A test of two models of probability judgment: quantum versus noisy probability

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Abstract

We test contrasting predictions of two recent models of probability judgment: the quantum probability model (Busemeyer et al., 2011) and the probability theory plus noise model (Costello and Watts, 2014). Both models assume that people estimate probability using formal processes that follow or subsume standard probability theory. The quantum probability model predicts people's estimates should agree with one set of probability theory identities, while the probability theory plus noise model predicts a specific pattern of violation of those identities. Experimental results show just the form of violation predicted by the probability theory plus noise model. These results suggest that people's probability judgments do not follow quantum probability: instead, they follow the rules of standard probability theory, with the systematic biases seen in those judgments due to the effects of random noise.

Introduction

Researchers over the last 50 years have identified a large number of systematic biases in people's judgments of probability. These biases are typically taken as evidence that people do not follow the normative rules of probability theory when estimating probabilities, but instead use a series of heuristics (mental shortcuts or 'rules of thumb') that sometimes yield reasonable judgments but sometimes lead to severe and systematic errors, causing the observed biases (Kahneman and Tversky, 1973). This 'heuristics and biases' view has had a major impact in psychology (Kahneman and Tversky, 1982, Gigerenzer and Gaissmaier, 2011), economics (Camerer et al., 2003), law (Sunstein, 2000), medicine (Eva and Norman, 2005) and other fields, and has influenced government policy in a number of countries (Oliver, 2013, Vallgårda, 2012).

The existence of these systematic biases in people's probabilistic reasoning is incontrovertible. The conclusion that these biases necessarily demonstrate heuristic reasoning processes is, however, less sure. Recent research has shown that many of these biases can be explained if we assume that people estimate probability using formal processes that follow or subsume standard probability theory. Two such formal models are the quantum probability model proposed by Busemeyer and colleagues (Busemeyer et al., 2011, Busemeyer and Bruza, 2012), and our own probability theory plus noise model (Costello and Watts, 2014, 2016a,b). Both models can account for a number of well-known biases seen in people's probabilistic reasoning. Both models predict systematic bias away from standard probability theory for a range of probalistic expressions: importantly, the models make contrasting predictions about the occurence and direction of these biases. In this paper we test these contrasting predictions.

The probability theory plus noise model

In standard probability theory the probability of some event A is estimated by drawing a random sample of events, counting the number of those events that are instances of A, and dividing by the sample size. The expected value of these estimates is $P(A)$, the probability of A; individual estimates will vary with an approximately normal distribution around this expected value. The probability theory plus noise model assumes that people estimate the probability of some event A in exactly the same way: by randomly sampling items from their memory, counting the number that are instances of A, and dividing by the sample size. If this counting process was error-free then people's estimates would have an expected value of $P(A)$ just as in probability theory. Human memory is subject to various forms of random error, however. To reflect this the model assumes a minimal form of random error such that items have some probability $d < 0.5$ of being counted incorrectly (we assume that this error term is constant within a given participant, but varies across participants: some participants are more prone to random error than others).

Because of this random error, there is a chance d that an item which is not truly an instance of A will be incorrectly counted as A, and the same chance d that an instance of A will be incorrectly counted as not A. Given this error, a randomly sampled item can be counted as A in two mutually exclusive ways: either the item truly is an instance of A and is counted correctly (this occurs with probability $P(A)(1-d)$, since $P(A)$ items are truly instances of A, and items have a $(1-d)$ chance of being read correctly), or else the item truly is not an instance of A and is counted incorrectly as A (this occurs with probability $(1-P(A))d$, since $(1-P(A))$ items are truly not instances of A, and items have a d chance of being read incorrectly). The expected value for a noisy estimate for the probability of A is thus

$$\langle P_E(A)\rangle = P(A)(1-d)+(1-P(A))d = (1-2d)P(A)+d \quad (1)$$

and we expect individual estimates $P_E(A)$ to vary independently around this expected value.

