Title
POLARIZATION AND ANGULAR CORRELATION IN THE PRODUCTION AND DECAY OF PARTICLES OF SPIN 1/2 AND SPIN 3/2

Permalink

Authors
Spitzer, Richard
Stapp, Henry P.

Publication Date
1957-06-05

UNIVERSITY OF CALIFORNIA

Radiation Laboratory

BERKELEY, CALIFORNIA

UCRL-3796

S. Y. FUNG

POLARIZATION AND ANGULAR CORRELATION IN THE PRODUCTION AND DECAY OF PARTICLES OF SPIN $\frac{1}{2}$ AND SPIN $\frac{3}{2}$

Richard Spitzer and Henry P. Stapp*

Radiation Laboratory, University of California
Berkeley, California

June 5, 1957

ABSTRACT

A general formalism describing the angular correlation and polarization effects in the production and subsequent decay of particles of arbitrary spins has been developed. It has been specialized to the cases of production and decay of particles of spin $\frac{1}{2}$ and $\frac{3}{2}$. Expressions for the angular distribution and polarization of the decay products have been reduced to tractable forms involving the physical vectors of the problem and a minimal number of parameters describing the production and decay interactions. The results are discussed for two particular production processes in order to determine what information on the spin of the hyperon and the production and decay mechanisms may be obtained from the analysis of the decay products.

---

* This work was performed under the auspices of the U.S. Atomic Energy Commission.

POLARIZATION AND ANGULAR CORRELATION IN THE PRODUCTION AND DECAY OF PARTICLES OF SPIN $\frac{1}{2}$ AND SPIN $\frac{3}{2}$

Richard Spitzer and Henry P. Stapp

Radiation Laboratory, University of California
Berkeley, California

June 5, 1957

Section I. Introduction

The angular distribution of the products of decay of a hyperon provides information regarding the hyperon spin. If this spin is one-half, then the probability that the direction of the final nucleon will lie in one of the two polar cones ($|\cos \Theta| > \frac{1}{2}$, where Θ is the center-of-mass angle between the hyperon velocity and the final nucleon velocity) must be exactly one-half. On the basis of recent measurements of the angular distribution of Σ -decay products the probability that the spin of the Σ is $\frac{1}{2}$ is 5%.[1] In view of this indication that the spin of the Σ may be greater than $\frac{1}{2}$, it is of interest to determine the detailed consequences of larger values for the hyperon spin. The purpose of this paper is to examine the correlation between the direction of the nucleon emitted in the decay of the hyperon on the one hand and the directions defined by the production process on the other hand. The polarization of the final nucleon is also treated. Some general formulas are quoted in this section and are applied to the case of spin-$\frac{3}{2}$ particles in the following sections. Some analogous results for the spin-$\frac{1}{2}$ case are given for comparison. We use an apparently nonrelativistic formulation, but the results may be applied to the relativistic case if appropriate interpretations and corrections are made. These are discussed in Section III.

In the analysis of polarization phenomena statistical mixtures of states must be considered, and a density matrix formulation is convenient.[2] The spin-space density matrix $\mathcal{U}(\theta\phi)$ is defined by the relation

$$\langle A \rangle_{\theta\phi} = \mathrm{Tr}\, A\, \mathcal{U}(\theta\phi) \tag{1.1}$$

where $\langle A \rangle_{\theta\phi}$ is the expectation value of a spin operator A if the measurement is made on a particle in the beam moving in the direction $\theta\phi$. The matrices A and $\mathcal{U}(\theta\phi)$ are square matrices of dimension $(2S + 1)$, where S is the spin quantum number. It is convenient to introduce a complete orthonormal set of matrices in this space. We use the matrices T^Q_κ defined as follows:[3]

$$\langle S' \ \mu' \mid T^Q_\kappa \mid S'' \ \mu'' \rangle = \left(\frac{2Q + 1}{2S' + 1}\right)^{\frac{1}{2}} C^{S'' \, Q \, S'}_{\mu'' \, \kappa \, \mu'}$$

$$= \left(\frac{2Q + 1}{2S' + 1}\right)^{\frac{1}{2}} C_{S''Q}(S' \mu', \mu'' \kappa) , \tag{1.2}$$

where the six-index symbols on the right are the usual Clebsch-Gordan coefficients.[4] The matrices T^Q_κ are real and their hermitian conjugates $\bar{T}^Q_\kappa$ are their respective transposes. By use of their completeness property the $\mathcal{U}(\theta\phi)$ may be expanded in the form

$$\mathcal{U}(\theta\phi) = \bar{\alpha}^Q_\kappa(\theta\phi)\, T^Q_\kappa = \alpha^Q_\kappa(\theta\phi)\, \bar{T}^Q_\kappa . \tag{1.3}$$

The coefficients $\alpha^Q_\kappa(\theta\phi)$ and $\bar{\alpha}^Q_\kappa(\theta\phi)$ defined by the above equations are complex conjugates owing to the hermiticity of the density matrix. In virtue of the orthonormality condition

$$\mathrm{Tr}\, T_{\kappa}^{Q}\, \bar{T}_{\kappa'}^{Q'} = \delta_{QQ'}\, \delta_{\kappa\kappa'} \,, \tag{1.4}$$

the $\alpha_{\kappa}^{Q}(\theta\phi)$ and $\bar{\alpha}_{\kappa}^{Q}(\theta\phi)$ may be expressed as

$$\begin{aligned} \alpha_{\kappa}^{Q}(\theta\phi) &= \mathrm{Tr}\ \mathcal{U}(\theta\phi)\, T_{\kappa}^{Q} \,, \\ \bar{\alpha}_{\kappa}^{Q}(\theta\phi) &= \mathrm{Tr}\ \mathcal{U}(\theta\phi)\, \bar{T}_{\kappa}^{Q} \,. \end{aligned} \tag{1.5}$$

We shall be interested in processes in which the initial states are described by the spin-orbit variables $(S', \mu', \theta', \phi')$ and the final states by the spin-orbit variables (S, μ, θ, ϕ). The spin-space characteristics of the initial system will be described by the coefficients $\alpha_{\kappa'}^{Q'}(\theta'\phi')$ and the final system will be similarily described by the coefficients $\alpha_{\kappa}^{Q}(\theta\phi)$. If the initial system is a plane wave moving in the direction $\theta'\phi'$ with a spin quantum number S', then the parameters $\alpha_{\kappa}^{Q}(S, \theta\phi)$, which describe the spin-space characteristics of the reaction products that emerge in the direction $\theta\phi$ and in the state with spin S, are given in terms of the parameters $\alpha_{\kappa'}^{Q'}(S', \theta'\phi')$, which describe the spin-space characteristics of the initial plane wave, by the equation[5]

$$I(\theta\phi)\ \alpha_{\kappa}^{Q}(S, \theta\phi) = \frac{N}{4\pi} \sum_{LL'L''L'''} \sum_{JJ'} R^{J}_{SL;S'L'}\ R^{J'\,*}_{SL'';S'L'''}$$

$$(2J+1)(2J'+1)\left[(2L+1)(2L'+1)(2L''+1)(2L'''+1)\right]^{\frac{1}{2}}$$

$$\times \sum_{\Lambda\Lambda'\lambda\lambda'} Y_{\Lambda}^{\lambda}(\theta\phi)\ Y_{\Lambda'}^{\lambda'}(\theta'\phi')\ C_{0\,0\,0}^{L\,L''\Lambda}\ C_{0\,0\,0}^{L'''L'\Lambda'}\ (-1)^{L+L'+\Lambda}$$

(Eq. (1.6) cont.)

$$\times \sum_{Q'\chi'} \alpha_{\chi'}^{Q'}(S', \theta'\phi')(2Q'+1)^{\frac{1}{2}} \sum_{Z\zeta} (2Z+1)^{\frac{1}{2}} C_{\lambda\zeta\chi}^{\Lambda Z Q} C_{\chi'\lambda'\zeta}^{Q'\Lambda' Z}$$

$$\times \quad X(L'' L \Lambda, J' J Z, S S Q)\, X(L''' L' \Lambda', J' J Z, S' S' Q') \,.$$

(1.6)

The X coefficient is the one defined by Fano,[6] the $Y_L^m(\theta\phi)$ are the usual spherical harmonics,[4] the $R^J_{SL;S'L'}$ are reaction matrix elements determined by the specific nature of the reaction,[7] and the coefficient N is a normalization factor. If the initial system is a plane-wave state with momentum k', and N is taken as $(2\pi/k')^2$, then $I(\theta\phi)$ is the differential cross section (see Appendix A). The value of $I(\theta\phi)$ may be determined by the condition (implied by Eqs. (1.1), (1.4) (1.5) and the requirement that the expectation value of a pure number is equal to that number)

$$\alpha_0^0(S, \theta\phi) = (2S+1)^{-\frac{1}{2}} \,. \tag{1.7}$$

If the initial system is unpolarized (i.e., only $\alpha_0^0(S' \theta'\phi') \neq 0$), Eq. (1.6) reduces to

$$I_0(\theta\phi)\, \alpha_{\kappa}^{Q}(S,\, 0\; 0) = \frac{\alpha_0^0(S',\, \theta'\phi')N}{(4\pi)^{3/2}} \sum_{LL'L''L'''} \sum_{JJ'} R^{J}_{SL;S'L'}\; R^{J'\,*}_{SL'';S'L'''}$$

$$\left[(2J+1)(2J'+1)(2L+1)(2L''+1)\right]^{\frac{1}{2}} (-1)^{2J'-J-S'+L''}$$

$$\times \sum_{\Lambda\Lambda'} Y_{\Lambda'}^{\kappa}(\theta'\phi')\; C_{0\,\kappa\,\kappa}^{\Lambda\,\Lambda'\,Q}\; C_{0\,0\,0}^{L\,L''\,\Lambda} \left(\frac{2\Lambda+1}{2S'+1}\right)^{\frac{1}{2}}$$

$$\times \quad X(L''\,L\,\Lambda,\; J'\,J\,\Lambda',\; S\,S\,Q)\,(i)^{L'''-L'-\Lambda'}\; Z(L'''\,J'\,L'\,J;\,S'\,\Lambda') \quad , \tag{1.8}$$

where for simplicity the z axis has been taken to lie along the direction of the outgoing fermion. The Z coefficient is the coefficient defined by Blatt and Biedenharn.[8]

In addition to processes in which the initial system is represented by a plane wave, we shall be interested in cases in which the initial state is an incoherent mixture of various orbital angular momentum states.[9] If the probability that the reaction is initiated in a state of orbital angular momentum L is W_L, and if there is no preferred direction for the initial system, then the $\alpha_{\kappa}^{Q}(S,\,\theta\phi)$ describing the final system are given by

$$I(00)\,\alpha^{Q}_{\kappa}(S,\,00) =$$

$$\frac{N}{4\pi}\,\frac{\delta_{0\kappa}}{(2S'+1)}\sum_{LL'L''J} R^{J}_{SL;S'L'}\,R^{J*}_{SL'';S'L'}\,\frac{W_{L'}}{(2L'+1)}\left(\frac{2J+1}{2S+1}\right)^{\frac{1}{2}}$$

$$\times\,(i)^{Q-L-L''}\,Z(Q\,S\,L''\,J;\,S\,L)\,, \qquad (1.9)$$

where the z axis has again been taken to lie along the velocity of the final fermion. If parity is conserved, the value of Q is restricted to even values. This is a consequence of the following relationship satisfied by the Z coefficient:

$$Z(Q\,S\,L''\,J;\,S\,L) = (-1)^{Q+L''-L}\,Z(Q\,S\,L''\,J;\,S\,L)\,.$$

By extracting from the general formula given by Eq. (1.6) the contributions from initial S states, one obtains the formula for the α's that describe the final system of the decay interaction in terms of the α's that describe the spin-space characteristics of the decaying particle:

$$I(\theta\phi)\,\alpha^{Q}_{\kappa}(S,\,\theta\phi) = \frac{N}{(4\pi)^{\frac{1}{2}}}(2S'+1)\sum_{LL''} R_{SL;S'}\,R^{*}_{SL'';S'}$$

$$\times\,(2L+1)^{\frac{1}{2}}\,(2L''+1)^{\frac{1}{2}}\sum_{\Lambda\lambda} C^{L\,L''\,\Lambda}_{0\;0\;0}\,Y^{\lambda}_{\Lambda}(\theta\phi)$$

$$\sum_{Q'\kappa'}\alpha^{Q'}_{\kappa'}(S')(2Q'+1)^{\frac{1}{2}}\,(-1)^{L''}\,C^{\Lambda\,Q'\,Q}_{\lambda\,\kappa'\,\kappa}\,X(L''L\Lambda;\,S'S'Q',\,S\,S\,Q)\,. \qquad (1.10)$$

The combined process of production followed by decay may be described, therefore, by first using Eq. (1.6), (1.8), or (1.9) to obtain the α's that describe the spin state of the intermediate particle, and then using Eq. (1.10) to obtain the α's that describe the polarization and angular distribution of the decay products.

