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Abstract

We prove an analogue of Fine's canonicity theorem for classes of monotonic neighborhood frames definable in the language of Coalgebraic Predicate Logic. The result holds true of classes definable relative to the classes of monotonic, quasi-filter, augmented quasi-filter, filter, or augmented filter neighborhood frames, respectively. Fine's original theorem follows as a special case concerning the classes of augmented filter neighborhood frames.

Keywords: modal logic, canonicity, Fine's theorem, neighborhood frames

1 Introduction

This article concerns a generalization of a classical result by Fine [7] on canonicity of normal modal logics. Canonicity is an important property of normal modal logics that implies Kripke completeness, which proved useful in establishing completeness of many familiar logics (see, e.g., [2]). Until a counterexample [13,12] was found in the early 21st century, all known canonical logics were generated by elementary classes of Kripke frames whereas all known non-canonical logics were not elementarily determined. Fine gave to this empirical fact an explanation: he proved that the normal modal logics determined by elementary classes of Kripke frames are canonical.

There are a number of reasons for relaxing the axioms of normal modal logics and considering monotonic modal logics. For instance, monotonic modal logics are considered more appropriate to describe the ability of agents or systems to make certain proposition true in the context of games and open systems [21,22,1]. The standard semantics for monotonic modal logics is provided by monotonic neighborhood frames (see, e.g., [14]), which are a particular kind of Set-coalgebras [15].

In this article we prove an analogue of Fine's canonicity theorem for monotonic modal logics. For this purpose, we need to define what it means for a

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monotonic modal logic to be canonical and for a class of monotonic neighborhood frames to be elementary. Our notion of canonicity of a monotonic modal logic is that of [14] derived from algebraic modal logic. By the elementarity of a class of monotonic neighborhood frames, we mean the definability of the class by a theory in a first-order-like language, namely, the language of *coalgebraic predicate logic* (CPL).

The language of coalgebraic modal logic was first proposed by Chang [3] for neighborhood frames and later generalized by Litak et al. [17] for general T-coalgebras for an endofucntor T over the category of sets. Chang's original language was motivated in the context of natural language semantics; coalgebraic predicate logic is also easy to motivate since it contains *coalgebraic modal logic* [5] in a way that admits an analogue [23] of the van Benthem-Rosen theorem (see, e.g., [2]), among other reasons (see [17]).

We will deal with a relativized notion of CPL-elementarity, relativized to subclasses of the class of monotonic neighborhood frames. There are several important subclasses to consider: the class of filter neighborhood frames, providing a more general semantics [11,10] for normal modal logics than relational semantics; the class of quasi-filter neighborhood frames, providing a semantics for regular modal logics; the class of augmented quasi-filter neighborhood frames, providing a less general semantics for regular modal logics; and the class of augmented filter neighborhood frames, which are Kripke frames in disguise [4,19].

Subclass	Each neighborhood family is closed under ...
monotonic	supersets
quasi-filter	supersets, intersections of nonempty finite families of neighborhoods
augmented quasi-filter	supersets, intersections of nonempty families of neighborhoods
filter	supersets, intersections of finite families of neighborhoods
augmented filter	supersets, intersections of families of neighborhoods

Table 1
Classes of monotonic neighborhood frames and their definitions

The analogue of Fine's theorem we will prove states that a sufficient condition for the canonicity of a monotonic modal logic is that it is determined CPL-elementarily relative to any of the classes of neighborhood frames in Table 1.

The relevance of coalgebraic predicate logic in this article is that many monotonic modal logics are determined by classes of monotonic neighborhood frames that are CPL-elementary. For instance, the monotonic modal logics axiomatized by formulas of the form

$$\langle\text{purely propositional positive formula}\rangle \to \langle\text{positive formula}\rangle \tag{1}$$

are determined by CPL-elementary classes of monotonic neighborhood frames (see Remark 2.4). In addition, relative to the class of augmented quasi-filter frames, all monotonic modal logics axiomatized by Sahlqvist formulas are CPL-elementarily determined [20] (see Example 2.5). Further discussion regarding the relevance of this language in the context of Fine's theorem is in Remark 4.4.

The paper is organized as follows. In § 2, we recall standard concepts in the semantics of monotonic modal logic and introduce the language for neighborhood frames. In § 3, we give an overview of the model theory of neighborhood frames for this language. We also introduce a two-sorted first-order language (Definition 3.15) and a translation of coalgebraic predicate logic into it (Proposition 3.17), which are used later to explain the existence of $\aleph_0$-saturated models of languages of coalgebraic predicate logic (Proposition 3.20). In § 4, we prove the variant of Fine's theorem for monotonic neighborhood frames.

Because of the dual correspondence between monotonic modal logics and varieties of monotonic Boolean algebra expansions (see § 2), our result can be stated in terms of varieties rather than logics. In fact, for the rest of this article, the technical matters are presented by means of varieties of monotonic Boolean algebra expansions for the sake of convenience. The presentation of the results in this article does not presuppose the reader's prior knowledge of coalgebras or coalgebraic predicate logic.

Proofs of the starred propositions, lemmas, and theorems are found in the Appendix.

2 Preliminaries

2.1 Languages and structures

In this subsection, we recall standard definitions in neighborhood semantics of modal logic and the language introduced in [3] and [17] to describe them.

We define languages of coalgebraic predicate logic relative to sets of nonlogical symbols here; this is so that we can use expansions of the smallest language in proofs in § 4.

Definition 2.1

(i) Let L_0 be a language of first-order logic. The *language of coalgebraic predicate logic L based on* L_0 is the least set of formulas containing L_0 and closed under Boolean combinations, existential quantification, and formation of formulas of the form $x \Box \lceil y \colon \phi \rceil$, where $\phi \in L$, and x and y are variables. To save space, we sometimes write $x \Box_y \phi$ or even $x \Box \phi$ for $x \Box \lceil y : \phi \rceil$.

(ii) Let L_0 be a language of first-order logic and L be the language of coalgebraic predicate logic based on L_0. An *L-structure* $F = (F, N^F)$ is an L_0-structure F with an additional datum $N^F : F \to \mathscr{P}(\mathscr{P}(F))$, a function that assigns to each element of F a family of subsets of F. The map N^F is called the *neighborhood function* of F. A set $U \in N^F(w)$ is called a *neighborhood around* w. If L_0 is the empty first-order language, the L-structures are exactly the *neighborhood frames*.

(iii) A neighborhood frame F is *monotonic* if for every $w \in F$ the family $N^F(w)$ is closed under supersets. F is a *quasi-filter* neighborhood frame if for every $w \in F$ the family $N^F(w)$ is closed under intersections of nonempty finite families of neighborhoods. F is is a *filter* neighborhood frame if it is a quasi-filter frame and for every $w \in F$ the family $N^F(w)$ is nonempty. F is an *augmented quasi-filter* neighborhood frame if for every $w \in F$ the family $N^F(w)$ is either empty or a *principal upset* in the Boolean algebra $\mathscr{P}(F)$, i.e., there exists $U_0 \subseteq F$ such that $U \in N^F(w) \iff U_0 \subseteq U$. A neighborhood frame F is an *augmented filter* frame if for every $w \in F$ the family $N^F(w)$ is a principal upset.