According to the model, the pattern of bias due to random noise seen in Equation 1 is predicted for all types of events, but will be more pronounced for complex events (such as conjunctions or disjunctions), which, because of their complexity, give more opportunity for error compared to simple events. For simplicity and clarity we ignore this effect here and assume the same rate of error for all events, so the expected value for conjunctive events $A \wedge B$ and disjunctive event $A \vee B$ are

$$\langle P_E(A \wedge B) \rangle = (1-2d)P(A \wedge B) + d \tag{2}$$

$$\langle P_E(A \vee B) \rangle = (1-2d)P(A \vee B) + d \tag{3}$$

We've previously shown how this model explains various biases frequently seen in people's probabilistic reasoning, such as underconfidence (people's tendency to overestimate the probability of low-probability events and understimate the probability of high probability events), subadditivity (people's tendency to give probabilities for the constituent events of a disjunction which, when added, are reliably higher than their probability for the disjunction as a whole) and the conjunction fallacy. The patterns of occurrence of these biases in experimental results match the model's predictions well (Costello and Watts, 2014, 2016a,b).

This model also predicts a range of other deviations from probability theory for people's probability estimates. Table 1 gives a series of probability theory identities: expressions which, in probability theory, are required to have a value of 0 for all events A and B. We can use our expected-value expression to obtain values for these identities as predicted by our model. For identity 5, for example, our model predicts an average value of

$$\begin{aligned}
&\langle P_E(A) \rangle + \langle P_E(\neg A \wedge B) \rangle - \langle P_E(A \vee B) \rangle \\
&\quad = (1-2d)P(A) + d + (1-2d)P(\neg A \wedge B) + d \\
&\qquad - (1-2d)P(A \vee B) - d \\
&\quad = d
\end{aligned}$$

(with values varying randomly around that average). Since d (the rate of random error) is always positive in this model, the model predicts a positive value for this identity for all events A and B: in other words, it predicts that, if we substitute people's probability estimates for A, $\neg A \wedge B$, $A \vee B$ and so on, into this identity, the values obtained will, on average, be positive for all events. Similar predictions hold for the other identities in table 1. As we see below, the quantum probability model makes a different prediction.

The quantum probability model

The quantum probability model assumes that people's probabilistic reasoning follows the mathematical principles of quantum probability. The primary theoretical distinction between quantum and standard probability lies in the idea of 'compatible' or 'incompatible' observables. An observable defines the set of all possible distinct outcomes for a given measurement. For example, if we are checking to see whether some event A has occurred or not, we are, in the terminology of quantum theory, measuring an observable **A**, which returns one of two distinct outcomes: A (the event has occurred) and $\neg A$ (the event has not occurred).[1] Two observables **A** and **B** are compatible, in quantum theory, if the outcome of a joint observation (such as $P(A \wedge B)$) does not depend on the order in which **A** and **B** were measured. Two observables are incompatible if the outcome of such a joint observation does depend on the ordering of measurement of **A** and **B**. In other words, if two observables are compatible then the equality $P(A \wedge B) = P(B \wedge A)$ must hold, while if the observables are incompatible, equality need not hold. This distinction is fundamental both in standard quantum theory and in the quantum probability model (see Busemeyer et al., 2011, p. 199). We use this difference between $P(A \wedge B)$ and $P(B \wedge A)$ as a measure of compatibility in our experiment, with the idea that the greater the difference between $P(A \wedge B)$ and $P(B \wedge A)$, the more confident we are that the observables in question are incompatible.

If two observables are compatible, then quantum probability expressions for all possible outcomes of those observables (that is, for $P(A)$, $P(\neg A)$, $P(B)$, $P(\neg B)$, $P(A \wedge B)$, $P(A \wedge \neg B)$, $P(A \vee B)$, and so on) are exactly equivalent to the standard probability theory expressions for those outcomes. In other words, if two observables are compatible then all the probability theory identities given in Table 1 should have the value of 0, as required in standard probability theory.