The above formulas relate the expectation values of operators in the initial and final states. It is sometimes convenient to consider the reaction matrix itself. According to the definitions given in Appendix A the matrix element $\langle S\ \mu | \mathcal{R}(\theta\phi;\ \theta'\phi') | S'\ \mu' \rangle$, when multiplied by $(2\pi/k')(v/v')^{\frac{1}{2}}$, where v' and v are the initial and final relative velocities, gives the reaction (or scattering) amplitude $f_{\mu}^{S}(\theta\phi)$ when the initial state is a plane wave of unit particle density in the spin state $\chi_{\mu}^{S'}$. For the case in which the z axis is chosen to lie along the outgoing direction the $\mathcal{R}(\theta\phi;\ \theta'\phi')$ matrix may be expressed in the form

$$\mathcal{R}(0\ 0;\ \theta'\ \phi') = \sum_{Q\,\kappa} a_{\kappa}^{Q}(S,\ 0\ 0)\ \bar{T}_{\kappa}^{Q}$$

where

$$a_{\kappa}^{Q}(S,\ 0\ 0) = \frac{(-1)^{S'-S}}{(4\pi)^{\frac{1}{2}}} \sum_{L\,L'\,J} R^{J}_{S\ L;\ S'L'}\ Y_{L'}^{\kappa}(\theta'\phi')$$

$$\times\ (2L+1)^{\frac{1}{2}}\ (2J+1)\ (-1)^{L}\ C^{L'\ L\ Q}_{\kappa\ 0\ \kappa}\ W(L\ J\ Q\ S';\ S\ L')\ , \qquad (1.11)$$

where W is the Racah coefficient.[10] If the initial and final spin quantum numbers, S' and S respectively, are equal then the matrices $\bar{T}_{\kappa}^{Q}$ are square matrices. Otherwise they are nonsquare, with $(2S' + 1)$ columns and $(2S + 1)$ rows.

Section II. Reaction Formulas for Spin $\frac{1}{2}$ and Spin $\frac{3}{2}$ Particles.

In this section explicit expressions are given for the angular distribution and polarization parameters for reactions in which the initial and final states are composed of one particle of spin 0 and one particle, which will be termed the fermion, of spin $\frac{1}{2}$ or $\frac{3}{2}$. The case in which the initial and final fermions are both spin $\frac{1}{2}$ is very simple and the general formulas given in Section I are not particularly useful. The results for this case will be quoted for comparison with the spin $\frac{3}{2}$ case.

For the case in which the initial and final fermions both have spin $\frac{1}{2}$ the $\mathcal{R}$ matrix can be written in the completely general form

$$\mathcal{R}(\underset{\sim}{K}, \underset{\sim}{K}') = (2\pi/k')^{-1}\left[f(\theta) + g(\theta)\sigma_N + h(\theta)\sigma_K + h'(\theta)\sigma_L\right], \tag{2.1}$$

where k' is the incident relative momentum, σ_A represents the Pauli spin matrix $\underset{\sim}{\sigma}\cdot\underset{\sim}{A}$, and the vectors $\underset{\sim}{N}$ and $\underset{\sim}{L}$ are unit vectors in directions $\underset{\sim}{K}' \times \underset{\sim}{K}$ and $\underset{\sim}{N} \times \underset{\sim}{K}$ respectively. The arguments of the $\mathcal{R}$ matrix have been given as $\underset{\sim}{K}'$ and $\underset{\sim}{K}$, unit vectors along the initial and final velocities respectively, rather than $\theta'\phi'$ and $\theta\phi$ as in Section I, because the dependence upon coordinate systems has been removed from the expression appearing on the right. The angle θ in Eq. (2.1), and in what follows, is the angle between $\underset{\sim}{K}$ and $\underset{\sim}{K}'$. The normalization is chosen so that the differential reaction cross section in the reaction center-of-mass frame is

$$I(\underset{\sim}{K}, \underset{\sim}{K'}) = |f|^2 + |g|^2 + |h|^2 + |h'|^2 + 2\,\mathrm{Re}(gf^*)(\underset{\sim}{P'}\cdot\underset{\sim}{N})$$
$$+ 2\,\mathrm{Re}(hf^*)(\underset{\sim}{P'}\cdot\underset{\sim}{K}) + 2\,\mathrm{Re}(h'f^*)(\underset{\sim}{P'}\cdot\underset{\sim}{L}) + 2\,\mathrm{Im}(gh^*)(\underset{\sim}{P'}\cdot\underset{\sim}{L})$$
$$- 2\,\mathrm{Im}(gh'^*)(\underset{\sim}{P'}\cdot\underset{\sim}{K}) + 2\,\mathrm{Im}(hh'^*)(\underset{\sim}{P'}\cdot\underset{\sim}{N}) \quad . \tag{2.2}$$

Here the vector $\underset{\sim}{P'}$ is the polarization vector of the incident particle. It is defined by the equation

$$\mathcal{U}(\underset{\sim}{k'}) = \tfrac{1}{2}(1 + \underset{\sim}{P'}\cdot\underset{\sim}{\sigma}) \quad . \tag{2.3}$$

The polarization vector $\underset{\sim}{P}$ of the final particle is defined in an exactly analogous way and is given by

$$\underset{\sim}{P}(\underset{\sim}{K}, \underset{\sim}{K'}) = I^{-1}(\underset{\sim}{K}, \underset{\sim}{K'}) \Big\{ 2\,\mathrm{Re}(gf^*)\underset{\sim}{N} + 2\,\mathrm{Im}(h'h^*)\underset{\sim}{N} + 2\,\mathrm{Re}(fh'^*)\underset{\sim}{L}$$
$$+ 2\,\mathrm{Re}(fh^*)\underset{\sim}{K} - 2\,\mathrm{Im}(gh^*)\underset{\sim}{L} + 2\,\mathrm{Im}(gh'^*)\underset{\sim}{K} + |f|^2\,\underset{\sim}{P'}$$
$$+ |g|^2\left[2(\underset{\sim}{P'}\cdot\underset{\sim}{N})\underset{\sim}{N} - \underset{\sim}{P'}\right] + |h|^2\left[2(\underset{\sim}{P'}\cdot\underset{\sim}{K})\underset{\sim}{K} - \underset{\sim}{P'}\right]$$
$$+ |h'|^2\left[2(\underset{\sim}{P'}\cdot\underset{\sim}{L})\underset{\sim}{L} - \underset{\sim}{P'}\right] + 2\,\mathrm{Im}(gf^*)(\underset{\sim}{P'} \times \underset{\sim}{N}) + 2\,\mathrm{Im}(hf^*)(\underset{\sim}{P'} \times \underset{\sim}{K})$$
$$+ 2\,\mathrm{Im}(h'f^*)(\underset{\sim}{P'} \times \underset{\sim}{L}) + 2\,\mathrm{Re}(gh^*)\left[(\underset{\sim}{P'}\cdot\underset{\sim}{K})N + (\underset{\sim}{P'}\cdot\underset{\sim}{N})\underset{\sim}{K}\right]$$
$$+ 2\,\mathrm{Re}(gh'^*)\left[(\underset{\sim}{P'}\cdot\underset{\sim}{L})\underset{\sim}{N} + (\underset{\sim}{P'}\cdot\underset{\sim}{N})\underset{\sim}{L}\right] + 2\,\mathrm{Re}(hh'^*)\left[(\underset{\sim}{P'}\cdot\underset{\sim}{L})\underset{\sim}{K} + (\underset{\sim}{P'}\cdot\underset{\sim}{K})\underset{\sim}{L}\right]\Big\} \tag{2.4}$$

If parity is conserved in the reaction, either $h(\theta)$ and $h'(\theta)$ are zero or $f(\theta)$ and $g(\theta)$ are zero. These two cases represent the possibilities that the relative intrinsic parities of the two initial particles are the same as, or alternately are opposite to, the relative intrinsic parities of the two final particles.

When the initial fermion is unpolarized the differential cross section is a function only of the scattering angle θ and of the reaction matrix elements. It will be written as $I_0(\theta)$. If only the contributions from final S, P, and D partial waves are included, Expression (2.1) for the $\mathcal{R}$ matrix becomes

$$\begin{aligned}
\mathcal{R}(K, K') &= \\
(4\pi)^{-1} \Big\{ & R_{00}^{\frac{1}{2}} - R_{22}^{3/2} - \tfrac{3}{2} R_{22}^{5/2} + (R_{11}^{\frac{1}{2}} + 2 R_{11}^{3/2}) \cos\theta \\
& + 3(R_{22}^{3/2} + \tfrac{3}{2} R_{22}^{5/2}) \cos^2\theta + i \sin\theta \, \sigma_N \Big[-R_{11}^{\frac{1}{2}} + R_{11}^{3/2} \\
& \qquad - 3(R_{22}^{3/2} - R_{22}^{5/2}) \cos\theta \Big] \\
& + \sigma_K \Big[R_{10}^{\frac{1}{2}} - R_{12}^{3/2} + (R_{01}^{\frac{1}{2}} + 2 R_{21}^{3/2} - \tfrac{9}{2} R_{23}^{5/2}) \cos\theta \\
& \qquad + 3 R_{12}^{3/2} \cos^2\theta + \tfrac{15}{2} R_{23}^{5/2} \cos^3\theta \Big] \\
& - \sin\theta \, \sigma_L \Big[R_{01}^{\frac{1}{2}} - R_{21}^{3/2} - \tfrac{3}{2} R_{23}^{5/2} + 3 R_{12}^{3/2} \cos\theta + \tfrac{15}{2} R_{23}^{5/2} \cos^2\theta \Big] \Big\}
\end{aligned} \tag{2.5}$$

