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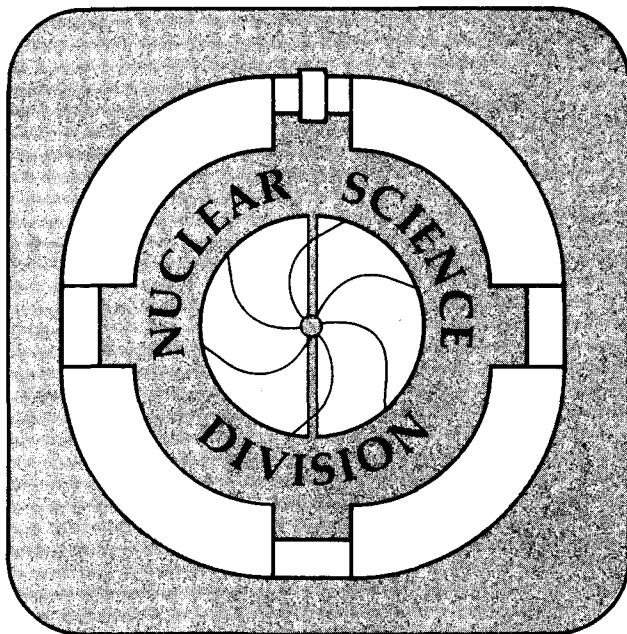
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Order, Chaos and Nuclear Dynamics: An Introduction

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ORDER, CHAOS AND NUCLEAR DYNAMICS.
AN INTRODUCTION

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ORDER, CHAOS AND NUCLEAR DYNAMICS. AN INTRODUCTION*

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ABSTRACT

This is an introductory lecture illustrating by simple examples the anticipated effect on collective nuclear dynamics of a transition from order to chaos in the motions of nucleons inside an idealized nucleus. The destruction of order is paralleled by a transition from a rubber-like to a honey-like behaviour of the independent-particle nuclear model.

1. AN IDEAL GAS: RUBBER, HONEY OR INVISCID FLUID?

Suppose you take a container in the shape of a shoe box and fill it with an ideal gas of non-interacting classical point particles, bouncing elastically off the walls of the container, assumed to be perfectly reflecting. Now try to slowly change the proportions of the box while keeping its volume fixed. You are dealing with an ideal gas being deformed at constant volume, so presumably there should be no work done, no resistance against a change in shape. Right? Wrong. What you will feel is an *elastic* resistance against changes of shape, as if the box were filled with rubber. You can try the same experiment with a container in the shape of a sphere, which you deform into a spheroid (an ellipsoid of revolution). Again what you will feel is a resistance like that of rubber: the energy of the gas will go up approximately quadratically with deformation, and will return to its original value as the container returns to its original shape.

Now put in some sizeable dents or ripples in the walls of the container and try deforming it, again at fixed volume. What you will feel now is a resistance to deformation as if the container were filled with viscous honey: the resistance is proportional to the speed with which you push, and if you cycle the shape back to its original configuration, you will find that you have expended a certain amount of energy, which will have gone irreversibly into heating up the gas.

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Next go back to the spherical container without dents and deform it along a sequence of shapes like those in nuclear fission: spheroidal at first, but then tending towards a necked-in, hourglass figure. You will first feel a rubber-like resistance, then the rubber will seem to melt and turn into honey. All this while what you really have inside the container is neither rubber nor honey, but an ideal gas of non-interacting mass points. What is going on?

Answering this question is the theme of this lecture. We will find that rubber gives place to honey because order in the motions of the gas particles gives place to chaos. What you are in fact experiencing is the destruction of so-called Poincaré tori in phase space and some of the consequences of the famous Kolmogorov, Arnold, Moser theorem in classical mechanics.

Before leaving the ideal gas container make one more experiment. Replace the mass points by little marbles with finite radii in order to make the mean free path of a marble small compared to the size of the container (but still large compared to the interparticle spacings). What you will now experience on deforming the container are the properties of an ordinary gas, whose energy is independent of the shape of the box and whose viscosity decreases with decreasing mean free path of the particles. The magic of the rubber-to-honey transition has been destroyed by the 'residual interactions' between the gas particles.

The relevance to nuclear physics of studying nearly independent particles in a container is, of course, that to a rough approximation a nucleus can be represented by the model of quantized independent particles in a mean-field potential well. Since quantization is expected to be an essential element, one has to study the order-to-chaos transition not only in the classical regime, but also for quantized particles. Eventually one has to include residual nucleon-nucleon interactions (which, however, are not strong enough to spoil entirely the independent-particle approximation). It is clear that in trying to understand this complex many-body problem of nuclear dynamics it cannot possibly hurt to understand thoroughly the order-to-chaos transition in the idealized model of independent particles in a time-dependent container. In the quantal version of this problem one is dealing with aspects of so-called 'Quantum Chaos,' which is today an active field of research with many ramifications. It is a problem full of subtleties and unanswered questions. But let us first return to the classical gas.

2. CLASSICAL POINT PARTICLES IN A CONTAINER

Let me substantiate what I said about the rubber-like or honey-like properties of an ideal gas.

2.1. The box

Let the sides of the box be a , b , c , and let the x , y , z components of a particle's velocity be u , v , w . The total kinetic energy of the gas is

$$T = \sum \left(\frac{1}{2} m u^2 + \frac{1}{2} m v^2 + \frac{1}{2} m w^2 \right) = T_x + T_y + T_z , \quad (1)$$

where the sum is over all particles, m being the particle mass. How does T change from its initial value T_0 , say, as the initial dimensions a_0, b_0, c_0 change slowly to a, b, c (with $a_0 b_0 c_0$ equal to abc by volume conservation)? Because of the symmetry of the box, each particle's motion separates into three independent motions u, v, w . So, in effect, one is dealing with three independent one-dimensional gases. It is trivial to derive an expression for the total energy of three such gases and calculate the response of the sum to slow changes in the one-dimensional 'volumes' a, b, c . A short-cut to the result, one which is instructive in the present context, is to recall the adiabatic invariance, with respect to slow changes of parameters, of the action integral $\int p dx$ in classical mechanics, the integral being over one cycle of a periodic motion (Goldstein 1981). In our case the action integrals over one period of the motions in the x, y, z directions are proportional to ua, vb, wc and the adiabatic theorem states that these products remain (almost) constant for slow changes of a, b, c . Thus $ua = u_0 a_0, vb = v_0 b_0, wc = w_0 c_0$. Hence the dependence of the kinetic energy on a, b, c is given by

