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We employ the hybrid soliton model of the nucleon consisting of a topological meson field and deeply bound quarks to investigate the behavior of the quarks in soliton matter as a function of density. To organize the calculation, we place the solitons on a spatial lattice. The model suggests the transition of matter from a color insulator to a color conductor above a critical density of a few times normal nuclear density. There is no latent heat associated with the transition.

I. INTRODUCTION

There has been a great deal of interest, in the last several years, in a remarkable notion that was advanced by Skyrme more than twenty years ago, that nucleons are soliton solutions of a non-linear field theory in which the only fields present are mesonic.[1] He established the existence of a soliton solution to a simple meson field theory, with which he constructed an anomalous conserved current, and conjectured that it represents the baryon current. His ideas were advanced long before QCD became a candidate as the theory of strong interactions. Curiously, They have received their justification and reinterpretation through developments in QCD. The generalization from SU(3) to SU(N) gauge theory by 't Hooft[2] and Witten[3] established the equivalence of QCD to an effective meson field theory. Balachandran *et al.*,[4] using the method of Goldston and Wilczek,[5] demonstrated that the topological meson configuration polarizes a quark field coupled to it so as to induce the same baryon charge, and Witten[6] showed that the Skyrme soliton has spin 1/2.

Although the correspondence of QCD to an effective meson field theory is established in the above quoted work, the effective fields and their lagrangian have not been derived. It is remarkable then that the properties of the Skyrme soliton, a solution to a very simple lagrangian having only four meson fields, the scalar sigma and the triplet of pions, have a close resemblance to the nucleon. These properties include the magnetic moment, charge radius, g−factors,[7] and a large number of resonance states,[8] that agree with experiment at the 30% level. Impressive progress is being made also in the degree of agreement of the soliton-soliton interaction in its various spin-isospin components with the N-N interaction.[9] It is conjectured that as additional fields are appropriately coupled, the agreement in all these nucleon properties and resonances will improve.[10] In any case the 30% level is already very interesting, in as much as it might have turned out that the soliton had little or no resemblance whatever to the nucleon.

What we find particularly appealing in these developments is that, having a lagrangian that describes the internal structure of the nucleon (soliton), one can investigate interesting questions concerning how the internal structure of free nucleons change when they are assembled to form nuclei or dense matter, and how the properties of matter reflect these internal changes. Several of the more interesting questions concern possible changes in quark behavior in free and bound nucleons, as suggested by the anomalous muon scattering on nuclei as compared to nucleons (EMC effect),[11] and the onset of deconfinement in dense matter. Of course we will not believe literally the predictions of the theory. It is in the large N-limit that the coupling of the meson fields become small and the mean field approximation is assured to be accurate. We live in a three color

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world and we know of no criteria by which we may judge how far from the limit we are, aside from the empirical success at the 30% level. In any case we do not expect that our model of matter as consisting of solitons, will rival the lattice gauge calculations for quantitative predictions of phase transitions. However, when a physical theory is very complex, as is the theory of strong interactions, it is always useful to have a model with which to form at least qualitative pictures of how the theory works. The model may suggest ways of probing nature that the exact theory, solved on large computers may not do. In this paper we will report on the start that we have made on such a program.[12] It is far from finished and there remain serious problems to be overcome. However the picture of color conductivity that emerges seems in itself to be a novel and interesting one.

II. REVIEW OF THE SKYRMION

Before introducing our model of matter, we will recall some of the salient features of soliton models of the nucleon. Briefly, a soliton is a solution to a non-linear theory whose energy density has a finite spatial extent, and which is stable in the sense that if several soliton solutions are constructed in different regions of space and allowed to come into proximity so that they interact, after they have moved apart they are restored to their original form. It was Skyrme who first suggested that baryons might be understood as soliton solutions of a field theory having only mesons as the fields.[1] How do baryons emerge from such a theory? We shall be content here to merely exhibit the conserved anomalous integer quantum number, which Skyrme conjectured to be the baryon number and which Witten recently confirmed.[3] Skyrme studied a theory based on a scalar and triplet of pseudoscalar pi mesons. Construct the two by two matrix,

$$U=\frac{1}{f_\pi}\Big(\sigma(x)+i\,\boldsymbol{\tau}\cdot\boldsymbol{\pi}(x)\Big)\quad,\ \sigma^2+\pi^2=f_\pi^2\quad, \tag{1}$$

and from this define,

$$L_\mu=U^\dagger\partial_\mu U\quad, \tag{2}$$

Skyrme constructed the lagrangian,

$$\mathscr{L}=-\frac{f_\pi^2}{4}\,\mathrm{tr}\Big[L_\mu L^\mu\Big]+\frac{\eta^2}{4}\,\mathrm{tr}\Big[L_\mu,L_\nu\Big]^2\quad, \tag{3}$$

The first term is an unfamiliar way of writing the kinetic term for scalar and pion fields. It turns out that there are no stable finite size soliton solutions for a theory possessing only the first term, and the second term was added to provide stability against collapse of the solution. We want to draw attention to the fact that it is of fourth order in derivatives of the fields, and that a sixth order term or higher would also stabilize the solution. This term plays no other essential role. In particular the quantum number is unaffected by it.

To show that there is a soliton solution, one makes the very peculiar ansatz that a solution of the form,

$$U_0=e^{i\boldsymbol{\tau}\cdot\hat{\mathbf{r}}\theta(r)}=\cos\theta(r)+i\,\boldsymbol{\tau}\cdot\hat{\mathbf{r}}\,\sin\theta(r)\quad, \tag{4}$$

exists. That is, that the isospin components of the pion field point in the radial direction,

$$\sigma=f_\pi\cos\theta(r)\ \ ,\ \ \boldsymbol{\pi}=\hat{\mathbf{r}}f_\pi\sin\theta(r)\quad. \tag{5}$$

For that reason the solution is called the hedgehog. That it is a solution can be shown easily by calculating the canonical form of the energy from the lagrangian, substituting the ansatz for the fields, and minimizing. This yields an equation for the chiral angle, $\theta(r)$, which has a solution that smoothly connects the boundary values

$$\theta(0)=0\ \ ,\ \ \theta(\infty)=n\pi\quad. \tag{6}$$

The energy is finite and can then be seen to be localized in the vicinity of the origin where $\theta(r)$ is non-vanishing.

What is very interesting, is that in addition to the Noether currents that correspond to the invariances of the lagrangian, the theory possess an anomalous current,

$$B_\mu=\frac{1}{24\pi^2}\,\epsilon_{\mu\alpha\beta\gamma}\,\mathrm{tr}\Big[L^\alpha L^\beta L^\gamma\Big]\quad, \tag{7}$$

where $\epsilon_{\mu\alpha\beta\gamma}$ is the antisymmetric tensor in all indices. By construction this quantity is divergenceless, independent of the equations of motion,

$$\partial_\mu B^\mu=0\quad, \tag{8}$$

and the charge, corresponding to the ansatz [Eq. (5)] is

$$B=\int_0^\infty d^3r\,B_0(r)=\frac{1}{\pi}\Big[\theta(r)-\frac{1}{2}\sin 2\theta(r)\Big]_0^\infty=n\ , \tag{9}$$

This soliton therefore has a conserved quantity which is integer and which Skyrme conjectured to be the baryon number. It is topological in nature, and in the case of the hegehog soliton [Eq. (4)] is also sometimes referred to as the winding number. This terminology can be understood by noting that Eq. (4) maps ordinary 3-space onto a unit 3-sphere in the 4 dimensional space $(\sigma,\pi_1\pi_2\pi_3)$. The origin is mapped onto the pole on the σ-axis, and all points at infinity either also onto this pole or the other, depending on the integer n in the boundary condition [Eq. (6)]. The magnitude of n determines how many times the surface of the sphere is traversed when r goes from 0 to ∞.

Our purpose in reviewing this material is to introduce the conserved topological charge, which is associated with the SU(2) character of the theory, and will carry over to modifications of the theory which leave this character intact. The Skyrmion as such is not interesting to us for the purpose set out in the beginning, because it has no quarks, and we want to see how the quarks begin to leak out of the baryons as the density of matter is increased. This is perhaps relevant both to the deconfinement phase transition as well as to anomalous lepton scattering from nuclei (EMC effect). Therefore, we would like to have a soliton with quarks that are confined, but not through the artificial mechanism of an impervious bag. In the absence of a known soliton solution possessing true confinement, we opt for a model in which the quarks are deeply bound in a topological soliton field. The hybrid soliton model fills this requirement.[13,14]

III. SOLITON WITH QUARKS

The hybrid soliton,[13,14] like the Skyrmion, is based on the chiral sigma model,[15] but now including the fermion sector, which here are quarks. In the limit of large scalar meson mass, the lagrangian is,

$$\mathscr{L}=\frac{1}{2}\left\{\partial_\mu\sigma\partial^\mu\sigma+\partial_\mu\boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi}\right\}$$
$$+\bar{\psi}(x)\left\{i\gamma_\mu\partial^\mu-g\left[\sigma(x)+i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}(x)\right]\right\}\psi(x) \quad . \tag{10}$$

This consists of the first term of Eq. (3) and in addition the lagrangian of the quarks, which are Yukawa coupled to the scalar and pion fields. The quarks have a constituent mass of $m=gf_\pi$. What Kahana *et al.*,[13] and Birse and Banerjee[14] showed is that there is a solution in which the quarks are deeply bound to the topological soliton field of the mesons. For a certain range of the coupling constant, g, this state has lower energy than the spatially uniform field solution. In this range of coupling, the soliton has the nucleon mass (Fig. 1).