If parity is conserved in the production and if the initial and final intrinsic parities are equal to each other then the contributions to $I_0(\theta)$ from final S, P, and D waves give

$$I_0(\theta) = \frac{\lambda\!\!\!^{-2}}{4}\left[A + B\cos\theta + C\cos^2\theta + D\cos^3\theta + E\cos^4\theta\right] , \tag{2.6}$$

where

$$A = \left|R_{00}^{\frac{1}{2}}\right|^2 + \left|R_{11}^{\frac{1}{2}}\right|^2 + \left|R_{11}^{3/2}\right|^2 + \left|R_{22}^{3/2}\right|^2 + \frac{9}{4}\left|R_{22}^{5/2}\right|^2$$

$$- 2\,\mathrm{Re}(R_{00}^{\frac{1}{2}} R_{22}^{3/2*} + R_{11}^{\frac{1}{2}} R_{11}^{3/2*}) - 3\,\mathrm{Re}(R_{00}^{\frac{1}{2}} R_{22}^{5/2*} - R_{22}^{3/2} R_{22}^{5/2*})$$

$$B = 2\,\mathrm{Re}(R_{00}^{\frac{1}{2}} R_{11}^{\frac{1}{2}*}) - 10\,\mathrm{Re}(R_{11}^{3/2} R_{22}^{3/2*}) - 9\,\mathrm{Re}(R_{11}^{\frac{1}{2}} R_{22}^{5/2*})$$

$$+ 4\,\mathrm{Re}(R_{00}^{\frac{1}{2}} R_{11}^{3/2*} + R_{11}^{\frac{1}{2}} R_{22}^{3/2*}) ,$$

$$C = 3\left|R_{11}^{3/2}\right|^2 + 3\left|R_{22}^{3/2}\right|^2 - \frac{9}{2}\left|R_{22}^{5/2}\right|^2$$

$$+ 6\,\mathrm{Re}(R_{00}^{\frac{1}{2}} R_{22}^{3/2*} + R_{11}^{\frac{1}{2}} R_{11}^{3/2*}) + 9\,\mathrm{Re}(R_{00}^{\frac{1}{2}} R_{22}^{5/2*})$$

$$- 36\,\mathrm{Re}(R_{22}^{3/2} R_{22}^{5/2*}) ,$$

$$D = 18\,\mathrm{Re}(R_{11}^{3/2} R_{22}^{3/2*}) + 15\,\mathrm{Re}(R_{11}^{\frac{1}{2}} R_{22}^{5/2*}) + 12\,\mathrm{Re}(R_{11}^{3/2} R_{22}^{5/2*}) \ ,$$

$$E = \frac{45}{4} \left| R_{22}^{5/2} \right|^2 + 45\,\mathrm{Re}(R_{22}^{3/2} R_{22}^{5/2*}) \ .$$

The polarization vector is, under the same conditions and in terms of the same reaction matrix elements, given by

$$I_0(\theta)\underset{\sim}{P} = \frac{\lambda\!\!\!^{-2}}{4} \sin\theta \left[F + G\cos\theta + H\cos^2\theta + K\cos^3\theta \right] \underset{\sim}{N} \qquad (2.7)$$

where

$$F = 2\,\mathrm{Im}(R_{11}^{\frac{1}{2}} R_{00}^{\frac{1}{2}*} + R_{00}^{\frac{1}{2}} R_{11}^{3/2*} - R_{11}^{\frac{1}{2}} R_{22}^{3/2*} - R_{22}^{3/2} R_{11}^{3/2*})$$

$$- 3\,\mathrm{Im}(R_{11}^{\frac{1}{2}} R_{22}^{5/2*} - R_{11}^{3/2} R_{22}^{5/2*}) \ ,$$

$$G = 6\,\mathrm{Im}(R_{11}^{\frac{1}{2}} R_{11}^{3/2*} - R_{00}^{\frac{1}{2}} R_{22}^{3/2*} + R_{00}^{\frac{1}{2}} R_{22}^{5/2*}) - 15\,\mathrm{Im}(R_{22}^{3/2} R_{22}^{5/2*}) \ ,$$

$$H = -18\,\mathrm{Im}(R_{11}^{3/2} R_{22}^{3/2*}) + 15\,\mathrm{Im}(R_{11}^{\frac{1}{2}} R_{22}^{5/2*}) + 3\,\mathrm{Im}(R_{11}^{3/2} R_{22}^{5/2*}) \ ,$$

$$K = 45\,\mathrm{Im}(R_{22}^{3/2} R_{22}^{5/2*}) \ .$$

The formulas for the case in which the relative intrinsic parities differ are the same as the formulas given above except that the numerical values of L', the initial orbital angular momentum, are replaced by $L' \pm 1$, the choice of sign being determined by the J value and the vector-addition law (e.g., $R_{11}^{\frac{1}{2}} \longrightarrow R_{10}^{\frac{1}{2}}$, $R_{11}^{3/2} \longrightarrow R_{12}^{3/2}$, etc.). The above formulas apply to the associated production of K particles and hyperons in pion-nucleon collisions (involving unpolarized nucleons) if the spins of K particle and hyperon are 0 and $\frac{1}{2}$ respectively, and if parity is conserved in this (strong) production reaction.

The form of the angular distribution and polarization of the reaction products of the subsequent decay of the assumed spin-$\frac{1}{2}$ hyperon (into one spin-zero particle and one spin-$\frac{1}{2}$ particle) may also be obtained from Eqs. (2.1) through (2.5) by dropping the contributions from all initial states with $L' \neq 0$. If the unit vector along the momentum of the fermion in the decay products is denoted by $\underset{\sim}{V}$ and the polarization vector of the initial system is denoted by $\underset{\sim}{P}_i$, the angular distribution of the decay products is given by

$$I(\underset{\sim}{V}) = \frac{N}{4\pi}\left[|R_0|^2 + |R_1|^2 + 2\,\mathrm{Re}(R_0 R_1^*)\underset{\sim}{P}_i\cdot\underset{\sim}{V} \right], \tag{2.8}$$

and the polarization is

$$\begin{aligned} \underset{\sim}{P}(\underset{\sim}{V}) &= I(\underset{\sim}{V})^{-1}\frac{N}{4\pi}\Big\{ 2\,\mathrm{Re}(R_0 R_1^*)\underset{\sim}{V} - 2\,\mathrm{Im}(R_0 R_1^*)(\underset{\sim}{P}_i \times \underset{\sim}{V}) + |R_0|^2 \underset{\sim}{P}_i \\ &\qquad + |R_1|^2\left[2(\underset{\sim}{P}_i\cdot\underset{\sim}{V})\underset{\sim}{V} - \underset{\sim}{P}_i\right]\Big\} \\ &= I(\underset{\sim}{V})^{-1}\frac{N}{4\pi}\Big[2\,\mathrm{Re}(R_0 R_1^*)\underset{\sim}{V} - 2\,\mathrm{Im}(R_0 R_1^*)(\underset{\sim}{P}_i \times \underset{\sim}{V}) + (|R_0|^2 + |R_1|^2)(\underset{\sim}{P}_i\cdot\underset{\sim}{V})\underset{\sim}{V} \\ &\qquad - (|R_0|^2 - |R_1|^2)(\underset{\sim}{P}_i \times \underset{\sim}{V}) \times \underset{\sim}{V}\Big] . \end{aligned} \tag{2.9}$$

If we take $N = 1$ and normalize the R_L so that $|R_0|^2 + |R_1|^2 = 1$, then $I(\underset{\sim}{V})d\Omega$ is the probability that the final nucleon will have its velocity in the solid angle $d\Omega$ about the direction $\underset{\sim}{V}$.

The case in which the initial fermion is a spin-$\frac{1}{2}$ particle and the final fermion is a spin-$\frac{3}{2}$ particle may be described in a form similar to the above. For this purpose we introduce the symbols

$$T(\underset{\sim}{u}_1) = \sqrt{\frac{4\pi}{3}}\; Y_1^{\kappa}(\underset{\sim}{u}_1)\,\overline{T}_{\kappa}^{\,1} ,$$

$$T(\underset{\sim}{u}_2, \underset{\sim}{u}_3) = \sqrt{\frac{4\pi}{5}}\; Y_2^{\kappa}(\underset{\sim}{u}_2, \underset{\sim}{u}_3)\,\overline{T}_{\kappa}^{\,2} ,$$

$$T(\underset{\sim}{u}_4, \underset{\sim}{u}_5, \underset{\sim}{u}_6) = \sqrt{\frac{4\pi}{7}}\; Y_3^{\kappa}(\underset{\sim}{u}_4, \underset{\sim}{u}_5, \underset{\sim}{u}_6)\,\overline{T}_{\kappa}^{\,3} . \qquad (2.10)$$

Here the $\underset{\sim}{u}_i$ are arbitrary vectors and the symbol $Y_N^{\kappa}(\underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_N)$ represents the function of the vectors u_i that is linear in each argument, is symmetric in all its arguments, and which becomes $Y_N^{\kappa}(\theta\phi)$ when all its arguments are set equal to the unit vector in the direction $\theta\phi$. The $\mathcal{R}$ matrix may be expressed as the following superposition of these T matrices:

$$\mathcal{R}(\underset{\sim}{K}, \underset{\sim}{K}') = \sqrt{2}\,(2\pi/k')^{-1}\Big[g_1(\theta)\,T(\underset{\sim}{N}) + g_2(\theta)\,T(\underset{\sim}{K}, \underset{\sim}{K}) + g_3(\theta)\,T(\underset{\sim}{K}, \underset{\sim}{K}')$$
$$+ g_4(\theta)\,T(\underset{\sim}{K}', \underset{\sim}{K}') + h_1(\theta)\,T(\underset{\sim}{K}) + h_2(\theta)\,T(\underset{\sim}{K}') + h_3(\theta)\,T(\underset{\sim}{N}, \underset{\sim}{K}) + h_4(\theta)\,T(\underset{\sim}{N}, \underset{\sim}{K}')\Big] . \qquad (2.11)$$

The explicit form of the g_i and h_i when only S- and P-wave final states contribute is given in Table I. The normalization factors in Eq. (2.11) have been chosen so that the differential reaction cross section for the case of an unpolarized initial fermion is

$$I_0(\theta) = \sum_i \left(|g_i|^2 + |h_i|^2 \right) + 2\,\mathrm{Re}(h_1 h_2^*)\cos\theta + \frac{3}{2}\,\mathrm{Re}(h_3 h_4^*)\cos\theta$$

$$+ 2\,\mathrm{Re}(g_2 g_3^*)\cos\theta + 2\,\mathrm{Re}(g_3 g_4^*)\cos\theta + \mathrm{Re}(g_2 g_4^*)(3\cos^2\theta - 1)$$

$$- \tfrac{1}{4}\left(|h_3|^2 + |h_4|^2 + \sin^2\theta\, |g_3|^2 \right) . \tag{2.12}$$

If parity is conserved in the reaction then the $h_i(\theta)$ will be zero for the case in which the relative intrinsic parity of the initial particles is the same as that of the final particles; the $g_i(\theta)$ will be zero if these relative intrinsic parities are opposite.

