


(b)

Figure 4.17: Simulation results of split from 1 to 60 m spacing: The initial velocity of both platoons was 25 m/s. The lead platoon applies maximum braking at $\Delta x = 5$ m. a) Results vs. time. b) Results in the phase plane Δx vs. $\Delta \dot{x}$.



(a)



(b)

Figure 4.18: Simulation results of the leader law. A platoon is traveling at 25 m/s and detects an stopped platoon 90 m in front of it. a) Results vs. time. b) Results in the phase plane Δx vs. $\Delta \dot{x}$.

Chapter 5

Conclusions

The safety of the regulation layer control laws for the hierarchical architecture of (Varaiya and Shladover, 1991) is analyzed. The notion of safety is that no platoon is allowed to collide with the platoon ahead of it at a relative velocity greater than a prescribed limit. The results show that for a safe normal mode operation of AHS, it is necessary to establish bounds on the parameters that determine the vehicle's behavior during the execution of the regulation layer maneuvers. These bounds allow to rule out the cases reported in (Lygeros, 1996*a*) in which safety can be compromised because of platoons' different braking capabilities. It is shown that, under the set of safety related constraints introduced in this report, the optimal safe strategy for the vehicles joining or splitting consists in applying full brakes when the vehicle ahead applies and holds maximum braking, as originally presented in (Frankel *et al.*, 1994; Puri and Varaiya, 1995; Li *et al.*, 1997*a*). Collision propagation in the highway is analyzed. It is concluded that, with a similar approach that the one used for the join and split control laws, this collision propagation can be avoided by constraining the behavior of platoons executing the leader control law. It is also shown that it is possible to design of feedback control laws for the regulation layer such that the overall safety of the AHS can be guaranteed, under the given notion of safety.

The results for safe platooning are also analyzed for the case in which no collisions are desired to occur during the execution of AHS maneuvers. It is concluded that to avoid collisions when platoons are maneuvering, vehicles' braking deceleration has to be controlled so as to make the braking capability of any vehicle larger than the braking capability of the vehicle ahead. Interplatoon distance for vehicles executing the leader law is also established. This distance is designed to satisfy the controlled braking requirements and to prevent collision even in the case in which a stopped platoon is detected in front. Combining the ideas presented in this report with the results presented in (Swaroop, 1994), expressions to guarantee this braking capability requirements are presented.

These results allow to decouple the design and verification of the regulation and coordination layers. The overall complexity of the design and verification of the AHS as an hybrid system is therefore greatly reduced.

Based on the safe platooning analysis, velocity profiles are derived for all the single lane maneuvers. These profiles are described in the state space of the platoons' relative motion and are therefore suited for feedback control implementation. A non-linear feed-

back velocity tracking controller is presented. This controller allows the maneuvers to be completed in minimum time and with comfort values of jerk and acceleration, whenever safety is not compromised. The simulation results presented illustrate the effectiveness of the designed control laws.

The approach here presented to design the control laws for the maneuvers in the normal mode of operation of the regulation layer is also being applied to the degraded mode maneuvers. Simulation results in SmartPath (Eskafi *et al.*, 1992) are presented.

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Appendix A

Implementation issues for the velocity tracking controller

A.1 Jerk control calculation

In the jerk control proposed in Eq. (3.17) it is possible to express the partial derivatives of e , in terms of known quantities and the partial derivatives of the desired velocity $v_d(\Delta x, v_{lead}, v_{trail})$.

First notice that

$$\dot{e} = \begin{pmatrix} \frac{\partial e}{\partial \Delta x} & \frac{\partial e}{\partial v_{lead}} & \frac{\partial e}{\partial v_{trail}} & \frac{\partial e}{\partial \hat{a}_{lead}} & \frac{\partial e}{\partial a_{trail}} \end{pmatrix} \begin{pmatrix} v_{lead} - v_{trail} \\ a_{lead} \\ a_{trail} \\ \hat{a}_{lead} \\ j_{trail} \end{pmatrix}. \quad (\text{A.1})$$

Defining $\mathbf{u}_1^T = (\Delta x \ v_{lead} \ v_{trail})$ and $\mathbf{u}_2^T = (\hat{a}_{lead} \ a_{trail})$, Eq. (A.1) can be rewritten as

$$\dot{e} = \frac{\partial e}{\partial \mathbf{u}_1} \frac{\partial \mathbf{u}_1}{\partial t} + \frac{\partial e}{\partial \mathbf{u}_2} \frac{\partial \mathbf{u}_2}{\partial t}, \quad (\text{A.2})$$

and Eq. (3.12) as

$$\dot{e} = -\lambda_1 e + \frac{\partial v_d}{\partial \mathbf{u}_1} \frac{\partial \mathbf{u}_1}{\partial t}. \quad (\text{A.3})$$