Definition 2.2 Let L be a language of coalgebraic predicate logic and F be an L-structure. We define the satisfaction predicate $F \models \phi$ for a sentence $\phi \in L$. It is convenient to define the predicate for the expanded language $L(F)$ of coalgebraic predicate logic. In general, for $A \subseteq F$, we define $L(A)$ to be the language of coalgebraic predicate logic that has all symbols of L and a constant symbol w that is intended to be interpreted as w itself for each $w \in A$.[2] Now, F is an $L(F)$-structure in the obvious way. We define the satisfaction predicate $F \models \phi$ for $\phi \in L(F)$. The predicate is defined by recursion on ϕ. For symbols of first-order logic in L, the predicate is defined in the ordinary way. For $\phi = w \,\Box_y\, \phi_0$, we define

$$F \models w \,\Box_y\, \phi_0(y) \iff \phi_0(F) \in N^F(w)$$

where

$$\phi_0(F) = \{v \in F \mid F \models \phi_0(v)\}$$

and $\phi_0(v)$ stands for a substitution instance of $\phi_0(y)$ with v substituted for y.

Example 2.3 Consider the B axiom $p \to \Box\neg\Box\neg p$. We see that this modal formula has a local frame correspondent relative to the class of monotonic neighborhood frames in the language $L_=$ of coalgebraic predicate logic based on the empty language, i.e., the language with just the equality symbol. Consider the validity of the B axiom for a monotonic neighborhood frame F and $w \in F$. By the monotonicity of F, the usual minimum valuation argument (see, e.g., [2]) applies: the B axiom is valid here if and only if its consequent is true under the minimum valuation that makes its antecedent true, which is the valuation that sends p to the set $\{w\}$. The latter condition is expressible by a formula in $L_=$:

$$w \,\Box_y\, \neg y \,\Box_z\, z \neq w.$$

Remark 2.4 It can be shown likewise that modal formulas of the form (1) have frame correspondents relative to the class of monotonic neighborhood

[2] This is standard practice in model theory (see, e.g., [18]); it makes the notation and the definitions much simpler, particularly in later parts of this article where we deal with types with parameters.

frames. Moreover, a formula of the form (1) is what is called a KW formula in [14] and thus axiomatizes a monotonic modal logic complete with respect to the class of monotonic neighborhood frames that it defines. Hence, the monotonic modal logics axiomatized by such formulas are determined by CPL-elementary classes (see Definition 4.1) of monotonic neighborhood frames.

Example 2.5 Consider the 4 axiom $\Box p \to \Box\Box p$. We show that this modal formula has a local frame correspondent relative to the class of augmented quasi-filter neighborhood frames in the same language $L_=$ as above. Consider the validity of the 4 axiom for an augmented quasi-filter neighborhood frame F and $w \in F$. If $w \in F$ is *impossible*, i.e., $N^F(w) = \emptyset$, then the 4 axiom is valid at w. Note that by monotonicity w is impossible if and only if $F \notin N^F(w)$, i.e., $F \models \neg w \Box_y y = y$. Otherwise, we can again use the minimum valuation argument. Here, the minimum interpretation of p that makes the antecedent true is $R[w]$ because F is an augmented quasi-filter neighborhood frame, where $R \subseteq F \times F$ is the binary relation defined by

$$xRy \iff \{z \in F \mid z \neq y\} \notin N^F(x) \quad (\iff F \models \neg x \Box_z z \neq y).$$

To summarize, the 4 axiom has the local frame correspondent

$$\neg w \Box \lceil y \colon y = y \rceil \vee (w \Box \lceil y \colon y = y \rceil \wedge w \Box \lceil y_1 \colon y_1 \Box \lceil y_2 \colon \neg y_2 \Box \lceil z \colon z \neq w \rceil \rceil \rceil).$$

In fact, since the accessibility relation R and the set of impossible worlds are definable in $L_=$ as we have seen above, the first-order frame correspondence language in [20] translates into $L_=$, and thus all Sahlqvist formulas have frame correspondents in $L_=$ relative to the class of augmented quasi-filter neighborhood frames.

Definition 2.6 Let F and F' be neighborhood frames. A function $f : F \to F'$ is a *bounded morphism* if for each $w \in F$:

$$f^{-1}(U') \in N^F(w) \implies U' \in N^{F'}(f(w)) \qquad \text{(``forth'')}$$

and

$$U' \in N^{F'}(f(w)) \implies f^{-1}(U') \in N^F(w) \qquad \text{(``back'')}$$

Lemma 2.7 ([6]) *Let F and F' be monotonic neighborhood frames and $f : F \to F'$ be a function that satisfies the "forth" condition. Suppose in addition that for all $U' \in N^{F'}(f(w))$ there exists $U \in N^F(w)$ such that $f(U) \subseteq U'$. Then f is a bounded morphism.*

Proof. By assumption, if $U' \in N^{F'}(w)$, then there exists U such that $f^{-1}(U') \supseteq U \in N^G(w)$; by monotonicity, we have $f^{-1}(U') \in N^G(w)$. □

2.2 Algebraic concepts

In this subsection, we recall some standard definitions from the algebraic treatment of modal logic; for more information, see [26].

Definition 2.8 A *monotonic Boolean algebra expansion* (*BAM* for short) $A = (A, \Box^A)$ is a Boolean algebra A with an additional datum $\Box^A : A \to A$, a function that is *monotonic*, i.e., for all $a, b \in A$ we have $a \leq b \implies \Box^A(a) \leq \Box^A(b)$.

Lemma 2.9 *Let F be a monotonic neighborhood frame. The function $\Box^F : \mathscr{P}(F) \to \mathscr{P}(F)$ defined by*

$$X \mapsto \{w \in F \mid X \in N^F(w)\}$$

is monotonic. $\Box$

Definition 2.10 [[6]] The *underlying BAM* F^+ of a monotonic neighborhood frame F is the BAM $(\mathscr{P}(F), \Box^F)$, where $\mathscr{P}(F)$ is the Boolean algebra of the powerset of F.

Proposition 2.11 *Let F and F' be monotonic neighborhood frames and $f : F \to F'$ be a bounded morphism. Then $f^+ : F'^+ \to F^+$ defined by $f^+(X) = f^{-1}(X)$ is a homomorphism.* $\Box$

Definition 2.12 Let B be a Boolean algebra. The *canonical extension* B^σ of B is the Boolean algebra of the powerset of the set $\mathrm{Uf}(B)$ of ultrafilters in B. An element of B^σ of the form $[a] := \{u \in \mathrm{Uf}(B) \mid a \in u\}$ for a fixed $a \in B$ is called *clopen*. Joins and meets of clopen elements of B^σ are *closed* and *open*, respectively. The sets of closed and open elements of B^σ are denoted $K(B^\sigma)$ and $O(B^\sigma)$, respectively.