If parity is conserved in the interaction and the initial fermion is unpolarized, the density matrix describing the spin of the final particle must be of the form

$$\mathcal{U}(\underset{\sim}{K}, \underset{\sim}{K'}) = \tfrac{1}{4} + b(\theta)\, T(\underset{\sim}{N}) + c(\theta)\, T(\underset{\sim}{KK}) + c'(\theta)\, T(\underset{\sim}{KK'}) + c''(\theta)\, T(\underset{\sim}{K'K'})$$

$$+ d(\theta)\, T(\underset{\sim}{NKK}) + d'(\theta)\, T(\underset{\sim}{NKK'}) + d''(\theta)\, T(\underset{\sim}{NK'K'}) . \tag{2.13}$$

The coefficients in this expression as functions of the $g_i(\theta)$ and $h_i(\theta)$ are given in Table II. When only S and P final states contribute, the differential reaction cross section reduces to the form

$$I_0(\underset{\sim}{k}, \underset{\sim}{k'}) = \frac{\bar{\lambda}^2}{4} \left[A' + B' \cos\theta + C' \cos^2\theta \right] , \tag{2.14}$$

where, for the case in which the relative intrinsic parities of the initial and final states are the same,

TABLE I

$$g_1(\theta) = \frac{1}{4\pi}\left[-\frac{i}{2}R_{11}^{\frac{1}{2}} - i\sqrt{\frac{5}{2}}\,R_{11}^{3/2}\right]\sin\theta$$

$$g_2(\theta) = 0$$

$$g_3(\theta) = \frac{1}{4\pi}\left[R_{11}^{\frac{1}{2}} - \sqrt{\frac{2}{5}}\,R_{11}^{3/2} + 2\sqrt{\frac{3}{5}}\,R_{13}^{5/2}\right]$$

$$g_4(\theta) = \frac{1}{4\pi}\left[-\sqrt{2}\,R_{02}^{3/2} - 5\sqrt{\frac{3}{5}}\,R_{13}^{5/2}\cos\theta\right]$$

$$h_1(\theta) = \frac{1}{4\pi}\left[R_{10}^{\frac{1}{2}} + \frac{1}{2}\sqrt{\frac{2}{5}}\,R_{12}^{3/2} + \frac{3}{2}\sqrt{\frac{3}{5}}\,R_{12}^{5/2}\right]$$

$$h_2(\theta) = \frac{1}{4\pi}\left[-\sqrt{2}\,R_{01}^{3/2} - \frac{3}{2}\sqrt{\frac{2}{5}}\,R_{12}^{3/2}\cos\theta - \frac{9}{2}\sqrt{\frac{3}{5}}\,R_{12}^{5/2}\cos\theta\right]$$

$$h_3(\theta) = 0$$

$$h_4(\theta) = \frac{1}{4\pi}\left[-i\,3\sqrt{\frac{2}{5}}\,R_{12}^{3/2} + i\sqrt{\frac{3}{5}}\,R_{12}^{5/2}\right]\sin\theta\ .$$

TABLE II

$$b(\theta) = I_0^{-1}(\theta) \Bigg\{ \frac{1}{4\sqrt{5}} \mathrm{Re}\left[3(h_1h_3^* + h_2h_4^*) - 2(g_1g_2^* + g_1g_4^*)\right] - \frac{\sqrt{5}}{2} \mathrm{Im}(h_1h_2^*)\sin$$

$$+ \frac{1}{4\sqrt{5}} \mathrm{Re}\left[3(h_1h_4^* + h_2h_3^*) - 2\, g_1g_3^*\right] \cos\theta$$

$$- \frac{9}{8\sqrt{5}} \mathrm{Im}(2\, g_2g_3^* + 2\, g_3g_4^* + h_3h_4^* + 4\, g_2g_4^* \cos\theta)\sin\theta \Bigg\}$$

$$c(\theta) = I_0^{-1}(\theta) \Bigg[\mathrm{Re}(g_2g_4^*) - \tfrac{1}{2} |g_2|^2 - \tfrac{3}{8} |g_3|^2 - \tfrac{1}{2} |h_1|^2$$

$$+ (2\sin\theta)^{-1} \mathrm{Im}(g_1g_3^* + h_1h_4^* - 2\, h_2h_3^*)$$

$$+ (2\sin^2\theta)^{-1}\left(|g_1|^2 + \tfrac{3}{4}|h_4|^2\right) - \tfrac{1}{2}\mathrm{Re}(g_2g_3^*)\cos\theta$$

$$+ \frac{\cos\theta}{\sin\theta} \mathrm{Im}(g_1g_2^* - \tfrac{1}{2} h_1h_3^*) + \frac{3}{4}\frac{\cos\theta}{\sin^2\theta} \mathrm{Re}(h_3h_4^*)$$

$$+ \frac{3}{8}\frac{\cos^2\theta}{\sin^2\theta} |h_3|^2 \Bigg]$$

TABLE II (Cont.)

$$c'(\theta) = I_0^{-1}(\theta) \Big[-\tfrac{1}{2}\,\mathrm{Re}(g_2g_3^* + g_3g_4^* + 2\,h_1h_2^* + \tfrac{3}{2}\,h_3h_4^*)$$

$$+ (\sin\theta)^{-1}\,\mathrm{Im}(g_1g_4^* - g_1g_2^* + \tfrac{1}{2}\,h_1h_3^* - \tfrac{1}{2}\,h_2h_4^*) + \tfrac{1}{4}\,|g_3|^2 \cos\theta$$

$$- 3\,\mathrm{Re}(g_2g_4^*)\cos\theta + \frac{3}{2}\,\frac{\cos\theta}{\sin\theta}\,\mathrm{Im}(h_2h_3^* - h_1h_4^*)$$

$$- \frac{\cos\theta}{\sin^2\theta}\left(|g_1|^2 + \tfrac{3}{4}\,|h_3|^2 + \tfrac{3}{4}\,|h_4|^2\right) -$$

$$- \frac{3}{2}\,\frac{\cos^2\theta}{\sin^2\theta}\,\mathrm{Re}(h_3h_4^*) \Big]$$

$$c''(\theta) = I_0^{-1}(\theta) \Big[\mathrm{Re}(g_2g_4^*) - \tfrac{3}{8}\,|g_3|^2 - \tfrac{1}{2}\,|g_4|^2 - \tfrac{1}{2}\,|h_2|^2$$

$$+ (2\sin\theta)^{-1}\,\mathrm{Im}(2\,h_1h_4^* - g_1g_3^* - h_2h_3^*) + (2\sin^2\theta)^{-1}\left(|g_1|^2 + \tfrac{3}{4}\,|h_3|^2\right)$$

$$- \tfrac{1}{2}\,\mathrm{Re}(g_3g_4^*)\cos\theta + \frac{\cos\theta}{\sin\theta}\,\mathrm{Im}(\tfrac{1}{2}\,h_2h_4^* - g_1g_4^*)$$

$$+ \frac{3}{4}\,\frac{\cos\theta}{\sin^2\theta}\,\mathrm{Re}(h_3h_4^*) + \frac{3}{8}\,\frac{\cos^2\theta}{\sin^2\theta}\,|h_4|^2 \Big]$$

$$d(\theta) = -\frac{3}{\sqrt{5}}\,\mathrm{Re}(g_1g_2^* + h_1h_3^*) - \frac{3}{2\sqrt{5}}\,\mathrm{Im}\Big[(\sin\theta)^{-1}\,h_3h_4^*$$

$$- g_2g_3^*\sin\theta\Big]$$

TABLE II (Cont.)

$$d'(\theta) = -\frac{3}{\sqrt{5}} \operatorname{Re}(g_1 g_3^* + h_1 h_4^* + h_2 h_3^*) + \frac{3}{\sqrt{5}} \operatorname{Im}\left(\frac{\cos\theta}{\sin\theta} h_3 h_4^* + g_2 g_4^* \sin\theta\right)$$

$$d''(\theta) = -\frac{3}{\sqrt{5}} \operatorname{Re}(g_1 g_4^* + h_2 h_4^*) - \frac{3}{2\sqrt{5}} \operatorname{Im}\left[(\sin\theta)^{-1} h_3 h_4^* - g_3 g_4^* \sin\theta\right] .$$

$$A' = \left| R_{11}^{\frac{1}{2}} \right|^2 + 2 \left| R_{02}^{3/2} \right|^2 + \frac{14}{5} \left| R_{11}^{3/2} \right|^2 + \frac{9}{5} \left| R_{13}^{5/2} \right|^2$$

$$+ \sqrt{\frac{2}{5}}\, \mathrm{Re}(R_{11}^{\frac{1}{2}} R_{11}^{3/2*}) + 3 \sqrt{\frac{3}{5}}\, \mathrm{Re}(R_{11}^{\frac{1}{2}} R_{13}^{5/2*})$$

$$- \frac{3}{5} \sqrt{6}\, \mathrm{Re}(R_{11}^{3/2} R_{13}^{5/2*}) ,$$

$$B' = -2\sqrt{2}\, \mathrm{Re}(R_{11}^{\frac{1}{2}} R_{02}^{3/2*}) + \frac{4}{\sqrt{5}} \mathrm{Re}(R_{11}^{3/2} R_{02}^{3/2*})$$

$$+ 6 \sqrt{\frac{6}{5}}\, \mathrm{Re}(R_{02}^{3/2} R_{13}^{5/2*}) ,$$

$$C' = -\frac{12}{5} \left| R_{11}^{3/2} \right|^2 + \frac{18}{5} \left| R_{13}^{5/2} \right|^2 - 3 \sqrt{\frac{2}{5}}\, \mathrm{Re}(R_{11}^{\frac{1}{2}} R_{11}^{3/2*})$$

$$- 9 \sqrt{\frac{3}{5}}\, \mathrm{Re}(R_{11}^{\frac{1}{2}} R_{13}^{5/2*}) + \frac{9}{5} \sqrt{6}\, \mathrm{Re}(R_{11}^{3/2} R_{13}^{5/2*}) . \quad (2.15)$$

When the contribution of the P final state is much smaller than that of the S final state, the parameters in Eq. (2.13) are given in terms of these same reaction matrix elements by

$$I_0(\theta)\, b(\theta) = \frac{\lambda^2}{4} \sin\theta\, \alpha_1 ,$$

$$I_0(\theta)\, c(\theta) = 0 ,$$

$$I_0(\theta)\ c'(\theta) = \frac{\lambda\!\!\!^{-2}}{4} \left\{ 2\alpha_3 + \alpha_4 \right\} ,$$

$$I_0(\theta)\ c''(\theta) = \frac{\lambda\!\!\!^{-2}}{4} \left\{ \alpha_2 - 5\cos\theta\ \alpha_3 \right\} ,$$

$$I_0(\theta)\ d(\theta) = 0 ,$$

$$I_0(\theta)\ d'(\theta) = 0 ,$$

$$I_0(\theta)\ d''(\theta) = \frac{\lambda\!\!\!^{-2}}{4} \left\{ \sin\theta\ \alpha_5 \right\} , \tag{2.16}$$

where the α_i are

$$\alpha_1 = \mathrm{Im}\left[-\sqrt{\tfrac{5}{2}}\ R_{11}^{\frac{1}{2}} R_{02}^{3/2^*} + \tfrac{2}{5} R_{11}^{3/2} R_{02}^{3/2^*} + \tfrac{9}{5}\sqrt{\tfrac{3}{2}}\ R_{02}^{3/2} R_{13}^{5/2^*} \right]$$

$$\alpha_2 = -\left| R_{02}^{3/2} \right|^2 ,$$

$$\alpha_3 = \mathrm{Re}\left[\tfrac{4}{5}\sqrt{\tfrac{6}{5}}\ R_{02}^{3/2} R_{13}^{5/2^*} + \frac{6}{5\sqrt{5}} R_{11}^{3/2} R_{02}^{3/2^*} \right] ,$$

$$\alpha_4 = \mathrm{Re}\left[\sqrt{2}\ R_{11}^{\frac{1}{2}} R_{02}^{3/2^*} + \frac{8}{5\sqrt{5}} R_{11}^{3/2} R_{02}^{3/2^*} - \tfrac{3}{5}\sqrt{\tfrac{6}{5}}\ R_{02}^{3/2} R_{13}^{5/2^*} \right],$$

$$\alpha_5 = \mathrm{Im}\left[\tfrac{18}{5} R_{02}^{3/2} R_{11}^{3/2^*} - \tfrac{3}{5}\sqrt{6}\ R_{02}^{3/2} R_{13}^{5/2^*} \right] . \tag{2.17}$$

The case in which the initial and final intrinsic parities are different is described by these same formulas modified by the substitutions described below Eq. (2.7).