Solving for the terms in Eq. (A.2)

$$\begin{aligned} \frac{\partial e}{\partial \mathbf{u}_1} &= -\lambda_1 \frac{\partial e}{\partial \mathbf{u}_1} + \frac{\partial \mathbf{u}_1^T}{\partial t} \frac{\partial^2 v_d}{\partial \mathbf{u}_1^2} + \frac{\partial v_d}{\partial \mathbf{u}_1} \frac{\partial}{\partial \mathbf{u}_1} \left(\frac{\partial \mathbf{u}_1}{\partial t} \right) \\ &= -\lambda_1 \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} + \lambda_1 \frac{\partial v_d}{\partial \mathbf{u}_1} + \frac{\partial \mathbf{u}_1^T}{\partial t} \frac{\partial^2 v_d}{\partial \mathbf{u}_1^2} + \frac{\partial v_d}{\partial \mathbf{u}_1} \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

and

$$\frac{\partial,}{\partial u_2} = -\lambda_1 \frac{\partial e}{\partial u_2} + \frac{\partial v_d}{\partial u_1} \frac{\partial}{\partial u_2} \left(\frac{\partial u_1}{\partial t} \right) = \frac{\partial v_d}{\partial u_1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Substituting the expressions for $\partial, / \partial u_1$ and $\partial, / \partial u_2$ into Eq. (A.2)

$$\begin{aligned} \dot{,} &= -\lambda_1 \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \frac{\partial u_1}{\partial t} + \lambda_1 \frac{\partial v_d}{\partial u_1} \frac{\partial u_1}{\partial t} + \frac{\partial u_1^T}{\partial t} \frac{\partial^2 v_d}{\partial u_1^2} \frac{\partial u_1}{\partial t} \\ &+ \frac{\partial v_d}{\partial u_1} \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial u_1}{\partial t} + \frac{\partial v_d}{\partial u_1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial u_2}{\partial t}. \end{aligned} \quad (\text{A.4})$$

Finally, using (A.3), (A.4) and (3.14), the jerk control in (3.17) can be written as

$$\begin{aligned} \left(1 - \frac{\partial v_d}{\partial v_{trail}} \right) j_{trail} &= -(\lambda_1 + \lambda_2) \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \frac{\partial u_1}{\partial t} - (\lambda_1 \lambda_2 + \beta) e + (\lambda_1 + \lambda_2) \frac{\partial v_d}{\partial u_1} \frac{\partial u_1}{\partial t} \\ &+ \frac{\partial u_1^T}{\partial t} \frac{\partial^2 v_d}{\partial u_1^2} \frac{\partial u_1}{\partial t} + \frac{\partial v_d}{\partial u_1} \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{\partial v_d}{\partial u_1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \hat{a}_{lead}^{\sim}, \end{aligned}$$

where $\partial u_1 / \partial t$ is approximated by $\partial u_1 / \partial t = (\Delta x \quad \hat{a}_{lead} \quad a_{trail})^T$. The error incurred by this approximation is already accounted for in \mathbf{g} .

A.2 Gains setting

To set the values of the velocity tracking controller gains consider, for example, the reduced order observer in which these gains are $\lambda_1, \lambda_2, \beta, \gamma, L_1, L_2$. The approximate linear dynamics of the combined system from (3.19) and (3.24) is

$$\frac{d}{dt} \begin{pmatrix} e \\ \tilde{,} \\ \tilde{a}_{lead} \end{pmatrix} = \begin{pmatrix} -\lambda_1 & 1 & -g_1(t) \\ -\beta & -\lambda_2 & -g_2(t) \\ \beta g_1(t)/\gamma & g_2(t)/\gamma & -L_2 \end{pmatrix} \begin{pmatrix} e \\ \tilde{,} \\ \tilde{a}_{lead} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} j_{lead}. \quad (\text{A.5})$$