The soliton carries its conserved topological charge. Kahana and Ripka[16] showed that the baryon charge on the soliton when the Dirac sea and the valence 0^- orbital are filled is equal to the topological charge. Indeed, according to Balachandran *et al.*,[4] the spatial distribution of baryon and topological charge should be identical. The topological meson configuration polarizes the vacuum in a very precise way.

For the hedgehog configuration [Eq. (5)], the meson field equations reduce to one non-linear differential equation for the chiral angle $\theta(r)$,

$$(r^2\theta')'-\sin 2\theta$$
$$=-\frac{3g^2m^2}{4\pi}\left(F^2-G^2)\sin\theta-2FG\cos\theta\right)r^2 \quad . \tag{11}$$

This is coupled to the Dirac equation for the quark spinors,

$$\left\{i\gamma_\mu\partial^\mu-m\left[\cos\theta(r)+i\gamma_5\boldsymbol{\tau}\cdot\hat{\mathbf{r}}\sin\theta(r)\right]\right\}\psi(x)=0 \tag{12}$$

or, using the expressions for the Dirac matrices in terms of the Pauli matrices,

$$\begin{pmatrix} m\cos\theta & -i\boldsymbol{\sigma}\cdot\nabla+i\,m\sin\theta\,\boldsymbol{\tau}\cdot\hat{\mathbf{r}} \\ -i\boldsymbol{\sigma}\cdot\nabla-i\,m\sin\theta\,\boldsymbol{\tau}\cdot\hat{\mathbf{r}} & -m\cos\theta \end{pmatrix}\begin{pmatrix}\psi_U\\ \psi_L\end{pmatrix}$$
$$=\epsilon\begin{pmatrix}\psi_U\\ \psi_L\end{pmatrix} \tag{13}$$

where ψ_U and ψ_L are the upper and lower components of the Dirac spinor. There is a zero "spin" solution ($\ell=0,(\mathbf{s}+\mathbf{t})^2=0$), having the form,

$$\psi(\mathbf{r})=\begin{pmatrix}F(r)\\ i\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}G(r)\end{pmatrix}|\nu> \quad , \tag{14}$$

where $|\nu>$ is a spinor eigenstate of the sum of spin and isospin, having eigenvalue zero,

$$(\mathbf{s}+\mathbf{t})|\nu>=0 \quad . \tag{15}$$

That such a peculiar combination comes in, follows from the coupling of the quarks to the hedgehog meson field in which the isospin components point in the radial direction. The differential equations for the F and G are,

$$-F'+m\,F\sin\theta \quad =(\epsilon+m\cos\theta)G$$

$$G'+\left[\frac{2}{r}+m\sin\theta\right]G=(\epsilon-m\cos\theta)F \tag{16}$$

We normalize the solutions so that

$$\int_0^\infty (F^2+G^2)r^2dr=(gf_\pi)^{-3} \quad . \tag{17}$$

We call the state [Eq. (14)] a positive parity state after the transformation of the large component. The state of opposite parity satisfies equations like Eq. (16) but with m → -m. This can be understood by noting that

$$\tilde{\psi} \equiv \gamma_5\psi=\begin{pmatrix} i\,\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}G(r) \\ F(r)\end{pmatrix}|\nu> \quad , \tag{18}$$

which according to the parity operator,

$$P=\gamma_0\,P(\mathbf{r}\rightarrow -\mathbf{r}) \quad , \tag{19}$$

has opposite parity to Eq. (14). Inserting γ_5^2 into the Dirac equation and commuting one of the γ_5 to the left, it is found that $\tilde{\psi}$ satisfies the same equation as ψ except that the sign of m is changed.

The eigenstates satisfying Eq. (15) are triply (color) degenerate. This can be seen as follows. Because of Eq. (15), $|\nu>$ has eigenvalue $(\mathbf{s}+\mathbf{t})^2=0$. Therefore,

$$|\nu>=$$

$$\frac{1}{\sqrt{2}}\left(|s,\tfrac{1}{2}>|t,-\tfrac{1}{2}>-|s,-\tfrac{1}{2}>|t,\tfrac{1}{2}>\right) \quad . \tag{20}$$

Thus $|\nu>$ is a combination of u and d quarks. We can assign color to wave functions like Eq. (20) in 9 ways; the first component can have any color, and so can the second. Choose any three of these. One then finds that any of the remaining such wave functions has either an u or d quark of a color already appearing in one of the first three. So only three such spin-isospin functions can be assigned to a level [Eq. (12)]. We may therefore introduce color by assigning both components of Eq. (20) the same color, and the states are therefore $|\nu,c>$ with c = r,b,g.

The solutions for the free soliton in this model have been discussed previously.[13,14] There are several quark levels that are bound by the soliton field in the energy interval between +m and -m. One of these, a 0^- level, is pulled down from +m, by interaction with the meson fields, and for a very extended soliton, migrates to -m. When this level and all below it are occupied, and the boundary conditions applied to the chiral angle requiring it to differ by π in the interval between r = 0 and r = ∞, the resulting soliton has baryon number unity.[16] In this theory, it represents the nucleon. Its mass is given by,

$$M=3\epsilon+\frac{2\pi}{g^2}\,m^2\int_0^\infty dr\left\{r^2\left(\frac{d\theta}{dr}\right)^2+2\sin^2\theta\right\} \quad , \tag{21}$$

where ϵ is the energy of the 0^- quark orbital, and the integral is the field energy of the mesons. At the stationary points of M, the field equations are satisfied. (Of course, to derive the field equations in this way, we must express the Dirac eigenvalue in Eq. (21) in terms of θ and the functions F and G.)

In this work, we solve the coupled equations [Eqs. (11,16)] variationally, by parameterizing the chiral angle. We note from M and the boundary conditions on θ, that θ should smoothly join its boundary values, with most of the change occurring at small r. Therefore we represent θ by four parameters, the radius R at which it becomes equal to π, and three other parameters, a, b, c, that measure its deviation from a straight line joining the points (0,0) and (R,π) at equally spaced intervals between r = 0 and R. The chiral angle is then represented by a cubic spline passing through these points. For a given such set of parameters, the Dirac equation is integrated under the eigenvalue condition that it decay exponentially at large r, and that it is everywhere finite. The minimum value of M is sought, at which point the field equations are satisfied. The same method can be used for the solution of the soliton in matter, except that the boundary conditions on the Dirac equation are different, as discussed below.

There is only one parameter in the theory, the value g of the coupling constant between quarks and the meson fields, since we take the experimental value for f_π = 94 MeV. The computed soliton mass as a function of g is shown in Fig. 1. Throughout the remainder of the calculations, we fix g = 5.96 to yield a soliton mass of 940 MeV. The Chiral angle as a function of radius for the free soliton, is shown in Fig. 2. The pion and sigma fields, and the large and small components of the quark valence orbital 0^- are shown in Fig. 3.

IV. CRYSTAL APPROXIMATION TO MATTER

Now we wish to assemble a large number of such solitons to form dense matter. Our problem is very different from the usual nuclear many-body problem. There the nucleons are regarded as point particles having no internal structure. In some cases an internal quantum number is associated with the baryons to account for the appearance in dense matter of isobars and hyperons. However no dynamics is associated with this degree of freedom. In reality, the nucleons do have an internal structure, the quarks and gluons, and their state of motion will be polarized by neigbouring nucleons when the density of matter is sufficiently high. In general therefore there is a sublevel to the one that is customarily treated in nuclear physics, and that level is the focus of our interest. However the general state of such a system must be extremely complicated to describe. It is a many-body problem in which the quarks within the individual solitons are moving in interaction with each other through the meson fields, while the solitons are moving about under the influence of the interaction of their constituent quarks with those of neighbouring solitons. It is possible that, at sufficiently high density, the solitons would arrange themselves into a crystalline lattice, because of the repulsion at short range. This would simplify the problem. In any case, we shall study this particular configuration. If the physical system does not arrange itself thus, we shall assume that the internal structure of the solitons (nucleons) that emerges under this assumption may describe the average, or typical, structure of nucleons in matter of the corresponding density.

Since the quarks are deeply bound in the soliton, by of the order of their constituent mass, they are relativistic. We have therefore a relativistic solid state problem.

As an initial orientation on what to expect, we solved the Dirac equation in one dimension for a particle in a periodic square potential.[17] This is a problem that had been solved long ago for the Schroedinger equation by Kronig and Penney.[18] The analytic solution for the eigenvalue spectrum is given by,

$$\frac{Q^2-K^2+V^2}{2QK}\sinh 2Qb \sin 2Ka + \cosh 2Qb \cos 2Ka = \cos 2k(a+b) \quad , \qquad (22a)$$

where,

$$Q^2=(m+V)^2-\epsilon^2 \quad , \quad K^2=\epsilon^2-m^2 \quad , \qquad (22b)$$

and k is the so-called crystal momentum. In the non-relativistic limit, this reduces to the formula of Kronig and Penney. The allowed values of the particle energy are those for which the left side does not exceed in absolute value, unity, so that the spectrum has the well known band structure. A typical spectrum as a function of the spacing between the attractive regions is shown in Fig. 4. The parameters of the problem are chosen so that the fermions become relativistic toward the top of the well. As in the non-relativistic case, the levels of the isolated wells, spread out into bands with each well contributing a level to the band. For close spacing, the bands tend to touch. The band structure persists into the positive energy spectrum above the top of the potential, with the gaps tending toward zero as the energy increases.