The expressions given above may be applied to the associated production of hyperons and K particles in pion-nucleon collisions if the K particle and hyperons are spin 0 and spin 3/2 respectively. In the subsequent decay of this hyperon into a pion plus nucleon, each term in the hyperon density matrix $\mathcal{U}(\underset{\sim}{K}, \underset{\sim}{K}') \equiv \mathcal{U}_H$ gives a characteristic angular distribution and also a characteristic angular dependence for the polarization of the final nucleon. In order to exhibit the angular dependences in a convenient way we first express $\mathcal{U}_H$ in its most general form,

$$\mathcal{U}_H = \tfrac{1}{4} + \sum_i \tau_1^{\,i}\, T(\underset{\sim}{u}_1^{\,i}) + \sum_j \tau_2^{\,j}\, T(\underset{\sim}{u}_2^{\,j}, \underset{\sim}{u}_3^{\,j}) + \sum_k \tau_3^{\,k}\, T(\underset{\sim}{u}_4^{\,k}, \underset{\sim}{u}_5^{\,k}, \underset{\sim}{u}_6^{\,k}) \quad . \tag{2.18}$$

In this formula the $\underset{\sim}{u}_N^{\,n}$ are vectors that are to be selected in a way that gives the desired form of $\mathcal{U}_H$. For example we obtain the form of $\mathcal{U}_H$ given in Eq. (2.13) by the choice $\underset{\sim}{u}_1^{\,1} = \underset{\sim}{N}$, $\underset{\sim}{u}_2^{\,1} = \underset{\sim}{K}$, $\underset{\sim}{u}_3^{\,1} = \underset{\sim}{K}$, $\underset{\sim}{u}_2^{\,2} = \underset{\sim}{K}$, $\underset{\sim}{u}_3^{\,2} = \underset{\sim}{K}'$, etc. The angular distribution of the decay products is given in terms of the general parameters introduced in Eq. (2.18) by

$$I(\underset{\sim}{V}) = (4\pi)^{-1} \Big\{ (|R_1|^2 + |R_2|^2) + \sum_i 5^{-\frac{1}{2}} \tau_1^{\,i}(\underset{\sim}{u}^i \cdot \underset{\sim}{V}) 4\,\mathrm{Re}(R_1 R_2^*)$$

$$- \sum_j \tau_2^{\,j} \left[3(\underset{\sim}{u}_1^{\,j} \cdot \underset{\sim}{V})(\underset{\sim}{u}_2^{\,j} \cdot \underset{\sim}{V}) - (\underset{\sim}{u}_1^{\,j} \cdot \underset{\sim}{u}_2^{\,j})\right] \left(|R_1|^2 + |R_2|^2\right)$$

$$- \sum_k 5^{-\frac{1}{2}} \tau_3^{\,k} \Big[5(\underset{\sim}{u}_1^{\,k} \cdot \underset{\sim}{V})(\underset{\sim}{u}_2^{\,k} \cdot \underset{\sim}{V})(\underset{\sim}{u}_3^{\,k} \cdot \underset{\sim}{V}) - (\underset{\sim}{u}_1^{\,k} \cdot \underset{\sim}{V})(\underset{\sim}{u}_2^{\,k} \cdot \underset{\sim}{u}_3^{\,k})$$

$$- (\underset{\sim}{u}_2^{\,k} \cdot \underset{\sim}{V})(\underset{\sim}{u}_3^{\,k} \cdot \underset{\sim}{u}_1^{\,k}) - (\underset{\sim}{u}_3^{\,k} \cdot \underset{\sim}{V})(\underset{\sim}{u}_1^{\,k} \cdot \underset{\sim}{u}_2^{\,k})\Big] \; 6\,\mathrm{Re}(R_1 R_2^*) \Big\} \quad . \tag{2.19}$$

The polarization vector of the nucleon in the final state is given by

$$I(\underset{\sim}{V})\vec{P} = (4\pi)^{-1} \Big[2\,\mathrm{Re}(R_1 R_2^*)\underset{\sim}{V}$$

$$+ \sum_i 5^{-\frac{1}{2}} \tau_1^{\,i} \Big[2\left(|R_1|^2 + |R_2|^2\right)(\underset{\sim}{u}^i \cdot \underset{\sim}{V})\underset{\sim}{V}$$

$$+ 4(|R_1|^2 - |R_2|^2)\underset{\sim}{V} \times (\underset{\sim}{u}^i \times \underset{\sim}{V})$$

$$- 8\,\mathrm{Im}(R_1 R_2^*)(\underset{\sim}{u}^i \times \underset{\sim}{V})\Big]$$

$$- \sum_j \tau_2^{\,j} \cdot 2\,\mathrm{Re}(R_1 R_2^*) \left[3(\underset{\sim}{u}_1^{\,j} \cdot \underset{\sim}{V})(\underset{\sim}{u}_2^{\,j} \cdot \underset{\sim}{V}) - (\underset{\sim}{u}_1^{\,j} \cdot \underset{\sim}{u}_2^{\,j})\right] \underset{\sim}{V}$$

$$- \sum_k 5^{-\frac{1}{2}} \tau_3^{\,k} \Big\{ \Big[(|R_1|^2 + |R_2|^2)(\underset{\sim}{V} \cdot \underset{\sim}{u}_1^{\,k})\underset{\sim}{V}$$

$$+ (|R_1|^2 - |R_2|^2)\underset{\sim}{V} \times (\underset{\sim}{u}_1^{\,k} \times \underset{\sim}{V}) - 2\,\mathrm{Im}(R_1 R_2^*)(\underset{\sim}{u}_1^{\,k} \times \underset{\sim}{V})\Big]$$

$$\cdot \left[5(\underset{\sim}{u}_2 \cdot \underset{\sim}{V})(\underset{\sim}{u}_3 \cdot \underset{\sim}{V}) - 3(\underset{\sim}{u}_2 \cdot \underset{\sim}{u}_3)\right] + \text{Sym.} \Big\} \Big] \tag{2.20}$$

The symbol Sym. in the above line represents the sum of the two terms needed to symmetrize the contents of the braces.

The expressions given above apply to the production of hyperons in pion-nucleon collisions. Under the conditions stated in Section I the density matrix of the hyperon produced by K-particle absorption from low-lying orbits may be obtained from Eq. (1.9). If parity is conserved in the production process the form of this density matrix is particularly simple. Of the coefficients that appear in Eq. (2.13) only $c(\theta)$ is different from zero. The coefficient $c(\theta)$, which is in this case independent of θ, completely determines the decay angular distribution. According to Eqs. (2.13) and (2.19), this angular distribution is given by

$$I(\underset{\sim}{V}) = (4\pi)^{-1} \left(\left| R_1 \right|^2 + \left| R_2 \right|^2 \right) \left[1 - c(\theta)(3 \cos^2 \Theta - 1) \right] , \tag{2.21}$$

where Θ is the angle between $\underset{\sim}{V}$ and $\underset{\sim}{K}$ as measured in the decay center-of-mass frame. When the K particle is captured from S and P states only, $c(\theta)$ has the form

$$c(\theta) = -\tfrac{1}{2}(\omega_1 W_0 + \omega_2 W_1)^{-1}(\omega_1 W_0 + \omega_3 W_1) , \tag{2.22}$$

where for the case in which the initial and final intrinsic parities are the same we have

$$\omega_1 = \left| R_{20}^{\frac{1}{2}} \right|^2 ,$$

$$\omega_2 = \frac{1}{3} \left| R_{11}^{\frac{1}{2}} \right|^2 + \frac{2}{3}\left(\left| R_{11}^{3/2} \right|^2 + \left| R_{31}^{3/2} \right|^2 \right) ,$$

$$\omega_3 = \frac{1}{3}\left|R_{11}^{\frac{1}{2}}\right|^2 + \frac{8}{15}\left(\left|R_{31}^{3/2}\right|^2 - \left|R_{11}^{3/2}\right|^2\right) - \frac{4}{5}\,\mathrm{Re}(R_{11}^{3/2} R_{31}^{3/2*}) ,$$

(2.23a)

and for the case in which the relative intrinsic parities are different we have

$$\omega_1 = \left|R_{10}^{\frac{1}{2}}\right|^2 ,$$

$$\omega_2 = \frac{1}{3}\left|R_{21}^{\frac{1}{2}}\right|^2 + \frac{2}{3}\left(\left|R_{01}^{3/2}\right|^2 + \left|R_{21}^{3/2}\right|^2\right) ,$$

$$\omega_3 = \frac{1}{3}\left|R_{21}^{\frac{1}{2}}\right|^2 - \frac{4}{3}\,\mathrm{Re}(R_{01}^{3/2} R_{21}^{3/2*}) .$$

(2.23b)

The polarization of the final nucleon is independent of $c(\theta)$ and is given by

$$\vec{P}(\underset{\sim}{V}) = 2\,\mathrm{Re}(R_1 R_2^*)\underset{\sim}{V}\left(\left|R_1\right|^2 + \left|R_2\right|^2\right)^{-1} . \qquad (2.24)$$

If parity is conserved in the decay either R_1 or R_2 must vanish. The polarization of the final nucleon must, therefore, also be zero unless parity is violated in either the decay or production process.

Section III. Discussion

In this section the angular correlation between the directions defined by the production and decay events is discussed. We consider specifically the associated production in a pion-nucleon collision of a spin-zero K particle with a hyperon of spin $\frac{1}{2}$ or spin $\frac{3}{2}$, and the subsequent decay of the hyperon into a pion-nucleon system. If the hyperon has spin $\frac{1}{2}$ the production-decay process is described by Eqs. (2.1) through (2.9). If parity is conserved in the production process and the initial fermion is unpolarized, then the deviation from isotropy in the angular distribution of the decay products is proportional to $\underset{\sim}{N}\cdot\underset{\sim}{V}$, as is shown by Eqs. (2.8) and (2.4). The amplitude of this term must be zero if parity is conserved in the decay, since parity conservation would require either R_0 or R_1 to vanish. The occurrence, experimentally, of this term would constitute proof that parity is violated in the decay process.[11] Parity nonconservation in the decay process can also be demonstrated by experiments measuring the polarization of the final nucleon. From Eq. (2.9) one sees that when $\underset{\sim}{V}$ is in the production plane the longitudinal (proper) polarization is equal to $2\,\mathrm{Re}(R_0 R_1^*)/(|R_0|^2 + |R_1|^2)$. The occurrence of this polarization would imply a parity violation. The magnitude of this effect does not depend upon the unknown amount of polarization of the hyperon as does the above-mentioned magnitude of the asymmetry in the angular distribution. This could be important if the hyperon polarization were small. If, on the other hand, the hyperon polarization is large we see from Eqs. (2.9) and (2.8) that the values of R_0 and R_1 can be determined up to an over-all phase by the knowledge of the nucleon angular distribution and polarization. These coefficients R_0 and R_1 provide the complete phenomenological characterization of the decay process; their values give all the information that can be deduced from the experimental study of the process.

The measurement of the final polarization also permits a direct test of invariance under time reversal. The term in Eq. (2.9) that is proportional to $\mathrm{Im}(R_0 R_1^*)$ will be zero in so far as the decay can be considered to be first order in the weak interaction, and invariant under time reversal, provided final-state interactions can be ignored. The inclusion of the final-state interactions changes this condition somewhat. For the case $\Sigma^- \longrightarrow N + \pi^-$ the upper limit on the absolute magnitude of the component of polarization along $\underset{\sim}{P_1} \times \underset{\sim}{V}$ for the case in which $\underset{\sim}{V}$ lies in the plane of production, is $\left| \sin(\delta_P - \delta_S) \right|$. The δ_P and δ_S are the $J = \frac{1}{2}$, isotopic spin-$\frac{3}{2}$ phase shifts of the pion-nucleon system. A similar limit may be obtained for the cases in which both isotopic spin states are involved.