Proposition 2.13 *For a Boolean algebra B, the map $[-] : B \to B^\sigma$ is an embedding.*

Proof. See, e.g., [26]. $\Box$

Definition 2.14 [see, e.g., [26]] Let $A = (A, \Box)$ be a BAM. The *canonical extension* $A^\sigma = (A^\sigma, \Box^\sigma)$ of A is the canonical extension of the Boolean algebra A expanded by the function $\Box^\sigma$, where

$$\Box^\sigma(u) = \bigvee_{u \supseteq x \in K(A^\sigma)} \bigwedge_{x \subseteq a \in A} \Box(a).$$

Proposition 2.15 *For a BAM $A = (A, \Box)$, the function $\Box^\sigma$ is monotonic, and thus the canonical extension $A^\sigma = (A^\sigma, \Box^\sigma)$ is again a BAM.*

Proof. See, e.g., [26]. $\Box$

Remark 2.16 Canonical extensions can be defined for larger classes of algebras. We stick to BAMs in this article since they admit the most natural definition for $\Box^\sigma$, among other reasons.

Definition 2.17 [[14]]

(i) Let A be a BAM. The *ultrafilter frame* of A is a neighborhood frame $(\mathrm{Uf}(A), N^\sigma)$ with N^σ defined by

$$U \in N^\sigma(w) \iff \exists X \subseteq U \, \forall a \in A([a] \supseteq X \Rightarrow \Box^F(a) \in w),$$

where $w \in \mathrm{Uf}(A)$, and X range over closed elements of $A^\sigma = \mathscr{P}(\mathrm{Uf}A)$. We denote the ultrafilter frame of A by $\mathrm{Uf}(A)$.

(ii) Let F be a monotonic neighborhood frame. The *ultrafilter extension* $\mathrm{ue}\,F$ of F is $\mathrm{Uf}(F^+)$.

Proposition 2.18 *Let A be a BAM.*

(i) $\mathrm{Uf}(A)$ *is monotonic.*

(ii) $(\mathrm{Uf}(A))^+ \cong A^\sigma$. □

Definition 2.19 A class of BAMs is *canonical* if it is closed under canonical extensions.

3 Model theory of neighborhood frames

In this section, we recall as well as develop results in the model theory of neighborhood frames and coalgebraic predicate logic.

3.1 Standard concepts in first-order model theory

Here, we define concepts that have counterparts in the classical first-order model theory.

Definition 3.1 Let L be a language of coalgebraic predicate logic, F be an L-structure, and $A \subseteq F$. A subset $X \subseteq F$ is *A-definable in F* if there is an L-formula $\phi(x;\bar{y})$ and a tuple $\bar{a}$ of elements of A (notation: $\bar{a} \in A$) such that $X = \phi(F;\bar{a})$. A subset X is *definable in F* if it is F-definable in F.

Definition 3.2

(i) A set of L-sentences is called an *L-theory*.

(ii) Let L be a language of coalgebraic predicate logic and F be an L-structure. The *full L-theory* $\mathrm{Th}_L(F)$ of F is the set of L-sentences ϕ such that $F \models \phi$.

(iii) Two L-structures F, F' are *L-elementarily equivalent*, or $F \equiv_L F'$, if $\mathrm{Th}_L(F) = \mathrm{Th}_L(F')$.

For the rest of this section, we fix a language L of coalgebraic predicate logic and a monotonic L-structure F. We also let $T = \mathrm{Th}(F)$.

Definition 3.3 Let $A \subseteq F$. We write $\mathrm{Def}(F/A)$ for the Boolean algebra of A-definable subset in F, its operations being the set-theoretic ones. We also think of $\mathrm{Def}(F/A)$ as a BAM whose monotone operation $\Box$ is defined[3] by

$$\Box(\phi(F)) = (\Box\phi)(F)$$

for an $L(A)$-formula $\phi(x)$, where $(\Box\phi)(x)$ is the L-formula $x \Box_y \phi(y)$.

It is easy to see that $\mathrm{Def}(F/A)$ is a subalgebra of F^+ as a BAM.

[3] It is easy to see that $\Box : \mathrm{Def}(F/A) \to \mathrm{Def}(F/A)$ is well-defined here. This is true of similar definitions that appear in later parts of the article.

Proposition* 3.4 *Assume $F' \models T$. Then* $\mathrm{Def}(F/\emptyset)$ *and* $\mathrm{Def}(F'/\emptyset)$ *are isomorphic as BAMs.*

Definition 3.5

(i) The *Stone space $S_1(T)$ of 1-types over $\emptyset$ for T* is the ultrafilter frame $\mathrm{Uf}(\mathrm{Def}(F/\emptyset))$ of $\mathrm{Def}(F/\emptyset)$. (This is defined uniquely regardless of choice of $F \models T$.) We consider $S_1(T)$ as a topological space whose open subsets are exactly the open elements of $(\mathrm{Uf}(\mathrm{Def}(F/\emptyset)))^+ = (\mathrm{Def}(F/\emptyset))^\sigma$. An element $p \in S_1(T)$ is called a 1-*type over $\emptyset$*.

(ii) Likewise, we let $S_1^F(A) = \mathrm{Uf}(\mathrm{Def}(F/A))$. An element $p \in S_1^F(A)$ is called a 1-*type over A*.

(iii) A set $\Sigma(X)$ of $L(A)$-formula with one variable, say, x, is called a *partial* 1-*type over A*. We write $\Sigma(F)$ for the set $\{w \in F \mid \forall \phi \in \Sigma\, F \models \phi(w)\}$.

Convention 3.6 *We identify a 1-type p over A with the partial 1-type*

$$\{\phi(x;\bar{a}) \mid \phi(F;\bar{a}) \in p, \bar{a} \in A\}$$

over A. In fact, this is closer to how types are usually defined in classical model theory and is what types are in [17]. Likewise, we write $[\phi]$ for the clopen set $[X]$ in a Stone space of 1-types if ϕ defines X.

Given a partial type $\Sigma(x)$, the intersection $\bigcap_{\phi\in\Sigma}[\phi]$ is a closed set in the Stone space of 1-types.

Definition 3.7

(i) A partial 1-type $\Sigma(x)$ over A is *deductively closed* if $[\phi] \subseteq \bigcap_{\phi\in\Sigma}[\phi] \implies \phi \in \Sigma$.

(ii) For a deductively closed partial 1-type $\Sigma(x)$, we write E_Σ for the closed set

$$\{p \mid p \supseteq \Sigma\} = \bigcap_{\phi\in\Sigma}[\phi].$$

Proposition 3.8 *Let $w \in F$ and $A \subseteq F$. The family* $\mathrm{tp}^F(w/A)$ *of A-definable subsets of F containing w is an ultrafilter in* $\mathrm{Def}(F/A)$ *and thus a 1-type over A.* □

Definition 3.9

(i) Let $A \subseteq F$. An element $w \in F$ *realizes* $p \in S_1^F(A)$, or $w \models p$, if $\mathrm{tp}^F(w/A) = p$. The 1-type p is *realized in F* if there is $w \in F$ with $w \models p$.

(ii) The L-structure F is $\aleph_0$-*saturated* if for every finite $A \subseteq F$, every $p \in S_1^F(A)$ is realized in F.

