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VORTEX OPERATORS IN GAUGE FIELD THEORIES

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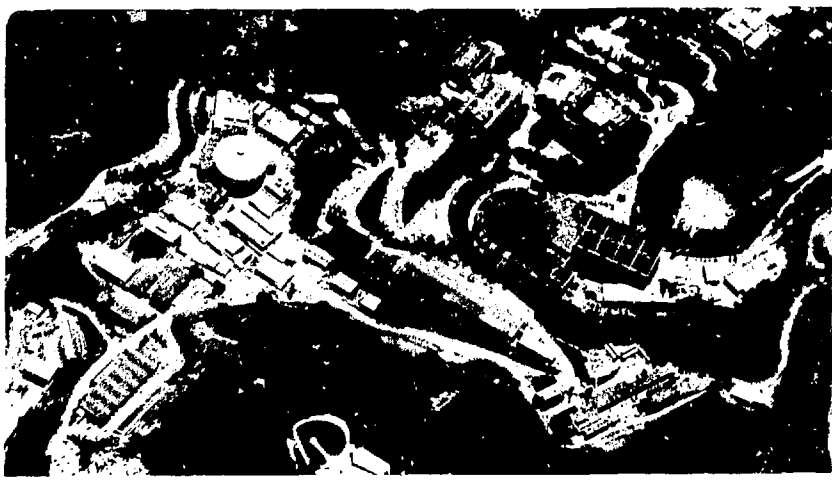
(Ph.D. thesis)

July 1980



Prepared for the U.S. Department of Energy under Contract W-7405-ENG-48

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ABSTRACT

We study several related aspects of the 't Hooft vortex operator. The first chapter reviews the current picture of the vacuum of quantum chromodynamics, the idea of dual field theories, and the idea of the vortex operator.

The second chapter deals with the Abelian vortex operator written in terms of elementary fields and with the calculation of its Green's functions. The Dirac veto problem appears in a new guise. We present a two dimensional "solvable model" of a Dirac string. This leads us to a new solution of the veto problem; we discuss its extension to four dimensions. We then show how the Green's functions can be expressed more neatly in terms of Wu and Yang's geometrical idea of "sections". The renormalization of the Green's functions of two kinds of Abelian looplike operators, the Wilson loop and the vortex operator, is studied. In each case the possible divergences are easily determined with the aid of the operator product expansion, and for both operators only an overall multiplicative renormalization is needed. In the case of the vortex this involves a surprising cancellation.

In the third chapter we discuss the dependence of the Green's functions of the Wilson and 't Hooft operators on the nature of the vacuum. We emphasize the cluster properties of the Green's functions rather than the vacuum expectation values. We explain 't Hooft's

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result relating the expectation values of the Wilson and 't Hooft operators. We then show that the vortex operator in a massive Abelian theory always has surface-like clustering, and we see how this appears in a graphical expansion. We emphasize that the form of Green's functions in terms of Feynman graphs is the same in Higgs and symmetric phases, and that the difference appears in the sum over all tadpole trees.

In the fourth chapter we consider systems which have fields in the fundamental representation, so that there are no vortex operators. When these fields enter only weakly into the dynamics, as is the case in QCD and in real superconductors, we would expect to be able to define a vortex-like operator. We show that any such operator can no longer be "local looplike", but must have commutators at long range. We can still find an operator with useful properties; its cluster property, though more complicated than that of the usual vortex operator, still appears to distinguish Higgs, confining and perturbative phases. to test this, we consider a U(1) lattice gauge theory with two matter fields, one singly charged (fundamental) and one doubly charged (adjoint). When the fundamental field is weakly coupled, we find the expected phase transitions. When it is strongly coupled, our operator still appears to be a good order parameter; a discontinuous change in its behavior leads us to find a new phase transition. We give some discussion of how operators can be good or bad order parameters.

Acknowledgements

I would like to thank my thesis advisor, Professor Stanley Mandelstam, for much guidance and instruction. I am grateful to Professor Korkut Bardakci for many helpful and enjoyable conversations, and to Bob Cahn for advice and encouragement. I would like to collectively thank everyone in the LBL and U. C. Berkeley theory groups, especially the members of the 1978-79 and 1979-80 study groups and softball teams, for fun times and for helping me to improve my understanding of physics.

I am grateful to the Fannie and John Hertz Foundation for five years of generous support, and to my parents for their continual support and encouragement. Finally, I would like to thank my fiancee and sweetheart, Dorothy Chun, for love and for sharing with me the various crises of graduate school.

This work was supported by the U. S. Department of Energy under Contract No. W-7405-ENG-48.

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CHAPTER 1

INTRODUCTION

A. The Vacuum of Quantum Chromodynamics

In the years since the discovery of asymptotic freedom, confidence that Quantum Chromodynamics might be the complete theory of the strong interactions has grown steadily. At high energy, although quantitative results have been limited by the size of the effective coupling constant and the need to factor out incalculable long-distance effects, the qualitative features of QCD appear to be correct[1]. A recent important success was the observation of three jet events, attributed to gluon bremstrahlung, at PETRA. At low energy, various theoretical approaches have shed light on the nature of the QCD vacuum[2], and there is evidence to support the long-standing hope that quarks are confined in QCD.

There is now a standard picture of the QCD vacuum. It appears that fluctuations involving the creation of quark-antiquark pairs can be neglected as a first approximation, so that hadrons can be treated as first-quantized quarks in interaction with a second quantized quarkless SU(3) gauge theory. The justification for this is empirical[3]: the success of the OZI rule and the fact, supported by hadron spectroscopy and by measurement of parton distributions, that baryons are predominantly qqq (not $qqqq\bar{q}$ or $qqqqq\bar{q}\bar{q}$) and that pions are predominantly $q\bar{q}$ (not $qq\bar{q}\bar{q}$, etc.). The theoretical basis for this is uncertain. It is a feature of SU(N) QCD in the large N limit, but the belief that SU(3) QCD resembles SU(N) QCD is in fact based on comparison with phenomenology, not on a calculation of the 1/N corrections.

The appearance of the pure SU(3) vacuum depends on the inverse distance scale k at which it is probed[4]. A process such as e^+e^- annihilation to hadrons at c.m. energy E probes the vacuum structure at scale k=E, while measurement of thermodynamic quantities in a gauge theory at temperature T probes the vacuum at scale k=T. At short distance, $k \gg \Lambda_{QCD}$, (Λ_{QCD} is the scale parameter of QCD, approximately 400 Mev), perturbation theory is accurate and the vacuum is dominated by configurations near the configuration of least classical action ($A_\mu = 0$ in most gauges). On a somewhat longer scale, other classical configurations (instantons) give non-negligible effects[5,6], but the functional integral is still dominated by a small class of configurations which are near classical solutions. On a still longer scale, $k < \Lambda_{QCD}$, both the perturbative and instanton calculations have uncontrolled divergences.

A different approach, the lattice approximation, gives a picture of the gauge theory vacuum at extremely long distance. At this scale, the vacuum looks like a coherent superposition of every possible configuration. In the Hamiltonian lattice formulation[7] this is signalled by the fact that the vacuum is an eigenstate of $\vec{E}$, the operator conjugate to the vector potential. In the Euclidean formulation[8] it is signalled by the fact that $\frac{1}{g^2}$, the coefficient of the action in the functional integral, goes to zero, so that configurations contribute purely according to entropy (volume in function space), without regard to action. In both formulations, the Wilson loop satisies the area law, implying a linear potential between quarks. The important recent advance in this picture has been the

improved calculation of the lattice beta function, by both analytic[9] and Monte Carlo[10] methods, indicating that the only UV fixed point is at zero coupling. This implies that as the lattice spacing is taken to zero, the lattice approximation turns smoothly into the continuum theory studied in perturbation theory. Qualitative features such as quark confinement would then persist in the continuum theory, and the ratio of the string tension to Λ_{QCD} has also been calculated and found to be of the correct order of magnitude.

The infrared divergences of the perturbative and instanton calculations can then be understood as a consequence of expanding around a qualitatively wrong vacuum, analogous to the infrared divergences which appear when one expands a two dimensional sigma model around the asymmetric vacuum[11], which is known by Coleman's theorem[12] to be incorrect. It would be desireable to have an approximation which starts with the correct sort of vacuum, that is, one dominated by entropy rather than by a few classical configurations in the infrared. The lattice approximation has this property, and is very useful, but the information it can give is limited by its crudeness. A different approach is suggested by the idea of duality. A wide class of lattice theories, namely Abelian (U(1) or Z_n) gauge and generalized gauge theories, in any number of dimensions, can be rewritten in terms of a dual gauge theory[13]. The coupling constant of the dual theory is roughly the inverse of the original coupling constant. Thus, if one theory is entropy dominated (large coupling) and therefore difficult to treat, the other will have a small coupling and can be expanded around the configurations of least action. This suggests a search for a dual description of continuum non-Abelian gauge

theories.

The idea of duality can also be reached from a different point of view[14]. Magnetic flux cannot penetrate a U(1) Higgs vacuum (such as a superconductor), but is squeezed into tubes, within which the vacuum is normal (unbroken) and flux can exist. If a monopole and antimonopole were placed in a superconductor, the flux between them would form a tube, with an energy proportional to its length, so that a linear potential would bind the monopole and antimonopole. This is exactly the behavior expected for the QCD vacuum, except that (color) electric rather than magnetic sources are to be confined. The QCD vacuum might then be expected to look like a colored superconductor, but with electric and magnetic quantities interchanged, so that it would be a coherent state of color-magnetic monopoles where the superconductor is a coherent state of electric charges (Cooper pairs). Duality in a gauge theory involves just this interchange of electric and magnetic quantities. Further, in non-Abelian gauge theories, monopoles[15] or monopole-like configurations[16] occur automatically; they need not be put in by hand as in the Abelian Dirac theory[17].

't Hooft[18] and Mandelstam[14] have suggested a kind of operator, the vortex operator, in terms of which this duality might be described. This thesis is concerned with various properties of these operators in continuum field theories. In the remainder of this chapter, we introduce the vortex operator and discuss some of its general properties. In Chapter 2 the Abelian vortex operator is written in terms of elementary fields. We emphasize a simple form in which it is the exponential of a surface integral, and we explore in

detail the requirement that the operator be independent of the particular surface chosen. The Dirac veto problem, known from magnetic monopole theory, is encountered, and we suggest a new solution to this problem. We then discuss the renormalization of Green's functions of vortex operators, and a nice cancellation of divergences is found. For comparison, the renormalization of another looplike operator, the Abelian Wilson loop, is also studied. Chapter 3 deals with the Green's functions of vortex operators at long distance, their cluster properties. We describe the various possible phases of a gauge theory in terms of the cluster properties of vortex and Wilson operators, and explain a result of 't Hooft[18] which restricts the possible phases. We then show that in any Abelian theory without magnetic monopoles and with no physical massless particle, the vortex operator must have a surface-like cluster property and so satisfy an area law. Finally, we show how this cluster property emerges from the graphical expansion in an Abelian Higgs theory.

For one to be able to define a vortex operator, the fields in the theory must satisfy a quantization condition (Π_1 for the gauge group must be non-trivial); QCD without quarks satisfies this condition but the introduction of quarks violates it. In Chapter 4 we examine the case where the quantization condition is not satisfied. We show that any vortex-like operator will no longer have the important property of being "local looplike". Nevertheless, in an Abelian theory we are able to find an operator with some reasonable properties: the energy that it creates is infrared finite, and its cluster property distinguishes a perturbative Higgs phase from a QED-like phase. We speculate on the properties of this operator in non-

Abelian and confining theories, and on the resolution of an apparent paradox involving a result of 't Hooft. We then examine a lattice model where the quantization condition is not satisfied. We find that our extension of the vortex operator is a good order parameter and correctly indicates that a Higgs/confining phase transition is taking place, when the field which violates the quantization condition is weakly coupled. Some preliminary results indicate that our operator is also a good order parameter when this field is strongly coupled, in the sense that discontinuous behavior of the operator implies a discontinuity in the phase diagram; it leads us to find a phase transition that might not have been expected. Chapter 5 summarizes our findings.

In summary, Chapter 2 deals with technical details related to the Green's functions of vortex operators. Chapter 3 discusses known relations between the behavior of the vortex operator and the nature of the vacuum, but with the emphasis on cluster property rather than vacuum expectation value. Chapter 4 then uses some of these elements to investigate systems where vortex operators were not known to be useful, and some light is shed on the phase structure of these systems.

B. Vortex Operators

The vortex operator, like the Wilson loop, is associated with a closed curve in spacetime. Let us consider a general "local loop-like" operator L associated with a closed curve C lying in the R^3 plane $t=t_o$, and let us consider its commutation relations with other operators at time t_o. By definition, L commutes with observable

(that is, gauge invariant) local operators associated with a point $\vec{x}$ not on C. This implies that for a gauge dependent field $\Psi(\vec{x})$

$$L\Psi(\vec{x}) = \Psi(\vec{x})^{g_L} \tag{1.1}$$

where $\Psi(\vec{x})^g$ is Ψ after a gauge transformation $g(\vec{x})$ and $g(\vec{x})$ is a gauge transformation associated with the operator L. For instance, in the usual gauges (axial, covariant, Coulomb), the gauge transformation associated with the Wilson loop is simply 1 everywhere.

It is important that $g(\vec{x})$ is not defined in all of space, R^3, but only in $R^3 - C$. $R^3 - C$ is multiply connected: curves are distinguished by their winding number around C. As a result there can be a non trivial effect, a topology, associated with $g(\vec{x})$. As $\vec{x}$ describes a path around C, $g(\vec{x})$ describes a path through the gauge group. If the gauge group is simply connected, there is nothing interesting about this, but if it is multiply connected the path may lie in a non-trivial element of $\Pi_1(G)$[20]. For different paths of the same winding number, continuity requires that this element be the same. Since paths of winding number other than one can be generated as a product of paths of winding number one (traversed backwards for negative winding number) the homotopy class associated with winding number one determines that for any other path. The class associated with L is gauge invariant: since any gauge transformation can be continuously deformed into one which is unity in an arbitrarily large volume (including the whole of L) so the gauge transformed $g(\vec{x})$ can be continuously deformed into its original value; homotopic invariants are therefore unchanged. Thus, there is a gauge invariant quantum number, a homotopy class, associated with any looplike operator.

Operators for which this class is not the trivial one are called vortex operators. The name comes from the Nielsen-Olesen vortex[21], a classical solution in a Higgs theory associated with twisted boundary conditions for the Higgs field. In a completely broken Higgs theory the set of classical vacua is identical to G, so that there will be solitons with quantum number in $\Pi_1(G)$[20]. These are the Nielsen-Olesen vortices, and when a vortex operator acts on the vacuum of the Higgs theory it produces a state containing such a soliton.

In an Abelian gauge theory, a vortex operator creates a loop of magnetic flux just as the Wilson operator creates a loop of electric flux. In the dual (Abelian) gauge theories mentioned earlier, the Wilson loops of one theory are mapped into vortex operators of the other. In the long distance, large coupling, limit of non-Abelian (as well as Abelian) gauge theories, the vacuum approaches an eigenstate of a simple vortex operator. Thus, they are attractive operators to consider. If one tries instead to consider duality in terms of pointlike, monopole creation, operators, one finds that there is no associated topological quantum number: $\Pi_2(G)$ is trivial for any Lie gauge group G. In this, there is an interesting analogy between electric and magnetic quantities: the pointlike operators in non-Abelian theories (gauge fields or monopoles) have no gauge or topological invariance, while the looplike operators (Wilson loops or vortices) do.

The quantum number associated with a vortex can be characterized in a different way. Consider the path described by $g(\vec{x})$ as $\vec{x}$ winds once around C. This path in G defines in a natural way $(g^{-1}\vec{\partial}g=\tilde{g}^{-1}\vec{\partial}\tilde{g})$ a path in the simply connected covering group $\tilde{G}$. By a

well-known connection between $\Pi_1(G)$ and the center of $\tilde{G}$[20], when $\vec{x}$ returns to its starting position $\tilde{g}(\vec{x})$ need not return to its original value but is multiplied by an element Z_L of the center of $\tilde{G}$. Because $g(\vec{x})$ is single valued, Z_L in $\tilde{G}$ is mapped into 1 in G by the usual homomorphism. Vortex operators can thus be considered to have quantum numbers in $Z(\tilde{G})/Z(G)$, the quotient of the centers of the two groups, which is isomorphic to $\Pi_1(G)$.

One can show from equation 1.1 that

$$L(C)W_s(C') = W_s(C')L(C)(z_{sL})^{w(C,C')} \tag{1.2}$$

where Z_L in representation s is z_{sL} times the identity, $w(C,C')$ is the winding number of C' through C, and $W_s(C')$ is the Wilson loop in representation s associated with curve C' (assumed here to lie in the $t=t_o$ hyperplane):

$$W_s(C') = \mathrm{Tr}\ P \exp(ig\oint_{C'} dx_i A_i^a(x) y_s^a) \tag{1.3}$$

with y_s^a the generator a in representation s, Tr representing the trace, and P representing the usual path ordering. Thus, although L(C) commutes with every <u>local</u> gauge invariant not on C, if it is a vortex operator it will not commute with certain Wilson loops linking C. Equation 1.2 characterizes completely the topological character of the vortex operator; it is only Z_L (or z_{sL} for all s), not $g(\vec{x})$ that can be defined in a gauge invariant way. Note that equation 1.2 is entirely dual between L(C) and W(C'); one cannot say that one is a topological operator and the other is not. The asymmetry arises only when one introduces gauge dependent quantities. The one genuine asymmetry is that the Hamiltonian is relatively simple in terms of

the Wilson loops (or the related vector potentials), but does not appear to have a simple form in terms of vortex operators; this asymmetry may or may not be permanent.

For a field ψ in representation s_ψ, equation 1.1 becomes

$$L(C)\psi(\vec{x}) = g_\psi(\vec{x})\psi(\vec{x})L(C) \qquad (1.4)$$

where $g_\psi(\vec{x})$ is $g(\vec{x})$ in representation s_ψ. In order that equation 1.4 be well-defined, it is necessary that $z_{s_\psi L}$ be 1. This is indeed the case, as Z_L in $\tilde{G}$ maps into 1 in G and s_ψ is single valued in G (by definition, the gauge group G is the smallest group with the Lie algebra of $\tilde{G}$ such that every field in the theory lies in a single valued representation). If we add new fields to the theory which are in representations s' not single valued in G (and thus, by definition, enlarge the gauge group to G') those vortices for which $z_{s'L}$ is not 1 no longer make sense. In other words, $\Pi_1(G')$ is smaller than $\Pi_1(G)$. $z_{s_\psi L}=1$ for all fields ψ is the quantization condition referred to above. Note that it is equivalent to saying that vortex operators must commute with W_{s_ψ} for all fields ψ in the theory.

We can illustrate these ideas for the group $SU(n)/Z_n$, an SU(n) gauge theory with all fields in the adjoint representation. Π_1 for this group is Z_n, so that the vortices carry an n-fold quantum number, p(L). The representions also carry an n-fold quantum number q(s), the "quark number" of the representation: if s lies in the product of m fundamental representations, $m=q(s)_{\text{modulo } n}$. Equation 1.2 becomes

$$L(C)W_s(C') = W_s(C')L(C)\exp\{2\pi i p(L)q(s)w(C,C')/n\} \qquad (1.5)$$

We see that the SU(n) gauge theory without quarks (whose gauge group is really $SU(n)/Z_n$) has vortex operators, while if quarks are added the gauge group becomes SU(n), which is simply connected, and vortex operators can no longer be defined. More will be said about this in Chapters 3 and 4.

The short distance properties of the vortex operator, near C, are still unspecified; the vortex operators are a class, not a single operator. Certain results, such as 't Hooft's concerning possible phases, are independent of the short distance properties. Others, in particular those related to dynamics and renormalization, depend on the details near C. In this thesis, Chapters 3 and 4 are primarily concerned with long distance properties, independent of the detailed form of the vortex operator. Chapter 2 deals with some of each.

CHAPTER 2

THE ABELIAN VORTEX OPERATOR

A. The Vortex Operator in Terms of Elementary Fields

We first review the ideas from Chapter 1 for a U(1) theory. The covering group of U(1) is R^1; a general element of R^1 is a real number y. In the representation of charge e, y becomes exp(iye). A general looplike operator L(C) is then associated, through equation 1.1, with a function $y(\vec{x})$, the R^1 version of $\tilde{g}(\vec{x})$. Like $\tilde{g}(\vec{x})$, $y(\vec{x})$ need not be single-valued, but $\exp\{iy(\vec{x})e\}$ must be if fields of charge e are present. It follows that when $\vec{x}$ winds once around C, $y(\vec{x})$ must change by $2\pi p/e_{min}$, where e_{min} is the unit of charge and p is any integer. Vortex operators are thus characterized by an integer p. Equation 1.2 is now

$$L(C)W_q(C') = W_q(C')L(C)\exp\{ipqw(C,C')/e_{min}\} \tag{2.1}$$

where W_q is the Wilson loop

$$W_q(C') = \exp\{iq\oint_{C'} dx_i A_i(x)\} \tag{2.2}$$

For those fields actually present, q is a multiple of e_{min} and the phase factor in equation 2.1 is 1.

Differentiating equation 2.1 with respect to q and setting q=0, and then using Stokes's theorem to relate the line integral of $\vec{A}$ to the surface integral of the magnetic field $\vec{B}$,

$$L(C)B_i(\vec{x}) = (B_i(\vec{x}) + \frac{2\pi ip}{e_{min}}\oint_C dx_i'\delta^3(\vec{x}-\vec{x}'))L(C) \tag{2.3}$$

Although equation 2.3 refers to a commutator directly on C, it fol-

lows from equation 2.1 and is therefore true for any vortex operator independent of its short distance details. This is the source of the statement that vortex operators create a loop of magnetic flux; the dual equation, replacing L by W_e, $\vec{B}$ by $\vec{E}$ and $2\pi p/e_{min}$ by e is also true. There is no corresponding local version of equation 1.2 for a non-Abelian theory. Equation 1.2 might then be taken as a definition of non-Abelian magnetic flux[18].

An operator satisfying equation 2.1 is

$$V_p(C) = \exp\{\frac{ip}{e_{min}}\int d^3x \frac{\hat{\theta}}{\rho}\cdot\vec{E}(\vec{x})+\theta j_0(\vec{x})\} \tag{2.4}$$

where θ,ρ, and z are cylindrical coordinates, $\hat{\theta}$ is a unit vector in the θ direction, and j_0 is the charge density. From the canonical commutators (we take the $A^0=0$ gauge for convenience, but V(C) is gauge invariant and so will be its commutators with gauge invariants such as equation 2.1)

$$[E_i(\vec{x}),A_j(\vec{y})] = i\delta_{ij}\delta^3(\vec{x}-\vec{y}) \tag{2.5a}$$

$$[j_0(\vec{x}),\psi(\vec{y})] = -e_\psi\delta^3(\vec{x}-\vec{y})\psi(\vec{y}) \tag{2.5b}$$

one finds

$$V_p(C)\psi(\vec{x}) = \exp[\frac{-ip\theta e_\psi}{e_{min}}]\psi(\vec{x})V_p(C) \tag{2.6a}$$

$$V_p(C)A_i(\vec{x}) = \{A_i(\vec{x}) + i\exp\{-ip\theta/e_{min}\}\partial_i\exp\{ip\theta/e_{min}\}\}V_p(C) \tag{2.6b}$$

$$V_p(C)E_i(\vec{x}) = E_i(\vec{x})V_p(C) \tag{2.6c}$$

This is the Abelian form of equation 1.1. The curve C is here the

z-axis; $g(\vec{x})$ is $\exp[-ip\theta]$. A vortex operator for any C and any $g(\vec{x})$ can be constructed in the same way.