If the hyperon is spin $\frac{3}{2}$, the correlation between the directions defined by the production process and those of the decay process are given by Eqs. (2.11) through (2.20). At production threshold, where only the S waves of the final state contribute, the angular distribution for the production is isotropic and the angular distribution of the decay products in the decay center-of-mass frame is of the form $(3 \cos^2 \Theta' + 1)$, where Θ' is the angle, measured in the decay center-of-mass frame, between the direction of the incident nucleon in the production process and the outgoing nucleon of the decay process. This may be compared to the case discussed by Treiman[12] in which it was the initial state of the production process that was an S state. In that case the angular distribution of the decay products was of the form $(3 \cos^2 \Theta + 1)$, where Θ labels the angle between the hyperon velocity and the velocity of the final nucleon. For this limit in which only S waves are produced there will be no asymmetry with respect to the normal to the plane of production. At somewhat higher energies, where the interference between the final S and P waves becomes important, the hyperon density matrix will contain nonvanishing contributions proportional to $T(\underset{\sim}{N})$, $T(\underset{\sim}{K}, \underset{\sim}{K}')$, $T(\underset{\sim}{K}', \underset{\sim}{K}')$, and $T(\underset{\sim}{N}, \underset{\sim}{K}', \underset{\sim}{K}')$. The form of the decay angular distribution associated with each of these terms may be obtained from Eq. (2.19). From the $T(\underset{\sim}{N})$ term one obtains a contribution proportional to $\cos \Theta_N$, where Θ_N is the angle between the normal to the production plane and the direction of the nucleon from the decay. This term is analogous to the one that appeared when the hyperon was considered to be spin $\frac{1}{2}$, and it must vanish if parity is conserved in the decay process. The contribution from the $T(\underset{\sim}{N}, \underset{\sim}{K}', \underset{\sim}{K}')$ term will also be nonzero only if parity is violated in the decay. The angular distribution associated with this term is obtained from the τ_3 contribution to Eq. (2.19) by setting $\underset{\sim}{u}_1^{1} = \underset{\sim}{N}$, $\underset{\sim}{u}_2^{1} = \underset{\sim}{K}'$, and $\underset{\sim}{u}_3^{1} = \underset{\sim}{K}'$. It is of the form $\cos \Theta_N \left[5 \cos^2 \Theta' - 1 \right]$. This gives an asymmetry with respect to the normal to the production plane that is greatest for particles that decay in the plane defined by the vectors $\underset{\sim}{N}$ and $\underset{\sim}{K}'$ and which reaches a maximum when $\Theta_N \simeq 58.9^\circ$. The maximum asymmetry from the $T(\underset{\sim}{N})$ term occurs, of course, at $\Theta_N = 0$.

In addition to these terms, which reveal parity violations, there is another new term in the angular distribution. This one is a consequence of the $T(\underset{\sim}{K}, \underset{\sim}{K}')$ contribution to the hyperon-density matrix. According to Eq. (2.19), the angular distribution characteristic of this term is $\left[3 \cos \Theta \cos \Theta' - \cos \theta\right]$. Each of these terms will also give its characteristic contribution to the polarization of the final nucleon. The form of these contributions is given by Eq. (2.20). At higher energies, where all the terms in the general form of the hyperon-density matrix given in Eq. (2.13) contribute, three additional terms may enter in the decay angular distribution. Two are present only if parity is violated, and have the forms $\cos \Theta_N \left[5 \cos \Theta \cos \Theta' - \cos \theta\right]$ and $\cos \Theta_N \left[5 \cos^2 \Theta - 1\right]$. The other has the form $(3 \cos^2 \Theta - 1)$.

We conclude this section with a few remarks. First, the contributions to the decay angular distribution that are present when parity is not violated give no information about the decay mechanism except its total strength. They are proportional to $(|R_0|^2 + |R_1|^2)$ for the spin-$\frac{1}{2}$ case and to $(|R_1|^2 + |R_2|^2)$ for the spin-$\frac{3}{2}$ case. This form does not allow the contributions from the two final angular-momentum states to be distinguished. For the same reason, however, these terms give information about the production process that is independent of the detailed nature of the decay reaction, and their measurement provides information useful in the study of the strong reactions. Second, if, in the decay angular distribution there should occur a term that is asymmetrical with respect to any direction that lies in the plane of production, then parity must be violated both in the decay and in the production. It is assumed here that the strange particles are single particles--not parity doublets. Third, it is of interest to determine whether the intrinsic parity of the K-hyperon system is the same as the intrinsic parity of the pion-nucleon system. In view of the great dissimilarity in the forms of the $\mathcal{R}$ matrices in these two cases (see Eqs. (2.5) and (2.11)), it might be thought that the correlations near threshold between the various angular distributions and polarizations would depend upon the relative intrinsic parities. However, no information about the relative intrinsic parities of the two systems can be obtained from the analysis of the angular distributions and polarizations discussed in this paper unless assumptions are made regarding the relative magnitudes of the contributions from various initial angular-momentum states in the production process. This is a consequence of the close similarity, which is discussed below Eq. (2.7), of the formulas that describe the two alternative possibilities.

Section IV. Relativistic Corrections

Although the expressions given above are nonrelativistic in form they may, if properly interpreted, be applied to relativistic problems. The fundamental idea is to apply the formulas to the proper polarization[13] of the fermions. The proper polarization is the polarization as observed in the rest frame of the particle, and it may be described by the nonrelativistic operators. If the covariant reaction matrix is multiplied by appropriate Lorentz transformations it acts directly upon the operators describing the initial covariant proper polarization to give the final covariant proper polarization. Specifically, if the reaction is treated in the center-of-mass frame, the reaction operator $\mathcal{R}_p$ that directly relates the initial and final proper polarizations is given in terms of the usual covariant reaction matrix $\mathcal{R}_r$ by the equation[14]

$$\mathcal{R}_p(k, k') = L(k)\, \mathcal{R}_r(k, k')L^{-1}(k') ,$$

where $L(k)$ is a Lorentz transformation that transforms spinors from their values in a frame in which the center of mass (of the reaction) is at rest to their values in a rest frame of the final particle whose four-momentum is k; the transformation $L(k')$ is defined in the same way but relative to the initial particle. The part of the matrix $\mathcal{R}_p$ that describes the transitions between initial and final states having energies of a well-defined magnitude and sign is a reduced matrix of the nonrelativistic form. Moreover, if the Lorentz transformations $L(k)$ and $L(k')$ are chosen to be pure timelike[15] transformations, then the vectors and spin matrices that appear in the reduced $\mathcal{R}$ matrix transform under spatial rotations in the usual nonrelativistic manner.

The nonrelativistic reaction matrix and density matrix of the earlier sections may therefore be identified with the reduced part of $\mathcal{R}_p$ and the proper density matrix respectively.[16]

If the center-of-mass frame of the reaction is not identical with the laboratory frame then there is an ambiguity in the definition of the proper polarization. The correspondence described above between the relativistic and the nonrelativistic formulations is valid specifically in the center-of-mass frame, and the components of proper polarization refer to those rest frames of the initial and final particles that are related to the center-of-mass frame by the transformations $L(k')$ or $L(k)$. In the usual definition of proper polarization the rest frame of the particle is taken to be one generated by the action upon the laboratory frame of a pure timelike Lorentz transformation. In order to obtain the usual proper polarizations from those proper polarizations appearing in our nonrelativistic expressions, the vectors describing the proper polarizations in the latter formalism must be transformed by the sequence of transformations that takes them first to the center-of-mass frame, then to the laboratory frame, and then to the usual rest frame. This sequence of transformations is equivalent to a pure rotation. If the center-of-mass frame is the one generated from the laboratory frame by a pure timelike Lorentz transformation, then the sequence of the three pure timelike transformations produces a rotation of the vectors describing the proper polarization by an amount specified in Eq. (48) of Reference 16.[17] A detailed treatment of the Dirac-particle case is given in that paper.

Acknowledgments

We would like to thank Dr. Warren Heckrotte for a discussion during the initial phase of this work.

Appendix A: Notation

The formal manipulations are most easily represented in the notation of Dirac[18] and Condon and Shortley.[19] The symbol $\langle S\mu Lm \mid S'L'JM \rangle$ represents $C_{SL}(JM,\mu m)\,\delta_{SS'}\,\delta_{LL'}$, where δ_{ij} is the Kronecker delta and C is the Clebsch-Gordan coefficient.[4] The expansion theorems are then represented by the equation

$$|\xi'\rangle = |\xi''\rangle\langle\xi''|\xi'\rangle, \tag{A.1}$$

where ξ' and ξ'' represent either set of the four parameters given above and a summation over indices that appear twice is implied. The ket $|S\mu Lm\rangle$ will, in this appendix,[20] represent the state with spin function χ_μ^S and space wave function $(2kv^{-\frac{1}{2}})Y_L^m(\theta\phi)i^L\,j_L(kr)$, where $Y_L^m(\theta\phi)$ is the usual spherical harmonic[4] and $j_L(kr)$ is the usual spherical Bessel function.[4] We consider the energy as well defined; k and v are the corresponding momentum and velocity (both for the reduced relative motion). With this normalization the outgoing flux density (number of particles per unit time per unit solid angle) of particles in the spin state χ_μ^S that move in the direction $\theta\phi$ is $I(S,\mu,\theta,\phi) = \left| Y_L^m(\theta\phi)\,\langle S\mu L\,m \mid\,\rangle \right|^2$, where $\langle S\mu L\,m \mid\,\rangle$ is the amplitude of the state $|S\mu L\,m\rangle$. It is convenient to define $\langle\theta\,\phi \mid L\,m\rangle = Y_L^m(\theta\phi)$. Then

$$\begin{aligned} I(S,\mu,\theta\,\phi) &= \left|\langle\theta\,\phi \mid L\,m\rangle\langle S\,\mu L\,m \mid\,\rangle\right|^2 \\ &\equiv \left|\langle S\mu\theta\,\phi \mid\,\rangle\right|^2 \\ &= \langle\,\mid S\mu\theta\,\phi\rangle\cdot\langle S\mu\,\theta\,\phi \mid\,\rangle \\ &\equiv \langle\mathcal{P}\,(S\mu,\,\theta\,\phi)\rangle \quad . \end{aligned} \tag{A.2}$$

Here we have used the definitions

$$\left| S \mu \theta \phi \right\rangle \equiv \left| S \mu L m \right\rangle \left\langle L m \middle| \theta \phi \right\rangle \tag{A.3}$$

and

$$\mathcal{P}(\xi') \equiv \left| \xi' \right\rangle \cdot \left\langle \xi' \right| . \tag{A.4}$$

The dot before $\langle \xi' |$ signifies that the sum over the repeated index ξ' is not to be performed. The operator $\mathcal{P}(\xi')$ will be referred to as the projection operator for the state labeled by the indices ξ', where ξ' may now represent the sets of parameters $| S L J M \rangle$, $| S \mu L m \rangle$, or $| S \mu \theta \phi \rangle$. For the discrete parameters,

$$I(\xi') \equiv \langle \mathcal{P}(\xi') \rangle \tag{A.5}$$

is the probability that the system will be found in the state labeled by ξ'. With our normalization it is also the outgoing flux in this state. The total outgoing flux density (i.e., summed over spin states) in the direction $\theta\phi$ is

$$\begin{aligned} I(\theta\phi) &= \left\langle \sum\nolimits_{S,\mu} \mathcal{P}(S\mu, \theta\phi) \right\rangle \\ &= \left\langle \sum\nolimits_{S,\mu} \left[| S \mu \theta \phi \rangle \cdot \langle S \mu \theta \phi | \right] \right\rangle \\ &\equiv \left\langle \left[| \theta \phi \rangle \cdot \langle \theta \phi | \right] \right\rangle \\ &\equiv \langle \mathcal{P}(\theta \phi) \rangle . \end{aligned} \tag{A.6}$$

The projection operators $\mathcal{P}(\xi')$ defined above are therefore of fundamental significance; their expectation values $\langle \mathcal{P}(\xi') \rangle$ are interpretable as probabilities and flux densities in the manner just described. The expectation value of an arbitrary spin-space operator A, if the measurement is made in the beam emerging in the direction $\theta\phi$, is

$$\langle \mathcal{P}(\theta\phi)A \rangle / \langle \mathcal{P}(\theta\phi) \rangle \quad . \tag{A.7}$$

The physically measured quantities are therefore directly related to expectation values of the form $\langle \mathcal{P}(\theta\phi)A \rangle$.