Definition 3.10 [[3,17]] Let L be a language of coalgebraic predicate logic based on L_0 and $(F_i)_{i\in I}$ be a family of monotonic L-structures. Suppose that D is an ultrafilter over I. Let $\prod_D F_i$ be the ultraproduct of $(F_i)_i$ as L_0-structures modulo D. A subset $A \subseteq \prod_D F_i$ is *induced* if for D-almost all i there exists a set $A_i \subseteq F_i$ such that

$$a \in A \iff a(i) \in A_i \text{ for all } i \text{ for which } A_i \text{ is defined.}$$

A *quasi-ultraproduct of* $(F_i)_i$ *modulo* D is a monotonic L-structure that is the L_0-structure $\prod_D F_i$ equipped with a neighborhood function N that satisfies

$$A \in N(w) \iff A_i \in N^i(w(i)) \text{ for all } i \text{ for which } A_i \text{ is defined,}$$

whenever $w \in \prod_D F_i$, A is induced, and A_i are defined as above. A class $\mathcal{K}$ of monotonic neighborhood frames *admits quasi-ultraproducts* if whenever $(F_i)_i$ is a family of neighborhood frames from $\mathcal{K}$, a quasi-ultraproduct of $(F_i)_i$ exists in $\mathcal{K}$.

Proposition* 3.11 ([17,3])

(i) *Each class of the classes in Table 1 admits quasi-ultraproducts.*

(ii) *Let $(F_i)_{i\in I}$ be as in the definition above. If F_i satisfies T for all $i \in I$, so does a quasi-ultraproduct of $(F_i)_i$.*

Remark 3.12 Since the class of filter frames is CPL-elementary relative to the class of quasi-filter frames (see Definition 4.1), and the class of augmented filter frames is CPL-elementary relative to the class of augmented quasi-filter frames, it suffices to prove the claim for the classes of monotonic, quasi-filter, and augmented quasi-filter neighborhood frames, respectively. This is also true of the main result (Theorem 4.7).

3.2 Model theory specific to neighborhood frames

In this section, we study the model theory of neighborhood frames while we relate it to the classical model theory.

Definition 3.13 Let L be a language of coalgebraic predicate logic based on L_0, F be an L-structure. The *essential part* F^{e} of F is the L-structure whose reduct to L_0 is the same as that of F and whose neighborhood function N^{e} is defined by

$$U \in N^{\mathrm{e}}(w) \iff U \text{ is definable in } F \text{ and } U \in N^F(w)$$

for $w \in F^{\mathrm{e}}$.

Proposition 3.14 ([3]) *Let L be a language of coalgebraic predicate logic and F, G be L-structures. Suppose $F^{\mathrm{e}} \cong G^{\mathrm{e}}$.*

(i) *$F \equiv_L G$.*

(ii) *If F is $\aleph_0$-saturated, so is G.* □

Definition 3.15 Let L_0 be a language of first-order logic and L be the coalgebraic predicate logic based on L_0. We define the language L^2 to be the two-sorted first-order language whose sorts are the *state sort* and *neighborhood sort* and whose nonlogical symbols are those in L_0, recast as symbols for the state sort and the relation symbols xNU and $x \in U$, where x and U are variables for the state sort and the neighborhood sort, respectively. (In general, we will use lowercase variables for the state sort and uppercase variables for the neighborhood sort.)

Definition 3.16 Let L be a language of coalgebraic predicate logic and F be an L-structure. Given a family $\mathcal{S} \subseteq \mathscr{P}(F)$ that contains all definable subsets of F, we can identify F with the following L^2-structure F'. The domain of the state sort of F' is that of F, and the domain of the neighborhood sort of F' is $\mathcal{S}$. The L^2-structure F' interprets all nonlogical symbols of L^2 but N and $\in$ in the same way as F. Finally, we have $(w, U) \in N^{F'} \iff U \in N^F(w)$ and $(w, U) \in \in^{F'} \iff w \in U$. A family $\mathcal{S}$ is *large for F* if $U \in \mathcal{S}$ whenever there is $w \in F$ with $U \in N^F(w)$. We write $(F, \mathcal{S})$ for F' and sometimes F for $(F, \mathscr{P}(F))$.

Proposition 3.17 ([3]) *Let L be a language of coalgebraic predicate logic. Let $(-)^2 : L \to L^2$ be the translation that commutes with Boolean combinations and satisfies*

$$(\exists x\, \phi)^2 = \exists x(\phi^2)$$
$$(x \Box_y \phi)^2 = \exists U[\forall y(y \in U \leftrightarrow \phi^2(y)) \wedge xNU].$$

Let $\mathcal{S} \subseteq \mathscr{P}(F)$ be a family that contains all definable subsets of F. Then for every L-formula ϕ and $\bar{a} \in F$ we have

$$F \models \phi(\bar{a}) \iff (F, \mathcal{S}) \models \phi^2(\bar{a}).$$

□

Remark 3.18 There is a third language for neighborhood frames used before as a model correspondence language [16,24] in regards to neighborhood and topological semantics of modal logic and to study model theory of topological spaces [8] in general. This is also a fragment of the two-sorted language introduced above. Coalgebraic modal logic, in fact, embeds into the third language.

Lemma* 3.19 *Let L be a language of coalgebraic predicate logic and F be an L-structure. Let G be an L^2-structure that is an elementary extension of F as an L^2-structure. There exists an L-structure G' whose domain is that of the state sort of G and a family $\mathcal{S} \subseteq \mathscr{P}(G')$ that satisfies the following:*

(i) *$\mathcal{S}$ contains all definable subsets in G'.*

(ii) *$\mathcal{S}$ is large for G'.*

(iii) *$G \cong (G', \mathcal{S})$.*

Proposition* 3.20 *Let L be a language of coalgebraic predicate logic and F be an L-structure. There exists an L-structure $G \equiv_L F$ that is $\aleph_0$-saturated.*

4 Canonicity

In this section we prove the main result. Our proof is an adaptation of that of the classical counterpart as presented in [2]. We do a case analysis, with the most interesting case treated in Lemma 4.3, which is a step analogous to [25, 8.9 Theorem]. There we follow the classical proof, by taking an expansion L of the correspondence language so that every subset of the given frame F will be definable and taking an $\aleph_0$-saturated extension in that language. However, we need to add more neighborhoods to the neighborhood frame G that is being constructed to make sure that the map from G to the ultrafilter frame of F is a bounded morphism. Much of the proof is dedicated to show that this construction preserves elementary equivalence in L.

Throughout the section, let $L_=$ be the language of coalgebraic predicate logic based on the empty language of first-order logic.

Definition 4.1 Let $\mathcal{K}_0$ be a class of monotonic neighborhood frames. A class $\mathcal{K}$ of monotonic neighborhood frames is *CPL-elementary relative to* $\mathcal{K}_0$ if there is an $L_=$-theory T with

$$\mathcal{K} = \{F \in \mathcal{K}_0 \mid F \models T\}.$$

Lemma* 4.2 *Let F, G, and G' be as in Lemma 3.19.*

(i) *If F is monotonic, $X, Y \subseteq G'$ are definable, $X \subseteq Y$, and $X \in N^{G'}(w)$ for $w \in G'$, then $Y \in N^{G'}(w)$.*

(ii) *If F is a augmented filter frame, then for every $w \in G'$ either $N^{G'}(w)$ is empty or has a minimum element.*

We are now ready to prove the key lemma used in the proof of our main result.