$$T = \sum \frac{1}{2} m \left[u_0^2 (a_0/a)^2 + v_0^2 (b_0/b)^2 + w_0^2 (c_0/c)^2 \right], \quad (2)$$

or

$$\frac{T}{T_0} = \frac{T_{x0}}{T_0} (a_0/a)^2 + \frac{T_{y0}}{T_0} (b_0/b)^2 + \frac{T_{z0}}{T_0} (c_0/c)^2, \quad (3)$$

where $T_{x0}/T_0, T_{y0}/T_0, T_{z0}/T_0$ are the fractions of the initial energy associated with the x, y, z directions. Assuming that the gas is originally isotropic (also in velocity space), the above fractions are one third each. Thus,

$$\frac{T}{T_0} = \frac{1}{3} \left[(a_0/a)^2 + (b_0/b)^2 + (c_0/c)^2 \right]. \quad (4)$$

This is the desired answer. To see what it implies, consider the special case where $b/b_0 = c/c_0$. Denoting a/a_0 by $1 + \alpha$, the relative stretching of the a -dimension, we find $b_0/b = c_0/c = \sqrt{1+\alpha}$ (from volume conservation) and, consequently,

$$\frac{T}{T_0} = \frac{1}{3} \left[(1 + \alpha)^{-2} + 2(1 + \alpha) \right] \approx 1 + \alpha^2 + \dots \quad (5)$$

Thus, for small α ,

$$T \approx T_0 + T_0 \alpha^2 = T_0 + \frac{1}{2} (2T_0) \alpha^2. \quad (6)$$

This equation says that the energy increase of the gas due to a stretching of the box by α is like that of an elastic material with a modulus of elasticity equal to twice the total energy content of the gas, i.e., it is a bulk property, like the elasticity of a piece of rubber. For a nucleus, the kinetic energy per particle is about 20 MeV, so $2T_0 = 40 A$ MeV, where A is the mass number. That's quite a large stiffness. To stretch a nucleus idealized as a box by 10% would require $20 A (0.1)^2$ MeV, i.e., 20 MeV for $A = 100$.

The basic reason for this unexpected rubber-like behaviour of the ideal gas is the existence of three constants of motion in the dynamics of each particle. These constants are the components of the particle's energy in the x,y,z directions. This causes the three-dimensional gas to separate into three one-dimensional gases, one of which is dilated by $1 + \alpha$, while the others are compressed by $1/\sqrt{1+\alpha}$. The energies associated with the changes in the one-dimensional volumes cancel to lowest order in α , but leave an elastic-like α^2 term at the next order.

2.2. Spheroidal distortions

For the motion of a particle in an axially symmetric cavity, the two obvious constants of motion are the particle's energy and the projection on the axis of symmetry of its angular momentum. When the cavity is an ellipse of revolution there exists a third, unexpected constant of motion. It is the scalar product $\vec{\ell}_1 \cdot \vec{\ell}_2$ of the particle's angular momenta, $\vec{\ell}_1$ and $\vec{\ell}_2$, calculated with respect to the two foci of the ellipse. Thus there are again as many constants of motion as there are degrees of freedom (three for each particle). When there are N constants of motion in a system with N degrees of freedom, the problem is said to be 'integrable' or 'solvable.' A striking consequence of integrability is that for a (bound) system with N degrees of freedom the motion of the representative point in its 2N-dimensional phase space takes place on an N-dimensional torus. (A 2-dimensional torus is a square with opposite sides identified. An N-dimensional torus is an N-dimensional cube with opposite faces identified.) Thus the N constants of motion force the time evolution in phase space, for each initial condition, to wind around an N-dimensional torus, rather than to explore the whole of the 2N-dimensional phase space. In the case of the spheroid, the consequence of integrability is to make the problem qualitatively similar to the integrable box problem, and the response of an ideal gas to slow spheroidal deformations is again rubber-like. The bottom panel in Fig. 1, the result of a computer simulation study, illustrates this behaviour (Blocki *et al* 1990, Swiatecki 1988). It gives the relative energy change, $(T - T_0)/T_0$, for a gas of several thousand point particles bouncing inside a spheroidal container during five complete cycles of oscillation. The equation specifying the container's shape is