A. Wigner-Seitz approximation

We turn now to the solution of the problem at hand, the spectrum of quarks in three dimensional soliton crystal matter. The hedgehog meson configurations are centered at lattice points thus generating a periodic field in which the quarks move. From solid state physics we know that the solution of the Hamiltonian for a periodic system must obey Bloch's theorem. Therefore the quark spinor must be of the form,

$$\psi_{\mathbf{k}}(\mathbf{r})=e^{i\mathbf{k}\cdot\mathbf{r}}u_{\mathbf{k}}(\mathbf{r}) \quad , \qquad (23)$$

where k is called the crystal momentum and $u_{\mathbf{k}}(r)$ is a periodic spinor function having the period of the lattice. That is to say, the solutions are plane waves with a periodic modulation.

To solve Eq. (16) on the lattice we employ the Wigner-Seitz approximation. Thus the actual problem is replaced by a spherically symmetric one which is solved for k = 0. This is the ground state of the band. The translational invariance by multiples of the unit cell also places a condition on the chiral angle. So that there be unit topological charge centered at each cell site, it must satisfy

$$\theta(0)=0 \ , \ \theta(R)=\pi \ . \tag{24}$$

The boundary condition for the Dirac spinor can be derived as follows. We note first that the Dirac gamma matrices in the three-current, $\bar{\psi}\gamma\psi$, connect upper and lower components of the Dirac spinor. For the ground state, we require that this current vanish, implying that one or the other component of the spinor should vanish on the cell boundary. We note also that upper and lower components have opposite parity. It is clear that to obtain a solution that is periodic in its relation to adjacent cells, the odd component must vanish at the cell boundary. There is no additional freedom, nor is any needed. The other component will take on the value dictated by the differential equations. For the spin zero case, [Eq. (14)], the odd component is G and we therefore require that G(R) = 0. According to the coupled Eqs. (16) this then yields

$$G(R)=0 \longleftrightarrow F'(R)=0 \tag{25}$$

The large component therefore satisfies the same condition as is required of the Schroedinger wave function in the non-relativistic theory. At the origin, it is evident from Eq. (16) that G(0) = 0. This in turn requires that F′(0) = 0. Therefore the boundary conditions

$$G(0)=G(R)=0 \ . \tag{26}$$

ensure that the Bloch theorem is satisfied, i.e. that both F and G are periodic.

B. Existence of a topological crystal solution

For the free soliton, with quark wave functions that decay exponentially at large r, Kahana, Ripka and Soni[12] showed that the non-linear equation for the chiral angle [Eq. (11)] admits of a solution that satisfies boundary conditions at the origin and infinity that correspond to integer topological charge [Eq. (9)]. Here we demonstrate for the crystal boundary conditions, that this is still so. The demonstration is necessarily more complicated. In the vicinity of the Wigner-Seitz boundary, r = R, we may approximate F by a polynomial and select the dominant term at R. For the present instance of zero spin quark orbital, we even have, from Eq. (16), that F(R) = constant. Hence, defining for convenience,

$$\phi(r)=\theta(r)-\pi \ , \tag{27}$$

the equation for the chiral angle near R, takes the form

$$r^2\phi''+2r\phi'=(2-cr^2)\phi \ , \tag{28}$$

where c is a constant whose value can be read from Eq. (11). Expand the solution in a power series,

$$\phi=\sum_{n=-\infty}^{\infty} a_n r^n \ , \tag{29}$$

and find the relation between coefficients,

$$(n^2+n-2)a_n=c\,a_{n-2} \ . \tag{30}$$

The coefficient of a_n vanishes for n = 1 or n = -2. This implies that a_{-1} and a_{-4} vanish, and that a_1 and a_{-2} are arbitrary. That is to say, the series breaks up into two pieces and ϕ is given by the sum of two series, the coefficients of one of them being proportional to a_1 and the other to a_{-2}. Hence, if the series converge, the function ϕ near r = R contains two arbitrary constants. The ratio of a_1 to a_{-2} can be chosen to make ϕ vanish at R, i.e., $\theta(R)=\pi$. The value of the remaining constant can then be used to aim in an inward integration "shooting" method to find the solution for which $\theta(0)$ = 0.

C. Band width

The Wigner-Seitz approximation allows us to calculate the eigenvalue of the ground state of each band (k = 0). Denote such an eigenvalue for a particular band by ϵ_0. We need to estimate the band width and we approximate the spinor of crystal momentum, **k**, by,

$$\psi_{\mathbf{k}}(\mathbf{r})=e^{i\mathbf{k}\cdot\mathbf{r}}u_0(r) \ . \tag{31}$$

In the Schroedinger theory the energy of such a level is calculated as the expectation of the hamiltonian. However the Dirac hamiltonian is linear in momentum and we have therefore to calculate the expectation of the square of the Dirac hamiltonian. This yields as an estimate of the energy of the level with crystal momentum, **k**,

$$\epsilon_{\mathbf{k}}=\sqrt{\epsilon_0^2+\mathbf{k}^2} \ . \tag{32}$$

Now consider a cubic crystal of N solitons along each direction, i.e. a cube of matter of dimensions L = 2RN. The allowed values of the component of crystal momentum in any of the three directions is

$$k=0 \ , \ \pm\frac{2\pi}{L} \ , \ \pm\frac{4\pi}{L} \ , \ \dots \ , \ \frac{N\pi}{L}=\frac{\pi}{2R} \ . \quad (33)$$

Therefore our estimate of the band width for a cell radius R, is,

$$\Delta=\left(\epsilon_0^2+(\pi/2R)^2\right)^{1/2}-|\epsilon_0| \ . \quad (34)$$

For the valence levels, and those above it, the Wigner-Seitz approximation locates the bottom of the band. However for the levels belonging to the sea, it locates the top of the band, just as the sea eigenvalues in the free case are - $\sqrt{m^2+\mathbf{k}^2}$.

An alternative approximation to the band width, that is valid for large separation of the solitons, is obtained by imposing the boundary condition,

$$F(R)=0 \ , \quad (35)$$

instead of Eq. (26). This corresponds to making the large component an odd function with respect to lattice sites, rather than an even one.

D. General spin

The form of the spinor having zero "spin," and the coupled equations for the two radial functions that appear in it were written above. Here we wish to write the equations for a general value of the "spin." It is evident that the eigenstates in the present problem are eigenstates, not of the total angular momentum, but of the total angular momentum plus the isospin. This is so because of the hedgehog configuration of the meson fields, in which the isospin is correlated with the radial direction. The solutions to the Dirac equation in this case are eigenfunctions of,

$$\boldsymbol{\lambda}=\boldsymbol{\ell}+(\mathbf{s}+\mathbf{t}) \ , \quad (36)$$

which we can call the grand spin or simply "spin." We introduce the eigenfunctions of "spin,"

$$\mathcal{Y}^{\lambda}_{\ell,\kappa}\equiv|\ell,(s,t)\kappa\,;\lambda> \ , \quad (37)$$

and for convenience we introduce the notation,

$$|1>=\mathcal{Y}^{\lambda}_{\lambda,0} \ , \ |2>=\mathcal{Y}^{\lambda}_{\lambda,1} \ ,$$

$$|3> \ = \mathcal{Y}^{\lambda}_{\lambda+1,1} \ , \ = |4> \ = \mathcal{Y}^{\lambda}_{\lambda-1,1} \ . \quad (38)$$

Then the matrix elements of $\boldsymbol{\sigma}\cdot\nabla$ and $\hat{\mathbf{r}}\cdot\boldsymbol{\tau}$ can be written in the space of these functions,[19]

$$(\boldsymbol{\sigma}\cdot\nabla)_{ij}=\begin{bmatrix} 0 & 0 & -\alpha D_2 & \beta D_{-1} \\ 0 & 0 & -\beta D_2 & -\alpha D_{-1} \\ -\alpha D_0 & -\beta D_0 & 0 & 0 \\ \beta D_1 & -\alpha D_1 & 0 & 0 \end{bmatrix} \quad (39)$$

$$(\hat{\mathbf{r}}\cdot\boldsymbol{\tau})_{ij}=\begin{bmatrix} 0 & 0 & \alpha & -\beta \\ 0 & 0 & -\beta & -\alpha \\ \alpha & -\beta & 0 & 0 \\ -\beta & -\alpha & 0 & 0 \end{bmatrix} \quad (40)$$

where,

$$\alpha=\left(\frac{\lambda+1}{2\lambda+1}\right)^{1/2} \ , \ \beta=\left(\frac{\lambda}{2\lambda+1}\right)^{1/2} \quad (41a)$$

and

$$D_2=\frac{d}{dr}+\frac{\lambda+2}{r} \ , \ D_{-1}=\frac{d}{dr}-\frac{\lambda-1}{r} \ . \quad (41b)$$

The upper and lower components of the Dirac spinor for general "spin" are written,

$$\psi^{\lambda}_U= f^{\lambda}_1(r)|1>+f^{\lambda}_2(r)|2>$$

$$\psi^{\lambda}_L=i\,g^{\lambda}_1(r)|3>+f^{\lambda}_2(r)|4> \ . \quad (42)$$

The parity of these components is opposite, and the parity of the Dirac function is customarily characterized by that of its upper component. Substituting this Dirac spinor into Eq. (13), and using the above matrix elements, we find the coupled radial equations,

$$\alpha(D_2+m\sin\theta)g_1-\beta(D_{-1}+m\sin\theta)g_2$$
$$=-(\epsilon-m\cos\theta)f_1$$

$$\beta(D_2-m\sin\theta)g_1+\alpha(D_{-1}-m\sin\theta)g_2$$
$$=-(\epsilon-m\cos\theta)f_2$$

$$\alpha(D_0-m\sin\theta)f_1+\beta(D_0+m\sin\theta)f_2$$
$$=(\epsilon+m\cos\theta)g_1$$

$$\beta(D_1-m\sin\theta)f_1-\alpha(D_1+m\sin\theta)f_2$$
$$=-(\epsilon+m\cos\theta)g_2 \ . \quad (43)$$