In the above we have considered systems represented by pure states. In the treatment of polarization phenomena it is necessary to consider statistical mixtures of states. Statistical mixtures are conveniently described by a density matrix ρ,[2] which is defined by the equation

$$\begin{aligned} \langle a \rangle = \mathrm{Sp}\ a\rho &\equiv \langle \xi' | a\rho | \xi' \rangle \\ &\equiv \langle \xi' | a | \xi'' \rangle \langle \xi'' | \rho | \xi' \rangle , \end{aligned} \tag{A.8}$$

where $\langle a \rangle$ is the expectation value of the arbitrary operator a.

The dynamics of the reactions that we consider may be represented by an operator $\mathcal{S}$ defined by $|\psi_f\rangle = \mathcal{S}|\psi_i\rangle$. Here $|\psi_i\rangle$ and $|\psi_f\rangle$ are eigenstates of an unperturbed problem and represent the system before and after the interaction, respectively. The effects of the reaction are contained in the difference of these states,

$$|\psi_r\rangle \equiv |\psi_f\rangle - |\psi_i\rangle = (\mathcal{S} - 1)|\psi_i\rangle \equiv (-i)\mathcal{R}|\psi_i\rangle \ . \tag{A.9}$$

If the density matrix that describes the initial state is ρ_i, then the density matrix for the reaction products is[21]

$$\rho_r = \mathcal{R}\,\rho_i\,\bar{\mathcal{R}} \tag{A.10}$$

where the bar denotes Hermitian conjugate.

The density matrix that represents a plane wave normalized to unit flux density and moving in the direction $(\theta'\phi')$ is, with our normalization conventions, represented by

$$\rho_i = (2\pi/k')^2\,\mathcal{P}(\theta',\phi')\,\mathcal{U}(\theta'\phi')\ , \tag{A.11}$$

where $\mathcal{U}(\theta'\phi')$ is a spin-space (density) matrix with unit trace that describes the spin-space characteristics--polarization in the general sense--of the system. If the initial system is a plane wave moving in the direction $\theta'\phi'$ with spin characteristics described by $\mathcal{U}$, then the expectation value in the final state of the operator $\mathcal{P}(\theta\phi)A$ is, according to (A.8), (A.10), and (A.11),

$$\langle \mathcal{P}(\theta\phi)A \rangle = (2\pi/k')^2\,\mathrm{Sp}\,\mathcal{P}(\theta\phi)A\,\mathcal{R}\,\mathcal{P}(\theta'\phi')\,\mathcal{U}(\theta'\phi')\,\bar{\mathcal{R}}\ . \tag{A.12}$$

Since with the assumption of rotational invariance J is a constant of the motion, it is convenient to express the spur in the J representation. Thus we write

$$\mathrm{Sp}\ \mathcal{P}(\theta\phi)A\ \mathcal{R}\ \mathcal{P}(\theta'\phi')\ \mathcal{U}(\theta'\phi')\ \bar{\mathcal{R}}$$

$$= \langle S''\ L''\ J''\ M'' \mid \mathcal{P}(\theta\phi)A \mid S\ L\ J\ M \rangle \langle S\ L\ J\ M \mid \mathcal{R} \mid S'\ L'\ J'\ M' \rangle$$

$$\times \langle S'\ L'\ J'\ M' \mid \mathcal{P}(\theta'\phi')\ \mathcal{U}(\theta'\phi') \mid S'''\ L'''\ J'''\ M''' \rangle$$

$$\langle S'''\ L'''\ J'''\ M''' \mid \bar{\mathcal{R}} \mid S''\ L''\ J''\ M'' \rangle$$

$$= \langle S'''\ L'''\ J''\ M'' \mid \mathcal{P}(\theta\phi)A \mid S\ L\ J\ M \rangle\ R^{J}_{SL;S'L'}$$

$$\langle S'\ L'\ J\ M \mid \mathcal{P}(\theta'\phi')\ \mathcal{U}(\theta'\phi') \mid S'''\ L'''\ J''\ M'' \rangle\ R^{J''}_{S'''L''';S''L''} \quad \text{(A.13)}$$

Here we have used the fact that $\mathcal{R}$ is diagonal in J and M and independent of M [22] to justify the abbreviation[23]

$$\langle S\ L\ J\ M \mid \mathcal{R} \mid S'\ L'\ J'\ M' \rangle = R^{J}_{SL;S'L'}\ \delta_{JJ'}\ \delta_{MM'}\ . \quad \text{(A.14)}$$

Since the matrices A and $\mathcal{U}(\theta\phi)$ are square matrices we have $S'' = S$ and $S''' = S'$; S' and S are the initial and final spin quantum numbers, which we assume to be fixed. The matrices A and $\mathcal{U}(\theta\phi)$ may be expanded in terms of the basis matrices T^{Q}_{κ}. It is sufficient to consider the A to be the various possible T^{Q}_{κ}. Using Eq. (A.13) we obtain, by performing the sums over the magnetic quantum numbers, for the quantity

$$N\, Sp\, \mathcal{P}(\theta\phi)\, T_{\kappa}^{Q}\, \mathcal{R}\, \rho(\theta'\phi')\, \mathcal{U}(\theta'\phi')\, \bar{\mathcal{R}} \quad ,$$

the expression on the right-hand side of Eq. (1.6). If N is set equal to $(2\pi/k_{in})^2$, then Eqs. (A.12) and (1.6) give

$$\langle \mathcal{P}(\theta\phi)\, T_{\kappa}^{Q} \rangle = I(\theta\phi)\, \alpha_{\kappa}^{Q}(S, \theta\phi) \quad , \qquad (A.15)$$

from which one obtains from Eqs. (1.4) and (1.7)

$$\langle \mathcal{P}(\theta\phi) \rangle = I(\theta\phi) \quad .$$

This justifies the interpretation of $I(\theta\phi)$ that was given in the text (see also Eq. (A.6)).

For processes in which the initial system is not represented by a plane wave, a different ρ_i is used. To represent processes that are initiated from an incoherent mixture of orbital angular momentum states the appropriate ρ_i is

$$\rho_i = \sum_{Lm} W_{L,m}\, \mathcal{P}(L, m) \quad ,$$

where $\mathcal{P}(L, m) = |L\, m\rangle \cdot \langle L\, m|$ is the projection operator for the state $|L\, m\rangle$ and $W_{L,m}$ is the probability that the reaction initiates from the state $|L\, m\rangle$.

The spins S and S' may be considered fixed in many problems. The reaction coefficients $R^{J}_{SL;S'L'}$ are, in these cases, abbreviated by $R^{J}_{LL'}$.

For the decay process the formalism is again changed only by a change of the incident density matrix. Since there is no orbital part of the state of the incident particle, the density matrix ρ_i becomes simply $\mathcal{U}$. Then $J = J'' = S'$ and the indices L''' and L' drop out. The reaction coefficients may therefore be abbreviated as R_L. This abbreviation suffices to distinguish the reaction coefficients (i.e., the R's) of the decay process from those of the production process.

It is sometimes convenient to consider the reaction matrix explicitly. That is, Eq. (A.13) may be expressed in the form

$$\begin{aligned} &\mathrm{Sp}\ \mathcal{P}(\theta\phi)\ A\ \mathcal{R}\,\mathcal{P}(\theta'\phi')\,\mathcal{U}(\theta'\phi')\ \bar{\mathcal{R}} \\ &= \langle s\mu L m \mid \theta\phi\rangle\cdot\langle\theta\phi \mid A\,\mathcal{R} \mid \theta'\phi'\rangle\cdot\langle\theta'\phi' \mid \mathcal{U}(\theta'\phi')\ \bar{\mathcal{R}} \mid s\mu L m\rangle \\ &= \mathrm{Tr}\cdot\langle s\mu\theta\phi \mid A\ \mathcal{R} \mid \theta'\phi'\rangle\cdot\langle\theta'\phi' \mid \mathcal{U}(\theta'\phi')\ \bar{\mathcal{R}} \mid s\mu\theta\phi\rangle\ , \end{aligned} \tag{A.16}$$

where the symbol Tr means a trace over the spin variables. (The symbol Sp is a diagonal sum over both spin and orbital variables.) The operators A and $\mathcal{U}(\theta'\phi')$ are matrices with respect to the spin variables alone, and are scalars with respect to the orbital part of space. Thus the above equation reduces to

$$\begin{aligned} &\mathrm{Sp}\ \mathcal{P}(\theta\phi) A\ \mathcal{R}\,\mathcal{P}(\theta'\phi')\,\mathcal{U}(\theta'\phi')\ \bar{\mathcal{R}} \\ &= \mathrm{Tr}\ A\ \mathcal{R}(\theta\phi;\theta'\phi')\ \mathcal{U}(\theta'\phi')\ \bar{\mathcal{R}}(\theta'\phi';\theta\phi)\ , \end{aligned} \tag{A.17}$$

where we have defined the spin-space reaction operator

$$\mathcal{R}(\theta\phi;\ \theta'\phi') \equiv \langle \theta\phi \mid \mathcal{R} \mid \theta'\phi' \rangle \ . \qquad (A.18)$$

This spin-space reaction operator is a matrix between the initial and final spin spaces. If $S \neq S'$ the matrix is nonsquare. It may be expanded in terms of the T_{χ}^{Q} defined in the text, which are also nonsquare if $S \neq S'$. The matrix elements of $\mathcal{R}(\theta\phi;\ \theta'\phi')$ are, according to Eq. (A.18),

$$\langle S\mu\,\theta\phi \mid \mathcal{R} \mid S'\mu'\,\theta'\phi' \rangle =$$

$$\langle \theta\phi \mid L\,m \rangle \langle S\mu L\,m \mid S''\,L''\,J\,M \rangle \langle S''\,L''\,J\,M \mid \mathcal{R} \mid S'''\,L'''\,J'\,M' \rangle$$

$$\times \langle S'''\,L'''\,J'\,M' \mid S'\mu'\,L'\,m' \rangle \langle L'\,m' \mid \theta'\,\phi' \rangle$$

$$= Y_L^m(\theta\phi)\ C_{\mu m M}^{SLJ}\ R_{SL;S'L'}^{J}\ C_{\mu' m' M}^{S'L'J}\ Y_{L'}^{m'\,*}(\theta',\phi') \ . \qquad (A.19)$$

Using the properties of the Racah and Clebsch-Gordan coefficients we can reduce this to the form given in Eq. (1.21) of the text.

Appendix B: Initial- and Final-State Interactions, Time Reversal, and Space-Time Inversion.