$$\frac{x^2}{a(t)^2} + \frac{y^2}{b(t)^2} + \frac{z^2}{b(t)^2} = 1, \quad (7)$$

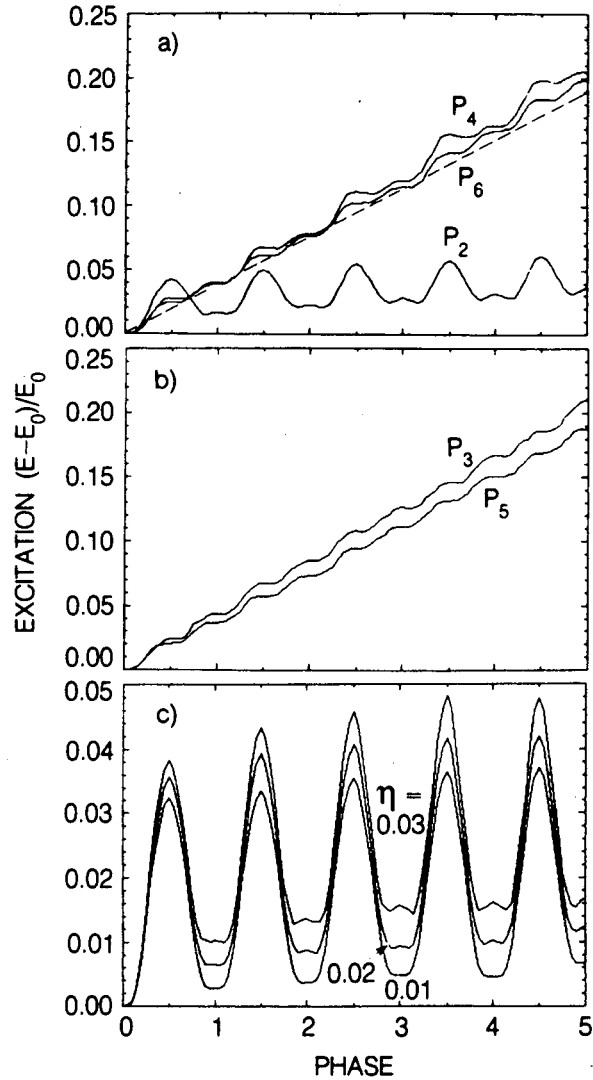
with

$$a(t) = R_0 (1 + \alpha \cos \omega t), \quad (8)$$

$$b(t) = R_0 (1 + \alpha \cos \omega t)^{-1/2}, \quad (9)$$

so that $ab^2 = R_0^3$ at all times. The initial positions of the gas particles inside the spheroid were chosen at random and their velocity vectors in velocity space were taken to be distributed

Figure 1. The relative energy change of a gas of particles inside a cavity undergoing five complete oscillations around the sphere. In the bottom panel the oscillations are spheroidal and, with decreasing frequency (as labeled by η), the response tends to a reversible, elastic one. The index η is the ratio of the maximum tip speed to the Fermi velocity v . The curves labeled P_n refer to oscillations proportional to Legendre polynomials. Except for P_2 the resulting monotonic, dissipative energy increase is close to that predicted by the wall formula, whose overall trend is indicated by the dashed line.



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randomly inside a Fermi sphere with radius v . The oscillation of the spheroid starts at $t = 0$, when its relative elongation is $1 + \alpha$. In the example shown, $\alpha = 0.2$. The maximum speed of the tip of the spheroid is $\dot{a}_{\max} = \alpha\omega R_0$ when $\omega t = \pi/2$, and the ratio of this maximum tip speed to the maximum particle speed v is the adiabaticity parameter η , i.e., $\eta = \alpha\omega R_0/v$. The three cases illustrated in Fig. 1 correspond to $\eta = 0.01, 0.02, 0.03$, so that the oscillations of the spheroid are in all cases quite slow compared to the speeds of most of the gas particles. (But there are always present some slow particles for which the adiabaticity condition is not satisfied.) Figure 1 shows that as η tends to zero, the changes in the energy of the gas appear to tend to perfect reversibility, i.e., the energy returns to its original value when the shape returns to its original elongation. Such behaviour is characteristic of an elastic solid. The amplitude of the energy oscillations is about 3%. Note that for a box whose a-dimension is changed from $1 + \alpha$ to $1 - \alpha$ and whose b- and c-dimensions, assumed equal, are increased from $1/\sqrt{1+\alpha}$ to $1/\sqrt{1-\alpha}$, Eq. (4) gives a relative energy increase of 19.4%, which is 6.5 times greater than the 3% deduced from Fig. 1. Thus a spheroid appears to be several times softer than the box against an axially symmetric adiabatic stretching.

2.3. Container with dents

Figure 1 also shows the energy of the same classical gas when the deformations are proportional to Legendre Polynomials $P_n(\cos\theta)$, which describe rippled distortions with n nodal meridians around the axially symmetric surface. The radius vector of the container was specified by

$$R(\theta,t) = \frac{R_0}{\lambda(t)} \left[1 + \sqrt{(2n+1)/5} \alpha P_n(\cos\theta) \cos \omega t + \alpha_1 P_1(\cos\theta) \cos \omega t \right]. \quad (10)$$

The quantity $\lambda(t)$ is a normalizing function ensuring volume conservation. The term $\alpha_1 P_1(\cos\theta) \cos \omega t$ ensures fixity of the center of mass of the container in the case of reflection asymmetric distortions (when odd values of n are present). The factor $\sqrt{(2n+1)/5}$ is included so that the RMS deviation of the container's surface from the sphere is the same for all values of n and given approximately by $\alpha/\sqrt{5}$. The maximum speed of the container's tip is now $\sqrt{(2n+1)/5} \alpha \omega R_0$ (when $\omega t = \pi/2$ and $\theta = 0$) and its ratio to v is $\sqrt{(2n+1)/5} \eta$. All the curves in the upper panels in Fig. 1 refer to $\alpha = 0.2$, as before, and to a slightly higher frequency corresponding to $\eta = 0.04$.

The appearance of the graphs of $(T-T_0)/T_0$ is now quite different than for the spheroid (except for $n = 2$, about which later). The excitation energy of the gas is seen to increase monotonically, the way it would for a container filled with viscous honey. After 5 oscillations (with a frequency implying a maximum tip speed of the container of the order of a few percent of the Fermi velocity) the excitation energy is about 20% of the original total kinetic energy T_0 . In the nuclear context, with $T_0 = 20$ A MeV, this translates into an excitation of 400 MeV for $A = 100$!

This unexpected honey-like behaviour of an ideal gas is actually quite easy to understand in terms of the wall formula for dissipation, derived in Blocki *et al.* (1978). Figure 2 illustrates the physics of this formula. Each surface element $d\sigma$ of the surface of the deforming container is assumed to be subjected to a pressure from the gas equal to the standard (static) pressure $\frac{1}{3} \rho \bar{v}^2$, decreased by a correction $\rho \bar{v} \dot{n}$ when $d\sigma$ is moving away from the gas with a normal speed \dot{n} :

$$p = \frac{1}{3} \rho \bar{v}^2 - \rho \bar{v} \dot{n} + \text{higher-order terms in } \dot{n}/v. \quad (11)$$