The coordinate θ must have a discontinuity of 2π on a semi-infinite surface S bounded by the z-axis (such as the half-plane $\theta=0=2\pi$). The exponent in the definition of $V_p(C)$ is therefore discontinuous but the operator itself has no discontinuity on the surface, as can be seen from its commutators. These are completely independent of where we choose to define the discontinuity of θ. Using $\partial\theta=\frac{\hat{\theta}}{\rho}$ + disc(θ), $V_p(C)$ can be rewritten

$$V_p(C) = \exp\{\frac{ip}{e_{min}}[\int d^3x\ (j_0(\vec{x})-\partial_i E_i(\vec{x}))\theta + 2\pi\int_S dn_i E_i\ ]\} \qquad (2.7)$$

$j_0(\vec{x})-\partial_i E_i(\vec{x})$ does not vanish as an operator in the $A^0=0$ gauge, but by Gauss's law it vanishes in gauge invariant Green's functions. The gauge invariant Green's functions are therefore the same for $V_p(C)$ as for

$$V_p'(C) = \exp\{\frac{2\pi ip}{e_{min}}\int_S dn_i E_i\} \qquad (2.8)$$

If one evaluates the commutator of $V_p'(C)$ with $\theta(\vec{x})=\psi^*(\vec{x})D_i\psi(\vec{x})$ (D_i is the covariant derivative $\partial_i-ie_\psi A_i(\vec{x})$), it does not appear to vanish on S; it must, however, because $[V_p(C),\theta(\vec{x})]=0$ on S and θ is gauge invariant. The problem is that V' is too singular for canonical commutators to be correct; if one evaluates the Green's functions of $V_p'(C)\theta(x)$ using the methods of the next section, one finds that they have no equal time discontinuity on S. V_p' is a more convenient form of the operator when one is discussing Green's functions.

One might wonder whether the commutators 2.6 are really correct in field theory, even for V, or whether some anomaly will develop due

to the discontinuity of Θ. V and V' are poorly defined because the discontinuity of Θ is sharp; if we define them by smearing the discontinuity and taking the smearing to zero, does the limiting operator satisfy 2.6? A simple example shows this to be a valid worry. Consider a theory with fermions, ψ, in two spacetime dimensions (they may even be free fermions), and consider the operator

$$\Omega(x) = \exp\{\int_{x_1}^{\infty} dx'_1 \; 2\pi i \bar{\psi}(x')\gamma_0\psi(x')\} \tag{2.9}$$

where x is the spacetime point (x_0,x_1) and the integral runs along the equal time path from (x_0,x_1) to (x_0,∞). From the commutator

$$[\bar{\psi}(y)\gamma_0\psi(y),\psi(z)]\delta(x_0-y_0) = -\delta^2(y-z)\psi(y) \tag{2.10}$$

we find, by the same canonical manipulations that lead from 2.5b to 2.6a, that $[\Omega(x),\psi(z)]=0$ at equal times, so that $\Omega(x)$ would be a c-number. On the other hand, from boson equivalence[22],

$$\begin{aligned}\Omega(x) &= \exp\{\int_{x_1}^{\infty} dx'_1 \; -2\sqrt{\pi}i\partial_1\phi(x')\} \\ &= \exp\{2\sqrt{\pi}i\phi(x)\} \\ &= \text{const}\cdot\bar{\psi}(x)(1+\gamma_5)\psi(x)\end{aligned} \tag{2.11}$$

If the fermions are massive, the leading piece of $\bar{\psi}(1+\gamma_5)\psi$ is in fact a c-number, but if they are massless, $\bar{\psi}(1+\gamma_5)\psi$ has no c-number piece and $[\Omega,\psi]$ is not identically zero.

One may also see this in a different way by evaluating

$$G(x,y,z) = \langle\, \Omega(x)\; \bar{\psi}(y)\; \psi(z)\,\rangle \tag{2.12}$$

for free fermions. G(x,y,z) is poorly defined because the support in

the exponent in equation 2.9 is too singular. If one smears the support (necessarily into the time direction) over a small distance λ (call the resulting operator Ω_λ and the Green's function G_λ), we can calculate $G_\lambda(x,y,z)$ directly. As λ goes to zero, for free massive fermions $G_\lambda(x,y,z)$ approaches $< \bar{\psi}(y)\,\psi(z) >$ times a constant, while for massless fermions it approaches

$$< \bar{\psi}(x)(1+\gamma_5)\psi(x)\;\bar{\psi}(y)\;\psi(z) >$$

times a constant.

This is not a serious problem. By regulating Ω in a slightly more complicated way, one may obtain the desired limit. Consider the operator

$$\bar{\psi}(x)(1+\gamma_5)\psi(x)\Xi_\lambda(x)$$

where

$$\Xi_\lambda(x) = \exp\{i\int d^2z\; f_\lambda(z)\bar{\psi}(x+z)\psi(x+z)\} \tag{2.13}$$

and $f_\lambda(z_0,z_1)$ is a family of functions with support in $|z| < \lambda$. If f_λ is defined so that as $\lambda\to 0$,

$$\int d^2z\; f_\lambda(z)/|z|^2 \to \infty \tag{2.14a}$$

$$\int d^2z\; f_\lambda(z)/|z| \to 0 \tag{2.14b}$$

then as $\lambda\to 0$, $\bar{\psi}(1+\gamma_5)\psi\,\Xi_\lambda$ approaches a c-number even for massless fermions. If one then defines a regulated $\Omega(x)$ as $\Omega_\lambda(x)\Xi_\lambda(x)$, its limit will be a c-number. Equations 2.14 are correct for free fermions or with a super-renormalizeable interaction; as one might expect, with a Thirring interaction there would have to be different

powers of z in the integrands.

It is interesting to repeat some of the above analysis for the Schwinger model, without using bosonisation. Using Gauss's law, we get

$$\Omega_\lambda(x) = \exp\{-\frac{2\pi i}{e}\int d^2z\ \delta_\lambda^2(z)E(x+z)\} \qquad (2.15)$$

where $\delta_\lambda^2(z)$ is the delta function, smeared over a distance λ in spacetime, and E, the electric field, is a (pseudo-)scalar in two dimensions. The analogy between this and equation 2.8 is clear.

We would like to compare

$$G(c,y) = \int_c dx_\mu \epsilon_{\mu\rho} < j_{5,\rho}(x)\ \Omega_\lambda(y) > \qquad (2.16)$$

with the same quantity with 1 in place of Ω_λ. Here $j_{5,\rho}$ is the axial current $\bar{\psi}\gamma_5\gamma_\rho\psi$ and c is a spacetime curve circling the spacetime point y at some distance large compared to λ but small compared to $1/e^2$ (which has dimensions of length). Because of the latter stipulation, we can neglect all but the leading term from perturbation theory.

Using the anamolous divergence equation,

$$\partial_\mu j_{5,\mu}(x) = -\frac{e}{\pi}E(x) \qquad (2.17)$$

we can rewrite equation 2.16 as

$$G(c,y) = \frac{e}{\pi}\int_r d^2x < E(x)\ \Omega_\lambda(y) > \qquad (2.18)$$

From $<E(x)E(y)>=\delta^2(x-y)+O(e^2)$, it follows that

$$< E(x)\ \Omega_\lambda(y) > = -\frac{2\pi i}{e}\ \delta_\lambda^2(x-y) + O(e) \qquad (2.19)$$

G(c,y) is then simply -2, independent of λ, whereas when Ω is

replaced by 1 in the matrix element, 2.18 is zero: the limit of Ω_λ as λ goes to zero is not a c-number. Ω has much in common with a Dirac string; equation 2.19 shows that there is a lump of flux which does not go away when λ goes to zero. Our result here is that a massive fermion does not feel this flux in the limit, whereas a massless fermion does, but that a massless fermion can be "shielded" from the flux by the additional regulator 2.9. Massive and massless bosons appear to behave like massive fermions: The Green's function analogous to 2.12 goes to the free propagator in the limit.

Our results for the four dimensional case are not so complete. We discuss them, and more of the analogy with the Dirac string, in the next section.

B. Green's Functions of Vortex Operators

In this section we will discuss the calculation of Green's functions of the V'(C) form of the vortex operator in Euclidean spacetime. For simplicity we shall assume only one charged field, scalars $\phi(x)$ with charge e, mass m, and four point interaction λ; the generalization to more fields and to fermions is straightforward. We can now consider a vortex operator associated with a general closed curve in spacetime. By analogy to equation 2.8 we define

$$V_p'(C,S) = \exp\{\frac{mp}{2e}\int_S d\sigma^y_{\alpha\beta} F_{\gamma\delta}(y) \epsilon_{\alpha\beta\gamma\delta}\}$$

$$= \exp\{\frac{mp}{e}\int_S d\sigma^y_{\alpha\beta} \tilde{F}_{\alpha\beta}(y)\} \qquad (2.20)$$

where S is any 2-surface whose boundary is C and $\epsilon_{1234}=1$. There is no factor of i in the exponent because $F_{j4}=iE_j$ in Euclidean spacetime. To show that 2.20 is correct, we would like to check that it

satisfies equation 2.1, and that its gauge invariant Green's functions are independent of the surface S.

Consider now a general Green's function of V':

$$< V_p'(C,S) >_{JKL} = \tag{2.21}$$

$$< V_p'(C,S)\ \exp\{\int d^4x J_\mu(x)A_\mu(x)+K(x)\phi(x)+L(x)\phi^*(x)\} >$$

where J_μ, K, and L are general external sources. The expansion of V' involves terms with arbitrarily large powers of 1/e, but owing to the exponential form of V', these sum up in a convenient way:

$$< V_p'(C,S) >_{JKL} = \exp \sum_{khmn} \frac{1}{k!h!m!n!} \tag{2.22}$$

$$< [\frac{ne}{2}\int_S d\sigma^\gamma_{\alpha\beta}\tilde{F}_{\alpha\beta}(y)]^k [\int d^4x\ J_\mu A_\mu]^h [\int d^4x\ K\phi]^m [\int d^4x\ L\phi^*]^n >_c$$

where $< >_c$ indicates the connected Green's function. The connected Green's function is of order at least e^{k+h} (e being the charge), except for k+h=2, m=n=0, for which it starts at order e^0. The sum in 2.22 therefore starts at order e^{-2}. Further, the graphs contributing at each order in e are readily classified: a graph with n_i internal photon lines (internal means both ends connected to charged lines) and n_e external lines (one end attached to a charged line and one attached to J_μ) is of order $e^{2n_i+n_e}$, independent of the number of vortex photon lines (defined as lines which run from S, the vortex operator, to a charged line). The connected graph with no charged lines (one photon line with both ends attached to S) is of order e^{-2}.

For example, the h=m=n=0 term, which is independent of J_μ, K, and L, and is the only term which contributes to $< V_p'(C,S) >$, is

$$< V_p'(C,S) > = \exp\{ \frac{2\pi^2 p^2}{e^2}\delta^2(0)A(S) \tag{2.23}$$

$$- \frac{p^2}{2e^2}\oint_C dx_\mu \oint_C dy_\mu (x-y)^{-2} - \ln \det\{\Delta^{vS}/\delta^2\} + O(e^2)\}$$

where A(S) is the area of the surface S, Δ^{vS} is $\{D_\mu^{vS} D_\mu^{vS}\}^{-1}$, D_μ^{vS} is $\partial_\mu - ieA_\mu^{vS}(x)$ and $A_\mu^{vS}(x)$ is given by the graph of figure 1a. The first two terms are from the one photon graph. The third is the sum of all graphs with one charged line connected to S by any number of photon lines. The first term is S dependent and quadratically divergent. We shall postpone further discussion of 2.23 until the end of this section. We will consider until then "reduced" Green's functions, with the h=m=n=0 term divided out.

To investigate the dependence on S of gauge invariant reduced Green's functions, consider first the order e^0 term of $< F_{\mu\rho}(x)\, V_p'(C,S) > / < V_p'(C,S) >$, which is equal to

$$\partial_{(\mu} A_{\rho)}^{vS}(x) = F_{\mu\rho}^{vS}(x) \tag{2.24}$$

(Parentheses on subscripts indicate antisymmetrization). A_μ^{vS} was defined above; 2.24 defines $F_{\mu\rho}^{vS}$ as its curl.

$$F_{\mu\rho}^{vS}(x) - F_{\mu\rho}^{vS'}(x) = \frac{\pi p}{2e}\int_{S-S'} d\sigma_{\alpha\beta}^y \epsilon_{\alpha\beta\gamma\delta} < F_{\gamma\delta}(y)\, F_{\mu\rho}(x) >_0$$

$$= -\frac{\pi p}{e}\int_{S-S'} d\sigma_{\alpha\beta}^y [\, \epsilon_{\alpha\beta\gamma\rho}\partial_\mu^y + \epsilon_{\mu\alpha\beta\gamma}\partial_\rho^y \,]\partial_\gamma^y \Delta_E(x-y) \tag{2.25}$$

where $< >_0$ indicates the free propagator and Δ_E is the Euclidean scalar propagator:

$$\partial^2 \Delta_E(x-y) = -\delta^4(x-y) \tag{2.26}$$

Using the fact that a completely antisymmetric 5-tensor vanishes in

four dimensions, the quantity in square brackets can be rewritten as

$$\epsilon_{\rho\mu\alpha\beta}\partial^y_\gamma + \epsilon_{\gamma\rho\mu\alpha}\partial^y_\beta + \epsilon_{\beta\gamma\rho\mu}\partial^y_\alpha$$

After surface integration the last two terms vanish, because S-S' has no boundary, while the remaining term gives, using 2.26,

$$F^{vS}_{\mu\rho}(x) - F^{vS'}_{\mu\rho}(x) = -\frac{n_p}{e}\int_{S-S'} d\sigma^y_\alpha \epsilon_{\alpha\beta\mu\rho}\delta^4(x-y)$$

$$= -\frac{2\pi p}{e}\int_{S-S'} d\tilde{\sigma}^y_{\mu\rho}\delta^4(x-y) \qquad (2.27)$$

We see that the Green's function is S-dependent, but only when x lies directly on S or S'. However, when x lies on S, $< F_{\mu\rho}(x)\ V_p'(C,S) >$ is not well defined due to problems of operator ordering. If we resolve this ordering problem by taking a limit as x approaches S, $F^{vS}_{\mu\rho}$ is S independent everywhere. In effect this is taking the T-product, whereas 2.24 defines the T*-product[23]:

$$T[F^v_{\mu\rho}(x)] = T^*[F^{vS}_{\mu\rho}(x)] + \frac{2\pi p}{e}\int_S d\tilde{\sigma}^y_{\mu\rho}\delta^4(x-y) \qquad (2.28)$$

is independent of S.

Now let us go one order further, and consider

$$G(P,C,S) = \qquad (2.29)$$

$$< \phi^*(x)\exp\{ie\int_P dx'_\mu A_\mu(x')\}\phi(y)\ V_p'(C,S) > / < V_p'(C,S) >$$

where P is some path from y to x. To lowest (e^0) order, this gauge invariant Green's function is given by all graphs of the form shown in figure 1b. These graphs sum up to give

$$G(P,C,S) = -\Delta^{vS}(y,x)\exp\{ie\int_P dx'_\mu A^{vS}_\mu(x')\} \qquad (2.30)$$

Using the path integral representation[24] for the propagator $\Delta^{vS}(y,x)$, we can write

$$G(P,C,S) = \exp\{ie\int_P dx'_\mu A^{vS}_\mu(x')\} \qquad (2.31)$$

$$\int \overline{Dz}\, ds\, \exp\{\int_0^s dt - \frac{m^2}{2} - \frac{1}{2}\dot{z}_\mu \dot{z}_\mu + ie\dot{z}_\mu A^{vS}_\mu(z)\}$$

with $\overline{Dz}$ indicating the integral over all paths z(t) such that z(0)=x and z(s)=y, and $\dot{z}_\mu$ is dz_μ/dt. As long as A^{vS}_μ is smooth (we must temporarily smear out the surface S) the path integral can be made mathematically precise[25]; at this level we simply have an external field problem and there are no short distance difficulties. The right hand side of 2.31 depends on A^{vS}_μ only through the phase factors

$$\exp\{ie\oint_{C'} dx'_\mu A^{vS}_\mu(x')\} = \exp\{\frac{ie}{2}\int_{R(C')} d\sigma_{\alpha\beta} T^*[F^{vS}_{\alpha\beta}]\} \qquad (2.32)$$

where C′ is the closed curve formed by joining P with z(t) and R(C′) is any 2-surface whose boundary is C′. To obtain equation 2.32 we had to use equation 2.20, which defines the S-dependent $T^*[F^{vS}_{\mu\rho}]$. We cannot try to simply define this S-dependence away by replacing $T^*[F^{vS}_{\alpha\beta}]$ with $T[F^{v}_{\alpha\beta}]$, as the result would in general depend on the choice of the arbitrary surface R.

From 2.27 and 2.32, it follows that changing the surface on which V_p' is defined changes the phase associated with the path C′ by a factor

$$\exp\{i\pi p\int_{S-S'} d\sigma^x_{\alpha\beta}\int_R d\sigma^y_{\alpha\beta}\delta^4(x-y)\} \qquad (2.33)$$

In four dimensions, a closed curve, C′, links a closed 2-surface, S-S′, a definite, integral, number of times, N_I (leaving out for now

those particle paths which actually intersect S or S'). Any surface R bounded by C' will then intersect S-S' at N_I points (intersections are defined in an oriented way, so that it is the net number of intersections that is counted). Setting up local coordinate systems on S-S' and on R near such an intersection, one finds that the double integral in 2.33 is exactly $2N_I$. The phase then changes by $\exp\{2\pi i p N_I\}$ under the change of surface, which is 1 if p is an integer as required by the earlier quantization condition.

The above argument is extremely familiar: it is just the argument that one can have magnetic monopoles in an Abelian gauge theory, if the monopole charge is quantized so that the charged particles don't "see" the monopole's Dirac string[26]. In fact, just as the Wilson loop can be regarded as the world-line of a classical charged particle, the vortex operator can be regarded as the world-line of a classical monopole, with the surface S as the world-sheet of the Dirac string. There is one problem with the Dirac string, and that problem is also present here: what of paths which actually pass through the surface S? If we smear the integral defining V_p' (or that for V_p) and let the smearing vanish as a limit, can we neglect paths passing through S because they are of "measure zero", or do they contribute in a sufficiently singular way that their effect does not go away?

This is exactly the question that was raised from a different point of view at the end of the last section. In our two dimensional model it was found, in effect, that when the support of our Dirac string (there it was a "Dirac lump") goes to zero with the total flux staying constant, under some conditions (massless fermions) the

effect of paths through the lump survives in the limit, while under others (massive fermions and bosons) the effect vanishes. Even in the massless fermion case it was possible to obtain the desired limit by adding additional regulators to the operator, which can be thought of as correcting the action for paths that pass through the lump.

The four dimensional case is not so easily analyzed. For S an infinite 2-plane (so that C is infinitely far off in some direction) the charged field propagator is easily calculated with the flux smeared, and one finds that it approaches the free propagator for massive or massless fermions, or for bosons. In the presence of this 2-plane plus an additional smooth A_μ-field, however, an argument similar to that for the Schwinger model (based on the anomalous divergence of the axial current) shows that the effect of the Dirac string survives in the limit if the fermions are massless. An extra regulator of the same form as Ξ in 2.15 (essentially giving the fermions a mass very near the string) corrects this particular problem. We have not been able to show, however, that this is the only additional operator needed in general, or that no correction is ever needed for massive fermions or bosons. It seems quite likely, however, that if we do need to correct the action for paths passing through the string, this correction will always take the form of local operators near the string. The whole point is, perhaps, moot, as we shall see that the Green's functions are uniquely determined without knowing the detailed form of the extra regulator (just as in the two dimensional case, we do not really need to know the form of Ξ; once we know that Ω is a c-number, all of its Green's functions are fixed up to an overall constant!).

Many other solutions for the Dirac veto problem have been given. Brandt, Neri, and Zwanziger[26] have rewritten a field theory of charges and monopoles as an integral over all numbers and configurations of particle paths. They then define the action associated with configurations having charge paths intersecting monopole strings as the limit of that for non-intersecting configurations. In a monopole field theory, our solution would have the form of a smeared Dirac string and an additional non-local charge monopole interaction along the string.

In the remainder of this thesis we assume that $V_p'(C,S)$ can be defined in such a way as to have the desired limit, so that $G(P,C,S)$ is independent of S. It then follows that there is a gauge transformation $g(S,S';x)$, defined except when x lies on S or S', such that

$$\Delta^{vS'}(y,x) = g(S,S';y)\Delta^{vS}g^*(S,S';x) \tag{2.34a}$$

$$A_\mu^{vS'}(x) = A_\mu^{vS}(x) + \frac{1}{e}g(S,S';x)\partial_\mu g^*(S,S';x) \tag{2.34b}$$

Any gauge independent quantity constructed from A_μ^{vS} and Δ^{vS} is therefore S-independent as well.

At least in the present case, where there is only a classical monopole, we can evaluate quantities such as $G(P,C,S)$ or $\Delta^{vS}(x,y)$ without using either an explicit form for $V_p'(C,S)$ or the path integral prescription. Equations 2.34 by themselves give sufficient information to determine the Green's functions we want, using a geometrical approach (in the sense of differential geometry) due to Wu and Yang[27]. Consider two non-intersecting surfaces, S_1 and S_2, each having C as its boundary. Take spacetime with $\tilde{C}$, a thin tube

containing C, removed. This space can be covered by two overlapping open regions, Q_1 and Q_2, such that S_1 lies entirely in Q_2, and S_2 lies entirely in Q_1. $A_\mu^{vS_1}(x)$ is defined for x in Q_1, and $A_\mu^{vS_2}(x)$ is defined for x in Q_2; in the overlap region these functions are related by 2.34b. They are given, as before, by the simple one photon graph, and $g(S_1,S_2;x)$ can be determined in terms of this graph. Any other gauge and S-dependent quantity can similarly be represented in as a pair of functions (or, for Δ^v as four functions, since it has two arguments), each defined only in a certain region but related in the overlap region by the gauge transformation $g(S_1,S_2;x)$. Gauge dependent quantities are thus "sections": sets of functions each defined only in an open region, but such that the regions cover all of spacetime (minus $\tilde{C}$ in this case) and such that the functions are related in the overlap of two or more regions.

Integrals of gauge (and therefore S-) independent products of sections over $R^4 - \tilde{C}$ can then be defined: in each region the integrand is defined in terms of the function which exists in that region; in overlap regions this is unambiguous because the integrand is S-independent. With an inner product based on this integral, $D_\mu^v D_\mu^v$ acting on sections can be made self-adjoint. It follows that Δ^v as a section exists and is unique. Δ^{vS} can be obtained from it as a function defined everywhere except on x. Power counting arguments, as developed in the next section, indicate that the limit as the cutoff $\tilde{C}$ is removed exists, at least order by order in p.

We may now consider higher order corrections to Green's functions. All graphs with a given configuration of charged lines and

internal and external photon lines are of the same order in e; summing over all numbers of vortex photon lines gives one graph of the same configuration with no vortex photon lines but with "effective" propagators and vertices. The photon propagator and $\phi'\phi A^2$ vertex are unchanged, and the charged propagator becomes $-\Delta^{vS}(x,y)$. The $\phi^*\phi A_\mu$ vertex picks up an extra term from graphs where a $\phi^*\phi A^2$ vertex is connected to one vortex photon line and one other photon line; it becomes $-ie\tilde{D}_\mu^{vS}(x)$. Equation 2.22 then becomes

$$< V_p'(C,S) >_{JKL} / < V_p'(C,S) > = \exp \sum_{hmn}{}' \frac{1}{h!m!n!} \qquad (2.35)$$

$$< [\int d^4x\ J_\mu A_\mu]^h [\int d^4x\ K\phi]^m [\int d^4x\ L\phi^*]^n >_{cvS}$$

where the prime on the sum excludes the term h=k=l=0 and the subscript cvS indicates all connected graphs constructed out of the effective vertices and propagators. Examples of higher order corrections to $< A_\mu(x)\ V_p'(C,S) > / < V_p'(C,S) >$ are shown in figure 2.

It is then evident that higher order corrections to gauge invariant reduced Green's functions are S-independent: Under a change of S, the phase factors from two propagators meeting at a $\phi^*\phi A^2$ vertex or an external source cancel; the total change in the propagators and vertex at a $\phi^*\phi A_\mu$ vertex vanishes. Actually, this is not strictly true if we have defined the vortex operator by smearing S and adding additional operators. The Feynman integrals in coordinate space include points lying in S, for which 2.34 do not hold. As in the case of the propagator, we would expect to be able to "repair" the Feynman integrals with additional corrections to the definition of $V_p'(C,S)$. Again, we need never know the form of these

corrections: regarding the propagators and vertex functions as sections, the Feynman integral for each graph can be written as the invariant integral discussed above, and the result is unique.