Lemma 4.3 *Let F be a monotonic neighborhood frame. There exists $G \equiv_{L_=} F$ such that there is a surjective bounded morphism $f : G \twoheadrightarrow \mathrm{ue}\, F$. Moreover, if $\mathcal{K}_0$ is either the class of monotonic neighborhood frames or the class of quasi-filter neighborhood frames, and $F \in \mathcal{K}_0$, then we can take $G \in \mathcal{K}_0$.*

Proof. Let L be the language of coalgebraic predicate logic based on $\{P_S \mid S \subseteq F\}$, the unary predicates for the subsets of F. The neighborhood frame F can be made into an L-structure naturally. Let $G_0 \succ_{L^2} F$ be an $\aleph_0$-saturated L^2-structure. By Lemma 3.19, we may assume without loss of generality that the domain of the neighborhood sort of G_0 is contained in the powerset of that of the state sort. Let G_1 be the L^2-structure obtained by restricting G_0 to the subsets of (the state sort of) G_0 definable with parameters from (the state sort of) G_0 in L. Let G_2 be the L-structure obtained from G_1 as follows: for each state $w \in G_1$, add as a neighborhood of w the set $\Sigma(G_1)$, where $\Sigma(x)$ is a partial type over a finite set $A \subseteq G_1$ such that $\Sigma(x)$ is deductively closed and that for every $\phi \in \Sigma$ we have $\phi(G_1) \in N^{G_1}(w)$ (we call such a partial type *good at* w). Let G be the L-structure obtained from G_2 by closing off the value of the neighborhood functions at each $w \in G_2$ by supersets, i.e., $U \in N^G(w) \iff \exists U_0 \subseteq U\, U_0 \in N^{G_2}(w)$.

We show that $G \equiv_L F$. By Proposition 3.14, it suffices to see that for every definable $X \subseteq G$ we have $X \in N^G(w) \iff X \in N^{G_1}(w)$. We show $\implies$ (the other direction is easy). By construction, there is either a definable set $Y \subseteq X$ with $Y \in N^{G_1}(w)$ or a partial type $\Sigma(x)$ over a finite set A good at w with $\Sigma(G_1) \subseteq X$. The former is a special case of the latter, so we assume the latter. Let A' be a finite set containing the parameters used in the definition of X and A. Let $f' : G_1 \twoheadrightarrow S_1^{G_1}(A')$ be defined by $f'(w) = \mathrm{tp}^{G_1}(w/\bar{a})$. By $\aleph_0$-saturation, f' is a surjection. We show that $f'(\Sigma(G_1)) = E_\Sigma \subseteq S_1^{G_1}(\bar{a})$. It is easy to show that $f'(\Sigma(G_1)) \subseteq E_\Sigma$; we show $f'(\Sigma(G_1)) \supseteq E_\Sigma$. Let $p \in E_\Sigma$ be arbitrary. By $\aleph_0$-saturation, take $w \in G_1$ with $f'(w) = p$. Since $p \supseteq \Sigma$, $w \in \Sigma(G_1)$. That $f'(X) = [X]$ can be easily shown as well. We have $E_\Sigma \subseteq [X]$. By the compactness of $S_1^{G_1}(A')$, we have a finite $\Sigma_0 \subseteq \Sigma$ for which $E_{\Sigma_0} \subseteq [X]$. Being the intersection of finitely many clopen sets,

$$E_{\Sigma_0} = \bigcap_{\phi \in \Sigma_0} [\phi] = \left[\bigwedge \Sigma_0\right]$$

is clopen. Since Σ is good at w, we have $(\bigwedge \Sigma_0)(G_1) \in N^{G_1}(w)$. We conclude that $X \in N^{G_1}(w)$ by Lemma 3.19 (i).[4]

Since $F^+ \cong \mathrm{Def}(F/\emptyset)$, we have $\mathrm{ue}\, F \cong S_1(T)$. We show $f : G \twoheadrightarrow S_1(T)$ defined by $f(w) = \mathrm{tp}^G(w/\emptyset)$, which is surjective by $\aleph_0$-saturation, is a bounded morphism. In the rest of the proof, we write N^σ for the neighborhood function of $S_1(T)$.

The "forth" condition

Suppose that $U \in N^G(w)$. We show that $f(U) \in N^\sigma(\mathrm{tp}^G(w))$. By construction, we have either (I) $U \supseteq \phi(G, \bar{a}) \in N^G(w)$ or (II) $U \supseteq \Sigma(G) \in N^G(w)$, where $\phi(x, \bar{y})$ is an L-formula, $\bar{a} \in G$, and $\Sigma(x)$ is a partial type over a finite set A good at w.

For (I), assume that $U \supseteq \phi(G, \bar{a}) \in N^G(w)$. Let

$$K = \{q \in S_1(T) \mid (\exists q' \in S_1^G(\bar{a}))\, \phi(x, \bar{a}) \in q'\}.$$

Being the image of a clopen set under the restriction map $S_1^G(\bar{a}) \twoheadrightarrow S_1(T)$, which is continuous and thus closed, K is a closed set. Note that $\Box\chi \in f(w) = \mathrm{tp}^G(w/\emptyset)$ if and only if $\chi(G) \in N^G(w)$. It suffices to show (i) that for every $\chi(x) \in L$ we have $[\chi] \supseteq K \implies \chi(G) \in N^G(w)$ and (ii) that $K \subseteq U'$. For (i), assume that $[\chi] \supseteq K$. For arbitrary $v \in \phi(G, \bar{a})$, since $\mathrm{tp}^G(v/\bar{a}')$ contains $\phi(x, \bar{a})$, $\mathrm{tp}^G(v/\emptyset)$ is in K and thus contains $\chi(x)$. Hence, $\chi(G) \supseteq \phi(G, \bar{a})$, and by monotonicity $\chi(G) \in N^G(w)$. For (ii), let $q \in K$ be arbitrary. It suffices

[4] We can replace the topological argument by the following, even though we have not defined the concepts used there. Suppose X is definable by $\psi(x; \bar{a})$ where $\psi \in L$ and $\bar{a} \in G$. By $\aleph_0$-saturation of G_1, we have $\mathrm{Th}_{L(\bar{a})} G_1 \cup \Sigma(x) \models \psi(x, \bar{a})$ (otherwise, realize the type $\Sigma(x) \cup \{\neg\psi(x, \bar{a})\}$ by some element in G_1, which would be in $\Sigma(G_1) \setminus X$.) By compactness, there is finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0(G_1) \subseteq \psi(G_1, \bar{a})$. Since $\bigwedge \Sigma_0(x)$ is a single formula of L, by deductive closure $\bigwedge \Sigma_0(x) \in \Sigma(x)$. Hence $\bigwedge \Sigma_0(G_1) \in N^{G_1}(w)$. By Lemma 3.19(i), we have $X = \psi(G_1, \bar{a}) \in N^{G_1}(w)$ as desired.

to show that q is realized by some $v \in U$. By the definition of K, there exists $q' \in S_1^G(\bar{a})$ that extends q with $\phi(x, \bar{a}) \in q'$. By $\aleph_0$-saturation, there is some $v \in G$ that realizes q'; since $\phi(x, \bar{a}) \in q'$, we have $v \in \phi(G, \bar{a}) \subseteq U$.