Here ρ is the mass density of the gas, \bar{v} is the average speed of the mass points and $\overline{v^2}$ is the average of v^2 . Equation (11) follows trivially from the kinetic theory of gases applied to a moving wall element or 'piston' (Gross 1975). The non-trivial assumption is that the gas particles are distributed chaotically in phase space, without the kind of order that follows from motion on tori for integrable systems. To calculate the work, δW , done by the gas on the walls of the container we multiply pressure times displacement (δn) and integrate over the container's surface:

$$\delta W = \oint \left(\frac{1}{3} \rho \bar{v}^2 - \rho \bar{v} \dot{n} \right) \delta n d\sigma. \quad (12)$$

The Wall Formula

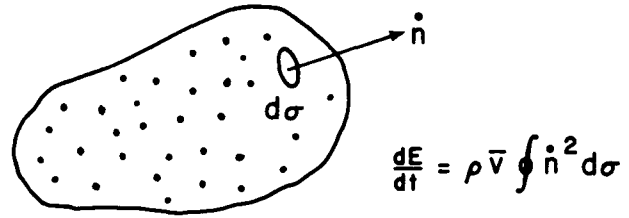
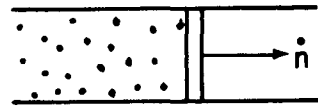


Figure 2. The essence of the Wall Formula: a container is filled with a gas of chaotically moving independent particles and its surface is deforming with normal speeds specified by \dot{n} . The correction to the static pressure leads to a dissipative flow of energy from the walls to the gas.

Physics:



$$\text{Pressure } p = \frac{1}{3} \rho \bar{v}^2 - \widetilde{\rho \bar{v} \dot{n}} + \dots$$

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For volume-preserving deformations the first term is obviously zero. Dividing the remainder by δt , we find for the rate of work being done by the walls on the gas, i.e., for $-dW/dt$, the result

$$-\frac{dW}{dt} = \rho \bar{v} \oint \dot{n}^2 d\sigma, \tag{13}$$

which is the wall formula for a gas without 'drifts' – see Blocki *et al* (1978).

Applying this formula to the case of a surface oscillating according to Eq. (10), one readily finds for the relative energy increase of the gas the expression

$$(T - T_0)/T_0 = \frac{3}{4} \alpha \eta \left(\omega t - \frac{1}{2} \sin 2 \omega t \right). \tag{14}$$

This represents a monotonic increase, consisting of a part linear in t , modulated by a term proportional to $\sin 2\omega t$ (making the rate of dissipation zero when the amplitude of the oscillation is at its turning point and the wall elements are instantaneously at rest). The linear term is shown in the upper panel of Fig. 1 as the dashed line. After 5 cycles, i.e., for $\omega t = 5(2\pi)$, Eq. (14) gives $(3/4) (0.2)(0.04)5(2\pi) = 0.189$. Including the modulations, one obtains a curve essentially indistinguishable from the curve for P_5 in the middle panel, and thus also close to the curves labeled P_3 , P_4 and P_6 . The curve for P_2 refers to deformations that, for the small value of α under consideration, are close to spheroidal. This is the case of a nearly integrable shape and the graph of the energy change is similar to the curves in the lowest panel. The P_2 distortion is considered further in the next subsection.

2.4. Fission-like deformation

A shape described by

$$R(\theta) = \frac{R_0}{\lambda} [1 + \alpha P_2(\cos\theta)] \quad (15)$$

is like a spheroid for small α and like an hourglass figure for α tending to 2. For α equal to 2 the neck radius has been reduced to zero and one has a scission-like shape of two equal, somewhat deformed fragments in contact. Figure 3 shows the energy increase of our classical gas when the originally spherical shape is made to scission by increasing α linearly in time, from 0 to 2. The constant rate of change, $\dot{\alpha}$, is such that, initially, the container's tip speed is, as before, given by ηv , with η taken to be 0.001, 0.01, 0.02, 0.03 and 0.1. Looking apart from the case $\eta = 0.1$, Fig. 3 shows that for deformations up to about $\alpha \approx 0.3$ the energy increase of the gas is essentially elastic, i.e., independent of the rate of deformation. Beyond $\alpha \sim 0.3$ the excitation of the gas shows a dependence on the speed of deformation qualitatively reminiscent of the behaviour expected from a viscous fluid. At the highest rate of deformation ($\eta = 0.1$) the response of the gas is not elastic even for small deformations.

Let us now look at the relation between the rubber-to-honey transition and the transition to chaos in the motions of the gas particles.

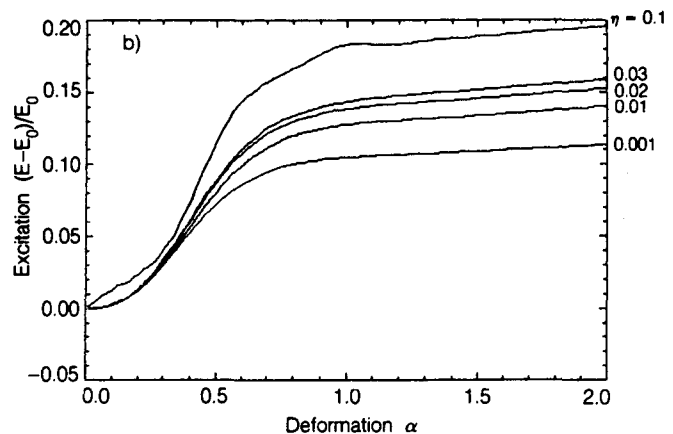


Figure 3. The relative excitation energy for a fission-like deformation proportional to $\alpha P_2(\cos\theta)$, where α increases linearly with time from $\alpha = 0$ to $\alpha = 2$. The deformation proceeds at different rates, as indicated by the adiabaticity index η .