They can be put into a more convenient form by defining two new combinations for the upper components,

$$F_1^\lambda=\alpha\, f_1^\lambda+\beta\, f_2^\lambda$$

$$F_2^\lambda=\beta\, f_1^\lambda-\alpha\, f_2^\lambda \quad . \tag{44}$$

Then the radial equations become,

$$\left[\frac{d}{dr}-\frac{\lambda}{r}-\frac{m\sin\theta}{2\lambda+1}\right]F_1-(\epsilon+m\cos\theta)g_1$$

$$=2\alpha\beta m\sin\theta\, F_2$$

$$\left[\frac{d}{dr}+\frac{\lambda+2}{r}+\frac{m\sin\theta}{2\lambda+1}\right]g_1+(\epsilon-m\cos\theta)F_1$$

$$=2\alpha\beta m\sin\theta\, g_2 \tag{45a}$$

$$\left[\frac{d}{dr}+\frac{\lambda+1}{r}+\frac{m\sin\theta}{2\lambda+1}\right]F_2+(\epsilon+m\cos\theta)g_2$$

$$=2\alpha\beta m\sin\theta\, F_1$$

$$\left[\frac{d}{dr}-\frac{\lambda-1}{r}-\frac{m\sin\theta}{2\lambda+1}\right]g_2-(\epsilon-m\cos\theta)F_2$$

$$=2\alpha\beta m\sin\theta\, g_1 \quad . \tag{45b}$$

We notice that as the chiral angle approaches its boundary values at both ends of the range of r, the right hand sides vanish, and the equations decouple to two pairs of coupled equations. Since the solution of this eigenvalue problem is very difficult, we will take advantage of this to solve the decoupled pairs, and then diagonalize in this basis, the hamiltonian containing the coupling terms that appear on the right sides of Eqs. (45). Notice that the four coupled equations [Eqs. (45)] can be written in matrix form as,

$$\begin{pmatrix} H & H' \\ H' & H \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}=\epsilon\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad , \tag{46}$$

where,

$$\psi_1=\begin{pmatrix} F_1[\alpha|1>+\beta|2>] \\ i\, g_1|3> \end{pmatrix} \quad ,$$

$$\psi_2=\begin{pmatrix} F_2[\beta|1>-\alpha|2>] \\ i\, g_2|4> \end{pmatrix} \quad , \tag{47}$$

while the decoupled pairs, with the right sides of Eqs. (45) set to zero, can be written as,

$$\begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}=\begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{48}$$

where H is the two by two matrix, [Eq. (13)], and the coupling matrix in Eq. (46) is,

$$H'=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} i\, m\, \boldsymbol{\tau}\cdot\hat{\mathbf{r}}\sin\theta(r) \quad . \tag{49}$$

For small r, where $\theta(r)\to 0$, the four functions are, to lowest order in r, according to Eqs. (44) and (45), given by,

$$F_1\to r^\lambda \;,\; g_1\to(m-\epsilon)/(2\lambda+3)r^{\lambda+1} \quad ,$$

$$F_2\to(\beta-\alpha)/(\beta+\alpha)r^\lambda \;,$$

$$g_2\to -(2\lambda+1)/(\epsilon+m)(F_2/r) \quad . \tag{50}$$

For the soliton in the Wigner-Seitz cell, we integrate the equations outward, and iterate on ϵ to find that value that yields the boundary value, $g(R) = 0$. Then the elements of the coupling matrix [Eq. (50)], are computed with the normalized functions [Eq. (48)],

$$V\equiv(\phi_1|H'|\phi_2)=2\alpha\beta m\int(F_1g_2-g_1F_2)\sin\theta\, r^2dr. \tag{51}$$

The eigenvalues of Eq. (46), approximated by diagonalizing on the basis [Eq. (48)], are therefore the roots of,

$$(\epsilon_1-\epsilon)(\epsilon_2-\epsilon)-V^2=0 \quad . \tag{52}$$

The boundary conditions corresponding to the crystal have been stated earlier. Here we can restate them in the form,

$$g(R)=0, \quad \text{if} \quad \lambda=\text{even}$$

$$\tag{53}$$

$$f(R)=0, \quad \text{if} \quad \lambda=\text{odd}$$

V. NUMERICAL RESULTS FOR SOLITON CRYSTAL

For a discrete set of lattice spacings, we find the solution to the coupled Dirac and meson field equations, subject to the crystal boundary conditions described in section IVA. The three parameters defining the chiral angle, and which minimize the M [Eq. (21)] are shown in Fig. 5 as a function of the Wigner-Seitz cell radius R. For that range of R, for which these parameters are positive, the chiral angle has a decreasing slope as a function of R. However for small crystal spacing, it is an increasing function. The meson fields, and components of the Dirac spinor are shown in Figs. 6 and 7 for two cell radii, and illustrate the boundary conditions required by the periodicity of the crystal and the requirement that the current vanish on the cell surface. One can see by comparing the quark wave functions of the free soliton, with those in successively denser matter, that the quark distribution becomes more concentrated near the cell surfaces, Fig 8.

The quark levels as a function of cell radius are shown in Fig. 9, for those orbitals that are bound at energies between +m and -m. The valence orbital, 0^- and all below it are fully occupied, the degeneracy of the levels being $3(2\lambda+1)$. For large R, the eigenvalues approach those of the free soliton. In the crystal, each soliton contributes to a band of levels, and since each level that is occupied in the free soliton is full, the band of levels that they spread into in the crystal, are also full. In this figure we show just the fundamental levels, those with $k = 0$. They are either the topmost or bottom-most levels of a band, depending on whether or not the level in question is the valence or higher level, or whether it belongs to the Dirac sea. The band width can be estimated from Eq. (34) or Eq. (35), and of course is narrow for large crystal spacing, becoming large for close spacing. Therefore, for moderate to large spacing, no bands intersect, and matter is an insulator. However as the crystal spacing is made smaller a first crossing of the valence level and the lowest level of the first unoccupied band will occur, as shown in Fig. 10. This occurs for a cell radius of about 0.65 fm, or intersoliton distance of 1.3 fm. At that density (0.45 fm^{-3}), the quarks are free to migrate throughout the crystal, as electrons in a metal.

In this model we interpret the intersection of an occupied band with an empty one as the onset of color conductivity of matter. At higher densities than the critical one, matter becomes increasingly a color conductor. This is a somewhat different picture of quark liberation in dense matter, than usually envisioned, in which the nucleons dissolve into a homogeneous quark matter. In the present model, the basic nucleon structure remains; only the quarks rather than being confined to the individual solitons (nucleons) become free to migrate throughout the matter; and would do so under the influence of an external color field. We believe that this picture is more general than the present assumption of a crystal structure, on account of the conserved topological quantum number carried by the meson fields.

The transition in phase between color insulator and color conductor is a second order one, in this model. There is no latent heat associated with the transition since it corresponds to the onset of a degeneracy between filled and empty quark orbitals. Instead, at each crossing of an occupied with an unoccupied band, the energy denstiy, as a function of soliton (nucleon) density, will have a discontinuity in slope.

VI. SUMMARY

We employed a soliton model of the nucleon to study the quark behavior in dense matter. To render tractable the many-body problem of interacting nucleons with a quark substructure, we assumed that dense matter can be approximated as a crystal. In this picture, the quark levels of the isolated solitons disperse into bands as the density increases. At a critical density of about three times nuclear density, matter under goes a transition from color insulator to color conductor, due to the intersection of occupied and empty quark bands which broaden with increasing density. There is no discontinuity in the energy density at this transition point, though there is a discontinuity of the slope of the energy as a function of density.

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FIGURE CAPTIONS

FIG. 1. Soliton mass as a function of coupling constant g for f_π = 94 MeV.

FIG. 2. Chiral angle as a function of radius for the free soliton.

FIG. 3. Dirac spinor components F and G and meson fields σ and π, for free soliton of mass 940 MeV.

FIG. 4. Band structure in a periodic square potential for relativistic fermions.

FIG. 5. Parameters of the chiral angle as a function of Wigner-Seitz cell radius R, for solitons in a crystal.

FIG. 6. Dirac spinor components F and G, and meson fields in crystal of cell radius R = 1 fm.

FIG. 7. Dirac spinor components F and G, and meson fields in crystal of cell radius R = 0.5 fm.

FIG. 8. Probability distribution of quarks in crystal of several spacings.

FIG. 9. Quark levels as a function of cell radius, corresponding to k = 0. These are the fundamental levels of the bands of which they are the top (for sea quarks) or bottom of the band.

Fig. 10. Bands are shown for several of the levels of Fig. 9 as a function of lattice spacing (2R).