We can now demonstrate equation 2.3:

$$\partial_\mu\, T\langle \tilde{F}_{\mu\rho}(x)\; V_p'(C,S)\rangle = \partial_\mu\,(T-T^*)\langle \tilde{F}_{\mu\rho}(x)\; V_p'(C,S)\rangle$$

$$= \partial_\mu\,(T-T^*)\langle \tilde{F}_{\mu\rho}(x)\; V_p'(C,S)\rangle_o$$

$$= \frac{2\pi p}{e}\oint_C dx'_\rho\,\delta^4(x-x')\langle V_p'(C,S)\rangle \quad (2.36)$$

The first equation follows because $T^*(\tilde{F}_{\mu\rho})$ is defined as $\epsilon_{\mu\rho\alpha\beta}\partial_\alpha T^*(A_\beta)$ and so its divergence vanishes identically. The second equality (the subscript indicates the lowest order graph, figure 1a) follows because any higher order graphs for the T^* Green's function, such those of figure 2, are S-independent and therefore continuous when x is at S: they do not contribute to $(T-T^*)$. The final equality follows from equation 2.28. Since $\partial_\mu \tilde{F}_{\mu\rho}(x)=0$ is true as an operator equation, 2.36 represents a commutator and is in fact the covariant version of 2.3. Equation 2.3 is an operator equation and we have only considered one matrix element of it. The same argument can be readily applied to any gauge invariant matrix element.

Equation 2.1 is also true. If we consider $\langle W_q(C')\; V_p'(C,S)\rangle$, only the graph of figure 3a has a discontinuity whenever C' crosses S: the Green's function jumps by a factor $\exp[\frac{2\pi i p q}{e}]$. This is the covariant form of 2.1. Figures 3b, 3c, etc. are S-independent and therefore continuous. Equation 2.3 thus exponentiates to give 2.1. Ordinarily this would not be true, because in general the commutators

of $\vec{B}$ do not determine those of the Wilson loop (although they do in naive canonical manipulations) because graphs such as 3c which depend on the composite nature of $W_q(C')$ have discontinuities; this is not a problem here. The location of the discontinuity of $< W_q(C') V_p'(C,S) >$ does depend on S, unlike the Green's functions of local gauge invariants. It seems, however, that the phase of this Green's function is not observable, but only the net change when C' is moved and then returned to its original position. The surface swept out by C' is then closed, so that it links C and intersects S a definite number of times. This fact will be used in the next chapter.

For later use we would like to examine the S-dependence of gauge-dependent quantities. From 2.34 and the effective Feynman rules, it follows that

$$< A_\mu(x)\ldots\phi(y)\ldots\phi^*(z)\ldots V_p'(C,S') > / < V_p'(C,S') > \qquad (2.37)$$

$$= < A_\mu(x)+\frac{i}{e}g(S,S';x)\partial_\mu g^*(S,S';x)\ldots g(S,S';y)\phi(y)\ldots$$

$$g^*(S,S';z)\phi^*(z)\ldots V_p'(C,S) > / < V_p'(C,S) >$$

We shall also be interested in the singularities of gauge dependent quantities near S. The singular behavior of $A_\mu^{vS}(x)$ is:

$$A_\rho^{vS}(x) \simeq \frac{p}{e}\alpha\beta\mu\rho n_1\alpha n_2\beta r_\mu/r^2 \qquad (2.38a)$$

where n_1 and n_2 are orthogonal unit vectors lying in S and r is the vector from x to the nearest point on S. From 2.34b and 2.38a we derive

$$g(S,S';x) \sim \exp[ip\theta(x)] \qquad (2.38b)$$

where S' is any surface distant from S and x. $\theta(x)$ is defined by taking a 2-plane normal to S and containing x: S will intersect this plane in one point and $\theta(x)$ is defined as the angle around this point, from x to an arbitrary fixed direction in the plane. It follows from 2.34a that for x near S, $\Delta_{vS}(x,y)$ is $\exp[-ip\theta(x)]$ times a non-singular function, and for y near S it is $\exp[ip\theta(y)]$ times a non-singular function.

We have found that reduced gauge invariant Green's functions are S independent; we return now to the factor that we divided out, equation 2.23. $< V_p'(C,S) >$ is given by the graph with one photon starting and ending on S, plus the sum of all vacuum bubbles constructed out of the effective propagators and vertices. For example, the determinant term in 2.23 is from the graph which is just one closed loop of the effective propagator Δ^v. As discussed earlier, graphs constructed from the effective propagators and vertices are all S independent. The only S dependence is that which we have found explicitly, the $\delta^2(0)$ term. We can take this term to be an artifact of the way we have defined V_p' when two of the fields in the expansion of the exponential are at the same point, and divide it out of the definition: all Green's functions are then S-independent. It is good that this term can be identified so unambiguously, so that artificial S dependence can be distinguished from a real, physical dependence of the vacuum expectation value of the vortex operator on the area of the minimal surface spanning C. Of the surviving terms in 2.23, the first is exactly the same as the leading term in the

expectation value of the Wilson loop, with the replacement of $\frac{2\pi}{e}$ for e. The next is of a form familiar from functional integrals. In fact, $< V_p'(C,S) >$ can be interpreted either in the normal way as a functional integral over continuous A and ϕ fields with $V_p'(C,S)$ inserted into the integrand, or as a functional integral with no insertion in the integrand but with the A_μ and ϕ fields fixed to have the discontinuities 2.37 on S. Since we want to be careful about divergence problems, we will stay with the first interpretation.

C. Renormalization of Looplike Operators

In the preceding section, we neglected renormalization. We did not specify whether quantities were bare or renormalized, we did not include graphs with counterterms, and we did not consider the convergence of the various graphs. These points are the subject of the present section. We include first, as an illustration of some of the ideas, a short section on the renormalization of the Wilson loop operator.

1. Renormalization of Green's Functions of the Wilson Loop

The Wilson loop is a composite operator involving products of arbitrarily many elementary fields. The associated divergences, however, turn out to be easily analyzed, at least in the Abelian case: all matrix elements can be made finite by one overall multiplication of the operator[28]. Gervais and Neveu[29] and Polyakov[30] have shown by the use of elegant methods that the same is true of the non-Abelian Wilson loop.

A general Green's function of the Wilson operator, $< W_e(C) >_{JKL}$, defined by analogy to equation 2.21, is given by

$$< W_e(C) >_{JKL} = \exp \sum_{khmn} \frac{1}{k!h!m!n!} \tag{2.39}$$

$$< [i\oint_C dy_\mu eA_\mu(y)]^k [\int d^4x\ J_\mu A_\mu]^h [\int d^4x\ K\phi]^m [\int d^4x\ L\phi^*]^n >_c$$

The quantity $eA_\mu(y)$ is invariant under renormalization, due to the Ward identity: $e_r A_r = e_o A_o$, where subscripts r and o represent renormalized and bare quantities, respectively. When it is necessary to take a particular renormalization scheme, we will use BPHZ at zero momentum (this is acceptable when the charged fields are all massive, which, for simplicity, we assume). If we then take the fields coupled to J, K, and L to be the renormalized ones, the connected Green's functions in equation 2.39 are all renormalized: they are finite when not evaluated at the same spacetime point, and they are integrable over spacetime regions that include coincident points. If we take $J_\mu(x)$, K(x), and L(x) to be smooth, the associated x integrals all exist.

The only possible divergence comes when k is greater than 1, so that there is a multiple integral over C. This may diverge when two or more integrands approach each other along the loop. To emphasize the region in the multiple loop integral where j of the arguments approach each other along the loop, we represent 2.39 as

$$(ie)^j \int_{-\infty}^{\infty} dx \oint_C dy_{1\alpha} dy_{2\beta} \cdots dy_{jy}$$

$$\delta\Big(\sum_{i=2}^{j} |y_i - y_1| - x\Big) < A_\alpha(y_1) \cdots A_y(y_j) \cdots >_c$$

$$= (ie)^j \int_{-\infty}^{\infty} dx\ x^{j-2} \oint_C dy_{1\alpha} dw_{2\beta} \cdots dw_{jy} \tag{2.40}$$

$$\delta\Big(\sum_{i=2}^{j} |w_i| - 1\Big) < A_\alpha(y_1)A_\beta(y_1+xw_2)\cdots A_\gamma(y_1+xw_j) \cdots >_c$$

where the final ellipsis in the Green's functions indicates the remaining operators, coupled to external sources or to distant points on C. To get the leading $x\to 0$ behavior of the Green's function we use the operator product expansion[31] to write the product of j A's in the Green's function as

$$g_{0\alpha\cdots\gamma}(xw_2,\cdots,xw_j)\cdot 1$$

$$+ g_{1\alpha\cdots\gamma\mu}(xw_2,\cdots,xw_j)\cdot A_\mu(y_1) + \text{higher operators}$$

This form places all of the x dependence in the coefficient functions g_i. We implicitly use a covariant gauge so as to avoid direction dependendent singularities, as found, for instance, in the axial gauge.

At fixed x, the j points cannot be coincident. We cannot assume, however, that the w-integrations converge, as there will be regions in integration space where some subset of the points come together. We will assume a coordinate-space version of Weinberg's theorem[32], which we have not proved but which seems quite plausible: that it suffices to consider just the x-integrations for each subset of points, and if naive power counting indicates that every one of these is convergent then the whole integral will be. In fact this is not necessary: at least when C is an infinite straight line the Green's functions can be written in momentum space. Weinberg's theorem and BPHZ subtraction may then be applied rigorously to verify the conclusions reached below. It is then very plausible that for a

smooth curve C the leading divergences are the same as for the straight line. The coordinate-space argument is shorter and perhaps more interesting.

Because A_μ is of dimension 1, at each order in perturbation theory (we are renormalizing order by order) $g_{0\alpha\ldots\gamma}(xw)$ is of order x^{-j} times logarithms and $g_{1\alpha\ldots\mu}(xw)$ is of order x^{-j+1} times logarithms; the higher coefficient functions are all smaller as $x\to 0$. The x-integration associated with the operator 1 is then linearly divergent. Performing the x integration with a distance cutoff Λ^{-1} and the w integration leaving out those subregions where subsets of points become coincident, the c-number piece of 2.40 becomes

$$\{ R(\Lambda)\textstyle\int_C dy_1 n_\alpha(y_1) + \text{finite terms} \} < \cdots >_c \qquad (2.41)$$

where R is $O(\Lambda)$, $n_\alpha(y_1)$ is a unit vector tangent to C at y_1, and the ellipsis in the matrix element is the same as in 2.40. If C were straight, it would be clear why n_α must appear: there is no other available vector. For C a smooth curve, the leading divergence is the same as for a straight line, since points close together don't see the curvature; therefore $n_\alpha(y_1)$ appears. To put that another way, curvature, the lowest dimensional measure of the actual shape of C, is of dimension two; its coefficient must be two powers less divergent than linear (that is, convergent). This argument fails if C has a kink, as the curvature there is infinite. In general there will be a composite divergence associated with a kink that is one power weaker than the leading divergence, from endpoint effects in the integration. There would therefore be an additional logarithmic divergence for each kink. This was also found by Gervais and

Neveu[29]. The integral in 2.41 is simply the perimeter P(C), and so the divergence corresponding to the c-number in the operator product expansion (which arises from graphs with no external lines) can be removed by multiplication of the Wilson loop by an overall factor $\exp[-R(\Lambda)P(C)]$.

The only other divergence is in the coefficient of $A_\mu(x)$. This is logarithmic and is given by graphs which have external lines but are such that cutting a single photon line separates all of the external lines from the Wilson loop. As one might expect, this divergence is actually absent due to the Ward identity. In fact, in our particular scheme (zero momentum BPHZ)[31],

$$g_{1\alpha}\cdots\gamma_\mu(xw_2,\cdots,xw_j) = G_\alpha\cdots\gamma_\mu(xw_2,\cdots,xw_j;0) \quad (2.42)$$

where G is the j-photon Green's function with one leg (the one with index μ) truncated and set to zero momentum. G satisfies a Ward identity

$$k_\mu G_\alpha\cdots\gamma_\mu(xw_2,\cdots,xw_j;k)=0 \quad (2.43)$$

From 2.43 and the fact that G is continuous at $k_\mu=0$ (for massive charged particles) it follows that G vanishes at zero momentum and so g_1 is identically zero and that there is no divergence coming from connected graphs with external lines. In fact, g_1 can be shown to vanish by gauge invariance in any renormalization scheme.

We find, then, that all matrix elements of the Abelian Wilson loop can be made finite by one overall multiplication.

2. Renormalization of Green's Functions of the Vortex Operator

The analysis of the vortex operator proceeds much like the analysis of the Wilson loop, and the result is the same: an overall multiplication makes all matrix elements finite. There is a potential logarithmic operator divergence, as for the Wilson loop, but its coefficient again turns out to vanish. The vanishing is here more intricate, involving cancellation of field renormalization divergences against composite operator divergences. Our nicest result, that $T[F^{V}_{\mu\rho}]$ is finite for an infinite straight vortex, can be obtained in a few lines (equations 2.48-2.50). The rest of this section is simply power counting to establish that this implies the cutoff independence of all matrix elements of vortex operators for any curve.

We first must ask whether the combinations $\frac{F_{\mu\rho}}{e}$ and $\frac{j_{\mu}}{e}$ appearing in 2.4 and 2.20 refer to the bare or the renormalized fields and charges, since these combinations are not invariant under renormalization. A canonical argument indicates that they must be bare. Equations 2.5 hold only for the bare quantities; if the renormalized quantities were to appear on the left hand side of 2.5, an extra factor of Z_3^{-1} would be needed on the right hand side. In order to have a finite commutator with the Wilson loop, as in 2.1, or with charged fields, as in 2.6, it is then clear that we must have $\frac{F_{o,\mu\rho}}{e_o}$ and $\frac{j_{o,\mu}}{e_o}$.

We can also reach this same conclusion from the Green's functions. The discussion of the last section did not include graphs with counterterms. This is correct if all fields and couplings, including those in the definition of V_p', are the bare ones (we must have a cutoff at this point). Otherwise, graphs with counterterms

enter and spoil the quantization condition. The exponent in the definition of V_p' must therefore be cutoff dependent. If we had started by defining V_p' with $F_{r\mu\rho}$ and e_r, we would have found that p/Z_3, not p, was an integer, so that the cutoff dependence would merely be shifted into p. Either way, writing V_p' in terms of cutoff independent p, e, and $F_{\mu\rho}$ gives

$$V_p'(C,S) = \exp\{\frac{\pi p Z_3}{e_r}\int_S d\sigma^y_{\alpha\beta}\tilde{F}_{r\alpha\beta}(y)\} \qquad (2.44)$$

with an explicitly cutoff dependent exponent. Note also that this implies that the total coefficient (all counterterms summed) of the one photon graph for $< e_o F_{o\mu\rho} V_p'(C,S) >$ or for $< e_r F_{r\mu\rho} V_p'(C,S) >$ is πp, with no factors of Z_3. This in turn implies that equation 2.36, which comes entirely from the one photon graph, is correct with $\tilde{F}_{\mu\rho}$ and e either both bare or both renormalized; explicitly,

$$\partial_\mu T< \tilde{F}_{r\mu\rho}(x)V_p'(C,S) > = \frac{2\pi p}{e_r}\int_C dx' \rho\, \delta^4(x-x')< V_p'(C,S) > \qquad (2.36')$$

We now investigate the cutoff dependence of the Green's functions of the vortex operator. Starting with the expression 2.22, we can analyze the divergences of the Green's functions very much as we did for the Wilson loop. There are two differences. One is that V, unlike W, contains a manifest cutoff dependence from field renormalization, as discussed above. The second difference is in the analysis of the composite divergences, where the vortex involves an integral over a two- rather than one-dimensional surface, and the field being integrated is of dimension two rather than one. This second difference is small: because the exponent is dimensionless for both opera-

tors, in each case there is only a small number of divergences. The analog of (2.40) when j points come together is

$$< V_p'(C,S) >_{JKL} = (\tfrac{np}{e})^j \int_{-\infty}^{\infty} dx\, x^{2j-3} \int_S d\sigma_{\alpha\beta}^{y_1} d\sigma_{\gamma\delta}^{w_2} \ldots d\sigma_{\mu\rho}^{w_j} \quad (2.45)$$

$$\delta\Big(\sum_{i=2}^{j} |w_i| - 1\Big) < \bar{F}_{\alpha\beta}(y_1)\bar{F}_{\gamma\delta}(y_1+xw_2)\ldots\bar{F}_{\mu\rho}(y_1+xw_j)\cdots >_c$$

The small x expansion for the operator product is now

$$h_{0\alpha\beta,\gamma\delta,\ldots,\mu\rho}(xw_2,\ldots,xw_j)\cdot 1 \quad (2.46)$$

$$+ h_{1\alpha\beta,\gamma\delta,\ldots,\mu\rho;\sigma\lambda}(xw_2,\ldots,xw_j)\cdot F_{\sigma\lambda}(y_1)$$

$$+ h_{2\alpha\beta,\gamma\delta,\ldots,\mu\rho}(xw_2,\ldots,xw_j)\cdot\phi^*(y_1)\phi(y_1) + \cdots$$

The expansion includes only gauge invariant operators, because the operator product is gauge invariant. h_0 is of order x^{-2j} and h_1 and h_2 are of order x^{-2j+2}, so that the x-integration for the coefficient of 1 is quadratically divergent, and those for the coefficients of $F_{\mu\rho}$ and $\phi^*\phi$ are logarithmically divergent.

We have neglected the extra operators in $V'_p(C,S)$ that we concluded were needed to insure S-independence. In the two dimensional case, equations 2.14 indicate that the effective dimension of $d^2z\, f_\lambda(z)$ in units of mass is greater than -2 but less than -1. The total dimension of the exponent of Ξ is then negative; it is a "soft" operator. We expect that this property will hold in general. The inclusion of these additional operators, then, will give additional contributions to the coefficient functions in 2.41, but will not lead to any stronger singularities.

The divergence proportional to 1 implies a common infinite factor of the form $\exp[R'(\Lambda)A(S)]$ in every matrix element of V_p', with $A(S)$ the area of the surface S, and with $R' \; O(\Lambda^2)$. Just such a divergence was found from the one photon graph in the last section, where it was also shown that such a term, being explicitly S dependent, could not arise from any higher order graph. It can therefore be unambiguously divided out. The operator 1 in the expansion also gives rise to a linearly divergent term proportional to the perimeter P(C), from edge effects in the surface integrals. This was also found from the one photon graph; for this divergence we would expect that higher order graphs will also contribute. At any rate, it can be divided out by a factor of the form $\exp[-R''(\Lambda)P(C)]$, with $R'' \; O(\Lambda)$.

The logarithmic divergence from the h_1 and h_2 can be removed by the addition of three counterterms to the exponent of $V_p'(C,S)$ (as in the case of the Wilson loop, this conclusion can be verified rigorously by use of Weinberg's theorem and BPHZ subtraction when C is a straight line):

$$c_1 = \int_S d\sigma^y_{\alpha\beta} \tilde{F}_{\alpha\beta}(y) \tag{2.47a}$$

$$c_2 = \int_S d\sigma^y_{\alpha\beta} F_{\alpha\beta}(y) \tag{2.47b}$$

$$c_3 = \int_S |d\sigma^y_{\alpha\beta}| \phi^*(y)\phi(y) \tag{2.47c}$$

$|d\sigma_{\alpha\beta}|$ appears in c_3 for the same reason that $dy_{1\alpha} n_\alpha(y_1)$ (which is just $|dy_1|$) appears in 2.41: there is no other Euclidean invariant form for the leading divergence.

The counterterm c_2 is forbidden by by CP invariance. $d\sigma_{\alpha\beta}\tilde{F}_{\alpha\beta}$ is CP even; as a result, so is any logarithmic composite divergence (because a smooth surface is CP invariant to the extent that its curvature can be neglected), while c_2 is CP odd. c_3 is forbidden by the S-independence of gauge invariant Green's functions. The easiest way to see this is to consider a surface S which doubles back on itself. It is important that because of S-independence, there is no new composite divergence associated with this doubling back. The doubled surface cancels out in the definition of $V_p'(C,S)$, because $d\sigma_{\alpha\beta}$ is oriented; it must therefore cancel out in any divergences. It does cancel in c_1 and c_2, but not in c_3, as $|d\sigma_{\alpha\beta}|$ is not oriented. We conclude that all of the composite divergence can be removed by a term of the form $f(\Lambda)\cdot c_1$.

There is still the second source of cutoff dependence in the Green's functions of $V_p'(C,S)$, the factor of Z_3 in 2.44. We see that this is of exactly the same form as the counterterm 2.47a; an appropriate choice of $f(\Lambda)$ can remove this divergence as well, leaving every Green's function of the vortex operator finite.

Since there is only one unknown function of the cutoff, we can determine it by calculating one single Green's function of one particular vortex operator. A convenient choice is

$$T< F_{r\mu\rho}(x)\ V_p'(Z,Y) > / < V_p'(Z,Y) > \qquad (2.48)$$

where Z is the line $y_0=y_1=y_2=0$ and Y is the half-plane $y_0=y_1=0$, $y_2>0$. By Euclidean invariance, this must be given by

$$a(x_z^2)\ \epsilon_{\mu\rho\alpha 3} x_{z\alpha} + b(x_z^2)\ x_{z(\mu}\delta_{\rho)3} \qquad (2.49)$$

where $x_{z\mu}$ is the vector from Z to x which is perpendicular to Z. Under parity, $V_p'(Z,Y)$ goes into $V_p'(Z,Y')$, where Y' is the half-plane $y_0=y_1=0$, $y_2<0$ (note carefully that p doesn't change sign). The gauge invariant Green's functions of these two operators are equal. Under $x_0\to x_0$, $x_i\to -x_i$, the Green's function 2.48 then has natural parity: space-space components are invariant and space-time components change sign. Only the first term is 2.49 has natural parity; it must be that $b(x_z^2)=0$. Equation 2.36' then requires that

$$a(x_N^2) = -\frac{p}{2e_r}(x_N^2)^{-3/2} \tag{2.50}$$

Remarkably, the Green's function 2.48 is completely determined, and it is cutoff independent without counterterms: $f(\Lambda) = 0$. The factor of Z_3 in 2.44 provides just the cutoff dependence to cancel that in the composite divergence[33].

In terms of graphs, the cancellation we have found is this: The order e contribution to 2.48 is figure 2a. Expanding the effective vertex and propagator in terms of the usual ones gives all graphs with one charged loop and no internal photons, such as those of figure 4. The graphs of figure 4a have the usual divergence; ordinarily this would be cancelled by a counterterm from the order e^{-1} graph, figure 1a. Here, this counterterm is absent owing to the factor of Z_3 in 2.44. However, the graphs of figure 4b are also divergent when the integration over the momenta of the photons attached to V_p' is included. What we have found above is that the divergences of 4b exactly cancel those of 4a. This cancellation has an interesting feature: the different graphs are proportional to different powers of p, depending on how many photon lines attach to V_p'. When we let p

vary, the full cancellation occurs only for integral p, as only then can we say that the Green's function is independent of S and that 2.49 is the only allowed form.

The fact that the matrix elements of the vortex operator turn out to be automatically finite is rather important. If we had had to add a counterterm 2.47a to the exponent of V_p', its commutation relations 2.1 and 2.3 would then have become cutoff dependent. We hope to find that there are relations between the commutation relations of V_p' (that is, the fact that it is a vortex operator) and its Green's functions; this would seem unlikely if cutoff independent commutators had been incompatible with cutoff independent Green's functions. This is remeniscent of the situation with Noether currents, generators of exact symmetries: there also we wish to ascribe physical significance to commutators, and there also the commutators and the Green's functions are simultaneously finite.

There is one weakness in the above analysis. What we have really shown with the operator product analysis is that the counterterm c_1 suffices to remove all divergences from Green's functions of V_p when they are expanded order by order in p (thus, in 2.44 we have isolated all graphs of order p^J). It would be preferable if we could first sum to all orders of p, getting effective propagators and vertices as before, and then analyze the divergences directly from the short distance properties of these effective propagators and vertices. We do not expect that our conclusions would change; however, we shall see in the next chapter that an expansion of the Green's functions in powers of p can sometimes lead to erroneous conclusions, so it would be good to have an analysis of the divergences which did

not rely on such an expansion. Note that the finiteness of 2.48 was independent of the expansion in p. It would also be interesting to see what effect hard P- or CP- violating interactions would have on the analysis.