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3. POINCARÉ TORI, POINCARÉ SECTIONS, KAM THEOREM

Consider first the integrable case of a particle in a spheroidal cavity and, for the sake of simplicity, start with motion in a plane containing the axis of symmetry (the x-axis). There are two degrees of freedom, x and ρ , where ρ is the distance from the axis of symmetry. The phase space $x, \rho, \dot{x}, \dot{\rho}$ is four dimensional. Energy conservation confines the motion to three dimensions and conservation of the quantity $\bar{l}_1 \cdot \bar{l}_2$ further reduces the accessible phase space to a two-dimensional surface. Textbooks on dynamics tell us that this surface is a torus and the representative point in phase space winds around this torus like the windings round a ring-like solenoid. Now make a so-called Poincaré cut through phase space by a plane

defined, for example, by $x = 0$ (i.e., a plane through the equator of the spheroid). The intersection of this plane with the torus in phase space will be a closed curve. This means that if, every time the particle trajectory crosses this $x = 0$ equatorial plane, you jot down the values of ρ and $\dot{\rho}$ and plot the result as a point in the $\rho-\dot{\rho}$ plane, successive such points should arrange themselves on a closed curve. That they really do is illustrated in the first frame of the top row in Fig. 4. This shows such $\rho, \dot{\rho}$ plots (in the figure $\dot{\rho}$ is denoted by v_ρ) for a particle followed numerically as it bounces about inside a (static) spheroid, whose axes are in the ratio of $(1+\alpha)$ to $(1+\alpha)^{-1/2}$, with $\alpha = 0.05$. Each curve corresponds to starting the particle off with certain given initial conditions. [In Fig. 4 the unit of ρ is the equatorial radius of the spheroid and the unit of v_ρ is the particle's (unchanging) speed.] The other panels in the first column in Fig. 4 refer to a similar study, but for planar trajectories in static containers deformed according to Legendre Polynomial deformations (Eq. (10)) with $n = 2, 3, \dots, 6$ and $\alpha = 0.05$ in all cases. For the more highly rippled, non-integrable containers ($n = 5, 6$) the Poincaré tori have been broken up into irregular swarms of points, indicating chaotic motions. For $n = 3$ and 4 traces of tori remain, and for $n = 2$ most of the tori appear intact.

Figure 5 shows a similar study but for larger distortions, specified by $\alpha = 0.1$. In this case chaos seems to dominate for $n = 3, 4, 5, 6$. The case $n = 2$ shows a mixture of chaos and order. In the case of the spheroid, which is an integrable system for all eccentricities, the tori remain intact, as they should. Figure 6 corresponds to a similar study with $\alpha = 0.01$. Now most of the tori are still present. This survival of tori in non-integrable dynamical systems is an illustration of the Kolmogorov, Arnold, Moser theorem, about which later.

The other columns in Figs. 4, 5, 6 correspond to non-planar trajectories, i.e., to particles with angular momentum around the symmetry axis. The values of the angular momenta in the four columns in each figure correspond to 0, 0.25, 0.5, 0.75, measured in units of the maximum possible angular momentum, i.e., the angular momentum that would correspond to starting the particle with its standard velocity just inside and along the tangent to the equator of the container. Note that increasing the angular momentum has the effect of restoring order. I believe the explanation is quite simple. Increasing angular momentum produces an increasing centrifugal force away from the symmetry axis, and this confines the particle to a decreasing domain in the vicinity of the equator. Now, a small segment of a smooth surface of revolution can be described with good accuracy as a portion of an osculating spheroid, and the particle has then almost no way of telling that it is not moving inside a spheroid. The spheroid being integrable, the motion of a particle with high angular momentum projection on the axis of symmetry tends to exhibit ordered motion on a torus.

Figures 1 and 3–6 illustrate the qualitative correlation between the presence of order or chaos in the particle trajectories in static containers on the one hand and, on the other, the elastic or dissipative response of a gas of such particles to slow deformations of the containers.

What are the rules governing the transition from order to chaos in dynamical systems, rules that may be expected to govern the transition from rubber-like to honey-like dynamics of idealized independent-particle models of many-body systems?