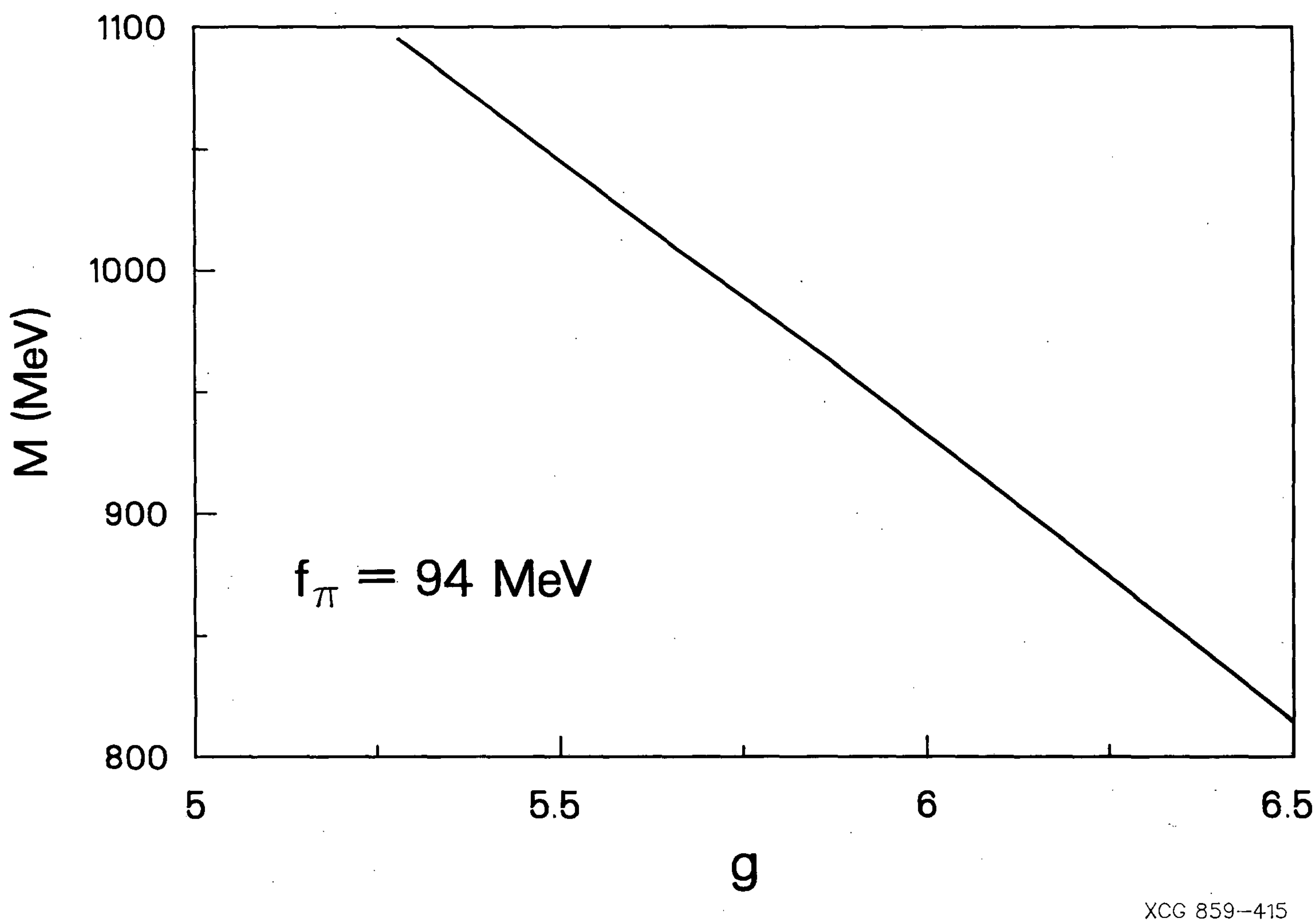
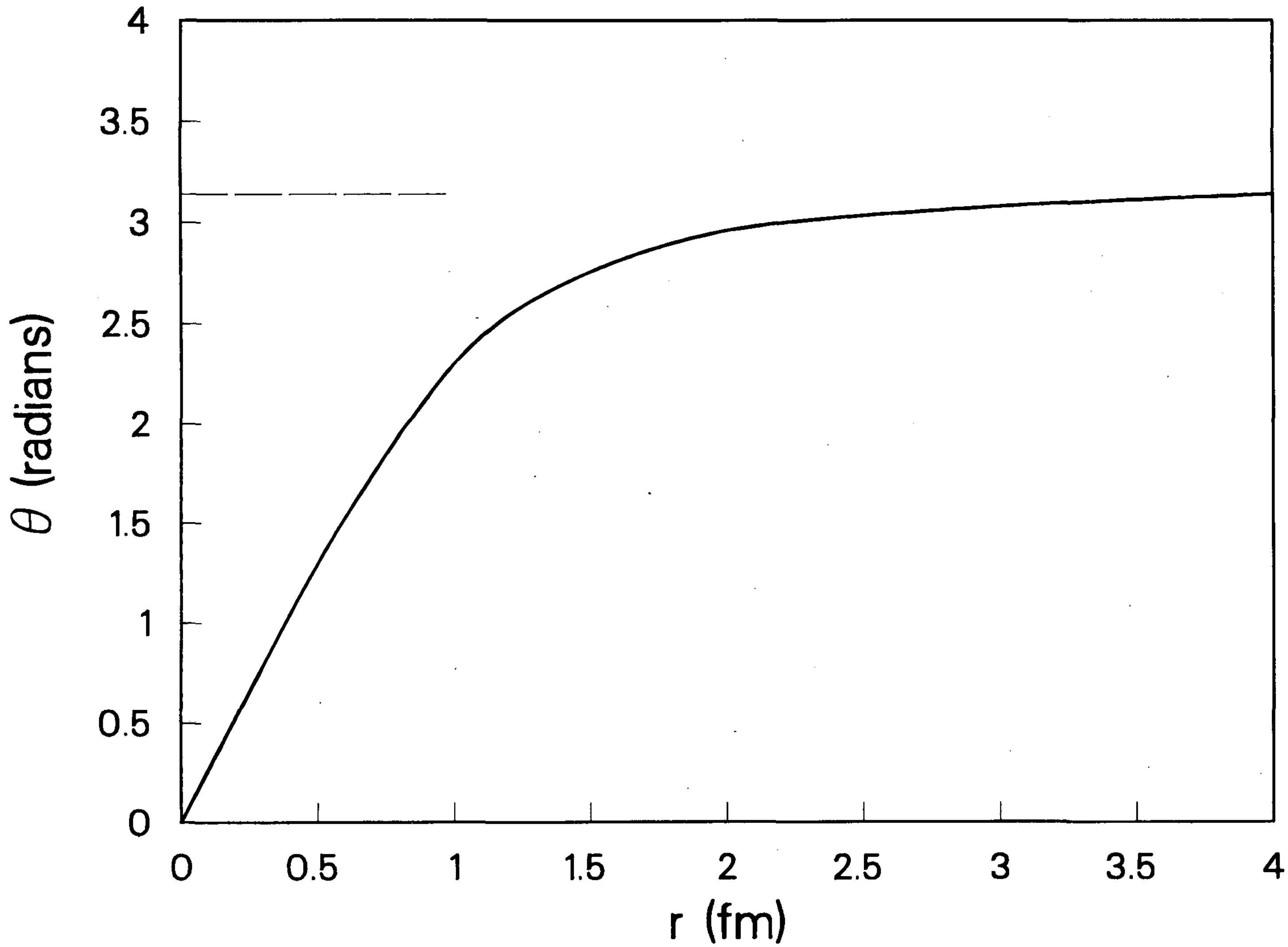
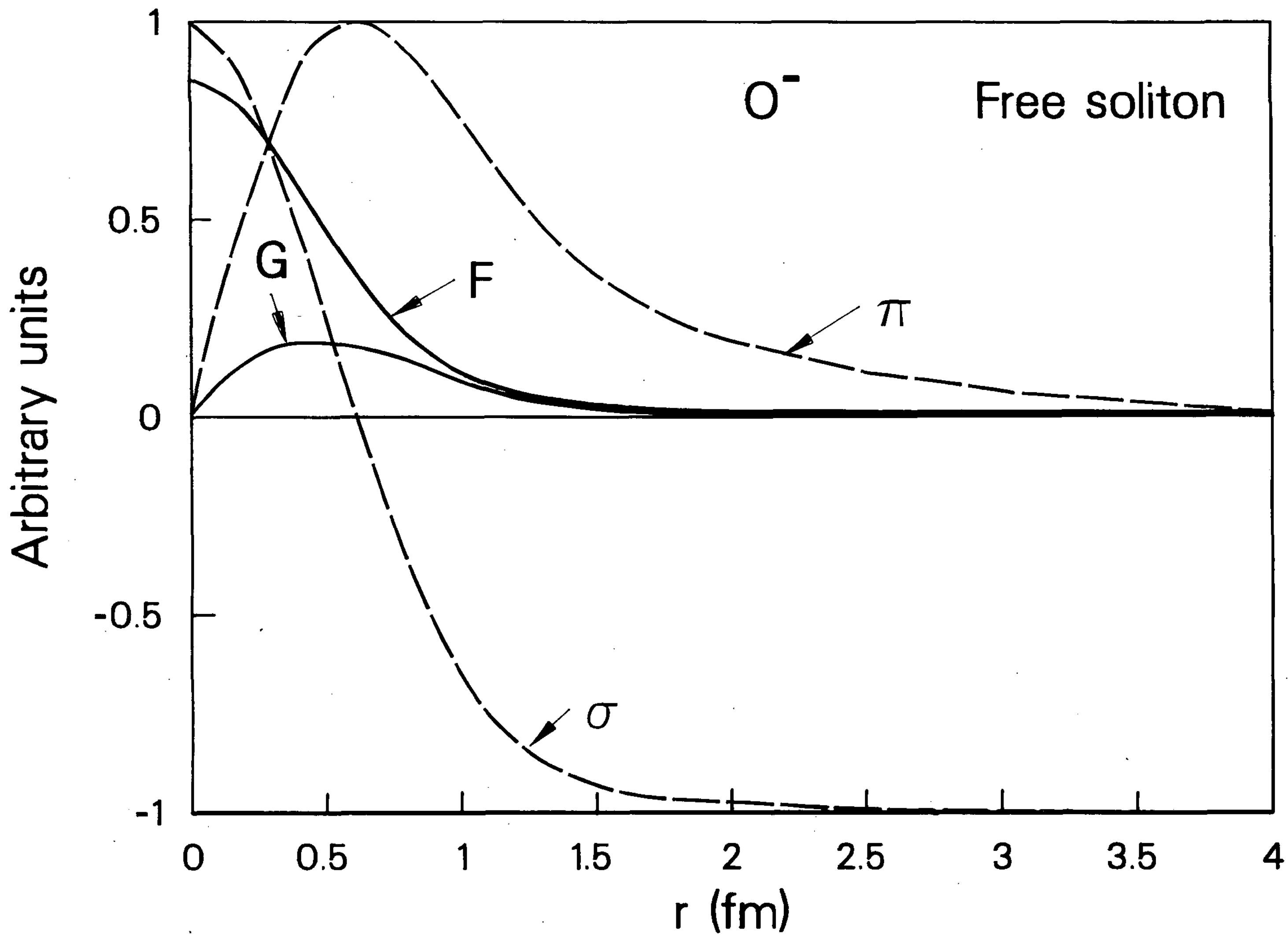


FIG. 1. Soliton mass as a function of coupling constant g for $f_\pi = 94$ MeV.