CHAPTER 3

CLUSTER PROPERTIES OF LOOPLIKE OPERATORS

A. Cluster Properties in a General Gauge Theory

In this chapter we discuss some of the long distance features of the Green's functions of operators associated with closed curves. The present section is quite general, and applies to an Abelian or non-Abelian theory, though we shall be particularly concerned with the non-Abelian case. In the next section we return to an Abelian theory and extend the results of the previous chapter to the case where the symmetry is spontaneously broken.

The spectrum of a given gauge field theory might have one of many different forms. It might be "QED-like", where the physical particles resemble closely those in the Lagrangian: massless vector particles and unconfined charged (that is, non-singlet) particles. It might be confining, with all physical particles gauge singlets. It might be Higgs-like, with the gauge particle massive and the gauge symmetry broken either by a fundamental or a dynamical Higgs field. It might be intermediate between these possibilities, or it may have some other form altogether. Which form, or phase, is realized will depend on the details of the dynamics.

It is desireable to have some order parameter, some precise distinction between the phases. For instance, the QED-like phase is distinguished by the presence of a massless vector particle. The confining phase of a non-Abelian theory is not so readily defined. As pointed out by Mandelstam[14], there is no global non-Abelian charge: the current is gauge covariant, so its integral over space

has little meaning. It is therefore imprecise to say that a physical particle is a gauge singlet. Nor is the identification of the non-Abelian Higgs phase obvious. The expectation value of the Higgs field is not gauge invariant, and there is no satisfactory gauge to use. The Coulomb and covariant gauges do not exist in a non-Abelian theory[34], and the axial gauge is too singular[14]. The confining and Higgs phases can be defined precisely by the Wilson[8] and 't Hooft[18] criteria, relating to the vacuum expectation values of large Wilson and vortex loops. We shall return to these after we show how the phases can be characterized by the cluster properties of the Wilson and vortex operators.

Consider the Euclidean Green's function

$$G(x,C) = \langle \Theta(x)\, W_S(C) \rangle_c$$

$$= \langle \Theta(x)\, W_S(C) \rangle - \langle \Theta(x) \rangle \langle W_S(C) \rangle \qquad (3.1)$$

with $\Theta(x)$ some local gauge invariant operator such as $F^a_{\alpha\beta}F^a_{\gamma\delta}$. In the phases we have mentioned above, this function will behave in one of three ways when C is very large:

Short distance clustering: G(x,C) falls off exponentially with d(x,C), the distance between x and C.

Surface clustering: G(x,C) is nonvanishing near the minimal surface, S_M, spanning C, and falls off exponentially away from that surface.

Long distance clustering: G(x,C) falls off as a power of d(x,C).

These alternatives may be better understood by considering a

typical state of the system in a 3-surface cutting perpendicularly through C. In this 3-surface, one sees a source/antisource pair, s and $\bar{s}$, where C intersects the surface. If the clustering is short range, there are only short range, Yukawa, fields around the sources, and vacuum elsewhere. If the clustering is surface-like, there is a tube of non-vacuum joining s and $\bar{s}$, whose energy per unit length gives rise to a linear potential between the external sources. If the clustering is long range, s and $\bar{s}$ have Coulomb-like fields with a power law fall-off. Long range clustering is only possible if there are massless particles. The other two types of clustering each have a characteristic scale (the range of the Yukawa field or the thickness of the tube) which are determined by the mass m_L of the lightest particle; for quarkless QCD this is presumably of the general order of Λ_{QCD}. As $m_L P(C)$ is taken to zero, either by shrinking C or by letting m_L go to zero, the first two cluster properties turn continuously into the third.

Short distance clustering is analogous to that for pointlike fields in a massive theory, where the general connected two-point Green's function falls off exponentially with distance. Long distance clustering is analogous to that for pointlike fields in a massless theory. Surface clustering is a new feature; it seems to arise when there is a flux which can neither spread nor be shielded. One might also image more general cluster properties, of course. These three, however, seem to cover all those which have arisen in various gauge theories and models.

We can also consider the cluster properties of Green's functions of the vortex operator $V_p(C)$ (p designating the homotopy class); the

same three possibilities seem to arise. The cluster property appears to be largely independent of the operator $\Phi(x)$. For instance, if a tube of non-vacuum runs between s and $\bar{s}$, we would expect most local operators to have within the tube an expectation value different from that which they have in vacuum. It may, however, depend on the representation s of the Wilson loop, or the homotopy class p of the vortex operator.

A general phase, then, may be characterized by which of the three cluster properties is realized for each representation and for each homotopy class. A confining theory is one in which the Wilson loop, at least in some representations, has surface clustering, as this implies linear confinement of external charges in those representations; this is the Wilson criterion[8]. The 't Hooft criterion defines a (completely broken) Higgs theory as one in which some of the vortex operators have surface clustering, as this implies that magnetic flux is forming into tubes.

This classification is closely related to the usual classification of phases in terms of the vacuum expectation values of the Wilson and vortex operators. For a very large curve C, the vacuum expectation value of a general looplike operator X(C) will be dominated by $\exp[-S_{cl}]$, where

$$S_{cl} = \int d^4x \, < L(x) \, X(C) >_c \, / \, < X(C) > \qquad (3.2)$$

and L(x) is the Lagrangian density. The connected Green's function in 3.2 is a special case of 3.1. For short distance clustering the integrand will be nonzero only for x near C, so that the whole integral is proportional to P(C): $< X(C) >$ follows a perimeter law.

For surface clustering, the integrand will be nonzero only for x near the minimal surface, so that the whole integral is proportional to $A(S_M)$. In phases without massless particles, there is a one-to-one correspondence between an area law and surface clustering, and between a perimeter law and short range clustering.

For long range clustering, the integrand is proportional to $d(x,C)^k$; the behavior of the integral depends on the particular value of k. In QED k is -4; the integral therefore is dominated by small values of $d(x,C)$ and is proportional to P. If k were -3, the integral over $d(x,C)$ would diverge logarithmically until it was cut off at roughly the linear scale R of the curve C; it is proportional to $R \cdot \ln(R)$. We use R rather than P because P implies a certain shape dependence, which need not hold here. If k were -2, the integral would be proportional to R^2, as it is for surface clustering. In this case there also be a linear potential between external sources, but without the formation of a flux tube. There would be, however, a strongly interacting massless particle, which is not observed. Furthermore, it is not clear that it is possible to find a consistent physical picture in which k is -2.

Restricting attention for now to phases without massless particles, the Wilson loop operator for each representation and the vortex operator for each homotopy class will have either short range or surface clustering. The cluster properties for the Wilson loop in different representations are related. If there are fields of representation s in the theory, it will always be energetically favorable for s and $\bar{s}$ external sources, when they are far apart, to pull an $s\bar{s}$ pair out of the vacuum and shield themselves rather than form a flux tube.

W_s will then always have short range clustering. Similarly, one can see that if for any representation r, W_r has short range clustering, this will also be true of any other representation r′ which is contained in the direct product of one r and any number of s representations. The cluster property is therefore a function only on the quotient, {r}/{s}, where {r} is the set of all representations of the gauge group, and {s} is the set of those representations carried by fields or products of fields. Incidentally, for any compact gauge group, the above quotient is isomorphic to $\Pi_1(G)$, so the set of distinct types of Wilson loops is isomorphic to the set of distinct vortex operators (though there is no unique natural mapping between the two sets). One further restriction on the clustering is that it must be the same for a representation and its conjugate (or for a homotopy element and its inverse), as the looplike operators for these differ only in the direction of their "flux", so that they can be turned into one another by a spacetime rotation.

't Hooft has shown that relations can be obtained between the cluster properties of Wilson loops and vortex operators in phases without massless particles. One result[18] is that if a Wilson loop and vortex operator do not commute, they cannot both have short range clustering. To see this, let C and C′ be large loops, C lying in the 3-plane t=0 and C′ lying in the 3-plane t=-R (we single out one Euclidean direction and call it t), where $R \gg m_L^{-1}$. Let them have the spatial orientation of figure 5a, linked but separated by the large distance R. Consider $\langle V_p(C)\, W_s(C') \rangle$, where p and s are such that the operators do not commute, when C′ is translated as follows: first, forward in time to t=R; then, in a spacelike direction to the

spatial configuration of figure 5b (unlinked, and separated by R); then, back to t=-R; finally, back to the position of figure 5a. The Green's function must now have returned to its original value. C and C′ are always separated by at least R. If both operators have short range clustering, there can be no correlation between them. In other words, C′ is essentially being moved through vacuum, so that $< V_p(C)\ W_s(C') >$ is constant by translational invariance. This is true except when C′ crosses S; the Green's function then jumps by a phase. C′ sweeps out a closed surface, which links C a definite number of times. It therefore crosses S this same number of times, independent of the choice of S. Taking the special case where S is the minimal surface spanning C, we see that C′ crosses this only once, on its first leg to t=R.

The only change in the Green's function during this round trip is one change of phase, when C′ crosses S, corresponding to the non-commutation of the two operators. This is inconsistent with the Green's function being single-valued. It is therefore impossible for both operators to have short range clustering. If either one has surface clustering, the Green's function changes not only when C′ crosses S, but also when one loop moves through the flux tube of the other. If there are massless particles and one or both loops have long range clustering, the Green's function can change even when the two loops are well separated. In either of the latter two cases, there is no contradiction with the single-valuedness of the Green's function. We have tried, above, to emphasize the distinction between the arbitrariness of phase of the Green's function associated with a change of S, and the fact that it is well defined, and single valued,

once S is fixed.

More recently, using a more powerful framework[35], 't Hooft has been able to place even stronger restrictions on the cluster properties in phases without massless particles. We shall refer to some of his results later.

We now illustrate the above ideas. In a free Abelian theory, one readily finds that all Wilson and vortex operators satisfy a perimeter law. For the vortex operator only the second term of 2.23 survives; an almost identical expression holds for the Wilson loop. External charges and monopoles have Coulombic fields, and connected Green's functions all satisfy long range clustering, with various powers of $d(x,C)$. For the special case of $\Theta(x)=L(x)$, $k=-4$, as mentioned above. For QED with massive charged fields, the infrared properties are essentially those of the free theory, and the above conclusions still hold.

In a pure SU(3) gauge theory (QCD with quark loops suppressed), there are three different kinds of representations, distinguished by "triality", the SU(3) version of the quark number discussed before equation 1.5. The representations of zero triality can all be formed as a product of adjoint representations; since there are always adjoint fields present (gluons) these can never have surface clustering. Assuming on phenomenological grounds the absence of massless glueballs, the Wilson loops for the representations of zero triality will always have short range clustering. The representations of trialities one and two will always have the same clustering, as they are conjugate to each other. The Wilson criterion, defining a confining

theory, is that this should be surface-like for quarkless QCD, so that quarks are bound by a linear potential until the flux tube breaks by creation of quark pairs. In this, the standard, picture, a linear potential can never form between gluons. Physical gluons must be assumed to be absent because they are shielded in some way: a high energy gluon resulting from an interaction would pull more gluons out of the vacuum and form a singlet glueball. As for the vortex operators, there are two homotopy classes, other than the trivial one, corresponding to the elements $\exp[\pm 2\pi i/3]$ of the center of SU(3). 't Hooft has shown that these will have short distance clustering[35], if the triality one and two Wilson loops have surface clustering.

When the contribution of quark loops is added, it becomes possible to shield any Wilson loop, so that all Wilson loops will satisfy a perimeter law. At the same time, the effective gauge group becomes SU(3) rather than $SU(3)/Z_3$. This is simply connected and does not permit vortex operators to be defined in a simple way. Thus, insofar as the Wilson and 't Hooft criteria are the only distinctions between phases, it appears that an SU(3) (or SU(n)) gauge theory with fields in the fundamental representation can exist only in a single phase, regardless of the dynamics. This phase may resemble a confining or a Higgs theory extremely closely, but it would be possible to change the theory in a continuous way from one into the other. This possibility has been much discussed, e.g. in references 36, 37 and 38. We shall have more to say about it in the next chapter.

Lastly, we consider a perturbative Higgs theory, one in which the coupling is quite small so that we may use a covariant gauge without worrying about the Gribov ambiguity[34]. This enables us to

discuss gauge dependent quantities such as the vacuum expectation value of the Higgs field. We shall take the gauge group G to be completely broken. M, the set of classical vacua (minima of the Higgs potential), is then identical to G and there are no massless particles. Let us consider

$$\phi^{vS}(x) = \langle \phi(x)\ V_p'(C,S) \rangle / \langle V_p'(C,S) \rangle \qquad (3.3)$$

where $\phi(x)$ is the Higgs field, C is a large curve and S is taken, for convenience, to lie far away from the minimal surface of C. In a three surface which is perpendicular to C and cuts it at two points we have figure 6. Near the small loop 1, which is far from C and from its minimal surface, gauge invariant connected Green's functions will vanish and $\phi^{vS}(x)$ will be position dependent but its values will lie in the set M.

If we have an Abelian theory, we know from 2.37 that $\phi^{vS}(x)$ will be $a \cdot \exp[ip\theta]$ as we traverse the infinitesimal loop 1 and θ goes from 0 to 2π. Here a is the magnitude of the vacuum expectation value of ϕ. M is the set of complex numbers of modulus a. On the loop 1, $\phi^{vS}(x)$ is seen to describe an element of the homotopy group $\Pi_1(M)$ identical to the element of $\Pi_1(G)$ associated with the vortex operator. This idea generalizes to a non-Abelian theory: there too there will be a gauge transformation $g(S,S';x)$ governing the S-dependence of gauge dependent quantities, which will give rise to a nontrivial element of the homotopy group when x circles S or S'; this causes $\phi^{vS}(x)$ trace out the corresponding element of $\Pi_1(M)$ when x travels on loop 1.

We may now imagine enlarging the loop, sliding it off S, taking it to the position of loop 2 and shrinking it to a point, without ever getting close to c or $\bar{c}$. Because the fields are singular only at S, $\phi^{vS}(x)$ must be essentially constant on loop 2: it maps out an element of the trivial homotopy class. By definition there is no way to continuously deform an element of one homotopy class into an element of another while staying within M. It follows that, somewhere between 1 and 2, $\phi^{vS}(x)$ took values outside of M; this must be the case at least within a tube between c and $\bar{c}$. This is precisely the argument by which one shows that when the Higgs field at spatial infinity in two space dimensions maps out a non-trivial element of the homotopy group, there must be a "lump", a Nielsen-Olesen vortex[20,21], somewhere in space. The tube between c and $\bar{c}$ is a Nielsen-Olesen vortex. Where $\phi^{vS}(x)$ does not lie in M, gauge invariant connected Green's functions such as that for the Higgs potential will be non-vanishing. It follows that the vortex operator in a completely broken small coupling Higgs phase has surface clustering. This is the source of the 't Hooft criterion. One may also check, order by order, that the Green's functions of the Wilson loop are short range, because all fields are massive.

B. Vortex Operators in an Abelian Higgs Phase

In the last section we saw that the vortex operator in a perturbative Higgs phase has a surface-like cluster property. For an Abelian theory we can obtain a stronger result: the vortex operator can never have short range clustering, so that in any Abelian theory without massless particles it will have a surface clustering and obey an area law. From 2.36' and Gauss's law,

$$\int_B d\sigma^x_{\mu\rho}\ T< \tilde{F}_{\mu\rho}(x)\ V_p'(C,S) > = \frac{4\pi p}{e} < V_p'(C,S) > \qquad (3.4)$$

where C is a large curve and B is a 2-sphere linking C (we might imagine figure 5a as showing the 3-surface containing C, with C' as the equator of B; the rest of B extends into the t-direction). All fields and charges in this section are taken to be renormalized. The integral over B is independent of the radius of B. On the other hand, since the Green's function in 3.4 is gauge invariant, short range clustering would require it to fall exponentially when the radius of B is greater than m_L; this is inconsistent with equation 3.4.

What we have shown is really quite simple: magnetic flux can never be shielded. If the absence of massless fields then makes it impossible to have a Coulomb field, magnetic flux can only form into tubes. One could see this directly from the operator equation $\vec{\partial}\cdot\vec{B}(x)=\rho_{mc}(x)$, where $\rho_{mc}(x)$ is a purely external magnetic charge density; this equation is one component of 2.36'. Given that magnetic flux is confined, one can extend 't Hooft's more recent results[35] to the Abelian case to show that all Wilson loops must obey a perimeter law. It follows that an Abelian theory (without magnetic monopole fields) never confines. This result was anticipated by Mandelstam[14] on the basis of the existence of the Abelian Coulomb gauge.

One might try to argue in a different way that the Green's functions of the vortex operator had to be short ranged in a phase without massless particles. Take a large curve C, with the surface S far away from the minimal surface, as it was in figure 6. Consider

$$< \Theta(x)\ V_p'(C,S) >_c \ / \ < V_p'(C,S) >$$

$$= \sum_{i=1}^{\infty} \frac{1}{i!} < \Theta(x)\ [\frac{np}{e}\int_S d\sigma^{\gamma}_{\alpha\beta} \tilde{F}_{\alpha\beta}(y)]^i >_c \qquad (3.5)$$

where again $\Theta(x)$ is any gauge invariant operator and where x is a point near the minimal surface of C but far from C itself. Because there are no massless particles, every Green's function in 3.5 falls exponentially for x distant from S. In particular they are vanishingly small when x is on the minimal surface, so that surface clustering is impossible. Further, since $< \Theta(x)\ V_p'(C,S) >_c$ is exponentialy small for x far from S, and also independent of S, it is exponentially small except for x near C. This is in direct disagreement with what was shown above. The problem must lie in the expansion 3.5: while this expansion is formally correct, the long distance behavior of the sum is not the same as that of the individual terms. This is the source of our statement, at the end of the section on renormalization, that the expansion in powers of p is not to be trusted.

In the remainder of this section, we shall extend to the case of a spontaneously broken Abelian theory the rules for calculating Green's functions involving $V_p'(C,S)$ in terms of effective propagators and vertices. We shall be particularly interested in seeing the surface-like cluster property emerge. We shall also be interested in the following question: In the presence of a vortex operator we expect the Higgs field to have a certain "twist", or singularity; yet we generally, in a Higgs theory, expand around a constant, non-singular Higgs field. How are these compatible? We shall see that

the twisted Higgs field arises in a natural way.

First, we will sketch the derivation of the Feynman rules in a Higgs phase. Consider the functional integral

$$Z(J) = \int D\psi \exp\{ -\tfrac{1}{2}M_{ij}\psi_i\psi_j - P(\psi) + J_i\psi_i \} \qquad (3.6)$$

Here, for convenience, we have joined all the fields into a single field ψ_i, where the index i includes quantum numbers, spacetime indices, and spacetime position. M_{ij} is the quadratic part of the action, $P(\psi)$ is the interaction Lagrangian, polynomial in ψ_i, and J_i is an external source, whose functional derivatives are the Green's functions of ψ. Writing $\psi_i = \rho_i + a_i$, where a_i is the value of ψ_i about which we wish to expand the functional integral, we may write 3.6 as

$$Z(J) = \int D\rho \exp\{ -\tfrac{1}{2}M_{ij}\rho_i\rho_j - M_{ij}a_i\rho_j - \tfrac{1}{2}M_{ij}a_ia_j - P(\rho)$$

$$- a_iP_{,i}(\rho) - \tfrac{1}{2}a_ia_jP_{,ij}(\rho) - \cdots - J_i\rho_i - J_ia_i\} \qquad (3.7)$$

$$= \exp\{ J_ia_i - M_{ij}a_i\frac{\delta}{\delta J_j} - P(\frac{\delta}{\delta J}) - a_iP_{,i}(\frac{\delta}{\delta J}) - \cdots \}$$

$$\exp\{ \tfrac{1}{2}(M^{-1})_{ij}J_iJ_j\} \qquad (3.8)$$

Here, we have Taylor expanded the interaction P, with the subscripts indicating derivatives with respect to ρ_i. The ellipses indicate the rest of the Taylor expansion, which, of course, has a finite number of terms. Overall constants in Z(J) are thrown out because we are always interested in Green's functions with the vacuum-to-vacuum amplitude divided out.

From 3.8 we may read off the Feynman rules, and compare them with the usual rules (obtained by taking 3.8 with $a_i=0$). The

propagator is the same, $(M^{-1})_{ij}$. There are new vertices, from the Taylor expansion of the interaction, which in graphs can be represented as interaction vertices with some number of external lines replaced with a "tadpoles" representing multiplication by a_i. Some of these new vertices are quadratic, and could also be summed into the propagator, rather than being treated as interactions. There is also a new one-point interaction $M_{ij}a_i$.

To illustrate these rules we consider

$$u_i = \langle \varphi_i \rangle = Z(J)^{-1} \frac{\delta}{\delta J} Z(J) \tag{3.9}$$

$$= a_i - (M^{-1})_{ij} M_{jk} a_k + (M^{-1})_{ij} \sum_{n=0}^{\infty} \frac{1}{n!} \Gamma^{n+1}_{ij_1 \cdots j_n} u_{j_1} \cdots u_{j_n}$$

This equation is represented in figure 7. The first term is from the action of the functional derivative on the term $J_i a_i$. In all other terms the derivative acts on the quadratic in the last line of 3.8, giving rise to a propagator and a connected graph. The second term of 3.9 is from the graph where this propagator ends on a one-point vertex $M_{ij}a_j$. In all other graphs this propagator ends on a vertex v_1 from P (which may have one or more tadpoles attached). We identify the one particle irreducible (1PI) part γ containing v_1 as v_1 plus all vertices which cannot be separated from v_1 by cutting a single line (v_1 may be the only vertex in γ), plus all lines connecting two vertices of γ. Note that all vertices of γ are from P; none are of the form $M_{ij}a_j$ as these can be separated from the rest of the graph by cutting a single line. γ is then independent of a_i, since by definition no tadpoles are included in it. Γ^{n+1} is defined as the sum of all such γ with n external lines besides the one which is

connected to the original propagator. Consider now the external lines of $\Gamma^{\sim}$: these may be a tadpole or a propagator, and the propagator may end on a $M_{ij}a_j$ vertex or a vertex from P (which then defines a new 1PI part). These are the same three choices we originally had, and sum up to give u_i again; summing over all numbers of external lines, and checking on the combinatorics, we get the last term of 3.9.

The first two terms of 3.9 sum to zero; only the last term, which is independent of a_i, contributes. Iterating figure 7 graphically, we see that u_i is given by the sum of all "tadpole trees"[39]. Throughout this section, "tree" refers to a term in the iteration of an equation such as figure 7, not to a graph without loops. Equation 3.9 can be rewritten

$$\frac{d}{dv_i}\Gamma(v)\Big|_{v_i=u_i} = -\sum_{k=0}^{\infty}\frac{1}{k!}\Gamma^{k+1}_{ij_1\cdots j_k}u_{j_1}\cdots u_{j_k} = 0 \quad (3.10)$$

where

$$\Gamma^2_{ij} = \Gamma^{\sim 2}_{ij} + M_{ij} \quad (3.11a)$$

$$\Gamma^n = \Gamma^{\sim n}, \qquad n\neq 2 \quad (3.11b)$$

are the 1PI Green's functions as usually defined (by the Legendre transformation)[40] and

$$\Gamma(v) = -\sum_{k=0}^{\infty}\frac{1}{k!}\Gamma^{k}_{j_1\cdots j_k}v_{j_1}\cdots v_{j_k} \quad (3.12)$$

is the quantum effective potential. Equation 3.10 indicates that u_i is an extremum of the effective potential; in fact it must be an absolute minimum or one will eventually run into pathologies in the

Green's functions indicative of an unstable vacuum[41].

It is not surprising that u_i is formally independent of a_i; formally, the expansion represents the whole integral, regardless of what value of the field we expand around. The sum of tadpole trees is, however, ambiguous when there is more than one minimum, as in the case of broken symmetry. If we try to sum the trees iteratively by gathering the last two terms of 3.9 into one term, which we treat as being small, and taking the first term, a_i, as a first appoximation to u_i, the result now depends on the value of a_i chosen. The value of a_i determines which extremum of the effective potential the sum converges to; for some values of a_i it diverges. The most efficient way to sum the trees is to calculate the effective potential and locate its absolute minima. Taylor[39] has emphasized the role of the effective potential in summing the tadpole trees. This is a rather trivial example of the idea that the convergence of a perturbation series depends on having the right sort of vacuum as a first approximation; one might hope, however that the effective potential would also be a powerful tool in the less trivial case where a composite operator is acquiring a vacuum expectation value[42].