For (II), assume that $U \supseteq \Sigma(G) \in N^G(w)$, where Σ is a partial 1-type over finite A good at w. We would like to show (i) and (ii) from above for $K = r(E_\Sigma)$, where $r : S_1^G(A) \to S_1(T)$ is the closed continuous map dual to the embedding $\mathrm{Def}(G/\emptyset) \hookrightarrow \mathrm{Def}(G/A)$. For (i), assume that $[\chi] \supseteq r(E_\Sigma)$, where $[\chi]$ denotes a subset in $S_1^G(A)$. This implies that $[\chi] \supseteq E_\Sigma$, where $[\chi]$ denotes a subset in $S_1(T)$. By deductive closure $\chi \in \Sigma$. By construction, $\chi(G) \in N^G(w)$. For (ii), it suffices to show that arbitrary $q \in E_\Sigma$ can be realized by an element of U. Since q is a type over a finite set, by saturation, we may take $v \models q$; this means $v \models \Sigma$, i.e., $v \in \Sigma(G) \subseteq U$.

The "back" condition.

Suppose that $U' \subseteq S_1(T)$ is in $N^\sigma(\mathrm{tp}^G(w/\emptyset))$. We show that there is $U \subseteq G$ in $N^G(w)$ such that $f(U) \subseteq U'$. By the definition of N^σ, there is a partial type $\Sigma(x)$ over $\emptyset$ good at w such that $E_\Sigma \subseteq U'$. By construction, $\Sigma(G) \in N^G(w)$. Let $U := \Sigma(G)$. Then for every $v \in U$, the type $\mathrm{tp}^G(w/\emptyset)$ extends Σ and thus is in $E_\Sigma \subseteq U'$.

Closure in relatively CPL-elementary classes.

By construction, G is monotonic.

Suppose that F is a quasi-filter neighborhood frame. Let $w \in G$ and $U, U' \in N^G(w)$ be arbitrary. By construction, there are deductively closed partial types $\Sigma(x), \Sigma'(x)$ over a finite set of parameters both of which are good at w with $\Sigma(G) \subseteq U$ and $\Sigma'(G) \subseteq U'$. The partial type $\Sigma \cup \Sigma'$ is also over a finite set, good at w. Moreover, $\Sigma \cup \Sigma'$ is deductively closed since F is a quasi-filter frame. Therefore, we have $(\Sigma \cup \Sigma')(G) = \Sigma(G) \cap \Sigma(G) \subseteq U \cap U'$, so $U \cap U' \in N^G(w)$. We have seen that G is a quasi-filter neighborhood frame. $\square$

Remark 4.4 In the proof above, we obtain G not only by compactness but also by altering the neighborhoods in an ad-hoc way while maintaining elementary equivalence in $L_=$. There is no reason for us to believe that the G has the same theory as F in ${L_=}^2$ or in the language described in Remark 3.18. This is why we find it difficult to extend our main result to the more expressive languages.

Lemma⋆ 4.5 *Let F be an augmented quasi-filter neighborhood frame. There exist an augmented quasi-filter neighborhood frame $G \equiv_{{L_=}^2} F$ and a surjective bounded morphism $f : G \twoheadrightarrow \mathrm{ue}\, F$.*

Lemma⋆ 4.6 *Let $\mathcal{K}$ be a class CPL-elementary relative to any of the classes in Table 1. Let $\mathcal{S} \supseteq \mathcal{K}^+$ be the least class of BAMs closed under subalgebras.*

(i) *$\mathcal{S}$ is closed under canonical extensions.*

(ii) *$\mathcal{S}$ is closed under ultraproducts.*

Theorem 4.7 *Let $\mathcal{K}$ be a class CPL-elementary relative to any of the classes in Table 1. The variety of BAMs generated by $\mathcal{K}^+$ is canonical.*

Proof. Recall Remark 3.12. Gehrke and Harding [9] showed that if $\mathcal{S}$ is a class of BAMs closed under ultraproducts and canonical extensions, then $\mathcal{S}$ generates a canonical variety. Apply this result for the class $\mathcal{S}$ in Lemma 4.6 to conclude that the variety generated by $\mathcal{K}^+$, which is identical to the variety generated by $\mathcal{S}$, is canonical. $\square$

Note that Fine's original theorem follows as a special case concerning the classes of augmented neighborhood frames.

Example 4.8 Consider the B axiom $p \to \Box\neg\Box\neg p$, which we considered earlier in this article. Recall that it defined a CPL-elementary class relative to the class of monotonic neighborhood frames. By [14, Proposition 6.5], the variety $\mathcal{V}$ defined by the B axiom is canonical and hence generated by $\mathcal{K}^+$. The canonicity of $\mathcal{V}$ is explained by the CPL-elementarity of $\mathcal{K}$.

Example 4.9 Consider the 4 axiom $\Box p \to p$, which we considered earlier in this article. Recall that it defined a CPL-elementary class relative to the class of augmented quasi-filter neighborhood frames. The usual argument [2] shows that the variety $\mathcal{V}$ of BAMs defined by the 4 axiom is canonical and hence generated by $\mathcal{K}^+$. The canonicity of $\mathcal{V}$ is explained by the CPL-elementarity of $\mathcal{K}$.

5 Conclusion

We proved an analogue of Fine's canonicity theorem relative to the classes in Table 1 of monotonic neighborhood frames. Our version of the theorem deals with a generalized notion of elementarity of classes of monotonic neighborhood frames, which is elementarity in coalgebraic predicate logic.

As we mentioned in Remarks 3.18 and 4.4, one could attempt to use a different notion of elementarity in stating and proving an analogue of Fine's theorem, but we stuck to coalgebraic predicate logic due to the limitation of the proof technique we used. A natural question to ask here would be whether there is a more expressive first-order-like logic that admits an analogue of Fine's theorem possibly by a different kind of proof. Another question would be to characterize classes of monotonic neighborhood frames that admit analogues of Fine's theorem in the same sense as in the main result of this article.

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Appendix

Proposition⋆ 3.4 *Assume $F' \models T$. Then* $\mathrm{Def}(F/\emptyset)$ *and* $\mathrm{Def}(F'/\emptyset)$ *are isomorphic as BAMs.*

Proof. It is easy to see that they are isomorphic as Boolean algebras. Indeed, the inclusion of a definable set in another is a CPL-elementary property (it is the universal closure of the implication from the definition of the first set to that of the other). Likewise, the two algebras are isomorphic also as BAMs because the value of the operation at a definable set being another is also CPL-elementary, which can easily be seen by using the operation $\Box$ for L-formulas in Definition 3.3. □

Proposition⋆ 3.11 ([17,3])

(i) *Each class of the classes in Table 1 admits quasi-ultraproducts.*

(ii) *Let $(F_i)_{i\in I}$ be as in the definition above. If F_i satisfies T for all $i \in I$, so does a quasi-ultraproduct of $(F_i)_i$.*

Proof.