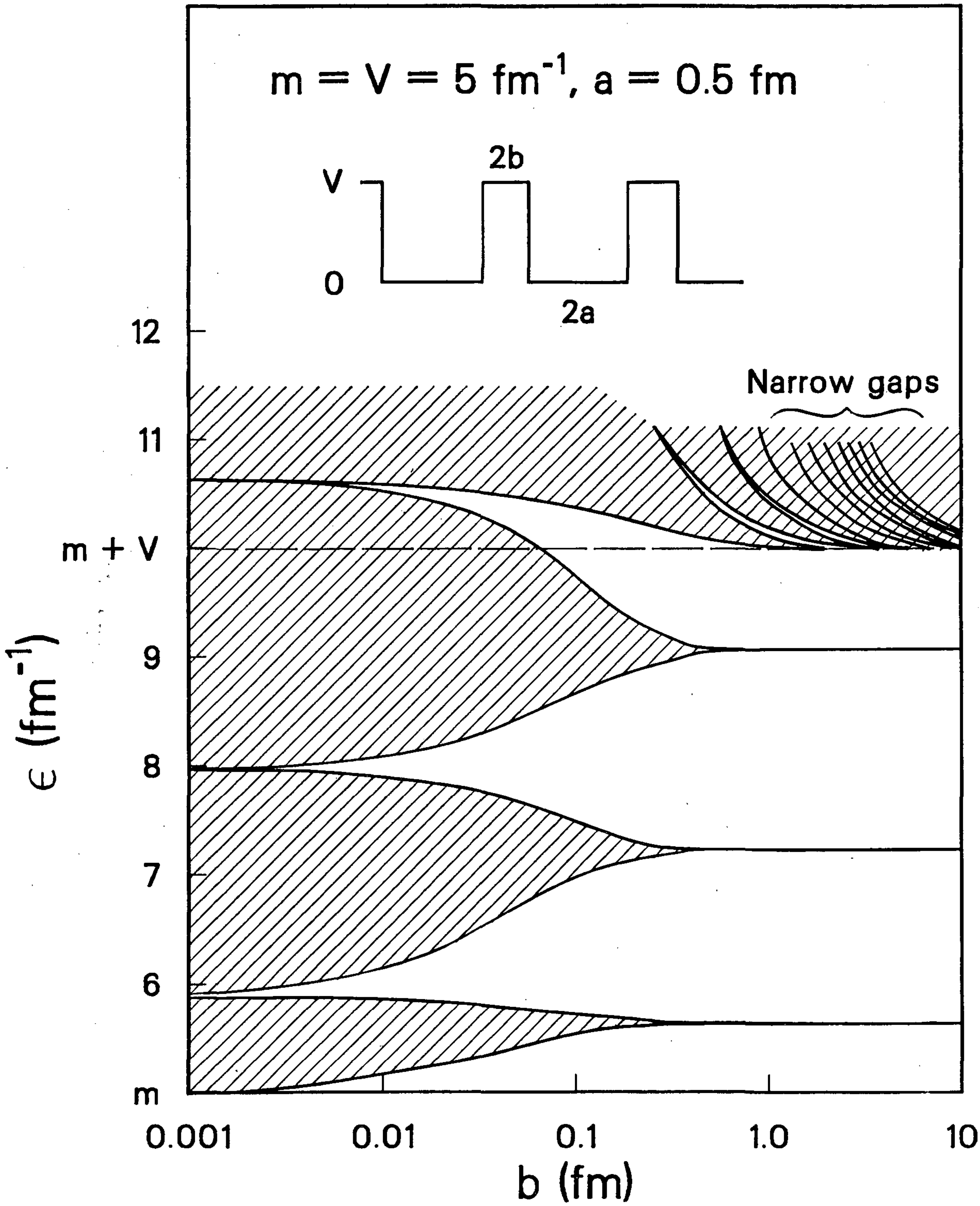
XCG 859–417

FIG. 2. Chiral angle as a function of radius for the free soliton.



XCG 859–418

FIG. 3. Dirac spinor components F and G and meson fields σ and π, for free soliton of mass 940 MeV.



XCG 851-10

FIG. 4. Band structure in a periodic square potential for relativistic fermions.

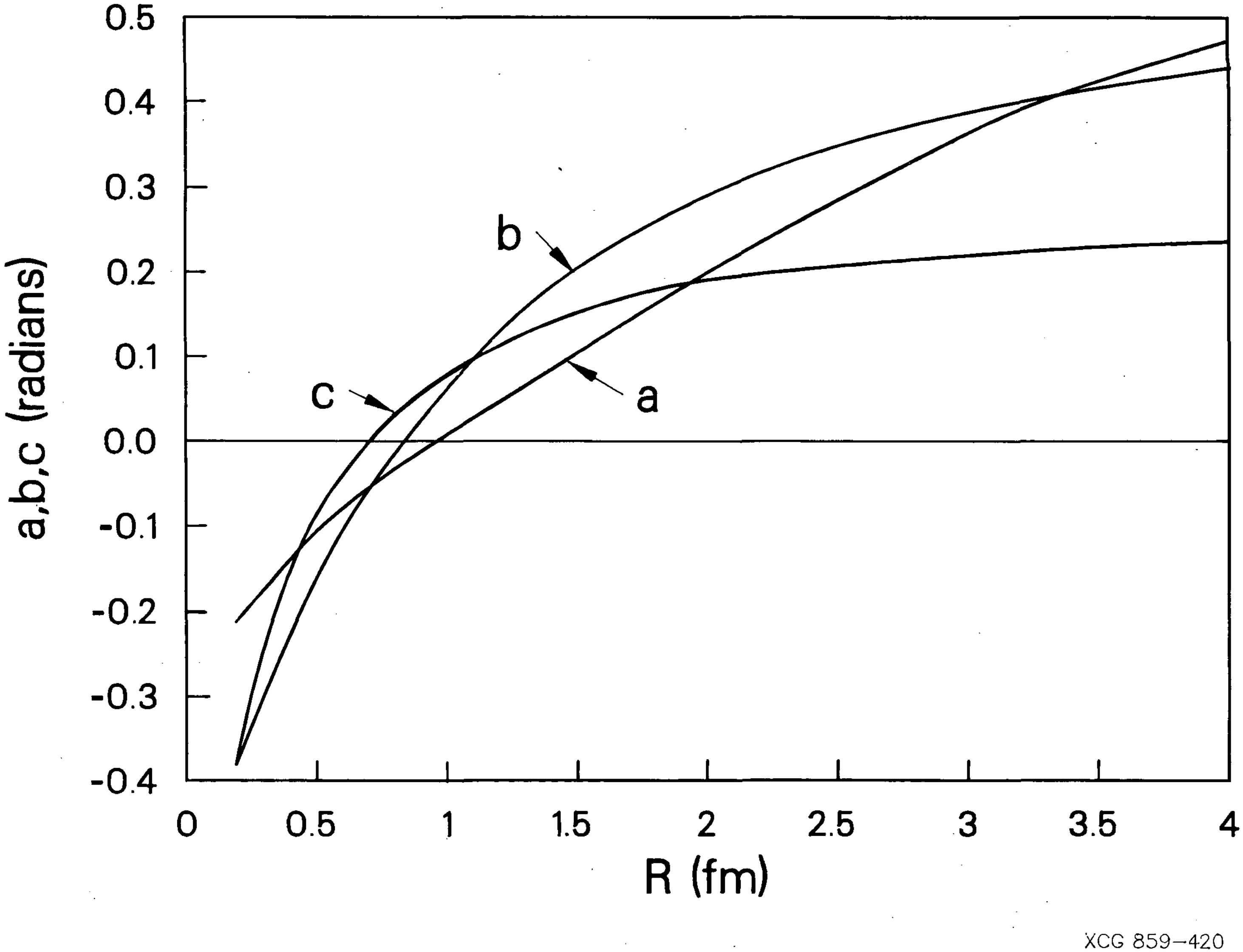
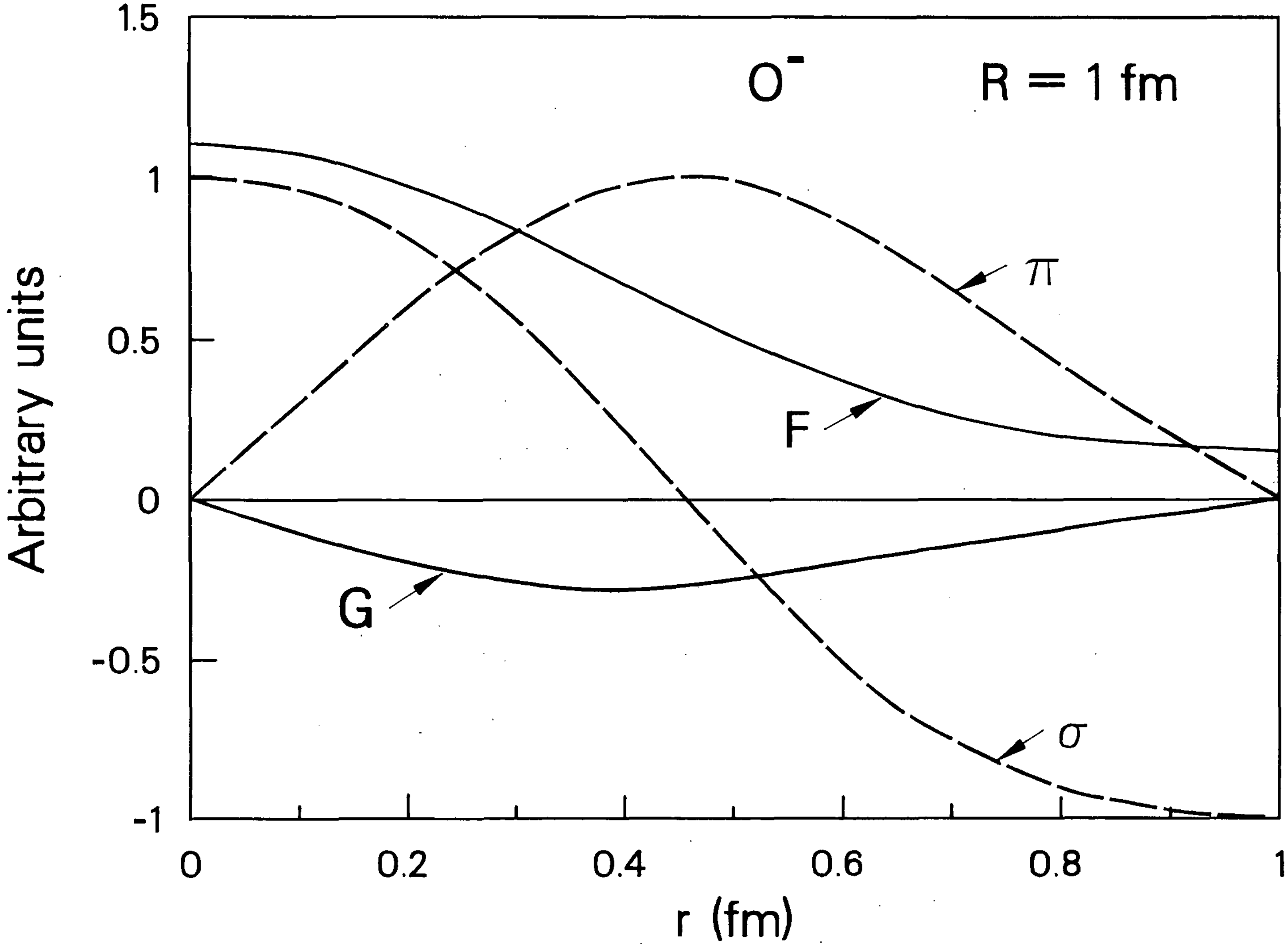