If two observables are incompatible, in quantum theory those observables cannot both be measured simultaneously: instead they must be measured separately, one after the other. If two incompatible observables are measured in the order **A** then **B**, then quantum probability expressions for outcomes of the second observable can deviate from the requirements of probability theory, giving, for example

$$P(B) = P(\neg A \wedge B) + P(A \wedge B) + \delta_B \tag{4}$$

where $P(B)$ is the probability obtained when **B** is measured with no prior measurement of **A**, $P(\neg A \wedge B)$ and $P(A \wedge B)$ are the probabilities obtained when **A** and **B** are measured sequentially, and where $\delta_{\mathbf{B}}$ is a 'quantum interference' term for observable B. This quantum interference term arises because, contrary to the 'macroscopic realism' view of the world and thus to the assumptions of standard probability theory, in quantum theory if **A** is not measured, it is not necessarily in either state A or state $\neg A$: it may be in some 'superposition' of states. This means that the two probabilities $P(\neg A \wedge B)$ and $P(A \wedge B)$ do not necessarily cover all possible cases arising when estimating $P(B)$ with no prior measurement of **A**, and so $P(B) = P(\neg A \wedge B) + P(A \wedge B)$ does not necessarily hold. Note that quantum interference is not an error term here: for a given observable **B** (and a given participant, in

[1] In quantum theory, each of these outcomes A and $\neg A$ would be referred to as an *eigenvalue* of the observable, because the observable defines the set of orthonormal vectors of unit length (*eigenvectors*) in a multidimensional state space. We don't need to use this detailed view of quantum theory in our discussion here, and so we avoid this more complex terminology.

Table 1: Predicted values of the noise model and the quantum model for a series of probability theory identities. Standard probability theory requires these identities to have a value of 0. The probability theory plus noise model predicts that the average value of each these identities will be positive for all events, deviating from 0 by more than d (the chance of random error). The quantum probability model makes different predictions for three mutually-exclusive situations: when observables **A** and **B** are compatible; when observables are incompatible and measured in the order **A** then **B**, or when they are incompatible and measured in the order **B** then **A**.

label	identity	noise model	quantum model		
			compatible	incompatible **A** then **B**	incompatible **B** then **A**
1	$P(A)+P(\neg A\wedge B)-P(A\vee B)$	d	0	0	δ_A
2	$P(B)+P(A\wedge\neg B)-P(A\vee B)$	d	0	δ_B	0
3	$P(A\wedge B)+P(A\wedge\neg B)-P(A)$	d	0	0	$-\delta_A$
4	$P(A\wedge B)+P(\neg A\wedge B)-P(B)$	d	0	$-\delta_B$	0
5	$P(A\wedge B)+P(\neg A\wedge B)+P(A\wedge\neg B)-P(A\vee B)$	$2d$	0	0	0

Busemeyer et al.'s model) this quantum interference term δ_B has a fixed value that specifies the relationship between $P(B)$ and $P(\neg A\wedge B)+P(A\wedge B)$ for that observable. The quantum interference term δ_B can take on different values for different observables **B** (and different participants): in some cases positive, in some negative, and in some cases 0.

If two incompatible observables are measured in the order **A** then **B** then $P(A)$ has no such interference term, and so remain exactly equivalent to the corresponding standard probability theory expression, giving, for example

$$P(A)=P(A\wedge\neg B)+P(A\wedge B)$$

This is because measurement of **A** causes the observable to collapse out of superposition and take on either state A or state $\neg A$: the two probabilities $P(A\wedge\neg B)$ and $P(A\wedge B)$ arising from subsequent measurement of **B** do cover all possible cases arising after the outcome A, and so their sum equals $P(A)$. If incompatible observables are measured in the opposite order **B** then **A**, there is a parallel interference δ_A for observable **A**, and no interference term for observable **B**. Note that deviations from probability theory in the quantum model arise from single event probabilities ($P(A)$ or $P(B)$) because only these single-event probabilities have associated quantum interference terms: there are no such interference terms associated with combined probabilities such as $P(A\wedge B)$.

How does the quantum probability model impose an ordering on incompatible events? The model assumes that incompatible observables **A** and **B** are ordered in terms of causal links between those observables. If observables **A** and **B** are causally linked such that the occurrence of event A in some way influences the subsequent occurrence or non-occurrence of B, Busemeyer et al. (2011) assume that observable **A** is measured first and **B** is measured second. To quote Busemeyer et al. (2011, page 199)

when asked to judge the likelihood that "cigarette tax will increase and a decrease in teenage smoking will occur," it is natural to assume that the causal event "increase in cigarette tax" is processed first.