We can now write down a general form for a connected k-point Green's function (k>1) in a Higgs phase:

$$< \psi_{i_1} \cdots \psi_{i_k} >_c = \sum_{n=0}^{\infty} \frac{1}{n!} \bar{G}_{i_1 \cdots i_k; j_1 \cdots j_n} u_{j_1} \cdots u_{j_n} \qquad (3.13)$$

where $\bar{G}$ is given by the sum of all connected Feynman graphs such that there is no way to cut a single line and be left with two pieces, one of which is connected to none of the external points $i_1, \ldots, i_k$. $\bar{G}$ is

1PI in the indices j, but not in the indices i. To see 3.13, note that $\bar{G}$ is essentially the general Feynman graph with all tadpoles and trees amputated, and the u_i in 3.13 then represent the sum of all possible attachments of tadpoles and trees.

In the case at hand, an Abelian Higgs theory, ψ_i stands for the fields $A_\mu(x)$ and $\phi(x)$. u_i is expected to be zero for A_μ and some nonzero value u (which can be chosen real) for ϕ and ϕ^*. That is, we draw graphs where either end of a charged line may end on a tadpole, whose value is u. The presence of $\bar{G}$ would complicate the graphical analysis for the vortex operator, so instead of 3.13 we use the equivalent

$$< \psi_{i_1} \cdots \psi_{i_k} >_c = \bar{\Sigma} \text{ (all connected graphs, with trees)} \quad (3.14)$$

The meaning of 3.14 is made clearer by comparing it with 3.13. The tadpoles u_i have each been replaced by the tree sum to which they are equivalent; there is therefore no longer a 1PI requirement on the graphs. We may see that 3.14 is correct at k=1, as it is there simply the sum of trees, which is u_i.

Equation 3.14 holds in a symmetric as well as in a Higgs phase; the only difference is that in a symmetric phase the fields as normally defined have vanishing expectation value, so the trees sum to zero and we can neglect them. To evaluate $<V_p'(C,S)>$, equation 2.22, in a Higgs phase, fix the configuration of charged lines, and internal and external photon lines, summing over all numbers of vortex photons. As before, we get a graph of the same configuration, but with no vortex photons and with all propagators and vertices replaced by the effective ones. Summing over all configurations, we simply

recover equation 2.35 (which includes graphs with trees).

The Higgs phase is distinguished, not by the formal expression for the Green's functions, but by the nature of the tree sum. We emphasize this by rewriting the Green's function in 2.35 in a form analogous to 3.13, that is, with the trees summed. We find, again in shorthand notation,

$$< \psi_{i_1} \cdots \psi_{i_k} >_{cvS} = \tag{3.15}$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \tilde{G}^{vS}_{i_1 \cdots i_k ; j_1 \cdots j_n} u^{vS}_{j_1} \cdots u^{vS}_{j_n}$$

where $\tilde{G}^{vS}$ is partially 1PI as before, but is now constructed from the effective vertices and propagators, and u^{vS}_j is the sum of trees with effective propagators and vertices. u^{vS}_j satisfies the iterative equation shown in figure 8. It therefore is the extremum of the "effective potential in the presence of the vortex"

$$\Gamma^{vS}(v_i) = \frac{1}{2} M^{vS}_{ij} v_i v_j - \sum_{k=1}^{\infty} \frac{1}{k!} \tilde{\Gamma}^{vS}_{i_1 \cdots i_k} v_{i_1} \cdots v_{i_k} \tag{3.16}$$

Here v_i represents A_μ, ϕ, and ϕ^*, $\tilde{\Gamma}^{vS}$ is the sum of 1PI graphs with the effective propagators and vertices, and the first term is explicitly

$$M^{vS}_{ij} v_i v_j = \int d^4x \, \frac{1}{2} \partial_\mu A_\rho (x) \partial_{(\mu} A_{\rho)}(x) + (D^{vS}_\mu \phi(x))^* (D^{vS}_\mu \phi(x)) \tag{3.17}$$

One may verify that under a change of surface

$$\Gamma^{vS'}(A_\mu, \phi, \phi^*) = \Gamma^{vS}(A_\mu, g(S,S')^* \phi, g(S,S') \phi^*) \tag{3.18}$$

From 3.18, it follows that

$$A_\mu^{tS'}(x) = A_\mu^{tS}(x) \tag{3.19a}$$

$$\phi^{tS'}(y) = g(S,S';y)\phi^{tS}(y) \tag{3.19b}$$

where the superscript tS indicates the extremum of the effective potential, that is, the sum of tree graphs constructed with the effective propagators and vertices for the operator $V_p'(C,S)$. The expectation value of ϕ is just ϕ^{tS}. For the expectation value of A_μ, there are the graphs where the external photon ends on a vertex--these sum up to A_μ^{tS}-- and there is the graph where it ends on S--this is simply A_μ^{vS}. It follows from 2.34b and 3.19a that the expectation value of A_μ, which is $A_\mu^{tS}+A_\mu^{vS}$, satisfies 2.37, as does the expectation value of ϕ.

The Green's functions are then given by graphs constructed from the effective propagators and vertices, with tadpoles corresponding to the tree sums for A_μ and ϕ in the presence of the vortex, rather than their free values. The tree sum for ϕ has the expected singularity on S, as we can see from 3.19b. This is equally true in the Higgs and symmetric phase; the difference, again, is in the nature of the extremum of the effective potential. If we take a vortex operator with C and S bounded, distant from C and S the propagators and vertices are essentially the same as in vacuum, and so will be the effective potential. The tadpole u_i^{vS} will therefore have its free value. For the symmetric theory this is 0 for all fields; in the Higgs theory it is a constant $ue^{i\alpha}$ for ϕ and zero for A_μ. Arguing as in the last section, in the Higgs phase there must be at least a surface (of some thickness) spanning C on which the modulus of the expectation value of the Higgs field is not u (it must in fact go

through zero). In the symmetric phase, however, $\phi^{VS}(x)$ may have its vacuum value of zero everywhere (except very close to C), as 3.19 is linear in ϕ and so is trivially satisfied by zero. The surface cluster property is seen to emerge from the sum of all trees, even though each term in the sum can be shown as before to have short range clustering.

CHAPTER 4

UNQUANTIZED VORTEX OPERATORS

A. Continuum Theories with Fields in the Fundamental Representation

In the foregoing chapters, a certain quantization condition has played a central role. This is, that if there are fields present in a representation s, the vortex operator must commute with the Wilson loop for the representation s. We consider here the case of a system which does not have vortex operators, but which closely resembles a different system which does.

At least two such systems are of interest. The first is an ordinary superconductor. The Cooper pairs, which form the Higgs field, have charge 2e. This field will support flux tubes quantized in units of $\frac{2\pi}{2e}$, and tubes with a single unit of flux are in fact observed experimentally[43]. We would expect to be able to define an operator which creates such a flux tube, and whose surface-like cluster property would be a signature of the stability of the tube and the existence of the Higgs phase. On the other hand, electrons and holes, with charge $\pm e$, are also present, and as a result we seem to be able to define the vortex operator only for even numbers of flux units. Since these "sea" electrons and holes do not appear to radically alter the properties of the single-unit flux tubes, the quantization condition does not seem to reflect the physics of the system.

The second system is QCD, where, as discussed in section 3A, it appears that the introduction of quarks has removed all distinction between phases. In an SU(3) gauge theory without fields in the fundamental representation, we would expect to be able to distinguish a

confining from a Higgs phase by the stability of electric or magnetic flux tubes. It is clear that the introduction of quarks makes it impossible to observe an electric flux tube, since they can shield an electric source. It is not at all clear, though, why they should make it impossible to observe a magnetic flux tube, and we might still expect to be able to define an operator whose cluster property would enable us to say definitely that QCD is or is not in a Higgs phase.

In this chapter, we first show that without the quantization condition, a vortex operator cannot be "local looplike", in the sense of commuting with every gauge invariant operator at a spacelike separation from the given closed curve. By analogy with QCD, we call representations which violate the quantization condition fundamental, and those which respect it adjoint. (This designation depends on which vortex we are considering, of course: electrons are fundamental with respect to odd-unit Abrikosov vortices but adjoint with respect to even-unit vortices.) We show that we can define a vortex-like operator which has a surface-like cluster property in a phase with adjoint Higgs field and fundamental non-Higgs field. We argue that in a confining phase with the particles in the fundamental representation weakly coupled the operator would not have surface-like clustering, so it does serve to characterize the Higgs phase. This implies that there can be a Higgs-confining phase transition when particles in the fundamental representation are present, provided the Higgs field is in the adjoint representation. We point out a possible paradox involving 't Hooft's restriction on the possible cluster properties, and suggest its resolution. We then discuss the

extension of these ideas to a non-Abelian theory. In a second section we give a short discussion of a lattice gauge theory with two matter fields, one fundamental and one adjoint, which provides an excellent model for our problem. We show that when the fundamental field is weakly coupled, there are the expected phase transitions, and that our modified vortex operator is a good order parameter. When the fundamental field is strongly coupled, it is not clear whether our operator should still be a good order parameter, as it is quite nonlocal. We investigate this question and find some indication that it is a good order parameter; we then discuss why this might be so, and discuss areas for future investigation.

Consider first the operator given by 2.4 or 2.8 when p is not an integer. The commutation relation 2.6a is then discontinuous on the surface S where the angle θ is discontinuous. It then follows that the gauge invariant operator

$$\Theta(\vec{x},\vec{y};P) = \phi^*(\vec{x})\exp\{ie\textstyle\int_P dx'_i A_i(\vec{x}')\}\phi(\vec{y}) \qquad (4.1)$$

will not commute with $V_p(C,S)$ or $V_p'(C,S)$ if P crosses S. This is true even if $\vec{x}$ and $\vec{y}$ are very close together but on opposite sides of S, so that $\Theta(\vec{x},\vec{y};P)$ is essentially local. The local operator $\phi^*(x)\vec{D}\phi(x)$ can be obtained from $\Theta(x,y;P)$ as a limit when $\vec{y}\to\vec{x}$; it fails to commute with V_p or V_p' when x is on S. We also note that a meson-like bound state of ϕ and ϕ^* at position z is created by a superposition of operators 4.1 with x and y near z. For z near S, the V_p and V_p' will commute with some of these operators (those which do not cross S) but not with others, so that it multiplies only certain components of the meson wavefunction by a phase, leaving

something which no longer looks like the same kind of meson. We conclude that when the quantization condition is satisfied, the operators 2.4 or 2.8 are surface-like rather than looplike; their Green's functions would be expected to cluster near the surface S, and they would obey an area law, regardless of the nature of the vacuum.

We might wonder whether by being cleverer we could find a local looplike operator which still satisfied equation 2.1, or, if not, just how nonlocal the operator must be. We can make a general argument that if there are fields present in some representation s, and if an operator L multiplies the Wilson loop for this representation by a nontrivial phase z_{Ls} (the essential feature of a vortex operator), then the commutators of L with certain gauge invariants can fall off no faster than $1/\rho$, where ρ is the distance from the vortex:

Note first that

$$[\pi_{s,i}(\vec{x})W_{s,ij}(\vec{x},\vec{y};P)\phi_{s,j}(\vec{y}) \ , \ \pi_{s,k}(\vec{w})W_{s,kl}(\vec{w},\vec{z};P')\phi_{s,l}(\vec{z})]$$

$$= i\pi_{s,i}(\vec{x})W_{s,il}(\vec{x},\vec{z};P+P')\phi_{s,l}(\vec{z})\ \delta^3(\vec{w}-\vec{y}) \qquad (4.2)$$

$$- i\pi_{s,k}(\vec{w})W_{s,kj}(\vec{w},\vec{y};P'+P)\phi_{s,j}(\vec{y})\ \delta^3(\vec{x}-\vec{z})$$

Here ϕ_s is a field of representation s and π_s is its canonical momentum; the operators in 4.2 are then a canonical momentum at one point joined by parallel transport along a path to a field at another point. Equation 4.2 says that the commutator joins two such operators into one whose path is the sum of the first two. From this it follows that a large Wilson loop of radius ρ can be written as the multiple commutator of $2\pi\rho/d$ short segments of length d (to join the

last two endpoints we actually need to take the difference of this multiple commutator with the same multiple commutator with all the orders of π_s and ϕ_s reversed). We can use the Jacobi identity to write the commutator of our vortex-like operator L with this large Wilson loop as the sum of $2\pi\rho/d$ terms each of which involves the commutator of L with a single short segment. Since the total commutator is of order 1 (that is, a phase, independent of ρ) the commutator of L with the short segment can be no smaller than $O(d/\rho)$ as d becomes small or ρ becomes large. The local operator $\pi_{s,i}\vec{D}_{ij}\phi_{s,j}$ with $\vec{D}$ the covariant derivative can be obtained from the segment operators as a limit; its commutator can fall off no faster than $1/\rho$.

The above argument relies heavily on canonical commutators, which are always suspect, but we believe that the principle is correct. Certainly it is correct for the lattice gauge theories we will consider later. In the language of the lattice, when the quantization condition is not satisfied the operators 2.4 and 2.8 introduce a "frustration" on the surface S; this frustration can only be spread out, never eliminated[44]. The resulting operator is then associated with a 3-surface, not a curve or 2-surface (it is not possible to spread it out still further, into the fourth direction). Here we will generally take this surface to be a fixed time-plane, but when we discuss Green's functions we may take any closed 3-surface containing C.

We now consider an Abelian theory again, with a single charged scalar field ϕ of charge e and mass m, and with the gauge symmetry unbroken. We look for an operator $V_p(C)$ satisfying 2.1 (with $e_{min}=e$) but with p non-integral. The spin of the charged field makes no

difference--a fermion works out the same way. We remain, for a while, in a single time slice. For convenience, we will now take C to be the z-axis, though everything we will do generalizes to any curve. We would like to construct an operator which has its frustration spread out, and see if it is useful as an order parameter. We will look for an *operator* which creates a minimum amount of energy; that is, we will choose V_p to minimize

$$E_v = \langle v| H |v\rangle = \langle 0|V_p^+ H V_p|0\rangle \tag{4.3}$$

where H is the Hamiltonian, $|0\rangle$ is the vacuum, and $|v\rangle = V_p|0\rangle$. This energy will be of interest both as a guide in constructing V_p and in its own right. If the $1/\rho$ falloff in the commutators implies a $1/\rho$ or $1/\rho^2$ falloff in the energy density $\langle v|H(\vec{x})|v\rangle$, the total energy per unit length will be infrared divergent. This would imply infinite energy for an isolated Abrikosov vortex, and a long range interaction between pairs of vortices.

The minimum of E_v can be determined without actually constructing V_p. We first smear out 2.1 near the z-axis, to remove the infinite energy associated with an infinitely thin flux tube. The condition on V_p will be

$$V_p W_q(C') = W_q(C') V_p \exp\{\frac{ipq}{e}\oint_{C'} dx_i x_i f(\rho^2)\} \tag{4.4}$$

where ρ is again a cylindrical coordinate and

$$f(\rho^2) = \frac{1}{\rho^2} \qquad \rho > a \tag{4.5a}$$

$$f(\rho^2) = \frac{1}{a^2} \qquad \rho < a \tag{4.5b}$$

The integral in 4.4 gives the winding number of C′ around the z-axis when C′ does not pass within distance a of that axis; this reproduces 2.1. Taking derivatives of 4.4 with respect to e, we find the commutators of V_p with products of magnetic field operators:

$$V_p B_{i_1}(\vec{x}_1)\cdots B_{i_m}(\vec{x}_m) \tag{4.6}$$

$$= [B_{i_1}(\vec{x}_1)+M_{i_1}(\vec{x}_1)]\cdots[B_{i_m}(\vec{x}_m)+M_{i_m}(\vec{x}_m)]V_p$$

with

$$\vec{M}(\vec{x}) = \frac{\hat{z}p}{e\rho}\frac{d}{d\rho}(\rho^2 f(\rho^2)) \tag{4.7}$$

Note that $\vec{M}$ is nonzero only for $\rho<a$. 4.6 and 4.7 are correct with $\vec{B}$ and e either both bare or both renormalized; we shall take all quantities to be renormalized until further notice. Equation 4.6 is equivalent to 4.4; it implies the following constraint on the state $|v\rangle$:

$$\langle v|\ B_{i_1}(\vec{x}_1)\cdots B_{i_m}(\vec{x}_m)\ |v\rangle \tag{4.8}$$

$$= \langle 0|\ [B_{i_1}(\vec{x}_1)+M_{i_1}(\vec{x}_1)]\cdots[B_{i_m}(\vec{x}_m)+M_{i_m}(\vec{x}_m)\ |0\rangle$$

For m=1 and 2 respectively.

$$\langle v|\ B_i(\vec{x})\ |v\rangle = M_i(\vec{x}) \tag{4.9a}$$

$$\langle v|\ B_i(\vec{x}_1)B_j(\vec{x}_2)\ |v\rangle = M_i(\vec{x}_1)M_j(\vec{x}_2) \tag{4.9b}$$

$$+ \langle 0|\ B_i(\vec{x}_1)B_j(\vec{x}_2)\ |0\rangle$$

If 4.9a were the only constraint on the state $|v\rangle$ (together with $\langle v|\phi(x)|v\rangle=0$, which follows because the symmetry is unbroken), the minimum energy would be given immediately by the effective potential[40,41],

$$\mathcal{E}_v = U(\vec{A}_M) \tag{4.10}$$

$$= -\sum_{n=1}^{\infty} \frac{1}{n!} \int d^3x_1 \cdots d^3x_n \, A_{M,i_1}(\vec{x}_1) \cdots A_{M,i_n}(\vec{x}_n)$$

$$\tilde{\Gamma}^n_{i_1 \cdots i_n}(\vec{x}_1, \cdots, \vec{x}_n)$$

where $\vec{A}_M$ is any vector potential such that $\vec{\partial} \times \vec{A}_M(\vec{x}) = \vec{M}(\vec{x})$ and

$$\tilde{\Gamma}^n_{i_1 \cdots i_n}(\vec{x}_1, \cdots, \vec{x}_n) \tag{4.11}$$

$$= \int_{-\infty}^{\infty} dt_2 \cdots dt_n \, \Gamma^n_{i_1 \cdots i_n}(\vec{x}_1, 0; \vec{x}_2, t_2; \cdots; \vec{x}_n, t_n)$$

In 4.11 the Γ^n are the 1PI Green's functions of the vector potential, and we have written out the time coordinate explicitly.

Satisfying 4.8 for all m requires a generalization of the effective potential[42]. Equation 4.10, however, is correct to the first two orders in e, e^{-2} and e^0. To see this, consider adding a term

$$\int d^3x \, J_i(\vec{x}) A_i(\vec{x})$$

to the Hamiltonian, with J_i an external source chosen so that

$$\langle \vec{B}(\vec{x}) \rangle_J = \vec{M}(\vec{x}) \tag{4.12}$$

where the subscript J indicates an expectation value in the ground state of the new Hamiltonian. This ground state is the state of

minimum energy subject to the constraint 4.12 (or 4.9a) and its energy is given by 4.10[41]. Such expectation values can be evaluated in terms of Feynman graphs; the graphs for the matrix element 4.12 are shown in figure 9. The graphs for the matrix element $\langle \vec{B}(\vec{x})\vec{B}(\vec{y})\rangle$ are shown in figure 10. We see that 4.9b is satisfied to order e^{-2} and to e^{0}, but fails at order e^{2} (note that $\vec{A}_M$ and J_i are of order e^{-1}). To produce a state satisfying 4.9b, we would have to add to the Hamiltonian a term of the form

$$\int d^3x\, d^3y\, K_{ij}(\vec{x},\vec{y})A_i(\vec{x})A_j(\vec{y})$$

with the bilinear source K_{ij} of order e^2; this would only affect the energy at order e^2. The same is true for the higher constraints.

Since the charged field contributes first at order e^0, we will only study the energy to this order, and 4.10 is satisfactory.

$$E_v = \frac{Z_3}{2}\int d^3x\, \vec{M}^2(\vec{x}) - \sum_{n=2}^{\infty} \frac{1}{n!}\int d^4x_1 \cdots d^4x_n$$

$$\delta(t_1)A_{M,i_1}(\vec{x}_1)\cdots A_{M,i_n}(\vec{x}_n)\Gamma^{n,1\ \mathrm{loop}}_{i_1\cdots i_n}(x_1,\cdots,x_n) + O(e^2)$$

The order e^0 term is from graphs with one charged loop propagating in the external field $\vec{A}_M$. The cutoff dependence from the Z_3 in the zero loop term cancels the divergence from the 1 loop term with n=2; the effective potential is well known to be ultraviolet finite. The four-dimensional integrals in 4.13 can be taken either in Minkowski or Euclidean space; this is just a contour rotation. It is convenient to take them in Euclidean space; the 1PI Green's function then falls off at least as $\exp(-m|x_i-x_j|_{max})$. We can write

$$\Gamma^{n,1\ \mathrm{loop}}_{\mu_1\cdots\mu_n}(x_1,\ldots,x_n) = \frac{\partial}{\partial x_1^\rho}\tilde{\Gamma}^{n}_{\rho\mu_1;\mu_2\cdots\mu_n}(x_1,\ldots,x_n) \qquad (4.14)$$

where

$$\tilde{\Gamma}^{n}_{\rho\mu_1;\mu_2\cdots\mu_n}(x_1,\ldots,x_n) \qquad (4.15)$$

$$= \int_1^\infty d\beta\ \beta^2\ (x_1-\bar{x})_{(\mu_1}\Gamma^{n,1\ \mathrm{loop}}_{\rho)\mu_2\cdots\mu_n}(\beta x_1+(1-\beta)\bar{x},x_2,\ldots,x_n)$$

and $\bar{x}$ is the mean of $x_2,\ldots,x_n$. Because $\beta>1$, the arguments of Γ on the right-hand side are no closer than those of $\tilde{\Gamma}$ on the left; $\tilde{\Gamma}$ then also falls at least as fast as $\exp(-m|x_i-x_j|_{max})$. Inserting 4.14 in 4.13 and integrating by parts gives

$$E_v = \frac{1}{2}\int d^3x\ \vec{M}^2(\vec{x}) - \sum_{n=2}^{\infty}\frac{1}{2\cdot n!}\int d^4x_1\cdots d^4x_n\, e_{hjk} \qquad (4.16)$$

$$\delta(t_1)M_h(\vec{x}_1)A_{M,i_2}(\vec{x}_2),\ldots,A_{M,i_n}(\vec{x}_n)\tilde{\Gamma}^{n}_{jk;i_2\cdots i_n}(x_1,\ldots,x_n) + O(e^2)$$

(There is no surface problem with the integration by parts, as can be seen by considering the limit of a large but finite curve C). $M_h(\vec{x}_1)$ is nonzero only for $\vec{x}_1$ within a distance a of the curve C, and $\tilde{\Gamma}$ falls off rapidly at large distance. Therefore, the integrand is small when any of the x_i is far from the origin: there is no infrared divergence in 4.16. In fact, the energy density in the state can be shown to fall off exponentially. Similarly it may be checked that the interaction between distant vortices due to the electron sea is exponentially small (to this order in e).

Equation 4.13 is a perturbation series in p, which we have learned to distrust. This particular series, however, can be shown to converge for any p. The reason is essentially dimensional: each

extra power of p in 4.13 comes with an extra propagator, $O(m^{-1})$, and an extra power of $\vec{A}_M$, which turns out to be $O(a^{-1})$, a being the smearing radius. As long as we take $a > \frac{p}{2m}$, the series converges; since we are only interested in the infrared divergence of the energy, we may take a to be as large as we like.