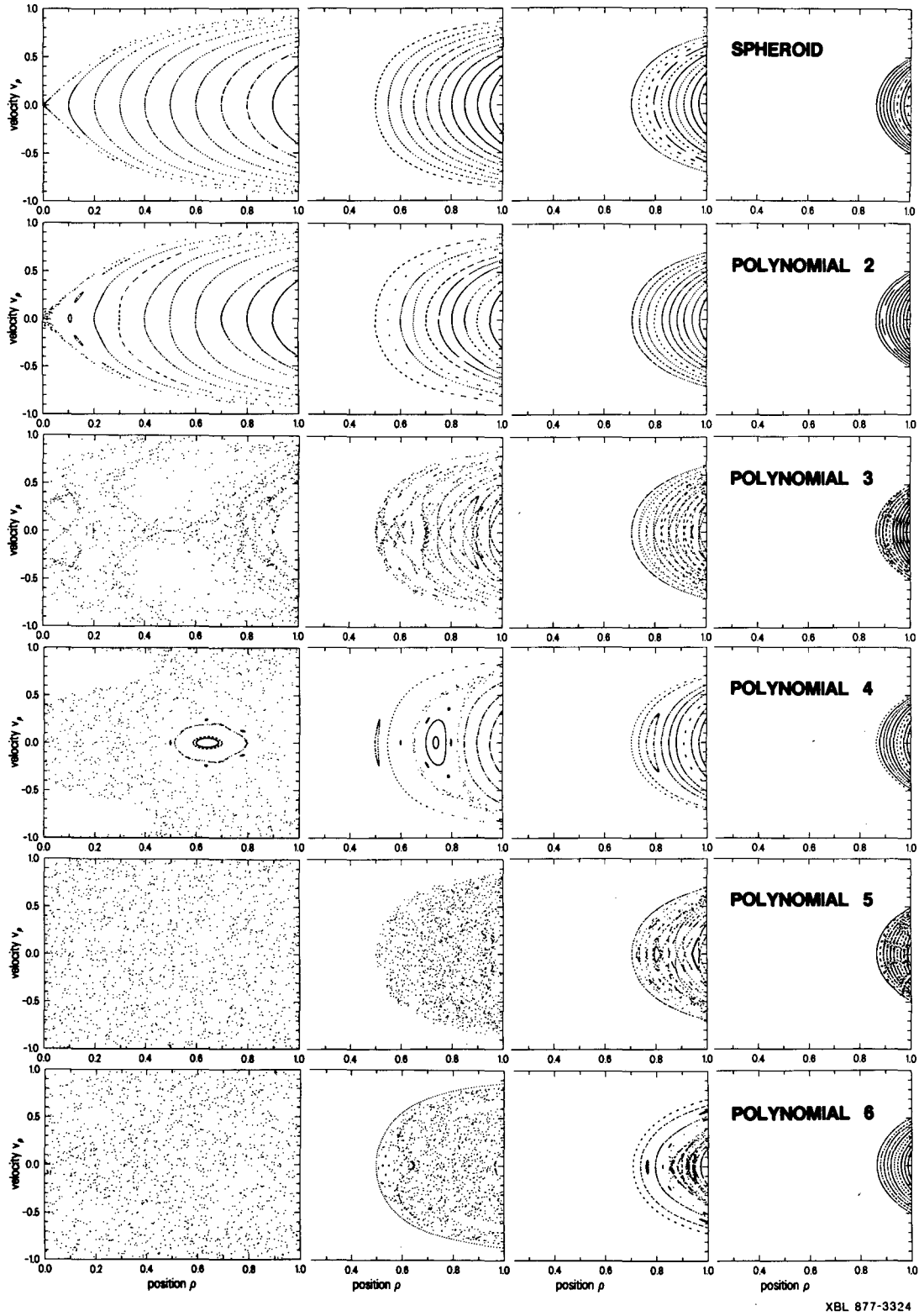
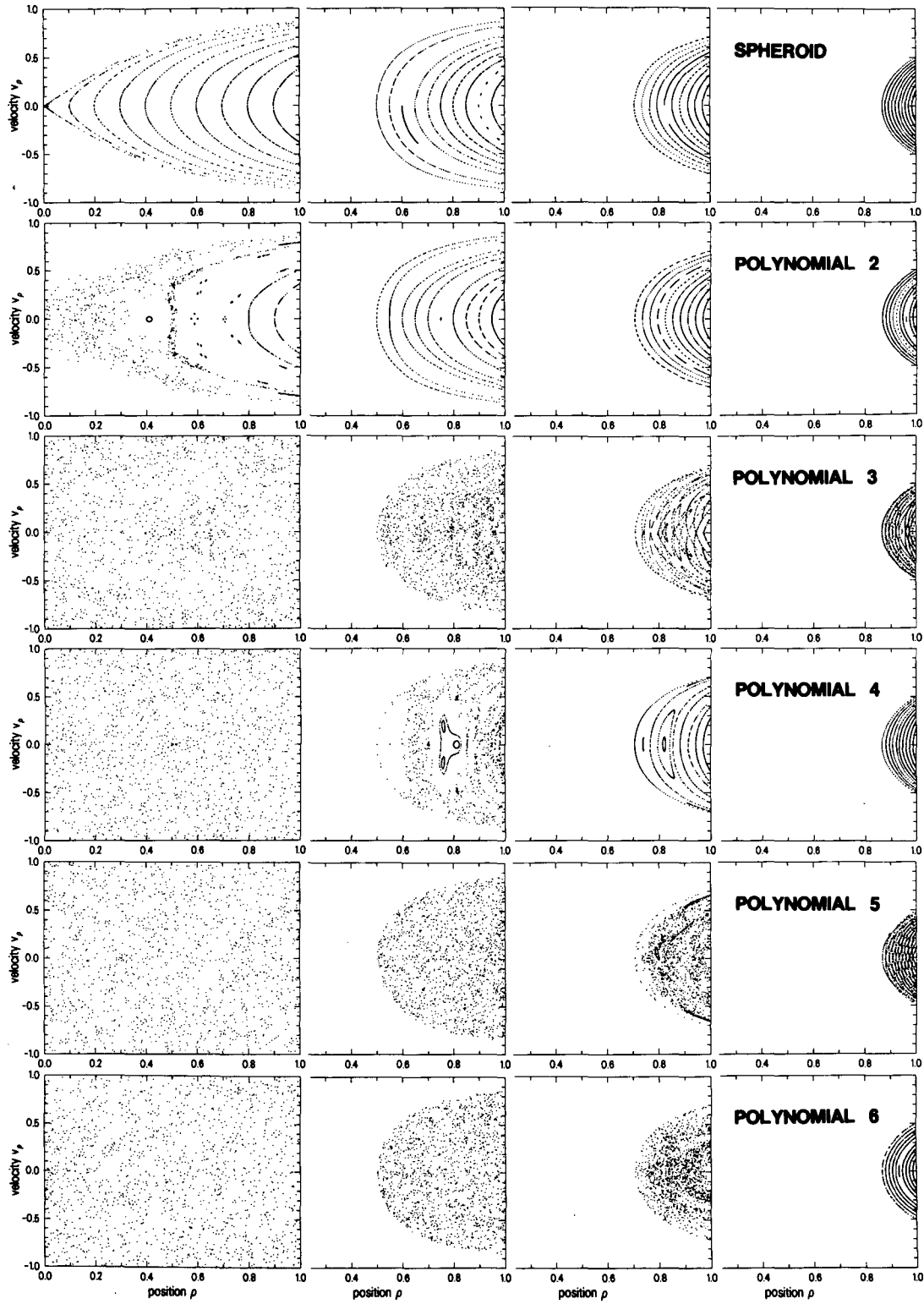
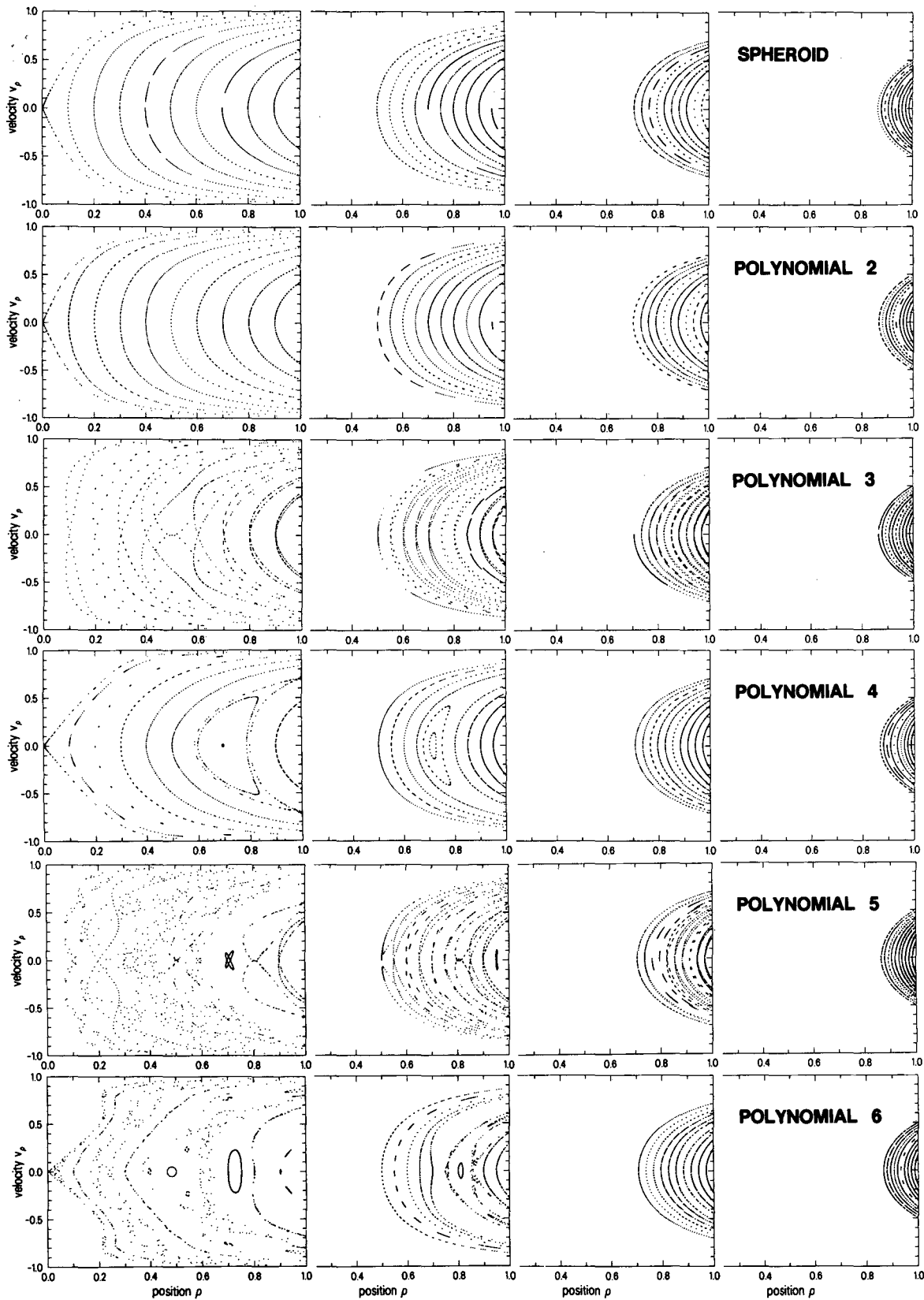