FIG. 5. Parameters of the chiral angle as a function of Wigner-Seitz cell radius R, for solitons in a crystal.



XCG 859–416

FIG. 6. Dirac spinor components F and G, and meson fields in crystal of cell radius R = 1 fm.

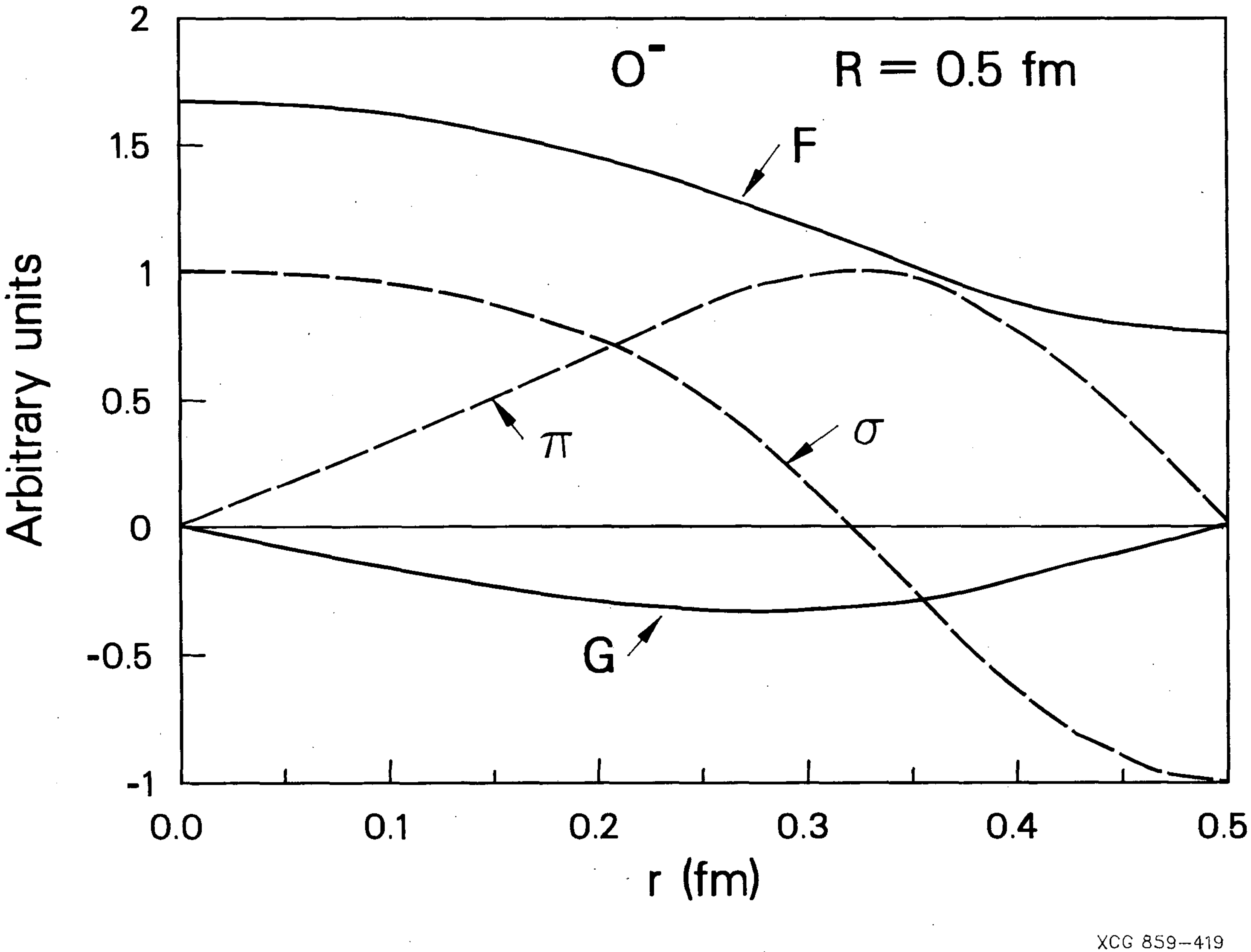
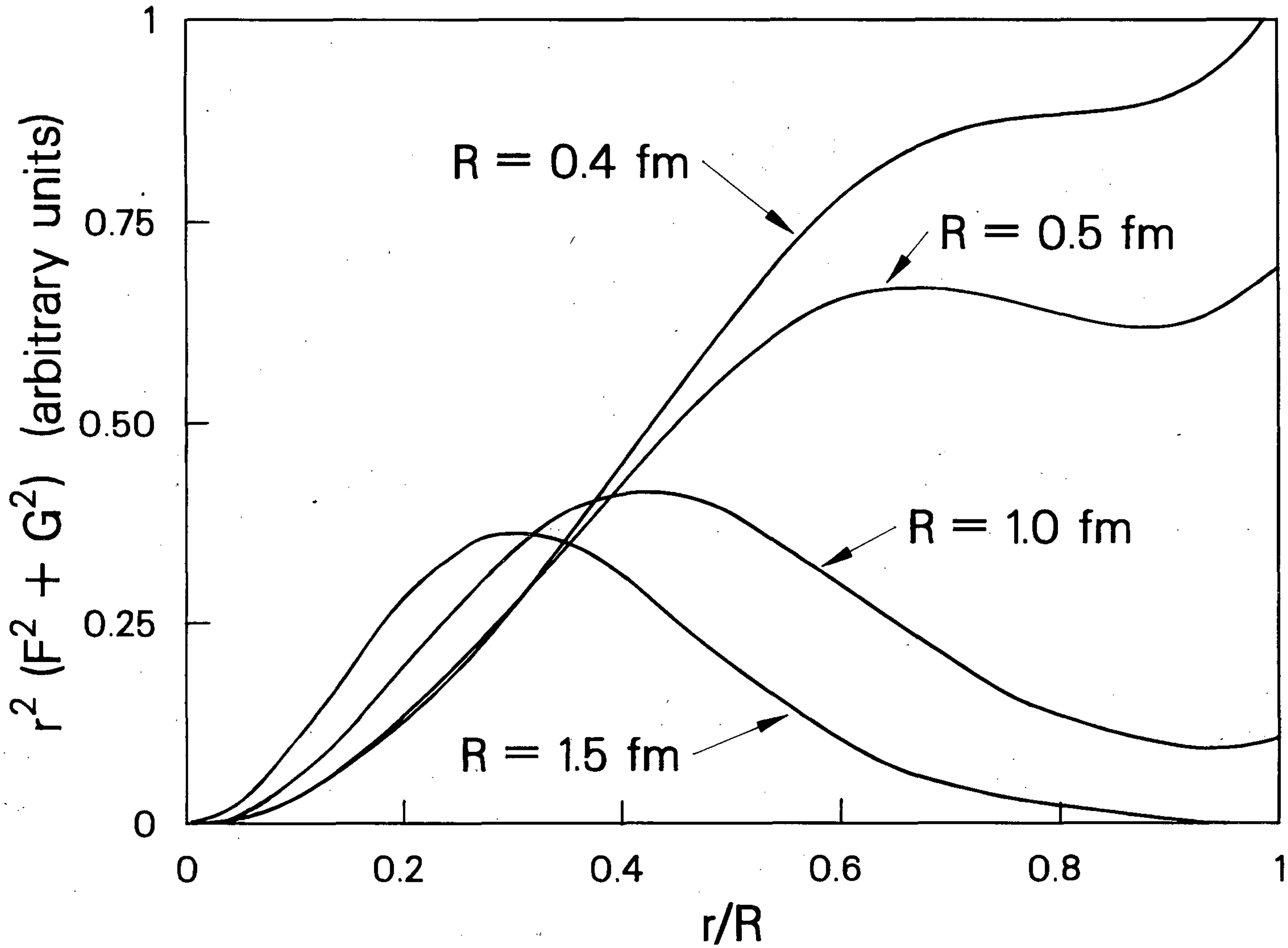