We now try to find a simple operator which satisfies 4.4 and which creates the minimum energy state. Our first try is

$$V_{1,p} = \exp\{i\int d^3x\, \vec{A}_M(\vec{x})\cdot\vec{E}(\vec{x})\} \tag{4.17}$$

This is also the form of the vortex operator when there are no charged fields at all. If we need to consider a specific form for $\vec{A}_M$, we expect the rotationally symmetric form $p\hat{\theta}f(\rho^2)/e$ to have the lowest energy. With this choice of $\vec{A}_M$, the commutators of $V_{1,p}$ have the minimal $1/\rho$ falloff discussed earlier; with other choices they fall off more slowly, at least in some directions. To calculate the energy 4.3, we use

$$V^+_{1,p}\, H\, V_{1,p} = H - i[\int d^3x\, \vec{A}_M(\vec{x})\cdot\vec{E}(\vec{x}),\ H\ ] \tag{4.18}$$

$$- \tfrac{1}{2}[\int d^3x\, \vec{A}_M(\vec{x})\cdot\vec{E}(\vec{x}),\ [\int d^3y\, \vec{A}_M(\vec{y})\cdot\vec{E}(\vec{y}),\ H\ ]] + \cdots$$

We will evaluate 4.18 using canonical commutators, so we must take all quantities to be bare. The second term on the right-hand side of 4.18 can be obtained directly from the equations of motion; it is

$$\int d^3x\, \vec{B}(\vec{x})\cdot\vec{M}(\vec{x}) - \vec{j}(\vec{x})\cdot\vec{A}_M(\vec{x}) \tag{4.19}$$

with $\vec{j}$ the current. The third term of 4.18, with the double commutator, can now be evaluated; it is

$$\int d^3x \, \frac{1}{2} \vec{M}^2(\vec{x}) + e^2 \vec{A}_M^2(\vec{x}) \phi^*(\vec{x}) \phi(\vec{x}) \tag{4.20}$$

It is important that because of gauge invariance, the Hamiltonian is not fully normal ordered (though it is permitted to add a constant to H so that the energy of the vacuum vanishes), and so the product $\phi^*\phi$ in 4.20 is not normal ordered. The triple and higher commutators vanish, so the energy E_1 of the state $V_{1,p}|0\rangle$ is the vacuum expectation value of H (zero by definition) plus the vacuum expectation value of 4.19 (zero by rotational invariance) plus the vacuum expectation value of 4.20, which is

$$E_1 = \int d^3x \, \frac{1}{2} \vec{M}^2(\vec{x}) + \vec{A}_M^2(\vec{x}) \langle 0 | \phi^*(\vec{x}) \phi(\vec{x}) | 0 \rangle \tag{4.21}$$

The vacuum expectation value of $\phi^*\phi$ is quadratically divergent, so $V_{1,p}$ appears to be a poor guess (with charged fermions, the same quadratic divergence arises from a Schwinger term). The problem is that it creates a coherent state; coherent states in interacting theories generally have divergent energies[20]. In particular, $V_{1,p}$ creates a photon field, but the vacuum fluctuations of the charged field are not correlated with the photons. To correct this we try

$$V_{2,p} = V_{1,p} \exp\{i \int d^3x \, d^3y \, F(\vec{x},\vec{y}) \phi^*(\vec{x}) \phi(\vec{y}) \tag{4.22}$$

$$+ G(\vec{x},\vec{y}) \phi(\vec{x}) \pi(\vec{y}) + G^*(\vec{x},\vec{y}) \phi^*(\vec{x}) \pi^*(\vec{y}) + H(\vec{x},\vec{y}) \pi^*(\vec{x}) \pi(\vec{y}) \}$$

$$= V_{1,p} \Delta$$

where $\pi(\vec{x})$ is again the canonical momentum and F, G, and H are unknown functions. The energy of the state $V_{2,p}|0\rangle$ is found to be

$$E_2 = \int d^3x \{\frac{1}{2}\vec{M}^2(\vec{x}) + \langle\Delta|H_0(\vec{x})+H_1(\vec{x})|\Delta\rangle\} + O(e^2) \qquad (4.23)$$

where $|\Delta\rangle = \Delta|0\rangle$ and

$$H_0(\vec{x}) = :\pi^*(\vec{x})\pi(\vec{x}) + \vec{\delta}\phi^*(\vec{x})\cdot\vec{\delta}\phi(\vec{x}): \qquad (4.24a)$$

$$H_1(\vec{x}) = ie\vec{A}_M\cdot[\phi^*(\vec{x})\vec{\delta}\phi(\vec{x}) - \phi(\vec{x})\vec{\delta}\phi^*(\vec{x})] \qquad (4.24b)$$

$$+ e^2\vec{A}^2(\vec{x})\,\phi^*(\vec{x})\phi(\vec{x})$$

H_0 and H_1 are quadratic in the fields and are both of order e^0. Interaction terms contribute at order e^2 and have been dropped. H_0 is normal ordered, essentially because the energy of the vacuum is defined to be zero, but H_1 is not normal ordered.

The functions F, G, and H can always be chosen so that $|\Delta\rangle$ is the ground state of H_0+H_1. we require $a'_i\,\Delta = \Delta\,a_i$, where the a_i annihilate eigenstates of H_0 (plane wave states) and the a'_i annihilate eigenstates of H_0+H_1. The particular form of Δ depends on how we pair the eigenstates of the two Hamiltonians (perturbation theory in H_1 gives a natural pairing). We will not determine an explicit form for Δ, but will simply work out the ground state energy. We do this using stationary state perturbation theory in H_1; the complete expression for the energy is[45]

$$E_\Delta = \langle 0|\, H_1\left(1 + \frac{1}{E_\Delta - H_0}[1-|0\rangle\langle 0|]H_1\right)^{-1}\,|0\rangle \qquad (4.25)$$

where $|0\rangle$ is the free field vacuum. Equation 4.25 can be expanded perturbatively in H_1. The first term is just the vacuum expectation value of H_1, which is the quadratic divergence found earlier. There is a second quadratic divergence from second order perturbation

theory in the first term of H_1. These two quadratic divergences cancel exactly (we use Pauli-Villars regulators to tell us how to add divergent terms) leaving a logarithmic divergence. This logarithmic divergence cancels against the divergence from the $\vec{M}^2$ term, which is present because the e which appears in the denominator of M is bare in this calculation.

These cancellations may be verified by direct calculation. A simpler way to see them, however, is to note that the first order term is exactly the contribution of the graph 11a for the effective potential, while the divergent second order term can be cast in the form

$$- \int d^3x\, d^3y \,\langle 0|\, \vec{A}_M(\vec{y})\cdot\vec{j}(\vec{y})\, \frac{1}{H_0}\, \vec{A}_M(\vec{x})\cdot\vec{j}(\vec{x})\,|0\rangle \tag{4.26}$$

$$= - \int d^3x\, d^3y \int_0^\infty dy_0 \langle 0|\, \vec{A}_M(\vec{y})\cdot\vec{j}(\vec{y})\, \exp(-H_0 y_0)\, \vec{A}_M(\vec{x})\cdot\vec{j}(\vec{x})\,|0\rangle$$

$$= - \frac{1}{2}\int d^4x\, d^4y\, \delta(x_0)\, T\langle 0|\, \vec{A}_M(\vec{y})\cdot\vec{j}(y)\, \vec{A}_M(\vec{x})\cdot\vec{j}(x)\,|0\rangle_I$$

where the subscript I indicates the interaction representation and $\vec{j}$ is the free field current. Equation 4.26 is just the contribution of the graph of figure 11b to the effective potential. The cancellation of the quadratic divergences of graphs 11a and 11b is well known; further, the $\vec{M}^2$ term in 4.21 is exactly the zero loop term of 4.13, so the logarithmic divergences cancel as well. In fact, the whole expansion of 4.25 can be cast in the form of the one loop effective potential: $V_{2,p}$ creates the state of minimum energy, to this order. The idea of $V_{2,p}$ can be extended to higher orders in e, but the form of the operator becomes more complicated in each order.

With additional charged fields present, the one loop contribution to the effective potential is just a sum of contributions from each charged field. In particular, when there is a Higgs field (of charge ne/p, n an integer, so that it respects the quantization) the energy of a flux tube is just the energy of a flux tube without the field of charge e, plus the one loop term found above. The operator which creates the minimum energy state, to this order, is simply a product of terms: the term $V_{1,p}$, times a term of the form of Δ for each field not respecting the quantization, times a term of the form $\exp(\frac{ip}{e}\int d^3x\ \theta j_0(\vec{x}))$ for each field respecting the quantization. As before there is a surface S from the discontinuity of θ. (Does Δ go over to this latter form as we vary the charge of a field from unquantized to quantized? It depends on how we choose to pair the states when we define Δ, but they can be so chosen.) We should note that when one of the fields which does not respect the quantization acquires an expectation value, the minimum energy becomes infrared divergent (the energy density goes as $1/\rho$) and $V_{2,p}$ no longer has any nice properties. This is in accord with expectations: we would not expect to have flux tubes then, even classically.

The commutator of $V_{2,p}$ with the Hamiltonian density can be checked to fall off as $1/\rho$, whereas the energy density falls off much faster:

$$\langle 0|V_{2,p}^{+}\ H(\vec{x})\ V_{2,p}|0\rangle \simeq \exp(-m\rho) \qquad (4.27)$$

The point is that the operator $V_{2,p}$ has been carefully tailored (by the minimum energy construction) so that the state $V_{2,p}|0\rangle$ very closely resembles the vacuum state when we move away from the curve

C. We might then expect that when we go to four dimensions and look at the connected Green's functions of $V_{2,p}$, there would be no anamolous clustering near the 3-plane t=0 (on which $V_{2,p}$ is defined) but far from the curve C. This is in fact the case. For instance, we have checked that in a QED-like phase, the connected Green's function $< \theta(x,y;P)\ V_{2,p}(C) >$ (see 4.1) falls off as $[|P|/d(x,C)]^2$ in all directions when the length of P, $|P|$, is small compared to $d(x,C)$. This is the same behavior as when the quantization condition is satisfied, and holds even though the commutator of θ with $V_{2,p}$ falls off only as $|P|/d(x,C)$. Similarly, in a Higgs phase, the same Green's function falls exponentially when x is not near the minimal surface. It can also be seen that when there is a Higgs field which respects the quantization and another field which does not, the vortex operator still has a surface-like cluster property: The Higgs field still has the singularity 2.37, so the discussion following equation 3.3 still holds, and there must be a surface spanning C on which the expectation value of the Higgs field is zero and the expectation value of the Higgs potential is greater than its vacuum expectation value. Cluster properties and vacuum expectation values of $V_{2,p}$ in perturbative phases are then the same as for the vortex operator in theories with only adjoint fields.

In order to have the correct commutation relation, we must have

$$V_{1,p} = \exp\{iZ_3 \int d^3x\ \vec{A}_{r,M}(\vec{x})\cdot\vec{E}_r(\vec{x})\} \qquad (4.28)$$

where $\vec{A}_{r,M}$ is cutoff independent: its curl is 4.7 with the denominator containing e_r. The Green's functions of $V_{1,p}$ are then ultraviolet divergent; to renormalize them requires adding counterterms to

the exponent in all of space, not just on the curve C. The operator Δ provides just these counterterms: improving the ultraviolet behavior of the energy has also improved the ultraviolet behavior of the Green's functions. To show this, we note that $V_{2,p}$ can be written

$$V_{2,p} = \exp\{i\int d^3x\, \vec{A}_{r,M}(\vec{x})\cdot\vec{E}_{phys}(\vec{x}) + O(p^2)\} \qquad (4.29)$$

where

$$\vec{E}_{phys}(\vec{x}) = Z_3\vec{E}_r(\vec{x}) + \vec{T}(\vec{x}) \qquad (4.30)$$

and

$$\vec{T}(\vec{x}) = ie\int_0^\infty du\, \exp(iH_0u)[\phi^*(\vec{x})\vec{\partial}\phi(\vec{x}) - \phi(\vec{x})\vec{\partial}\phi^*(\vec{x})]\exp(-iH_0u) \qquad (4.31)$$

Performing the u integration, one sees the connection between equation 4.31 and the wavefunction correction in stationary state perturbation theory[53]. The higher order corrections in the exponent of 4.29 can be neglected because they are less singular ($\vec{A}_M$ has dimensions of mass). $\vec{T}$ is bilinear in the charged field (the u integral smears out the two charged fields) and so has the correct form 4.22. The matrix elements of Δ will have a composite divergence from the graph of figure 12a if $F(\vec{x},\vec{y})$, $G(\vec{x},\vec{y})$, or $H(\vec{x},\vec{y})$ is too singular as $\vec{y}\to\vec{x}$; this divergence has the form of figure 12b, which is the same as the cutoff dependence in the first term of $\vec{E}_{phys}$. We have calculated the ultraviolet behavior of the graph of figure 12a, and find that its cutoff dependence exactly cancels that from the Z_3. $\vec{E}_{phys}$ therefore has finite Green's functions as well as the finite commutator

$$[E_{phys,i}(\vec{x}),B_{r,j}(\vec{y})] = i\epsilon_{ijk}\frac{\partial}{\partial x^k}\delta^3(\vec{x}-\vec{y}) \qquad (4.32)$$

The "physical field" $\vec{E}_{phys}$ thus has certain desireable characteristics of the bare field (finite commutators), and certain desireable characteristics of the renormalized field (finite Green's functions), at the cost of being nonlocal. The Green's functions of our particular $\vec{E}_{phys}$ are divergent in the next order in e; it seems likely, though, that a suitable operator can be constructed order by order in e.

Equation 4.27, the exponential falloff of the energy density, is not so much a property of the operator $V_{2,p}$ as it is a combined property of this operator and the state $|0>$; as we have noted, $V_{2,p}$ has been tailored to leave the vacuum almost invariant at large distance. As we change parameters in the theory (such as charged field masses), the necessary $V_{2,p}$ will change. As an "order parameter" it might be more appropriate to have one operator which we can apply to any vacuum. This also is possible: we can simply take

$$V_{3,p} = \exp\{i\int d^3x\, \vec{A}_M(\vec{x})\cdot\vec{E}_{phys}(\vec{x})\} \qquad (4.33)$$

where $\vec{E}_{phys}$ is chosen to have finite Green's functions and the right commutator, and to be of finite range. $\vec{T}(\vec{x})$ as given by 4.31 does not have finite range, but if we simply cut it off beyond some distance, it will be of finite range and still have finite commutators and Green's functions. In a QED-like phase, the connected Green's function $< \theta(x,y;P)\, V_{3,p}(C) >$ falls off as $[|P|/d(x,C)]^2$, except within $O(m^{-1})$ of the 3-plane t=0, where it only falls off as $|P|/d(x,C)$. Its Green's functions differ from those of $V_{2,p}$, and

from those of the quantized vortex operator, by an additional piece which is large only near the 3-plane on which the operator is defined and which falls off only as fast as the commutator. This is what we might expect for an operator not chosen to "fit onto" the vacuum state just right. The same thing happens to $V_{3,p}$ in a Higgs phase: the connected Green's function is still exponentially small away from the 3-surface of the operator and away from the minimal 2-surface of C (which need not lie in the 3-surface) but it now has a piece $O(|P|/d(x,C))$ near the 3-surface. The arguments requiring the Higgs field to have a surface of zeroes and a large energy density on the surface still hold. The connected Green's function of the Lagrangian or Hamiltonian density with $V_{3,p}$ is the same as in the case of the quantized vortex operator, plus an additional piece near the 3-surface which is of order $d(x,C)^{-2}$; the extra piece is not of order $d(x,C)^{-1}$ because $\langle \vec{E}_{phys}(x)L(y)\rangle$ vanishes by C-invariance. The integral of this over d(x,C) diverges logarithmically and is cut off at the size of C, roughly P; the integral around the perimeter then gives a total of $O(P\cdot\ln(P))$, and the expectation value is the exponential of this.

Since we do not have a perturbative model of confinement we must be more heuristic when discussing a confining phase. Imagine a confining theory, such as an Abelian theory with a magnetic monopole Higgs field, to which we add a charged field which violates the quantization for some of the vortex operators. Suppose that the additional field affects the dynamics only weakly, as quarks are believed to in QCD, so that we may expand in the number of loops of the new field. Assume also that there is no absolutely massless

pion, so that we may take the loops to be localized in spacetime. The new vortex operator, $V_{3,p}$, is the old one times $\exp\{i\int d^3x\, \vec{A}_M(\vec{x})\cdot\vec{T}(\vec{x})\}$. Consider the vacuum expectation expectation value of $V_{3,p}$. Loops near C can only contribute O(P). Loops far from the 3-surface (and therefore far from C) are unaffected by the presence of the vortex operator (since the operator about which we are expanding has short range clustering) and do not contribute to the expectation value. Loops near the three surface but far from C give contributions of the form

$$\int d^3x_1 \cdots d^3x_n A_{M,i_1}(\vec{x}_1)\cdots A_{M,i_n}(\vec{x}_n) < T_{i_1}(\vec{x}_1)\cdots T_{i_n}(\vec{x}_n) >_c \quad (4.34)$$

This vanishes by C-invariance when n=1. When n=2 it is logarithmically divergent at large ρ, as $\vec{A}_M$ is $O(1/\rho)$, and for larger n there is no divergence. This is exactly as in the other phases, and leads to the same $P\cdot\ln(P)$ in the exponent. In the QED and confining phases this piece dominates the usual perimeter term, but in a Higgs phase the area term still dominates. In the same way, one may see that the cluster property is now short range plus a new piece near the t=0 plane, falling off slowly with ρ.

We appear to have a general rule for the cluster property when fields in the fundamental representation are added: to the original cluster property is added a piece which is non-zero only near the 3-plane of the operator and which falls off rather slowly, like the commutators. The cluster properties for the three kinds of phase are still distinct. Note again that we may choose, for ease in identifying the cluster property, to spread out the "frustration" on a 3-surface other than t=0, so that we may take one which does not

contain the minimal surface. We are led to conjecture that in systems such as a superconductor or QCD, with the particles in the fundamental representation having only a small effect on the dynamics, we can distinguish a Higgs phase from a confining phase based on the expectation value or cluster property of our modified vortex operator.

There is a possible paradox here. As usually stated, the result of 't Hooft is that either the vortex operator or the Wilson loop must satisfy an area law. We have conjectured that in a QCD-like theory, our modified vortex operator follows a P·ln(P) law, while the Wilson loop in a confining theory with particles in the fundamental representation satisfies a perimeter law because of screening. Expressed in terms of cluster properties, there is no paradox. 't Hooft's result was that both loops could not have short range clustering. The vortex operator here does not have short range clustering; spacetime near the t=0 surface does not look like vacuum. When the Wilson loop in 't Hooft's argument moves through this region, it can give back the phase that it gets from the canonical commutator.

However, in QED-like and Higgs phases we were able to find an operator which did not have this extra clustering. If this were also possible in a confining phase, there might then be a confict with 't Hooft's result; the proof is too heuristic for us to be sure whether there is. It may be that the vortex operators considered in this chapter evade the proof automatically because of their nonlocal commutators. We are inclined to believe that this is not the case, however. The proof considered only the Green's function $\langle W(C')V(C)\rangle$, with C' lying entirely before or entirely after C. This Green's

function depends only on the states $\langle 0|V(C)$ and $V(C)|0\rangle$, not on any other properties of V(C). Short range clustering for V(C) then implies that these states are locally vacuum away from C; this should be enough for the proof to go through.

We cannot be sure, then, whether short range clustering is in fact excluded for any vortex operator when there are fields in the fundamental representation, but we conjecture that it is. (In our earlier construction of a vortex operator with short range clustering ($V_{2,p}$), we had to treat the dynamics perturbatively; it is not at all clear that we could define such an operator for a confining vacuum.) This leads to the related conjecture that it is not possible to satisfy equation 4.8 (or its non-Abelian analogue) with a state of finite energy in a confining theory. This would provide an additional means of distinguishing a confining theory.

We should mention that all of this chapter can be extended to a non-Abelian theory. In that case, there are special problems with defining a gauge invariant operator. In a weak coupling phase, though, we can simply take all fields to be in the Coulomb gauge. We can then define the vortex operator exactly as in the Abelian case, except that we must add the "gluon color density" to the charge density in the exponent of V, and we must consider all of the fields and currents in the exponent to lie in some arbitrary U(1) subgroup of the gauge group. The perturbative analysis in this chapter then changes only by the addition of a few color matrices, and the physical arguments and conclusions are all the same.

B. A Lattice Model

We have recently begun to consider an Abelian lattice model in which the above ideas apply and in which they can be tested in a more precise way. Although our study of this model is far from complete, we present some of the results. They support many of the conclusions reached above, while opening up interesting new questions about the nature of order parameters and phase transitions[46].

The model contains an Abelian gauge field U_{ab} defined on links, a singly charged matter field φ_{1a} defined on sites, and a doubly charged matter field φ_{2a} defined on sites. The subscripts a, b, ... refer to sites, so a link is labelled by two sites, its endpoints (in a directed way), and a plaquette is labelled by its four corners. The action is

$$S = \sum_P S_P + \sum_L (S_{1,L} + S_{2,L}) \tag{4.35}$$

where the sums run over all plaquettes and all links respectively. The gauge field action is

$$S_P = -K\,\mathrm{Re}(U_P) \tag{4.36a}$$

U_P is the product of link variables around the plaquette, $U_{ab}U_{bc}U_{cd}U_{da}$ for the plaquette abcd. The matter field actions are

$$S_{1,L_{ab}} = -\beta_1 \mathrm{Re}(\varphi^*_{1a} U_{ab} \varphi_{1b}) \tag{4.36b}$$

$$S_{2,L_{ab}} = -\beta_2 \mathrm{Re}(\varphi^*_{2a} U^2_{ab} \varphi_{2b}) \tag{4.36c}$$

The variables U_{ab}, φ_{1a}, and φ_{2a} are complex numbers of modulus 1. U_{ba}, φ^*_{1a}, and φ^*_{2a}, respectively, are their complex conjugates, not

independent fields. The action is gauge invariant; the gauge transformation is

$$U_{ab} \rightarrow g_a U_{ab} g_b^* \tag{4.37a}$$

$$\phi_{1,a} \rightarrow g_a \phi_{1,a} \tag{4.37b}$$

$$\phi_{2,a} \rightarrow g_a^2 \phi_{2,a} \tag{4.37c}$$

where g_a is an arbitrary field of modulus 1.

With no matter fields ($\beta_1=\beta_2=0$) this is a pure Abelian gauge theory. On the lattice, however, this theory is not free and in fact it is known to have two phases[47,48]. When K is small (corresponding to large coupling in a continuum theory) it is confining; when K increases beyond a certain critical value K_c it changes to a Coulombic theory, with a massless photon.

A vortex operator for a curve C can be defined as follows: choose a surface S whose boundary is C, and identify all plaquettes intersecting S but orthogonal to it. For instance, consider C to be an infinite line in the 3-direction plus another infinite line running in the (-3)-direction and displaced in the 2-direction from the first; S can be taken to be the flat surface (in the 2,3 direction) lying between the two lines. Part of this is drawn in figure 13a; we have suppressed the 3-direction entirely, so that all the links and plaquettes shown are perpendicular to the 3-direction, and the full spacetime is obtained by stacking the figure on top of itself (in the 3-direction) an infinite number of times. The plaquettes we have identified above are the 0,1 plaquettes drawn. The vortex operator

is defined by replacing the action S_p with

$$S_{\alpha,P} = -K\mathrm{Re}(e^{i\alpha}U_P) \tag{4.38}$$

on the indicated plaquettes[49]. It appears that this is a surface-like operator, but by changing variables

$$U \rightarrow Ue^{-i\alpha} \tag{4.39}$$

on certain links, we can change the apparent surface S (see figure 13b). Only the curve C is invariant. The expectation value of the vortex operator satisfies a perimeter law for both phases of this pure gauge theory, for all α[49]. Note that because there are no charges as yet, there is a continuum of vortex operators $0<\alpha<2\pi$.

The Abelian theory with either β_1 or β_2 non-zero, but not both, was studied by Fradkin and Shenker[36] and others[37,38]. Even though the matter fields are constrained to have unit magnitude, when β is small they act like linear fields, of zero vacuum expectation value. When β becomes large, though, the matter action is more and more strongly constrained to stay near its minimum, and the fields become Higgs-like. When there is a doubly charged matter field, the vortex operator is only useful for $\alpha=\pi$. It is only for this value that 4.39 leaves the matter action invariant; for other values the vortex operator is surface-like rather than looplike, and always has a surface-like cluster property. When the matter field is singly charged, there is no nontrivial value of α for which 4.39 leaves the matter action invariant, and we can define no vortex operator. We might then expect that in the doubly charged case we can make an absolute distinction between Higgs and non-Higgs phases on the basis

of the cluster property of the $\alpha=\pi$ vortex operator, while in the singly charged case this distinction may be purely quantitative, unless we find another order parameter. In fact, Fradkin and Shenker showed that the doubly charged system has three phases, separated by phase transitions, while the singly charged case has only two, with the vacuum energy proven to be analytic between the regimes that resemble Higgs and confining phases. The phase diagrams for the doubly and singly charged systems are shown in figure 14.