Figure 4. Twenty four Poincaré sections for ten particles bouncing about in a spheroidal container (top row) or a sphere deformed according to a Legendre Polynomial $P_n(\cos\theta)$. The particle's distance ρ from the axis of symmetry and its associated velocity $\dot{\rho}$, denoted by v_ρ , are noted at the moment when the particle crosses the equatorial plane of the container. The result is entered as a dot in the v_ρ vs ρ plot. The dots lie on a curve (a section through a Poincaré torus) if the motion is regular, or from a random swarm if the motion is chaotic. The four columns correspond to four increasing values of the projection of the particle's angular momentum on the symmetry axis, starting with zero in the first column. The RMS deviation of the surface of the container from the sphere is, in all cases, $\alpha/\sqrt{5}$, with $\alpha = 0.05$.



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Figure 5. This is like Fig. 4, but for a distortion specified by $\alpha = 0.1$.



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Figure 6. This is like Fig. 4, but for a smaller distortion, specified by $\alpha = 0.01$.

Mathematically, this is the question of analyzing the dynamics of a system governed by a Hamiltonian in the form

$$H = H_0 + \epsilon H_1 , \tag{16}$$

where H_0 is integrable and ϵH_1 is a perturbation that, with increasing ϵ , spoils the integrability more and more strongly. Poincaré called this “the fundamental problem of dynamics.” It has turned out to be an unbelievably intricate problem. In the 19th century it was the focus of interest in connection with the dynamics of the idealized solar system, which is integrable if the interplanetary forces are disregarded, but not otherwise. A milestone in the analysis of the problem represented by Eq. (16) was a theorem, dating from about 1954, that bears the names of A.N. Kolmogorov, V.I. Arnold, and J. Moser. Crudely paraphrased, the KAM theorem states that the vast majority of Poincaré tori of the integrable problem are tough against generic non-integrable perturbations. Thus, when a small (but finite) non-integrable perturbation is switched on, it is not that the trajectories in phase space slowly wander away from the tori and the tori disappear after a sufficiently long time, but that most of the tori stay forever as exact tori. The only thing that typically happens to most of the tori is that they get somewhat distorted. The KAM theorem has far-reaching consequences. For example, it says that quite often you may be presented with a generic non-integrable Hamiltonian, i.e., no extra constants of motion besides the energy, and yet the phase space may be dominated by tori, as if the problem were integrable. Of course, this is not always the case: if you crank up the deviation from integrability, more and more tori will get destroyed and phase space will be an intricate mixture of regular solenoidal trajectories wound on the surviving tori and chaotic trajectories randomly weaving about in between. For certain shapes all tori may be destroyed and complete chaos can take over. Whether that will happen in any given case cannot be decided by just examining the Hamiltonian, except for a few very special cases.

Hundreds of papers have been written in the past years on the theory of the transition from order to chaos in dynamical systems. They include at one extreme rigorous mathematical theorems, on the other computer simulations of idealized dynamical systems, as well as ingenious experimental investigations.