(i) This can be proved by using the machinery introduced in Litak et al. [17], but it is easy to prove it in the following elementary way.

Let $\mathcal{K}_0$ be either the class of monotonic neighborhood frames or the class of quasi-filter neighborhood frames. Let $(F_i)_i$ be a family of neighborhood frames in $\mathcal{K}_0$. Let N_i be the neighborhood function of F_i. Define the neighborhood function N on $\prod_D F_i$ as follows: A subset $U \subseteq \prod_D F_i$ is in $N(w)$ if and only if there is an induced set $A \subseteq \prod_D F_i$ with $A_i \in N^i(w_i)$ for all i for which A_i is defined. It is easy to see that this indeed defines a quasi-ultraproduct and that if each F_i is in $\mathcal{K}_0$ then so is the quasi-ultraproduct.

Let $\mathcal{K}_0$ be the class of augmented quasi-filter frames. Each F_i can be thought of as a first-order structure in the language $L_0 \cup \{R, P\}$, where R and P are binary and unary predicate symbols, respectively,

$$P^{F_i} = \{w \in F_i \mid N^{F_i}(w) \neq \emptyset\},$$

and

$$R^{F_i} = \{(w, w') \in {F_i}^2 \mid w \in P^{F_i} \text{ and } w' \in \min N^{F_i}(w)\},$$

where min is with respect to $\subseteq$. Take the ultraproduct $\prod_D F_i$ as an $L_0 \cup \{R, P\}$-structure. Consider this as an L-structure by definining its neighborhood function N by

$$U \in N(w) \iff w \in P\left(\prod_D F_i\right) \text{ and } U \supseteq R\left(w, \prod_D F_i\right).$$

It is easy to see that this is a quasi-ultraproduct and that it is an augmented quasi-filter neighborhood frame.

(ii) The usual argument by induction works; see Litak et al. [17].

□

Lemma⋆ 3.19 *Let L be a language of coalgebraic predicate logic and F be an L-structure. Let G be an L^2-structure that is an elementary extension of F as an L^2-structure. There exists an L-structure G' whose domain is that of the state sort of G and a family $\mathcal{S} \subseteq \mathscr{P}(G')$ that satisfies the following:*

(i) *$\mathcal{S}$ contains all definable subsets in G'.*

(ii) *$\mathcal{S}$ is large for G'.*

(iii) *$G \cong (G', \mathcal{S})$.*

Proof. Note that F satisfies extensionality:

$$F \models \forall U \, \forall V [\forall x (x \in U \leftrightarrow x \in V) \to U = V].$$

By CPL-elementarity, so does G. Let G', S^G be the domain of the state sort and the neighborhood sort of G, respectively. Let $i : S^G \to \mathscr{P}(G')$ be defined by

$$i(U) = \{w \in G' \mid G \models w \in U\}.$$

By the extensionality of G, i is injective. Let $\mathcal{S}$ be the range of i. Define the neighborhood function $N^{G'}$ by

$$U \in N^{G'}(w) \iff G \models wNU.$$

Let $\phi(x; \bar{y})$ be an L-formula and $X := \phi(G', \bar{a})$ be a definable set in G', where $\bar{a} \in G'$. Note that the L^2-structure F satisfies comprehension:

$$F \models \forall \bar{y} \, \exists U \, \forall x (\phi(x; \bar{y}) \leftrightarrow x \in U).$$

So does G. Let U witness the satisfaction by G of the existential formula $\exists U \, \forall x (\phi(x; \bar{a}) \leftrightarrow x \in U)$. It can easily be seen that $i(U) = \phi(G', \bar{a})$.

It is easy to see that $\mathcal{S}$ is large for G' and that $G \cong (G', \mathcal{S})$. □

Proposition⋆ 3.20 *Let L be a language of coalgebraic predicate logic and F be an L-structure. There exists an L-structure $G \equiv_L F$ that is $\aleph_0$-saturated.*

Proof. Consider the L^2-structure $F = (F, \mathscr{P}(F))$, and take an elementary extension G_0 of F. By Lemma 3.19(iii), take an L-structure G and $\mathcal{S} \subseteq \mathscr{P}(G)$ with $G_0 \cong (G, \mathcal{S})$. Suppose that $A \subseteq G$ is finite. Let $p \in S_1^G(A)$ be arbitrary. We show that p is realized in G. There is a surjective continuous map π from the Stone space in L^2 onto that in L. Let $p^2 \in \pi^{-1}(p)$. By the $\aleph_0$-saturation of G_0, we can take $w \in G_0$ realizing p^2. By Proposition 3.17, we have $w \models p$. □

Lemma⋆ 4.2 *Let F, G, and G' be as in Lemma 3.19.*

(i) *If F is monotonic, $X, Y \subseteq G'$ are definable, $X \subseteq Y$, and $X \in N^{G'}(w)$ for $w \in G'$, then $Y \in N^{G'}(w)$.*

(ii) *If F is a augmented filter frame, then for every $w \in G'$ either $N^{G'}(w)$ is empty or has a minimum element.*

Proof. For (i), let $L(G')$-formulas $\phi(x;\bar{a})$ and $\psi(x;\bar{b})$ define X and Y, respectively. Since F is monotonic, we have

$$F \models \forall\bar{y}\,\forall\bar{z}\,\forall v[\,\forall x(\phi(x;\bar{y}) \to \psi(x;\bar{z})) \\ \wedge\, v \,\Box_x\, \phi(x;\bar{y}) \to v \,\Box_x\, \psi(x;\bar{z})],$$

where F is seen as an L-structure. Since F as an L^2-structure satisfies the $(-)^2$-translation of the right-hand side of the displayed formula by Proposition 3.17, so does G. Again by Proposition 3.17,

$$G' \models \forall x(\phi(x;\bar{a}) \to \psi(x;\bar{b})) \wedge w \,\Box_x\, \phi(x;\bar{a}) \to w \,\Box_x\, \psi(x;\bar{b}).$$

By assumption, we have $\psi(G',\bar{y}) \in N^{G'}(w)$.

For (ii), first observe that the L^2-structure F satisfies the sentence

$$\forall x[\neg\,\exists U\, xNU \vee \exists U_0\,\forall U(xNU \to U_0 \subseteq U)],$$

where $\subseteq$ is an abbreviation of the obvious L^2-formula. Since G' satisfies the same L^2-formula, the claim follows. $\Box$

Lemma* 4.5 *Let F be an augmented quasi-filter neighborhood frame. There exist an augmented quasi-filter neighborhood frame $G \equiv_{L_=^2} F$ and a surjective bounded morphism $f : G \twoheadrightarrow \mathrm{ue}\,F$.*

Proof. Expand the language as before to obtain the two-sorted language L. Take a 2-saturated $G \succ_L F$. We show that $f = \mathrm{tp}^G : G \twoheadrightarrow S_1(T)$ is a bounded morphism, where $T = \mathrm{Th}_L(F)$.