The above work, relating to theories with only a single matter field does not bear directly on what we have done. Restricting attention to the $\alpha=\pi$ vortex, the doubly charged field is what we have identified as adjoint, and the singly charged field is what we have called fundamental. Our analysis of the previous section only applies to theories where the fundamental field is weakly coupled, so that β_1 must be small. When β_1 is small, we do not expect to see a Higgs phase if there are no other matter fields. When β_1 is large, we do not expect to see vortices even at the classical level, and the analysis of the previous section does not apply. To test our ideas, we must consider the gauge theory with both matter fields, with β_1 small.

The phase diagram for this theory is a cube, the three directions representing the couplings K, β_1, and β_2. The face of the cube on which $\beta_1=0$ is the doubly charged theory (see figure 14 again); this face has a confining, a Coulombic, and a Higgs phase. The arguments of the last section suggest that if we consider a slice of the cube at finite but small β_1, we should still have three phases: the transitions on the $\beta_1=0$ face shown in figure 14 should

be walls that extend into the cube, not lines etched on the surface. The small β_1 region can be analyzed easily with the ideas of Fradkin and Shenker, based on ideas of Wegner[50]. If we expand the part of the functional integral involving the singly charged field, we get a sum over products of $S_{1,L}$ for various sets of links. Each product will involve various powers of $\phi^*_{1,a}$ and $\phi_{1,a}$ at each site. If the powers of ϕ^*_1 and ϕ_1 are not equal at each site, the functional integral over ϕ_1 gives zero. If they are equal on each site, the integral gives 1, leaving just the link variables from $S_{1,L}$. The condition from the ϕ_1 integration forces the remaining link variables to be formed into closed curves. The lowest term is of order β_1^4, coming from the product $S_{1,ab}S_{1,bc}S_{1,cd}S_{1,da}$. The integral over ϕ_1 leaves just link product $U_{ab}U_{bc}U_{ca}U_{da}$: this term just changes the effective coefficient of the plaquette action from K to $K+\frac{1}{8}\beta_1^4$. Higher orders in β_1 give traces of link variables around longer curves. Very long curves are suppressed by large powers of β_1. Fradkin and Shenker argue that these interactions cannot destabilize the phase transitions when β_1 is small. These arguments are based more on lore than on rigor, but phase diagrams derived from them have been verified nicely by numerical work[51]. The conclusion is that for small β_1 the effect is just a shift in the effective coupling; as we take our slices in the cube (at small fixed β_1) we see the same structure as on the face, shifted to the left by the change in the coupling. The small β_1 expansion on the lattice closely resembles our continuum arguments about the expansion in number of loops of the fundamental field, though on the lattice there is at least the hope of more rigor.

What is our modified vortex operator on the lattice? The definition 4.38 will no longer do. 4.38 replaces K with -K on the links orthogonal to S (for $\alpha=n$); we can change S by changing variables of integration. The integration over the singly charged field adds $\frac{1}{8}\beta_1^4$ to the coefficient on every plaquette (the matter action doesn't know about S). The coefficient is then $K+\frac{1}{8}\beta_1^4$ on most links but $-K+\frac{1}{8}\beta_1^4$ on S. These differ in magnitude and not just sign, so no change of integration variables can now shift S. The vortex operator is surfacelike (as we also argued earlier). Nor can we remedy this simply by changing the sign of the matter action on certain links: that just moves the problem elsewhere.

We can only spread the problem out: pick out some 3-surface X of links, in which S lies. Such a surface is illustrated in figure 15. On these links, change the charge 1 action to

$$S'_{1,L_{ab}} = -\beta_1 \mathrm{Re}(e^{i\Theta/2}\phi^*_{1a}U_{ab}\phi_{1b}) \qquad (4.40)$$

where Θ is illustrated in figure 15, and the discontinuity of the exponential is defined to lie on the surface S. When we now change variables on some set of links, changing the apparent surface S in the gauge field action, the apparent discontinuity in the matter action moves as well. X may be any 3-surface containing C; S may be any 2-surface bounded by C and lying in X. No change of variables will change the surface X (it is distinguished by the fact that the product of the arguments of the "Re" in the matter actions on the four links of a plaquette does not equal the argument of the "Re" for the gauge field action on the plaquette); the operator depends on X but not on S.

When we integrate over the charge 1 field, the effective gauge action that we obtain is unchanged on plaquettes not lying in X. On plaquettes such as A, shown in the figure, the effective action becomes, to leading order in β_1,

$$- \mathrm{Re}[(K+\frac{1}{8}\beta_1^4 \exp\{i\frac{\theta-\theta'}{2}\})U_p] \qquad (4.41)$$

$$= (K+\frac{1}{8}\beta_1^4)\mathrm{Re}(U_p) - \frac{1}{8}\beta_1^4[1-\cos\frac{\theta-\theta'}{2}]\mathrm{Re}(U_p) + \frac{i}{8}\beta_1^4 \sin\frac{\theta-\theta'}{2}\mathrm{Im}(U_p)$$

On plaquettes such as B which lie in S, the effective action is exactly the negative of this: the K term has the opposite sign by definition, while the induced terms have the opposite sign because o the discontinuity of $e^{i\theta/2}$.

The expectation value of our modified vortex operator in the full theory is then equal to the expectation value of the operator

$$\exp\{\sum_{P<X} \theta_P\}$$

times the usual vortex operator, in the effective gauge theory after integration of the charge 1 field. θ_P stands for the second and third terms on the right hand side of 4.41. At small β_1, the expectation value of the exponential is the exponential of the expectation value. The expectation value of $\mathrm{Im}(U_p)$ is zero by C invariance (complex conjugation here); the expectation value of $\mathrm{Re}(U_p)$ is some constant. $\theta-\theta'$ is of order a/ρ, so the integration over ρ is logarithmically divergent as in the continuum case. The expectation value of the modified vortex operator is then $\exp(-P\cdot\ln(P))$ times the expectation value of the normal vortex operator for the effective gauge theory. The total expectation value therefore changes from

exp(-P·ln(P)) behavior to exp(-A) behavior on just those curves where the effective gauge theory (and the full theory as well) change from confining to Higgs-like. The cluster properties are also the same as expected from the continuum case. It appears that we have found a good order parameter for the small β_1 region. We have not yet studied the conjecture about finite energy states which arose from the paradox with 't Hooft's result. This is probably studied most readily in a Hamiltonian lattice theory.

As we travel into the cube which represents theory space, do the phase walls we have found ever just end, so that the different phases can be connected by a continuous path through the middle of the cube, or are some of the phase regions completely walled off? Our modified vortex operator has different behavior in the Higgs and confining regions; on any path connecting them its behavior must change discontinuously. If we have a "good" order parameter, this will also imply a nonanalyticity in the vacuum energy and Green's functions. What makes a good order parmeter is not well understood; one can find many examples of an operator whose behavior is discontinuous even though the physics is not: the phase transition is "in the operator", not in the vacuum. As a simple example, consider a scalar theory whose Lagrangian density is $L_1(x)+m^2\phi^2$ and which has a phase transition at $m^2=m_c^2$. As an "order parameter" take

$$\exp\{k\int d^4x\ \phi^2(x)\}$$

The expectation value of this operator will not be discontinuous at $m^2=m_c^2$, where the vacuum is discontinuous, but at $m^2=m_c^2+k$. This is a rather trivial example, but there are much more subtle ones; these

"bad" order parameters disturb the vacuum rather than diagnosing it. Bad order parameters tend to be very nonlocal, as was our simple example. Since our modified vortex operator is more nonlocal than the usual vortex operator, we must be suspicious of it.

We have started to look at the behavior of our operator at large β_1. On the $\beta_2=0$ face, shown in figure 14, the behavior of the theory is well known; there is a region of analyticity connecting $\beta_1=0$ with $\beta_1=\infty$. When β_1 becomes very large, the functional integral for the expectation value of the vortex operator is dominated by the configurations which minimize the charge 1 action, 4.40. One such configuration is given by $\phi_1=1$ everywhere, $U_{ab}=1$ for links not on X, and $U_{ab}=e^{-i\theta/2}$ for links on X (ϕ_2 is decoupled). This gives 4.40 its minimum value, -2, everywhere. The gauge field action, 4.38 on S and 4.36a elsewhere, falls off as $1/\rho^2$ on X for this configuration. The action of this configuration is not large on any plaquette; the total action is again $O(P\cdot\ln(P))$. Our modified vortex operator does not have surface-like behavior anywhere near $\beta_1=\infty$, $\beta_2=0$; its behavior appears to be continuous in the region in which analyticity has been proven.

The acid test for our operator comes on the $\beta_2=\infty$ face. This face is best understood in the $\phi_2=1$ gauge, where the charge 2 action, 4.36c, constrains U to be 1 or -1 on every link. We then have a Z_2 gauge theory coupled to a U(1) matter field (the charge 1 field), which is still a fundamental representation in our terminology. Because the matter field is fundamental, we might expect this face to have a phase diagram like the $\beta_2=0$ face, with a phase in the small β_1, large K corner having free Z_2 excitations (this would still be a

Higgs phase from the point of view of U(1)) and one continuous phase elsewhere. This is shown in figure 16. Near the left side of the lower edge of the diagram, we can integrate out the charge 1 field and we have simply a Z_2 gauge theory at small K. This is a confining theory[49]; the vortex operator satisfies a P·ln(P) law. Near the $\beta_1=\infty$ face, where we must minimize the charge 1 action, we can no longer find a configuration which makes the matter action small everywhere; our earlier configuration required U to take values other than -1 and 1. Any configuration will have at least a surface of plaquettes (bounded by C) on which the action is large. Our operator must therefore make a discontinuous change to surface-like clustering somewhere between edge 1 and edge 2.

This leads us to a closer examination of the $\beta_2=\infty$ face. When the gauge field and the matter field are the "same size", U(1) and U(1) or Z_2 and Z_2, the phase diagram of figure 16 is known to be correct, but here the matter field has more structure than the gauge field. We therefore look at the K=0 edge, where only the action $S_{1,L}$ survives (the whole effect of the charge 2 field is the constraint $U=\pm 1$). Summing over U on each link, we find that the functional integrand (leaving out the vortex operator now) is a product of a factor of

$$\exp\{-I_{ab}\} = \exp\{-\beta_1 \mathrm{Re}(\phi^*_{1a}\phi_{1b})\} + \exp\{\beta_1 \mathrm{Re}(\phi^*_{1a}\phi_{1b})\} \qquad (4.42)$$

for each link. This factor is invariant under $\phi_{1a} \to -\phi_{1a}$; there is a surviving Z_2 gauge invariance. The action is therefore a function on $U(1)/Z_2$, which is just a different U(1); we write it in terms of a new field θ_a defined by

$$\exp\{i\theta_a\} = \phi_a^2 \tag{4.43}$$

The action is now a sum of terms for each link

$$I_{ab} = -\ln\{2\cosh(\beta_1\cos\frac{\theta_a-\theta_b}{2})\} \tag{4.44a}$$

We shall see that this is a close approximation to the action of the xy-model[52],

$$J_{ab} = -\beta_{xy}\cos(\theta_a-\theta_b) \tag{4.44b}$$

At small β_1,

$$I_{ab} = \text{const} - \frac{1}{2}\beta_1^2\cos^2(\frac{\theta_a-\theta_b}{2}) + O(\beta_1^4)$$

$$= \text{const}' - \frac{1}{4}\beta_1^2\cos(\theta_a-\theta_b) + O(\beta_1^4) \tag{4.45}$$

This is an xy model at $\beta_{xy}=\frac{1}{4}\beta_1^2$.

At large β_1, the action grows quickly away from its minimum, and $(\theta_a-\theta_b)^2$ is constrained to be $O(\beta_1^{-1})$. The argument of the hyperbolic cosine is then very large, so it is essentially an exponential. We then get

$$I_{ab} = -\beta_1\left|\cos\frac{\theta_a-\theta_b}{2}\right|$$

$$= \text{const} + \frac{1}{8}\beta_1(\theta_a-\theta_b)^2 + O(\beta_1(\theta_a-\theta_b)^4) \tag{4.46}$$

Performing the same expansion for the xy model at large β_{xy}, we find that J_{ab} is the same as 4.46 when $\beta_{xy}=\beta_1/4$.

At small β_1, the K=0, $\beta_2=\infty$ edge is an xy model at small β_{xy}, which is known to be in a disordered phase; at large β_1, it is an xy model at large β_{xy}, which is known to be in an ordered phase[52]. We

conclude that there is a phase transition on this edge. We have not yet studied the stability of this transition at finite K; we have no reason to believe, however, that is does not join with the other phase boundary, giving the phase diagram of figure 17. We note that the phase we have labelled "total Higgs" is walled off on the K=∞ face (seen in figure 17) and on the $\beta_1=\infty$ face (shown in figure 18). The K=0 face is still terra incognita to us. There are known to be phase transtions on exactly two of its edges, but we have not yet analyzed the stability of these transitions away from the edge.

Our operator seems to be a good vortex operator for the whole cube, or at least as much of it as we have been able to study. Is there a physical basis for this? We believe that strong evidence for it is provided by the nature of the discontinuity of the operator. If the phase transition were "in the operator" we might the action density to change discontinuously <u>within</u> the support of the operator. In our case, if we choose X to lie away from the minimal surface, we see a discontinuous change from clustering (action density) near the 3-surface X to clustering near the minimal 2-surface. This seems as if it should reflect a true change in the nature of the vacuum.

What we have found indicates that the "free" and "confining" phases, with P·ln(P) behavior for our operator, should be totally walled off from the "Higgs" (also known as "free Z_2") and "total Higgs" phases, with area law behavior. It may be difficult to show this analytically, as the center of the cube is amenable to few approximations; numerical studies should provide the answer. What we can do, at the least, is to examine the stability of our new transition against small nonzero K and large non-infinite β_2.

Everything we have done has been directed at four dimensions, but it can all be extended to three or to more than four; it would be interesting to study the phase structures of these theories as well. We note that our result implies that the corresponding operator in QCD should also be able to diagnose phase changes; it should enable us to say that QCD is not in a Higgs phase, and that any path through parameter space which bring it to a Higgs phase must pass through a phase transition. A careful study of this problem might reveal something of the nature of phase transitions and the meaning of order parameters.

CHAPTER 5

CONCLUSIONS

In this thesis, we dealt first with some technical aspects of the vortex operator. Our resolution of the Dirac string problem, and our two dimensional "Dirac lump", may be more novel than they are useful, at least for the present problem where the methods of Wu and Yang can be used. For a field theory of electrons and monopoles, it may be helpful to use ideas akin to ours, as the monopole is no longer classical and the method of writing a monopole field theory as a sum over monopole paths is rather formal. We have also shown that the divergences of looplike operators can be analyzed in a straightforward way by the use of the operator product expansion. We believe that any attempt to treat QCD as a theory of looplike operators (Wilson loops or vortex operators) must include a careful treatment of short-distance questions, along these lines.

In the next chapter we discussed the relation between the Green's functions of the vortex operator and the structure of the vacuum. We have placed emphasis on the cluster properties of these Green's functions as providing more detailed information than the vacuum expectation value. We then showed two correct ways (divergence equation and tree sum) to find the cluster property in an Abelian Higgs phase, and one wrong way (expansion in p). Our discussion of the graphical expansion in a Higgs phase emphasized the fact that the Higgs phase differs not in the form of the expansion but in the nature of the tree sum; the extension of this should be borne in mind when dealing with spontaneous breakdown through a composite

operator.

In the fourth chapter we showed that the idea of the vortex operator can sometimes be extended to gauge groups with fields in the fundamental representation, groups which are simply connected. Continuum arguments indicated that this is true when the field in the fundamental representation enters weakly into the dynamics. Study of a lattice model confirmed this, and also suggested that our operator is a good order parameter at all values of the couplings. More study of the physics here is required, but it seems possible that the analogous operator would distinguish the QCD vacuum from a Higgs phase with quarks in a qualitative rather than merely quantitative way.

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FIGURE CAPTIONS

1. a. The graph for $A_\mu^{vS}(x)$. The double line represents the surface S.

   b. A typical graph for G(P,C,S). The curved line is the path P of the line integral; the straight segments are scalar propagators.

2. a. The order e graph for $< A_\mu(x)\ V_p'(C,S) > / < V_p'(C,S) >$. The heavily circled v's are effective vertices, the lightly circled v's indicate effective propagators.

   b. An order e^3 graph for the same matrix element.

3. a. The discontinuous graph for $< V_p'(C,S)\ W_q(C') >$. The single heavy line represents C'.

   b. Another graph for the same Green's function.

   c. Another graph, connected to C' by three photons.

4. Graphs in the expansion of figure 2a.

   a. The two graphs with field renormalization divergences.

   b. Two of the graphs which have composite divergences.

5. a. The curves C and C', linked and separated by a large distance R.

   b. C and C' unlinked and separated by R.

6. The Green's function 3.3, considered in a 3-surface. C and $\bar{C}$ are the intersections of C with the 3-surface; S is the intersection of S with the 3-surface. The function 3.3 maps loop 1 into a nontrivial path in M; it maps loop 2 into a trivial path.

7. The graphical equation for u_i, represented by the u-tadpole; a_i is the a-tadpole. The first two terms cancel.

8. The graphical expansion for u_i^{vS}, represented by the v-tadpole.

9. Graphs for $<\vec{B}(x)>$.

   a. The order e^{-1} graph.

b. The order e^1 series.

10. Graphs for $\langle \vec{B}(x_1)\vec{B}(x_2)\rangle$

a. The order e^{-2} graph.

b. The order e^0 series.

c. An order e^{-2} graph not satisfying 4.9b.

11. Order p^2 graphs for the effective potential $U(\vec{A}_M)$.

12. a. A graph for a matrix element of Δ with a potential composite divergence.

b. The form of the divergence of figure 12a. The X represents a cutoff dependent function.

13. a. The lattice vortex operator: the plaquette product is multiplied by a phase on the indicated plaquettes. The whole picture lies in a 3-plane perpendicular to the 3-direction. C is an infinite line in the 3-direction plus an infinite line in the (-3)-direction; these lines intersect the pictured 3-plane only at the points C and $\bar{C}$. The surface S intersects the 3-plane in a line, shown running through the middle of the plaquettes. The pictured plaquettes lie in the 0,1-direction; C and $\bar{C}$ are separated in the 2-direction.

b. Changing variables on the nine heavily drawn links, we change the apparent surface S.

14. The phase cube for our lattice theory; the little dreibein shows the directions in which the various couplings increase. The upper face, with $\beta_1=0$, is the doubly charged theory; β_2 increases as we go upward on this face. The lower face, $\beta_2=0$, is the singly charged theory; β_1 increases as we go downward. The phase diagrams for these two theories are found in reference 36 or 51; the shaded region is where analyticity has been proven. The face $K=\infty$ is seen obliquely.

15. Part of the plane of links X on which we spread out the frustration in the singly charged matter field. The plaquettes of S are drawn in. The arcs indicate the angles θ and θ'; their difference is $O(1/\rho)$. The angles are defined to be

discontinuous on S; the action induced on B is therefore the negative of that induced on A.

16. The cube of figure 14, tilted forward to show the $\beta_2=\infty$ face, with one possible phase diagram. Our operator has $P\cdot\ln(P)$ behavior near the left side of the lower edge, 1, and area law behavior near the upper edge, 2.

17. The X indicates the phase transition we have found on the $K=0$, $\beta_2=\infty$ edge; the dashed line is its expected continuation.

18. The cube tilted forward again, showing the $\beta_1=\infty$ face. The $K=0$ face, not yet fully analyzed, is also exposed; a plausible phase transition is shown.