Let $(\bar{g}_1 \quad \bar{g}_2)$ be average values of $\mathbf{g}(\cdot)$, then the transfer function from $j_{lead}(\cdot)$ to $e(\cdot)$ is:

$$\frac{E(s)}{J_{lead}(s)} = -\frac{-\bar{g}_1 \left(s + \lambda_2 + \frac{\bar{g}_2}{\bar{g}_1} \right)}{\left((s + \lambda_1)(s + \lambda_2) + \beta \right) (s + L_2) + \frac{\beta \bar{g}_1^2 + \bar{g}_2^2}{\gamma} \left(s + \frac{\beta \bar{g}_1^2 \lambda_2 + \bar{g}_2^2 \lambda_1}{\beta \bar{g}_1^2 + \bar{g}_2^2} \right)}. \quad (\text{A.6})$$

To evaluate the frequency content in the jerk for the lead platoon j_{lead} , consider that j_{lead} is subject to the following constraints:

- $\|j_{lead}(\cdot)\|_{\infty} \leq j_{max}$
- the jerk has to be such that the acceleration $a_{lead} \in [-a_{min}^{lead}, a_{max}^{lead}]$.

From these constraints, it is possible to estimate the frequency contents of $j_{lead}(\cdot)$. First, notice that at low frequencies the magnitude of $\|j_{lead}(\cdot)\|$ must be constrained. For example, consider a square wave of frequency $\nu = \omega/2\pi$. Then, because of the saturation constraints of the acceleration, the magnitude of the square wave must satisfy

$$M(\omega) \leq \frac{\omega(a_{max}^{lead} - a_{min}^{lead})}{\pi}.$$

At high frequency, $M(\omega) \leq j_{max}$ should also be satisfied. The frequency at which the latter constraints becomes active is

$$\omega_j = \frac{\pi j_{max}}{a_{max}^{lead} - a_{min}^{lead}}.$$

Using $a_{max}^{lead} - a_{min}^{lead} = 7.5m/s^2$, and $j_{max} = 50m/s^3$ gives $\omega_j \approx 20rad/s$.

Thus, the frequency content of $j_{lead}(t)$ can be approximated by the magnitude of the following transfer function:

$$M(s) = \frac{s}{s + \omega_j}$$

Given a set of values for $\lambda_1, \lambda_2, \beta, \gamma, L_1, L_2$ it is possible to find the magnitude of the maximum velocity error. This value can be then subtracted from Eqs. (3.2), (3.4) and (3.7).

A.3 Cubic splines derivation

The desired velocity profiles for the trail platoon are composed with smooth sections. If transitions between these sections are also desired to be smooth, cubic splines can be used. Calculations to derive these cubic splines in terms of the relative displacement, Δx , follow.

The desired velocity v_d in the cubic spline region is

$$v_d = S(\Delta x) = a_0 + b_0(\Delta x - \Delta x_0) + c_0(\Delta x - \Delta x_0)^2 + d_0(\Delta x - \Delta x_0)^3; \quad (A.7)$$

$$\forall \Delta x \in [\Delta x_0, \Delta x_0 + \Delta x_S],$$

where $\Delta x_0 = \Delta x_{cr} - \Delta x_S/2$, Δx_{cr} is the intersection of any two smooth sections of the velocity profiles and Δx_S is the length of the cubic spline.

From the boundary requirements for the cubic splines

$$S(\Delta x_0) = a_0, \quad (A.8)$$

$$\left. \frac{\partial S(\Delta x)}{\partial \Delta x} \right|_{\Delta x = \Delta x_0} = b_0. \quad (A.9)$$

The values of c_0 and d_0 can be derived from

$$\left(\begin{array}{c} S(\Delta x_0 + \Delta x_S) - S(\Delta x_0) \\ \frac{\partial S(\Delta x)}{\partial \Delta x} \Big|_{\Delta x = \Delta x_0 + \Delta x_S} - \frac{\partial S(\Delta x)}{\partial \Delta x} \Big|_{\Delta x = \Delta x_0} \end{array} \right) = \begin{pmatrix} \Delta x_S^2 & \Delta x_S^3 \\ 2\Delta x_S & 3\Delta x_S^2 \end{pmatrix} \begin{pmatrix} c_0 \\ d_0 \end{pmatrix}. \quad (\text{A.10})$$