Let Σ be the partial type $\{\phi \mid \Box\phi \in \mathrm{tp}^G(w)\}$ over $\emptyset$. We show that E_Σ is the minimum element of $N^\sigma(w)$. First, we show $E_\Sigma \in N^\sigma(w)$. Suppose that for a formula χ we have $[\chi] \supseteq E_\Sigma$. Since F is an augmented quasi-filter neighborhood frame, and Σ is the set of (the definitions of) definable sets in F including the minimum set of $N^F(w)$, the partial type Σ is deductively closed. Thus $\chi \in \Sigma$, and $\Box\chi \in \mathrm{tp}^G(w)$. Hence, $E_\Sigma \in N^\sigma(\mathrm{tp}^G(w))$. Secondly, we show that if $U \subsetneq E_\Sigma$, then $U \notin N^\sigma(\mathrm{tp}^G(w))$. Consider an arbitrary closed set included in U; without loss of generality we may assume that the closed set is of the form $E_{\Sigma'}$ for some deductively closed Σ'. By the proper inclusion, $\Sigma' \supsetneq \Sigma$. Take $\chi \in \Sigma' \setminus \Sigma$; then $[\chi] \supseteq E_{\Sigma'}$ and $\Box\chi \notin \mathrm{tp}^G(w)$. Hence, $U \notin N^\sigma(\mathrm{tp}^G(w))$.

We show that $f^{-1}(U') \in N^G(w) \iff U' \in N^\sigma(\mathrm{tp}^G(w))$, where N^σ is the neighborhood function of $S_1(T)$. Since G and $S_1(T)$ are augmented quasi-filter neighborhood frames, if w and $\mathrm{tp}^G(w)$ are impossible, then there are $\{w\}$- and $\{\mathrm{tp}^G(w)\}$- definable sets which are the minimum elements of $N^G(w)$ and $N^\sigma(\mathrm{tp}^G(w))$, respectively. We write $R^G[w]$ and $R^\sigma[\mathrm{tp}^G(w)]$ for these sets. They are defined by (the ^{2}-translation of) the (same) formula

$$\neg a \,\Box_z\, z \neq x,$$

where the parameter a is either w or $\mathrm{tp}^G(w)$. It suffices to show that

$$w \text{ is possible and } f^{-1}(U') \supseteq R^G[w] \\ \iff \mathrm{tp}^G(w) \text{ is possible and } U' \supseteq R^\sigma[\mathrm{tp}^G(w)].$$

By the definition of N^σ,

$$\begin{aligned}\mathrm{tp}^G(w) \text{ is impossible} &\iff G \not\models (\Box\chi)(w) \text{ for every } L\text{-formula } \chi \\ &\iff w \text{ is impossible.}\end{aligned}$$

Hence, we may assume that both w and $\mathrm{tp}^G(w)$ are possible. It suffices to show

$$R^\sigma[f(w)] \subseteq f(R^G[w]), \tag{back}$$
$$R^\sigma[f(w)] \supseteq f(R^G[w]). \tag{forth}$$

The (forth) condition is clear. We show the (back) condition. Suppose that $q \in R^\sigma[\mathrm{tp}^G(w)]$ for $q \in S_1(T)$. Consider the partial type $\Sigma(x) = q(x) \cup \{R(w,x)\}$ over $\{w\}$, where $R(w,x)$ is the definition of $R^G[w]$. This is finitely satisfiable. Indeed, consider the finite partial type $\{\phi_1, \dots, \phi_n\} \cup \{R(w,x)\}$ where $\phi_i \in q$. Note that $\bigwedge_i \phi_i \in q$. Since $q \in R^\sigma[\mathrm{tp}^G(w)]$, we must have $(\neg\Box\neg \bigwedge_i \phi_i) \in \mathrm{tp}^G(w)$, i.e., $R^G[w] \cap (\bigwedge_i \phi_i)(G) \neq \emptyset$. Any element in this nonempty set realizes the finite partial type. We have shown $\Sigma(x)$ is satisfiable; we may take $v \models \Sigma(x)$ by 2-saturation. We have $v \in R^G[w]$ as desired. □

Lemma⋆ 4.6 *Let $\mathcal{K}$ be a class CPL-elementary relative to any of the classes in Table 1. Let $\mathcal{S} \supseteq \mathcal{K}^+$ be the least class of BAMs closed under subalgebras.*

(i) *$\mathcal{S}$ is closed under canonical extensions.*

(ii) *$\mathcal{S}$ is closed under ultraproducts.*

Proof.

(i) Let $A \in \mathcal{S}$. For some $F \in \mathcal{K}$ we have $A \hookrightarrow F^+$. By duality theory [9, Theorem 5.4], we have $A^\sigma \hookrightarrow (F^+)^\sigma$. By Lemma 4.3, there is $G \in \mathcal{K}$ with $(F^+)^\sigma \hookrightarrow G^+$. Thus, we have $A^\sigma \in \mathcal{S}$ by definition.

(ii) It suffices to do the following: given an ultraproduct $\prod_D F_i^+$ where I is an index set, D is an ultrafilter over I, and $(F_i)_i$ is a family of neighborhood frames in $\mathcal{K}$, we show that the ultraproduct embeds into $(\prod_D F_i)^+$, where $\prod_D F_i$ is a quasi-ultraproduct of $(F_i)_i$ modulo D. In fact, we show that $\iota : \prod_D F_i^+ \to (\prod_D F_i)^+$ defined by

$$s \in \iota(a) \iff s(i) \in a(i) \text{ for } D\text{-almost all } i,$$

where $s \in \prod_D F_i$ and $a \in \prod_D F_i^+$ is a BAM embedding (we do not write equivalence classes modulo D explicitly; it is easy to see that ι is well-defined). It can easily be seen that ι is a Boolean algebra embedding. We show that $\iota \circ \Box^{\mathrm{pu}} = \Box^{\mathrm{cm}} \circ \iota$, where $\Box^{\mathrm{pu}}$ and $\Box^{\mathrm{cm}}$ are the operations of the domain and the target of ι, respectively. Let N be the neighborhood function of the quasi-ultraproduct. We write $\Box^i$ and N^i for the operation of F_i^+. Note that for all a the set $\iota(a)$ is an induced subset of the quasi-ultraproduct; if we let $\pi_i(A)$ be the projection of an induced subset A of the quasi-ultraproduct onto the coordinate i, then $\pi_i(\iota(a)) = a(i)$ for

D-almost all i. We now have

$$\begin{aligned}
s \in (\iota \circ \Box^{\mathrm{pu}})(a) &\iff s(i) \in (\Box^{\mathrm{pu}}(a))(i) \text{ for } D\text{-almost all } i \\
&\iff s(i) \in \Box^{i}(a(i)) \text{ for } D\text{-almost all } i \qquad (*) \\
&\iff s(i) \in \Box^{i}(\pi_i(\iota(a))) \text{ for } D\text{-almost all } i \\
&\iff \iota(a) \in N^{i}(s(i)) \text{ for } D\text{-almost all } i \\
&\iff \iota(a) \in N(s) \\
&\iff s \in (\Box^{\mathrm{cm}} \circ i)(a),
\end{aligned}$$

where we have the equivalence (*) since $(\Box^{\mathrm{pu}}(a))(i) = \Box^{i}(a(i))$ for D-almost all i.

$\Box$