FIG. 7. Dirac spinor components F and G, and meson fields in crystal of cell radius R = 0.5 fm.



XCG 859-428

FIG. 8. Probability distribution of quarks in crystal of several spacings.

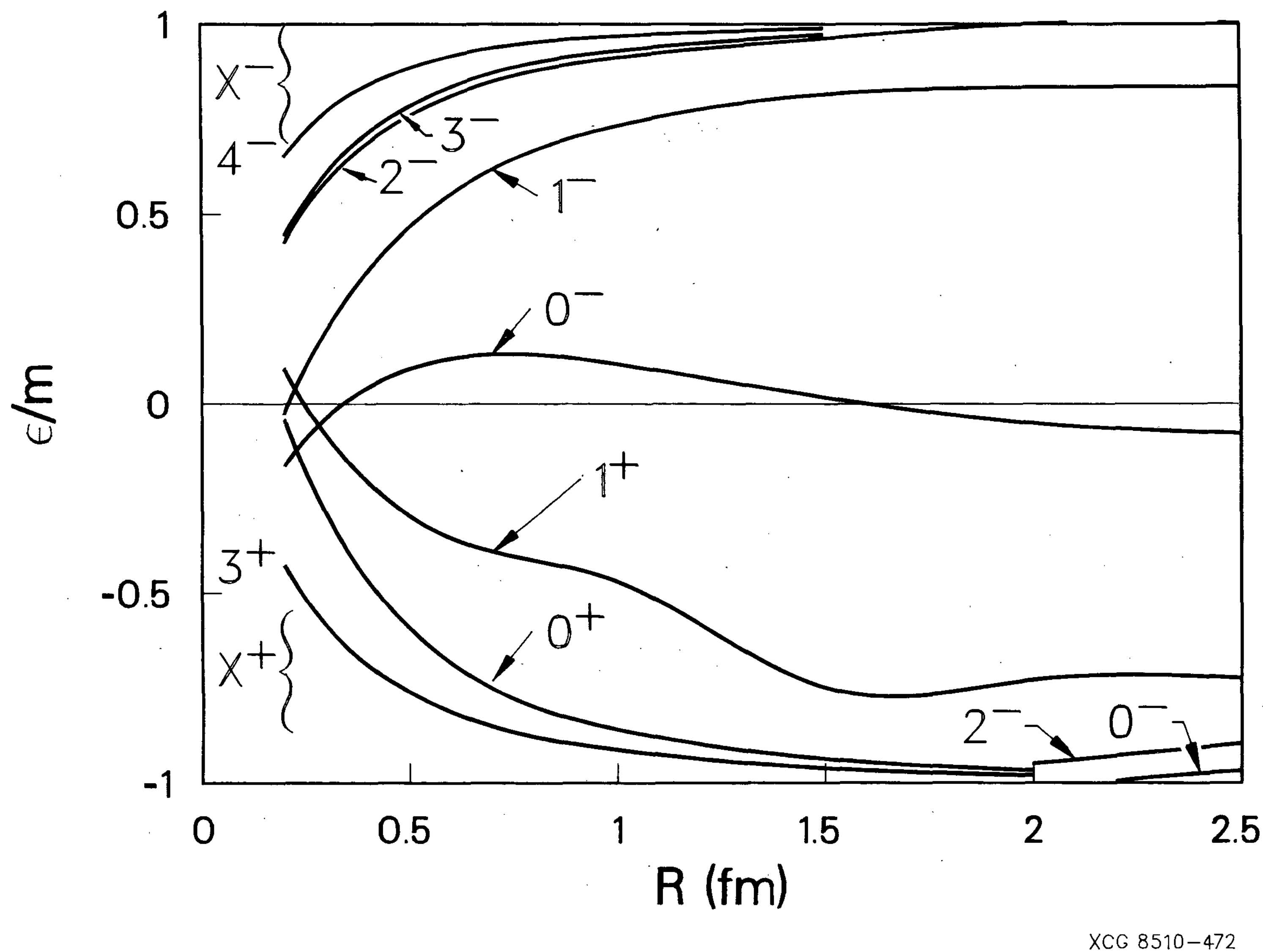


FIG. 9. Quark levels as a function of cell radius, corresponding to k = 0. These are the fundamental levels of the bands of which they are the top (for sea quarks) or bottom of the band.

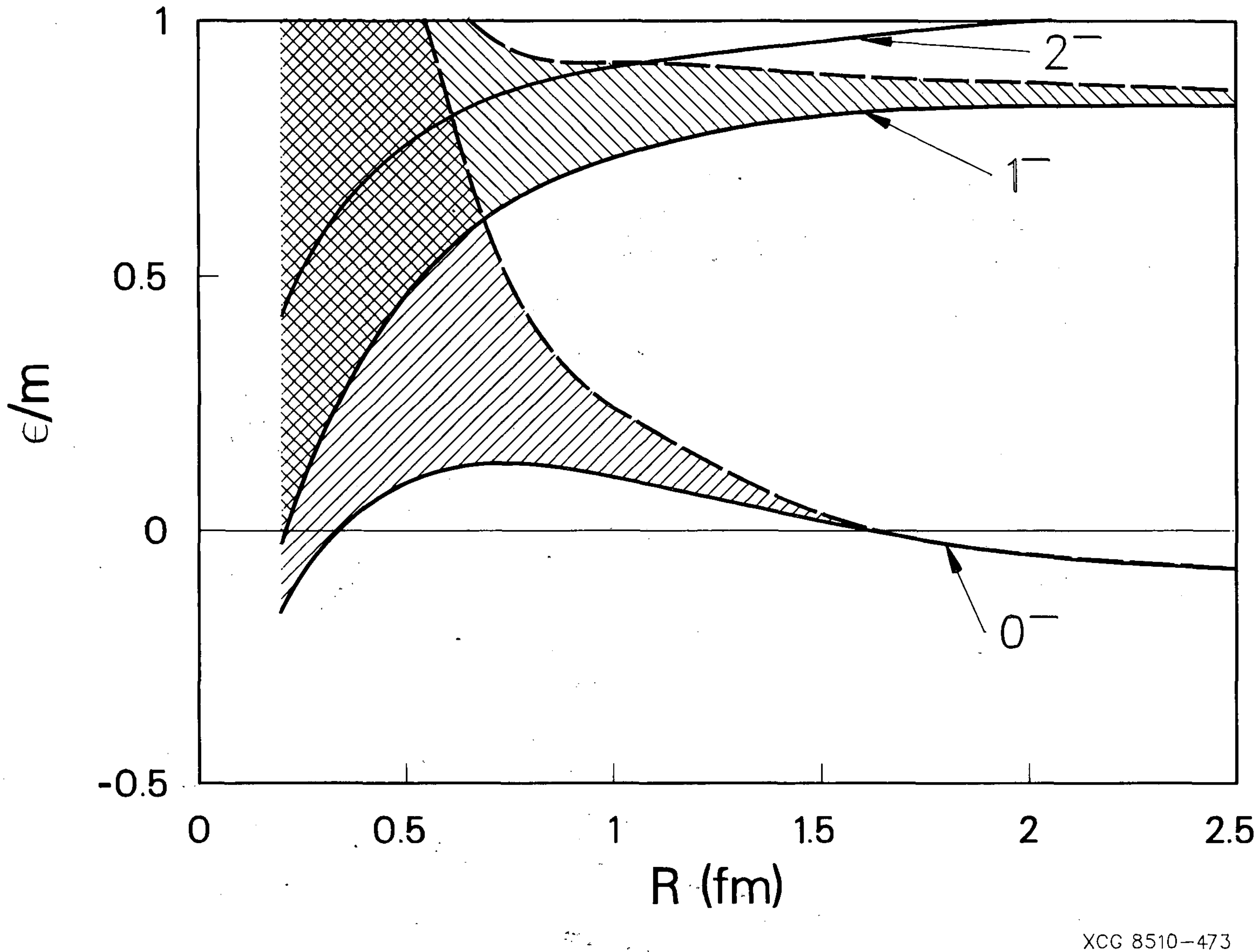


FIG. 10. Bands are shown for several of the levels of Fig. 9 as a function of lattice spacing (2R).

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