This approximately matches the approach taken in standard quantum theory, where the observable that occurs first is, naturally enough, the first measured observable. In this case the quantum probability model would allow deviations from probability for the second observable (the caused event 'a decrease in teenage smoking') but not for the first observable (the causing event 'an increase in cigarette tax'). If **A** and **B** have no causal link, Busemeyer et al. (2011) take a different approach and assume that the most probable of the two events is measured first: that is, if $P(A)>P(B)$ then the conjunctive probability $P(A\wedge B)$ is measured, but if $P(B)>P(A)$ then the conjunctive probability $P(B\wedge A)$ is measured. This represents a significant deviation from quantum theory because it connects ordering not to observables but to outcomes of observations. This also introduces other problems to the model. One problem concerns the ordering of the 'decision' measurements of $P(A)$ and $P(B)$. To decide whether the conjunction is measured as $P(A\wedge B)$ or $P(B\wedge A)$, we must first measure $P(A)$ and $P(B)$ to see which is more probable. If **A** and **B** are incompatible, we will get different results if we measure $P(A)$ first and then $P(B)$, or $P(B)$ first and then $P(A)$. We cannot decide which ordering to use by assuming the most probable event is measured first: before measurement, we don't know which event is most probable. It is not clear how this issue is resolved in the quantum probability model; given this, we don't address this 'ordering by probability' approach here.

Predictions: probability theory identities

All of the identities in Table 1 have a value of 0 in standard probability theory. In the quantum model, probability judgments necessarily agree with standard probability theory when the observables **A** and **B** are compatible; and so in this model, all of these identities should have a value of 0 for compatible observables and will only deviate from 0 for incompatible observables.

If observables are incompatible, then the values of these identities depend on the ordering of observables **A** and **B** and on whether the identity contains $P(A)$ or $P(B)$. If an identity contains a probability expression for the first measured observable, there is no interference term and the identity has a value of 0, as in standard probability theory. If an identity contains the second observable, however, there is an interference term for that observable, and the identity's value will be related to the value of that term. Consider, for example, identity 2 in Table 1. This identity contains $P(B)$. If observables are incompatible and **B** is measured first, then there is no interference term for **B** and identity 2 has a value of 0. If observables are incompatible and **A** is measured first, however, there is an interference term for **B** and identity 2 has the value

$$
\begin{aligned}
&P(B)+P(A\wedge\neg B)-P(A\vee B)\\
&=[P(\neg A\wedge B)+P(A\wedge B)+\delta_B]+P(A\wedge\neg B)-P(A\vee B)\\
&=\delta_B+P(\neg A\wedge B)+P(A\wedge B)+P(A\wedge\neg B)-P(A\vee B)\\
&=\delta_B
\end{aligned}
$$

(from Equation 4): this identity is predicted to have a value equal to the interference term δ_B.

Next consider identity 4 in Table 1. This identity again contains $P(B)$. If observables are incompatible and **B** is measured first, then there is no interference term for **B** and identity 4 has a value of 0. If observables are incompatible and **A** is measured first, however, there is an interference term for **B** and identity 4 has the value

$$
\begin{aligned}
&P(A\wedge B)+P(\neg A\wedge B)-P(B)\\
&=P(A\wedge B)+P(\neg A\wedge B)-[P(A\wedge B)+P(\neg A\wedge B)+\delta_B]\\
&=-\delta_B
\end{aligned}
$$

(from Equation 4): this identity is predicted to have a value equal to the negative of the interference term, or in other words, equal to that of identity 2 but with the opposite sign.

Parallel predictions hold for identities 1 and 3. These identities are expected to have values of 0 if **A** is measured first, while if **B** is measured first these identities have values equal to the interference term δ_A but with opposite signs. Note that, since identities 2 and 4 can only have values δ_B and $-\delta_B$ when **A** is measured first, while identities 1 and 3 can only have values δ_A and $-\delta_A$ when **B** is measured first, these two cases are mutually exclusive. This means that, if identities 2 and 4 have values significantly different from 0 for a given pair of events A and B, then the quantum probability model requires that identities 1 and 3 have a value equal to 0 (no interference term) for those events, and vice versa.