The implementation of the jerk controller requires the calculation of the partial derivatives of the desired velocity v_d . Then, from (A.7)

$$\begin{aligned} \frac{\partial v_d}{\partial \Delta x} &= b_0 + 2c_0(\Delta x - \Delta x_0) + 3d_0(\Delta x - \Delta x_0)^2, \\ \frac{\partial v_d}{\partial v_{lead}} &= - (b_0 + 2c_0(\Delta x - \Delta x_0) + 3d_0(\Delta x - \Delta x_0)^2) \frac{\partial \Delta x_0}{\partial v_{lead}} \\ &\quad + \frac{\partial a_0}{\partial v_{lead}} + (\Delta x - \Delta x_0) \frac{\partial b_0}{\partial v_{lead}} + (\Delta x - \Delta x_0)^2 \frac{\partial c_0}{\partial v_{lead}} + (\Delta x - \Delta x_0)^3 \frac{\partial d_0}{\partial v_{lead}}, \\ \frac{\partial v_d}{\partial v_{trail}} &= - (b_0 + 2c_0(\Delta x - \Delta x_0) - 3d_0(\Delta x - \Delta x_0)^2) \frac{\partial \Delta x_0}{\partial v_{trail}} \\ &\quad + \frac{\partial a_0}{\partial v_{trail}} + (\Delta x - \Delta x_0) \frac{\partial b_0}{\partial v_{trail}} + (\Delta x - \Delta x_0)^2 \frac{\partial c_0}{\partial v_{trail}} + (\Delta x - \Delta x_0)^3 \frac{\partial d_0}{\partial v_{trail}}, \\ \frac{\partial^2 v_d}{\partial \Delta x^2} &= 2c_0 + 6d_0(\Delta x - \Delta x_0), \\ \frac{\partial^2 v_d}{\partial \Delta x \partial v_{lead}} &= - (2c_0 + 6d_0(\Delta x - \Delta x_0)) \frac{\partial \Delta x_0}{\partial v_{lead}} \\ &\quad + \frac{\partial b_0}{\partial v_{lead}} + 2c_0(\Delta x - \Delta x_0) \frac{\partial c_0}{\partial v_{lead}} + 3d_0(\Delta x - \Delta x_0)^2 \frac{\partial d_0}{\partial v_{lead}}, \\ \frac{\partial^2 v_d}{\partial \Delta x \partial v_{trail}} &= - (2c_0 + 6d_0(\Delta x - \Delta x_0)) \frac{\partial \Delta x_0}{\partial v_{trail}} \\ &\quad + \frac{\partial b_0}{\partial v_{trail}} + 2c_0(\Delta x - \Delta x_0) \frac{\partial c_0}{\partial v_{trail}} + 3d_0(\Delta x - \Delta x_0)^2 \frac{\partial d_0}{\partial v_{trail}}, \\ \frac{\partial^2 v_d}{\partial v_{lead}^2} &= - \left(2 \frac{\partial b_0}{\partial v_{lead}} + 4(\Delta x - \Delta x_0) \frac{\partial c_0}{\partial v_{lead}} + 6(\Delta x - \Delta x_0)^2 \frac{\partial d_0}{\partial v_{lead}} \right) \frac{\partial \Delta x_0}{\partial v_{lead}} \\ &\quad + (2c_0 + 6d_0(\Delta x - \Delta x_0)) \left(\frac{\partial^2 \Delta x_0}{\partial v_d^2} \right)^2 \\ &\quad + \frac{\partial^2 a_0}{\partial v_{lead}^2} + (\Delta x - \Delta x_0) \frac{\partial^2 b_0}{\partial v_{lead}^2} + (\Delta x - \Delta x_0)^2 \frac{\partial^2 c_0}{\partial v_{lead}^2} + (\Delta x - \Delta x_0)^3 \frac{\partial^2 d_0}{\partial v_{lead}^2} \\ &\quad - (b_0 + 2c_0(\Delta x - \Delta x_0) + 3d_0(\Delta x - \Delta x_0)^2) \frac{\partial^2 \Delta x_0}{\partial v_{lead}^2}, \end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 v_d}{\partial v_{lead} \partial v_{trail}} &= - \left(\frac{\partial b_0}{\partial v_{trail}} + 2(\Delta x - \Delta x_0) \frac{\partial c_0}{\partial v_{trail}} + 3(\Delta x - \Delta x_0)^2 \frac{\partial d_0}{\partial v_{trail}} \right) \frac{\partial \Delta x_0}{\partial v_{lead}} \\
&+ (2c_0 + 6d_0(\Delta x - \Delta x_0)) \frac{\partial \Delta x_0}{\partial v_{lead}} \frac{\partial \Delta x_0}{\partial v_{trail}} \\
&+ \frac{\partial^2 a_0}{\partial v_{lead} \partial v_{trail}} + (\Delta x - \Delta x_0) \frac{\partial^2 b_0}{\partial v_{lead} \partial v_{trail}} + (\Delta x - \Delta x_0)^2 \frac{\partial^2 c_0}{\partial v_{lead} \partial v_{trail}} \\
&+ (\Delta x - \Delta x_0)^3 \frac{\partial^2 d_0}{\partial v_{lead} \partial v_{trail}} \\
&- \left(\frac{\partial b_0}{\partial v_{lead}} + 2(\Delta x - \Delta x_0) \frac{\partial c_0}{\partial v_{lead}} + 3(\Delta x - \Delta x_0)^2 \frac{\partial d_0}{\partial v_{lead}} \right) \frac{\partial \Delta x_0}{\partial v_{trail}} \\
&- (b_0 + 2c_0(\Delta x - \Delta x_0) + 3d_0(\Delta x - \Delta x_0)^2) \frac{\partial^2 \Delta x_0}{\partial v_{lead} \partial v_{trail}} ,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 v_d}{\partial v_{trail}^2} &= - \left(2 \frac{\partial b_0}{\partial v_{trail}} + 4(\Delta x - \Delta x_0) \frac{\partial c_0}{\partial v_{trail}} + 6(\Delta x - \Delta x_0)^2 \frac{\partial d_0}{\partial v_{trail}} \right) \frac{\partial \Delta x_0}{\partial v_{trail}} \\
&+ (2c_0 + 6d_0(\Delta x - \Delta x_0)) \left(\frac{\partial^2 \Delta x_0}{\partial v_d^2} \right)^2 \\
&+ \frac{\partial^2 a_0}{\partial v_{trail}^2} + (\Delta x - \Delta x_0) \frac{\partial^2 b_0}{\partial v_{trail}^2} + (\Delta x - \Delta x_0)^2 \frac{\partial^2 c_0}{\partial v_{trail}^2} + (\Delta x - \Delta x_0)^3 \frac{\partial^2 d_0}{\partial v_{trail}^2} \\
&- (b_0 + 2c_0(\Delta x - \Delta x_0) + 3d_0(\Delta x - \Delta x_0)^2) \frac{\partial^2 \Delta x_0}{\partial v_{trail}^2} .
\end{aligned}$$