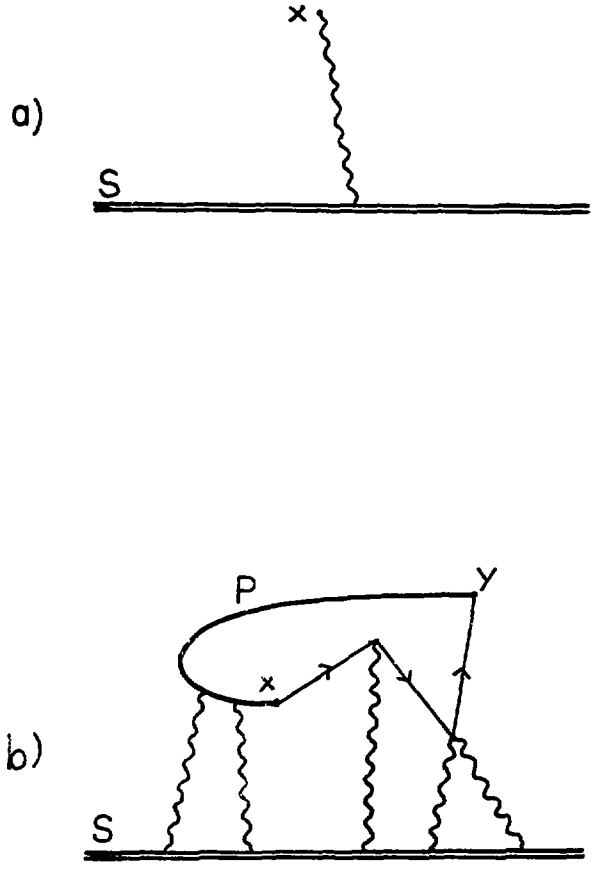


Figure 1

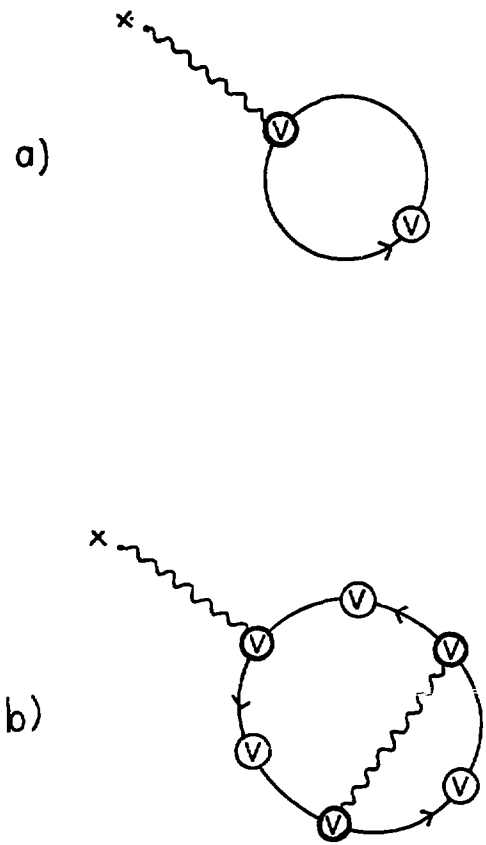


Figure 2

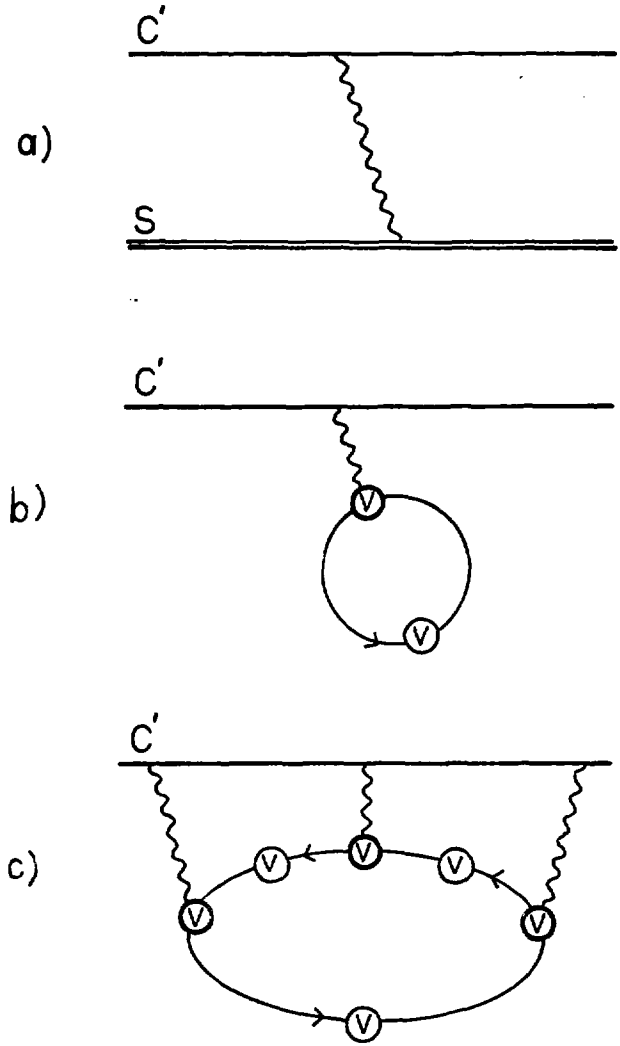


Figure 3

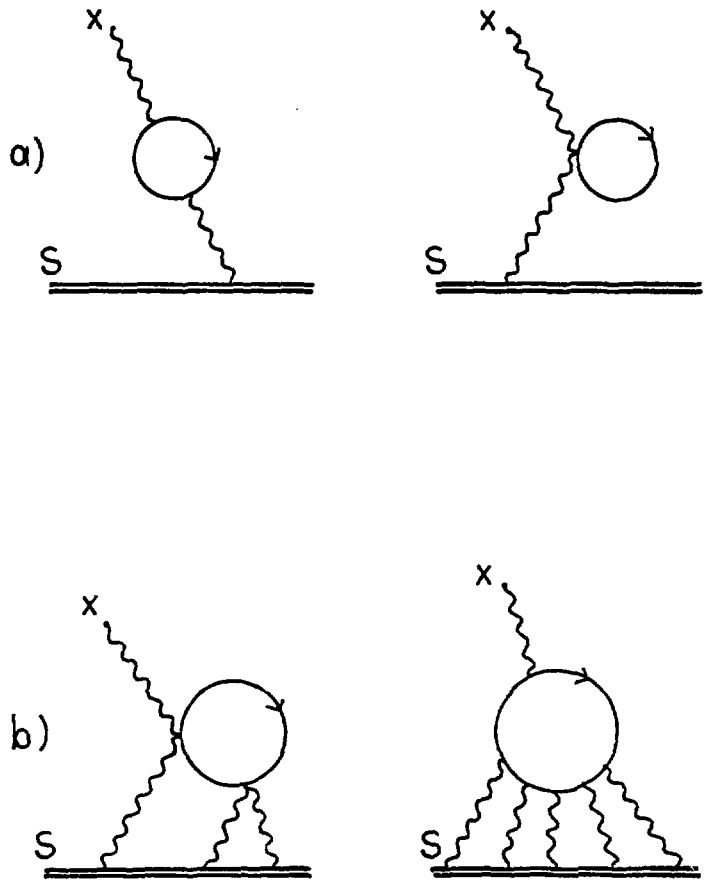


Figure 4

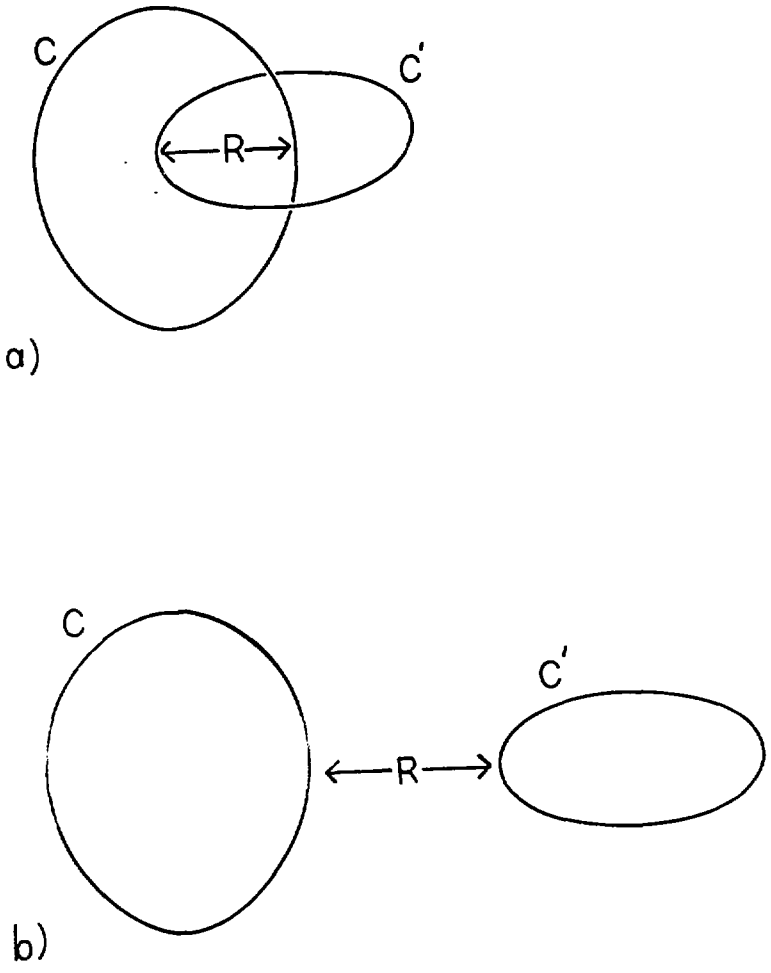
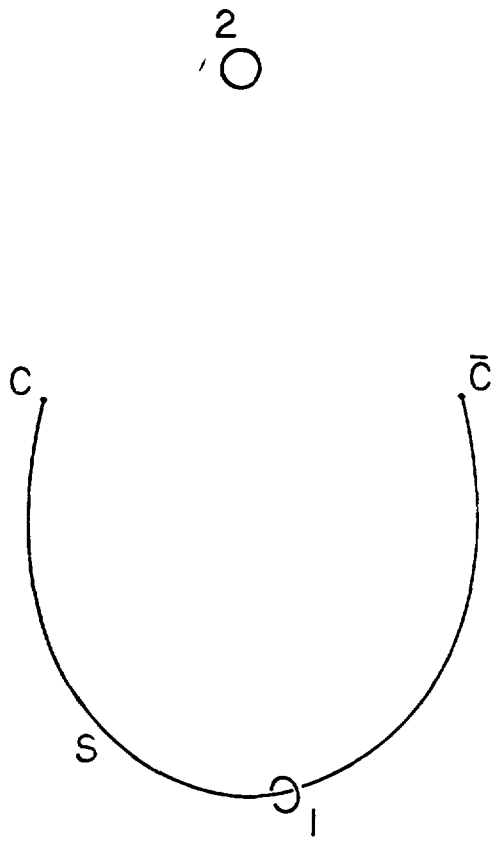


Figure 5



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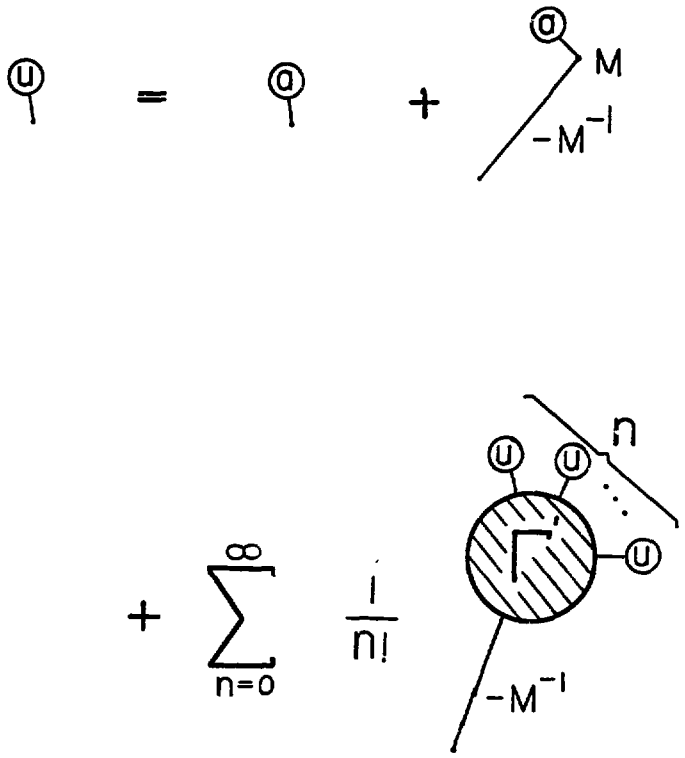


Figure 7

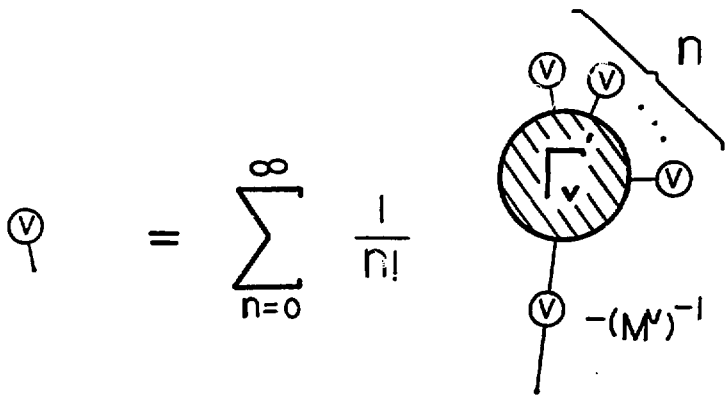
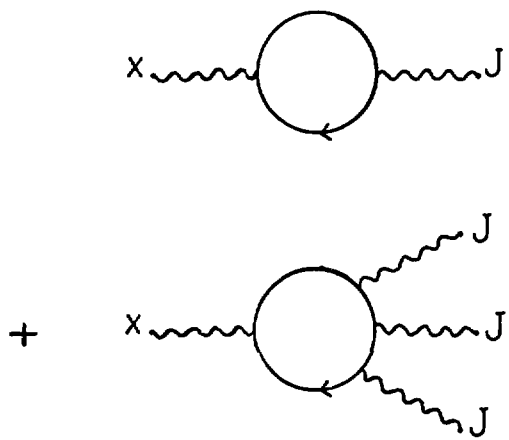


Figure 8

a)



b)

Figure 9

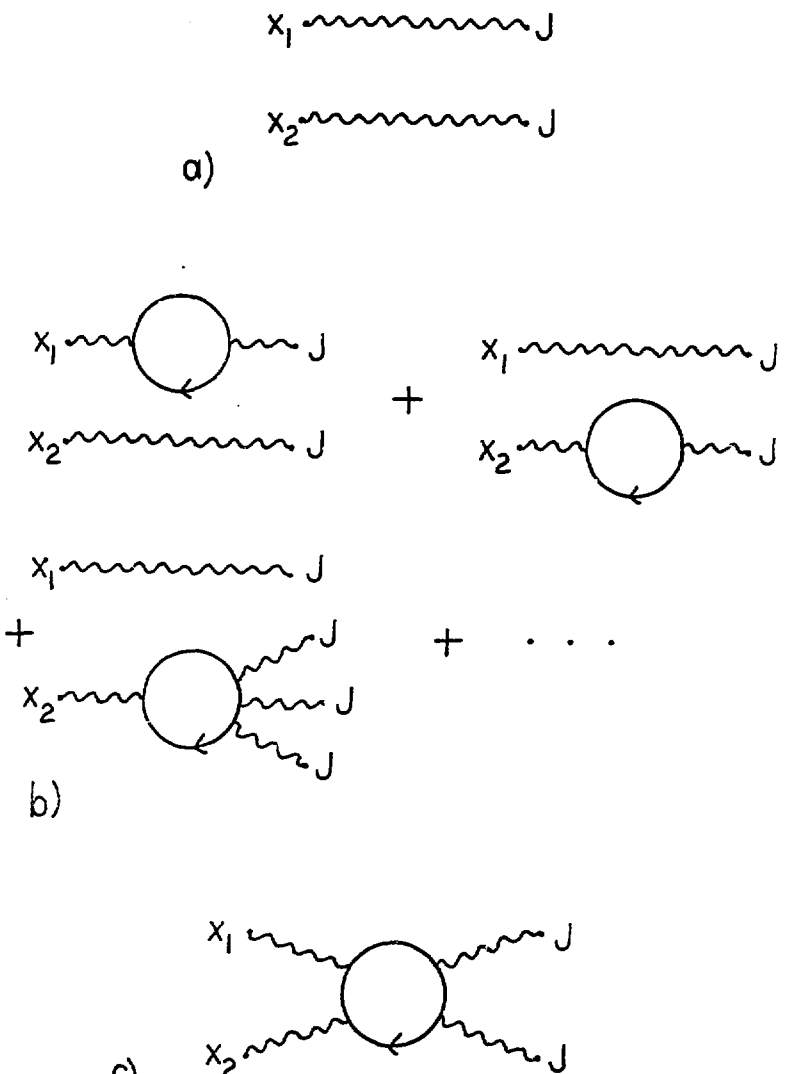


Figure 10

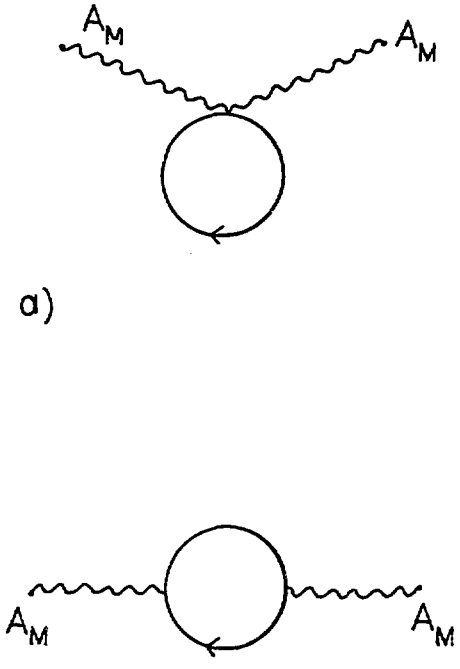


Figure 11

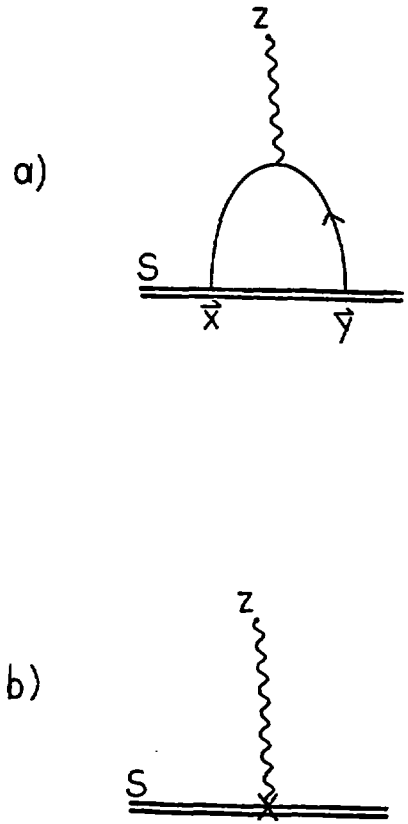
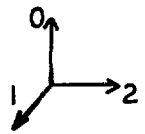
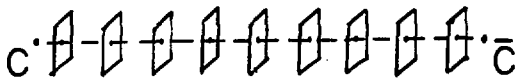
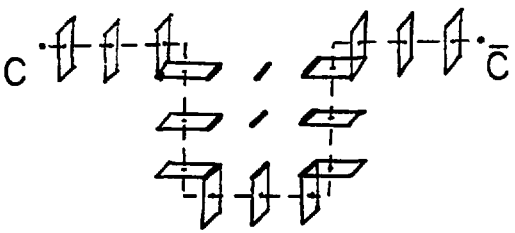


Figure 12





a)



b)

Figure 13

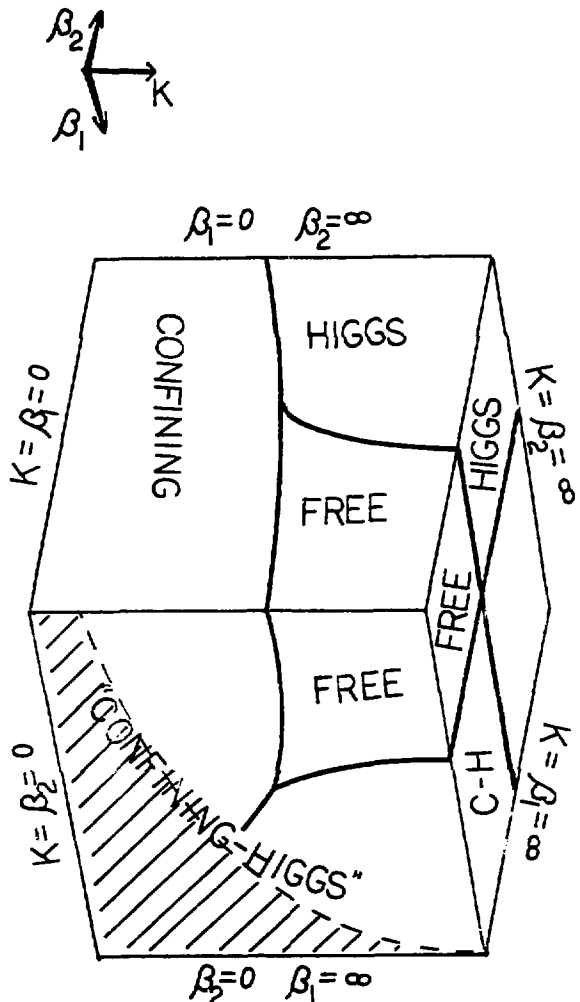

β_2
K
β_1
$\beta_1=0$ $\beta_2=\infty$
CONFINING
HIGGS
FREE
FREE
HIGGS
FREE
"CONFINING-HIGGS"
C-H
$K=\beta_1=0$
$K=\beta_2=\infty$
$K=\beta_2=0$
$K=\beta_1=\infty$
$\beta_2=0$ $\beta_1=\infty$

Figure 14

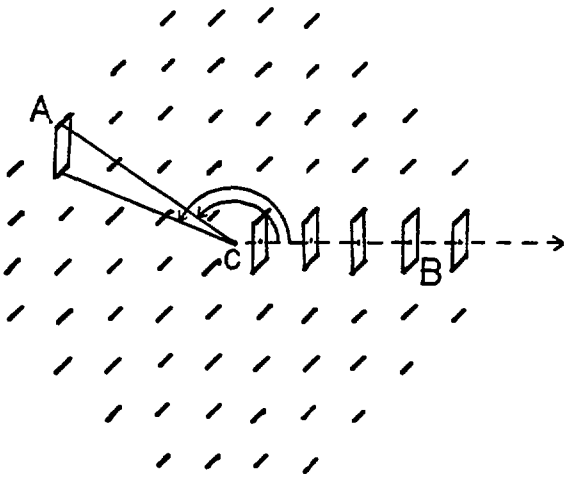


Figure 15

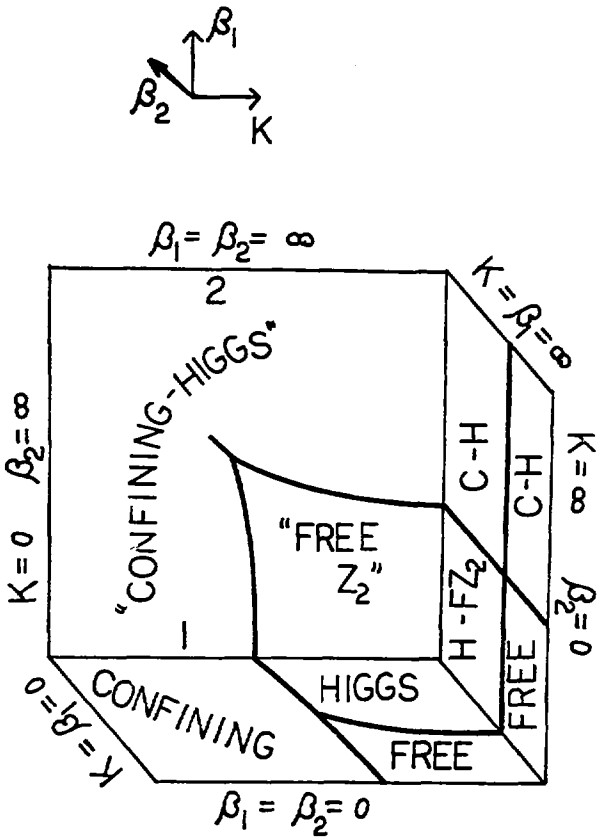

β_1
β_2
K
$\beta_1 = \beta_2 = \infty$
2
"CONFINING-HIGGS"
$K = \beta_1 = \infty$
$\beta_2 = \infty$
K = 0
"FREE Z_2"
C-H
C-H
K = ∞
$\beta_2 = 0$
1
H-FZ_2
FREE
HIGGS
CONFINING
FREE
$K = \beta_1 = 0$
$\beta_1 = \beta_2 = 0$

Figure 16

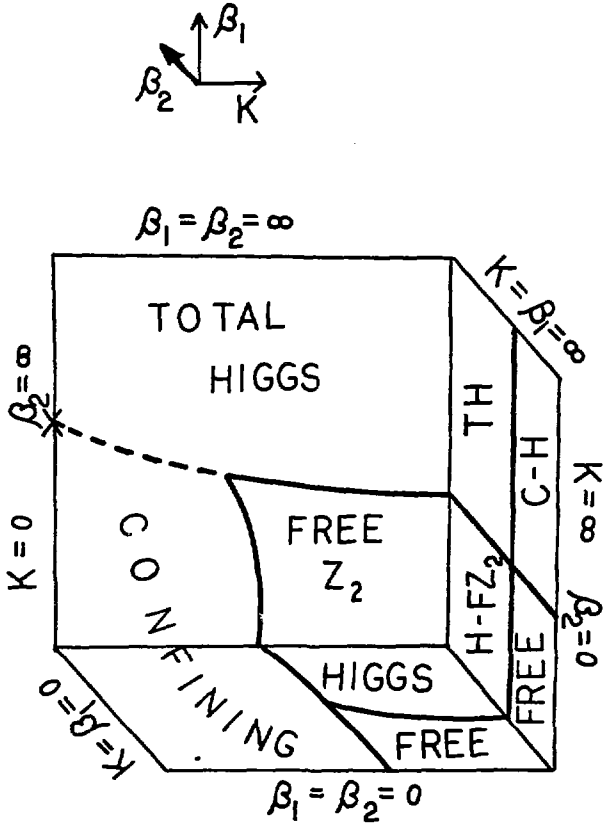

β_1
β_2
K
$\beta_1 = \beta_2 = \infty$
TOTAL
HIGGS
$K = \beta_1 = \infty$
$\beta_2 = 0$
K = 0
CONFINING
FREE
Z_2
TH
C-H
H-FZ_2
HIGGS
FREE
FREE
$K = \beta_1 = 0$
$\beta_1 = \beta_2 = 0$

Figure 17

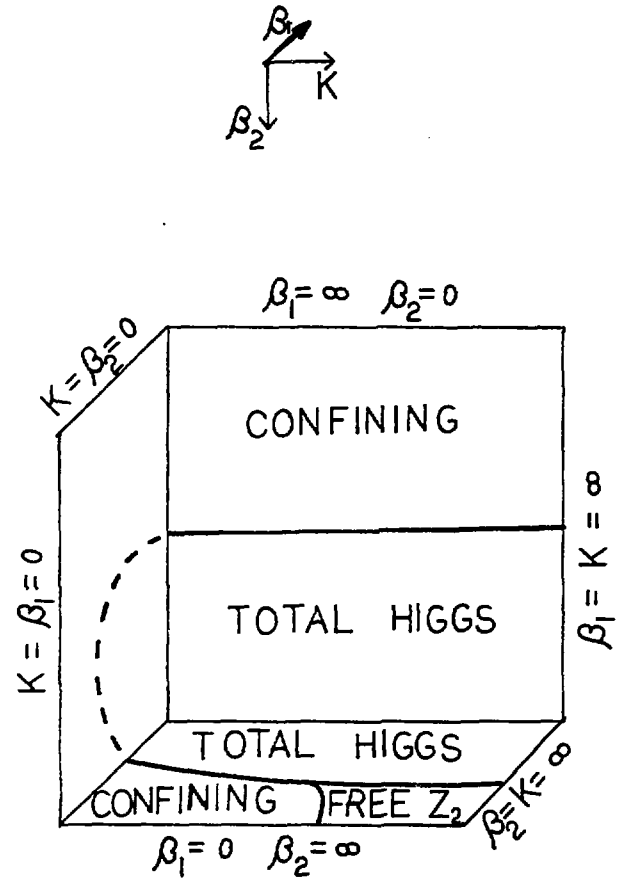

β_1
K
β_2
$\beta_1 = \infty$ $\beta_2 = 0$
$K = \beta_2 = 0$
CONFINING
$K = \beta_1 = 0$
TOTAL HIGGS
$\beta_1 = K = 8$
TOTAL HIGGS
CONFINING
FREE Z_2
$\beta_2 = K = \infty$
$\beta_1 = 0$ $\beta_2 = \infty$

Figure 18