Because of the relation between the order-to-chaos transition and the rubber-to-honey transition in the behaviour of an ideal gas of independent particles, theories of nuclear dynamics find themselves closely coupled with research in many other fields. But since quantization of the nucleonic motions is expected to play an essential role in the nuclear problem, let us have a look how quantization might affect the behaviour of our ideal gas.

4. QUANTAL ASPECTS OF ORDER AND CHAOS

First let us look again at the point particles in the shoe box, but this time quantized. The total kinetic energy of the gas will be the sum over all particles of the expectation value of the kinetic energy operator:

$$T = \sum_i \iiint \psi_i^*(x,y,z) \left[-\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \right] \psi_i(x,y,z) = T_x + T_y + T_z \quad (17)$$

Now slowly stretch the sides of the box by the factors a/a_0 , b/b_0 , c/c_0 . The wave functions will get stretched correspondingly. The curvature (second derivative) of a wave function stretched by a/a_0 is decreased in the ratio $(a_0/a)^2$, so the new kinetic energy will be

$$T = T_{x0}(a_0/a)^2 + T_{y0}(b_0/b)^2 + T_{z0}(c_0/c)^2 \quad , \quad (18)$$

where T_{x0} , T_{y0} , T_{z0} are the initial x,y,z components of the energy. This is exactly the same as Eq. (3) for classical particles.

For volume preserving deformations we may write

$$T = \lambda^{-2}T_{x0} + \mu^{-2}T_{y0} + \lambda^2\mu^2T_{z0} \quad , \quad (19)$$

where $\lambda = a/a_0$, $\mu = b/b_0$, $\lambda\mu = c_0/c$. Consider the case where the original kinetic energy is stationary with respect to deformations, viz.

$$\left. \frac{\partial T}{\partial \lambda} \right|_{\lambda=\mu=1} = -2T_{x0} + 2T_{z0} = 0 \quad , \quad \left. \frac{\partial T}{\partial \mu} \right|_{\lambda=\mu=1} = -2T_{y0} + 2T_{z0} = 0 \quad . \quad (20)$$

From this it follows again that $T_{x0} = T_{y0} = T_{z0} = \frac{1}{3} T_0$, exactly as in the classical case, so that, once again,

$$T/T_0 = \frac{1}{3} [a_0/a)^2 + (b_0/b)^2 + (c_0/c)^2] \quad . \quad (21)$$

In particular, for deformations where $b/b_0 = c/c_0$ and $a/a_0 = 1 + \alpha$, we shall again find that

$$T \approx T_0 + T_0\alpha^2 \quad . \quad (22)$$

Thus, the quantized gas in a box responds to slow deformations exactly like the classical gas, i.e., like rubber, with an elastic constant equal to twice the total energy of the gas.

The reason for this identity is that, for the box, the semi-classical quantization rule of setting the action integrals equal to multiples of Planck's constant happens to be exact. Now, for adiabatic deformations, the classical integrals are already constants, even before quantization, so as regards the response of the system to slow deformations, nothing has changed. Quantization restricts the allowed values of a particle's energy to certain discrete amounts, but the response of each allowed energy to slow shape changes is exactly the same as before quantization.

Incidentally, note that the form of Eq. (17) says nothing about ψ being the wave function of a particle in a box, or about the number of particles in the sum, whether one or more. So the following result is quite general: the kinetic energy associated with a wave function of any number of particles, in any kind of potential well, responds to stretchings away from equilibrium with an elastic constant equal to twice its original energy. But note that (unlike for

the box) a stretched wave function is not, in general, a solution of the Schrödinger equation in the correspondingly stretched potential well.

In the case of integrable motions in containers of a type more general than the box, e.g., for the spheroidal cavity, the correspondence between the classically and quantally calculated responses to slow shape changes is not expected to be exact, but only as good as the semi-classical quantization prescription of equating the relevant action integrals to constants.

This prescription is often very good, but it has its limitations, for example when penetration of the wave function into classically forbidden regions is important. For classical, semi-classical and quantal treatments of a particle in a spheroidal container, see especially Arvieu *et al.* (1987) and references therein. My expectation is that in the case of integrable situations the correspondence between the response of a classical and a quantal gas in a container should be generally quite close, i.e., in both cases the response to slow shape changes should be rubber-like.

In the opposite extreme of strongly non-integrable container shapes, when the particle motions are chaotic and the wall formula describes the dissipative response of a classical system, the corresponding quantal behavior is not well understood. Fragmentary numerical studies indicate that sometimes the wall formula continues to give a quantitative description, other times it fails badly (Blocki *et al.* 1990). There are several reasons for a qualitatively different behavior of a quantal system. An obvious one is the existence of a new unit of energy or frequency in the quantal problem, related to the spacing between the discrete energy levels. For small oscillation frequencies of the container, such that even the first level above the ground state is rarely excited, there obviously can be no correspondence with a formula that relies on the classical limit of a high (infinite) level density. A second, much more subtle effect, has to do with the saturation of the excitation energy that certain quantal systems (such as the quantal kicked rotor) are able to absorb when subjected to periodic perturbations. This is where some of the unsolved problems of 'Quantum Chaos' make their appearance (see Wilkinson 1988–90; Wilkinson and Austin 1990). This is outside the scope of the present introductory lecture, but I would like to draw your attention to two excellent review articles. The first, entitled "Simple Models of Quantum Chaos: Spectrum and Eigenfunctions," soon to appear in *Physics Reports* (Izrailev 1990), contains 197 references, most of them from the last few years, but none (as far as I could tell) concerned with nuclear dynamics. The second, Bohigas and Weidenmüller 1988, with 123 references, stresses stationary and statistical aspects of nucleonic motions. To integrate the progress being made in Quantum Chaos with collective nuclear dynamics is still an outstanding task for the future.

5. CONCLUSION

The purpose of this lecture was to illustrate by the simplest possible examples the relation that exists between the order-to-chaos transition in the dynamics of independent gas particles, and the rubber-like or honey-like collective response of the gas to shape changes. The existence of this relation suggests a fundamental connection between theories of collective nuclear dynamics and general studies of order-to-chaos transitions in classical and quantal systems.

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