Notice that if $u_3 = [v_{lead} \ v_{trail}]^T$ then

$$\frac{\partial \Delta x_0}{\partial u_3} = \frac{\partial \Delta x_{cr}}{\partial u_3} ; \quad \frac{\partial^2 \Delta x_0}{\partial u_3^2} = \frac{\partial^2 \Delta x_{cr}}{\partial u_3^2} .$$

The values of $\partial a_0 / \partial v_{lead}$ and $\partial b_0 / \partial v_{lead}$ can be calculated taking partial derivatives on the boundary conditions (A.8) and (A.9), i.e.,

$$\begin{aligned}
\frac{\partial S(\Delta x_0)}{\partial v_{lead}} &= \frac{\partial a_0}{\partial v_{lead}} , \\
\frac{\partial^2 S(\Delta x)}{\partial \Delta x \partial v_{lead}} \Big|_{\Delta x = \Delta x_0} &= \frac{\partial b_0}{\partial v_{lead}} .
\end{aligned}$$

The value of the partial derivatives $\partial c_0/\partial v_{lead}$ and $\partial d_0/\partial v_{lead}$, can be calculated by taking partial derivative of Eq. (A.10) with respect to v_{lead} , i.e.

$$\left(\begin{array}{c} \frac{\partial S(\Delta x_0 + \Delta x_S)}{\partial v_{lead}} - \frac{\partial S(\Delta x_0)}{\partial v_{lead}} \\ \frac{\partial S(\Delta x)^2}{\partial \Delta x \partial v_{lead}} \Big|_{\Delta x = \Delta x_0 + \Delta x_S} - \frac{\partial S(\Delta x)^2}{\partial \Delta x \partial v_{lead}} \Big|_{\Delta x = \Delta x_0} \end{array} \right) = \begin{pmatrix} \Delta x_S^2 & \Delta x_S^3 \\ 2\Delta x_S & 3\Delta x_S^2 \end{pmatrix} \begin{pmatrix} \frac{\partial c_0}{\partial v_{lead}} \\ \frac{\partial d_0}{\partial v_{lead}} \end{pmatrix}.$$

The remaining terms are calculated in a similar fashion.