The matrix element $R^{J}_{SL;S'L'}$ was defined in Appendix A as $\langle S\,L\,J\,M \,|\, \mathcal{R} \,|\, S'\,L'\,J\,M \rangle$, where the basis vector $|\,S\,L\,J\,M\rangle$ was the vector representing the wave function

$(2kv^{-\frac{1}{2}})\, i^{L}\, j_L\,(kr)\, C^{SLJ}_{\mu mM}\, Y^{M}_{L}(\theta\phi)\, \chi^{S}_{\mu}$. This definition is appropriate if the unperturbed Hamiltonian H_o is the free-particle Hamiltonian. If H_o is a more general unperturbed Hamiltonian, the basis vectors should be defined as eigenstates of this new H_o. Consider the generalization to a case in which H_o again commutes with the orbital angular momentum operator but may be identified with the free-field Hamiltonian only at large radial distances. A definition of $|\,S\,L\,J\,M\rangle$ that is suitable in this case is obtained by replacing in the above definition the spherical Bessel function $j_L(kr)$ by $f_L(kr)$, a real[24] solution of the (new) unperturbed radial equation for the eigenvalue L. The normalization of $f_L(kr)$ will be chosen so that at large r it approaches $(kr)^{-1}\sin\left(kr - (\pi L/2) + \delta_L\right)$. The outgoing part of the asymptotic wave function in the spin state χ^{S}_{μ} is then given by

$$(irv^{\frac{1}{2}})^{-1} \exp ikr\, e^{i\delta_L}\, Y_L^{m}(\theta\phi)\, \langle S\,\mu\,L\,m\,|\,\rangle$$

$$\equiv (irv^{\frac{1}{2}})^{-1} \exp ikr\, e^{i\delta_L} \langle \theta\phi\,|\,L\,m\rangle\langle S\,\mu\,L\,m\,|\,\rangle$$

$$= (irv^{\frac{1}{2}})^{-1} \exp ikr\, \langle \theta\phi\,|\,L\,m\rangle\langle S\,\mu\,L\,m\,|\,e^{i\delta_L}\,|\,\rangle$$

$$\equiv (irv^{\frac{1}{2}})^{-1} \exp ikr\, \langle S\,\mu\,\theta\,\phi\,|\,e^{i\delta_L}\,|\,\rangle ,$$

where in the last line δ_L is considered to be an operator (i.e., in Dirac notation $\delta_L \,|\, L' \, m' \rangle \equiv |\, L' \, m' \rangle \, \delta_{L'}$). The operator whose expectation value is the total outgoing flux density becomes, therefore, $e^{-i\delta_L} \, \rho'(\theta\phi) \, e^{+i\delta_L}$, where the prime on $\rho'(\theta\phi) \equiv |\, \theta\phi \rangle \cdot \langle \theta\phi \,|$ signifies that the basis vectors represent the states whose wave functions have the radial dependence $f_L(kr)$. Similarly the form of the incident-density matrix depends upon whether the basis vectors represent states with wave functions having the radial dependence $f_L(kr)$ or $j_L(kr)$. In particular the density matrix representing an incident plane wave of unit flux density moving in the direction $\theta'\phi'$ is $\rho'_i = (2\pi/k')^2 \, e^{i\delta_L} \, \rho'(\theta'\phi') \, e^{-i\delta_L} \, \mathcal{U}(\theta'\phi')$. In this representation, where the basis vectors are eigenstates of the generalized H_o, the matrix element $\langle S \, L \, J \, M \,|\, \mathcal{R} \,|\, S' \, L' \, J \, M \rangle$ is denoted by $R'^{J}_{SL;S'L'}$. The effect of the initial- and final-state interactions is to replace $R^{J}_{SL;S'L'}$ in the formulas obtained with the free-particle Hamiltonian H_o by $e^{i\delta_L} \, R'^{J}_{SL;S'L'} \, e^{i\delta_{L'}}$. Thus, for the case in which the unperturbed Hamiltonian includes initial- and final-state interactions, the unprimed quantities $R^{J}_{SL;S'L'}$ that appear in the various equations in the main body of the text should be interpreted as $e^{i\delta_L} \, R'^{J}_{SL;S'L'} \, e^{i\delta_{L'}}$.

The requirement of invariance under time reversal imposes certain conditions upon the $R'^{J}_{SL;S'L'}$. The fundamental consequence of this requirement is the equality of a matrix element of the $\mathcal{S}$ matrix and the transposed matrix element between time-inverse states. The time inverse of the state represented by $|\, S \, \mu \, L \, M \rangle$ is the state represented by $(-1)^{S-\mu+L-m} \mid S, -\mu, L, -m \rangle$. The factor $(-1)^{S-\mu}$ is part of the definition of time reversal for a spin state.[25] The $(-1)^{L-m}$ comes from the complex conjugation of the space part of the wave function. From the properties of the Clebsch-Gordan coefficients the states represented by $\mid S\,L\,J\,M \rangle = \mid S \mu L\, m \rangle \langle S \mu L\, m \mid S\,L\,J\,M \rangle$ have as their time inverses the states $(-1)^{J-M} \mid S\,L\,J-M \rangle$. One then readily obtains the symmetry relation $R'^{J}_{SL;S'L'} = R'^{J}_{S'L';SL}$. The same relationship is true for $R^{J}_{SL;S'L'}$.

A second consequence of time-reversal invariance is obtained if the reaction is considered to be first-order in the interaction term. In virtue of its unitarity the $\mathcal{S}$ matrix may be expressed as $\mathcal{S} = (1 - \frac{i}{2}\mathcal{K})(1 + \frac{i}{2}\mathcal{K})^{-1}$, where $\mathcal{K}$ is a Hermitian operator. To first order in $\mathcal{K}$ we have $\mathcal{R} \equiv i(\mathcal{S} - 1) \cong \mathcal{K}$. To this order the $\mathcal{R}$ operator is Hermitian, and its matrix elements $R'^{J}_{SL;S'L'}$ are real, owing to the symmetry property. The $R^{J}_{SL:S'L'}$, though symmetric, are not real in general.

For the consideration of the consequences of invariance under the product of time reversal and space inversion, it is convenient to remove the factor of i^{L} in the definition of the basis vectors. Then the space-time inverse of the state represented by $\mid S \mu L\, m \rangle$ is $e(-1)^{S-\mu+L-m} \mid S, -\ \ , L, -m \rangle$, where e is the intrinsic parity of the state--the product of the intrinsic parities of the elementary particles that are represented in the state. The reaction matrix elements in this representation will be denoted by $R''^{J}_{SL;S'L'}$. If the interaction is invariant under space-time inversion these matrix elements satisfy the symmetry property $R''^{J}_{SL;S'L'} = e'^{*} R''^{J}_{S'L';SL}\, e$, where e' and e are the intrinsic parities of the initial state and the final state respectively. If the reaction is considered only to first order in the interaction, the Hermiticity of the $\mathcal{R}$ matrix, together with the above symmetry property, gives the reality condition

$$R''^{J}_{SL;S'L'} = e^{*}(R''^{J}_{SL;S'L'})^{*} e' \quad .$$

When expressed in terms of the amplitudes of the basis vectors just introduced, the observable quantities must be represented by new functions. Arguments analogous to the ones used in the case of time reversal lead now to the identification

$$R^{J}_{SL;S'L'} = \exp i(\delta_{L} - \frac{\pi L}{2})\, R''^{J}_{SL;S'L'} \exp i(\delta_{L'} - \frac{\pi L'}{2}) \quad .$$

REFERENCES

1. From a compilation of available emulsion and bubble chamber data, made at the 1957 Rochester conference by G. Snow.

2. P. A. M. Dirac, The Principles of Quantum Mechanics, Third Edition (Clarendon Press, Oxford, 1947), p. 130;
L. Wolfenstein and J. Ashkin, Phys. Rev. 85, 947 (1952);
U. Fano, Revs. Modern Phys. 29, 74 (1957).

3. A more general matrix representation of these operators has been given by C. Eckart, Revs. Modern Phys. 2, 305 (1930). We have chosen the normalization differently from that of most authors.

4. Blatt and Weisskopf, Theoretical Nuclear Physics (John Wiley and Sons, New York, 1952).

5. We have assumed $S + S'$ is integral. For a derivation of this equation and a much more detailed discussion of general formalism for the treatment of production and decay see Richard Spitzer and Henry P. Stapp, Polarization and Angular Correlation in the Production and Decay of Particles of Spin $\frac{1}{2}$ and Spin $\frac{3}{2}$, University of California Radiation Laboratory Report No. UCRL-3796 Rev, June 1957 (unpublished). A formula essentially equivalent to Eq. (1.5) is given by A. Simon, Phys. Rev. 92, 1050 (1953).

6. U. Fano, National Bureau of Standards Report 1214, p. 48. Algebraic formulas and tables of the X coefficients are given by H. Matsunobu and H. Takebe, Progr. Theoret. Phys. Japan 14, 589 (1955). The phase and normalization of our X coefficient are the same as those of the U coefficient of Matsunobu and Takebe.

7. See Appendix A for a definition of the reaction coefficients.

8. J. M. Blatt and L. C. Biedenharn, Revs. Modern Phys. 24, 258 (1952). The values of the Z coefficient have been tabulated by L. C. Biedenharn, Tables of the Racah Coefficients, Oak Ridge National Laboratory Report No. ORNL-1098, 1952.

9. If a system has made transitions to the low-lying orbital angular momentum states by γ emission or by collision processes, then contributions to the reaction processes that initiate in these states will be incoherent.

10. G. Racah, Phys. Rev. 62, 438 (1942). See ORNL-1098 for tables of the Racah coefficient.

11. This has been discussed in a preprint, Possible Detection of Parity Nonconservation in Hyperon Decay, by Lee, Steinberger, Feinberg, Kabir, and Yang.

12. S. B. Treiman, Phys. Rev. 101, 1216 (1956).

13. The term polarization is interpreted to include tensor-type polarizations.

14. The light-face vector arguments are four-vectors. We have suppressed the dependence of the operators in this equation upon the total energy-momentum vector for the reaction. Since we are considering the center-of-mass frame, this vector is pure timelike. The operator $\mathcal{R}_p$ may be obtained formally by introducing into Eq. (A.12) the unit operators $L^{-1}(k)L(k)$ and $L^{-1}(k')L(k')$ and identifying $L(k')\ \rho_i\ L^{-1}(k')$ as the proper initial density matrix and making a similar identification for the operators related to the final density matrix. See reference No. 16 for a detailed discussion.

15. A pure timelike Lorentz transformation will mean one that leaves unchanged the space components along directions perpendicular to the relative velocity of the two frames. The transformations $L(k)$ and $L(k')$, when restricted by the conditions we have imposed, are pure timelike transformations.

16. H. P. Stapp, Phys. Rev. 103, 425 (1956). This reference gives a detailed justification for the nonrelativistic treatment of spin $\frac{1}{2}$ particles. For higher half-integral spins it is convenient to consider the formulation of the relativistic wave equation as given by W. Rarita and J. Schwinger, Phys. Rev. 60, 61 (1941). In this formulation the wave function $\psi_{\mu_1 \cdots \mu_k}$ of a particle of spin $k + \frac{1}{2}$ has the mixed transformation properties of a Dirac four-component spinor (the spinor indices have been suppressed) and of a symmetric tensor of rank k. Because the proof of the validity of the nonrelativistic treatment for the spin $\frac{1}{2}$ case depends essentially on the transformation properties of the Dirac spinors, and each tensor component of $\psi_{\mu_1 \cdots \mu_k}$ does

16. (Cont.)

transform like a spinor, it follows that for our case the $\mathcal{R}$ matrix can also be put into a form the reduced part of which can be expanded in terms of nonrelativistic spin operators.

17. The velocity vectors $\underset{\sim}{V}_a$ and $\underset{\sim}{V}_b$ that occur in Eq. (48) of this reference are the relativistic or covariant velocities $(d\underset{\sim}{x}/d\tau) = \gamma(d\underset{\sim}{x}/dt)$, where t, τ, and γ are time, proper time, and relativistic contraction factor respectively. This fact is not made sufficiently clear in the reference.

18. P. A. M. Dirac, ibid.

19. Condon and Shortley, The Theory of Atomic Spectra (University Press, Cambridge, 1951).

20. This definition is used if the unperturbed Hamiltonian is taken as the free-particle Hamiltonian. For the more general case see Appendix B.

21. L. Wolfenstein and J. Ashkin, Phys. Rev. **85**, 947 (1952).

22. J. M. Blatt and L. C. Biedenharn, Revs. Modern Phys. **24**, 258 (1952).

23. If initial- and final-state interactions are present the interpretation of the states $|\xi\rangle$ are slightly altered. In this case it is $R'^{J}_{SL;S'L'}$ that is defined by Eq. (A.14). (See Appendix B).

24. It is sufficient that $f_L(kr)$ be real in the asymptotic region. This can certainly be arranged if there is conservation of particles in each angular momentum state (when the interaction Hamiltonian is neglected) and if only elastic processes are generated by the Hamiltonian H_o.

25. L. C. Biedenharn and M. E. Rose, Revs. Modern Phys. 25, 729 (1953).