Finally, consider identity 5. This identity does not contain an expression $P(A)$ or $P(B)$, and so does not contain a quantum interference term. The quantum probability model thus predicts that, for any fixed ordering of observables **A** and **B** , this two identity will always a value of 0 irrespective of whether A and B are compatible or incompatible.

We can summarise the quantum model's predictions for the identities in Table 1 as follows. For a given pair of events A and B, there are three possible situations: First, **A** and **B** are compatible, in which case the quantum model predicts a value of 0 for all identities. Second, **A** and **B** are incompatible and measured in the order **A** then **B**, in which case the quantum model predicts a value of 0 for all identities but 2 and 4: these two identities are predicted to have opposite signs (one positive, one negative). Third, **A** and **B** are incompatible and measured in the order **B** then **A**, in which case the quantum model predicts a value of 0 for all identities but 1 and 3: these two identities are predicted to have opposite signs (one positive, one negative). The probability theory plus noise model, by contrast, predicts that every one of these identities will deviate from 0, and all will have positive values.

Experiment: incompatibility and probability theory identities

In this experiment we assess the role of incompatibility in the values of the probability theory identities in Table 1. Recall that the probability theory plus noise model predicts that values for these identities should be reliably positive for all pairs of events A and B. The quantum probability model, by contrast, predicts that for any pair of events, either all identities are 0, all are 0 but identities 2 and 4 (and these have opposite signs), or all are 0 but identities 1 and 3 (and these have opposite signs).

We tested these predictions using data from an experiment on conjunction and disjunction fallacies (Experiment 2 in Costello and Watts, 2014). This experiment gathered 68 participants' estimates for $P(A)$, $P(B)$, $P(A\wedge B)$, $P(A\vee B)$, $P(A\wedge\neg B)$ and $P(\neg A\wedge B)$ for 9 different pairs A,B of weather events (see Table 2). These pairs were selected so that they contained events of high, medium and low probabilities. Conjunctive and disjunctive weather events were formed by placing 'and'/'or' between the elements of each pair as required, generating weather events such as 'windy and cold', 'windy or cold', and so on. One group of participants ($N = 34$) were asked questions in terms of probability, of the form

- What is the probability that the weather will be W on a randomly-selected day in Ireland?

for some weather event W. The second group ($N = 34$) were asked questions in terms of frequency, of the form

- Imagine a set of 100 different days, selected at random. On how many of those 100 days do you think the weather in Ireland would be W?

where the weather events were as before. These two question forms were used because of a range of previous work showing that frequency questions can reduce fallacies in people's probability judgments; the aim was to check whether

Table 2: This table shows the degree of incompatibility of constituent events A and B in the Experiment, measured as the absolute difference between estimates for a conjunction in the ordering $A \wedge B$ and estimates in the ordering $B \wedge A$. Events are are listed in order of increasing incompatibility. This table also shows the average values for identities from Table 1, computed from individual participant's estimates for each event pair. The quantum model predicts that these identities should have a value of 0 when events A and B are compatible; when events are incompatible the quantum model predicts that most identities will have a value of zero, but some pairs of identities will have opposite signs (one positive, one negative) . The probability theory plus noise model predicts that all of these will have positive values, in all pairs of events.

A	B	$\lvert P_E(A \wedge B) - P_E(B \wedge A) \rvert$	Identity				
			1	2	3	4	5
windy	cold	0.005	0.56 (0.31)	0.25 (0.24)	0.26 (0.27)	0.29 (0.29)	0.55 (0.4)
icy	windy	0.007	0.17 (0.28)	0.21 (0.24)	0.2 (0.21)	0.16 (0.27)	0.38 (0.38)
rainy	windy	0.03	0.34 (0.32)	0.27 (0.31)	0.33 (0.27)	0.39 (0.35)	0.67 (0.47)
sunny	icy	0.03	0.11 (0.27)	0.16 (0.27)	0.27 (0.29)	0.22 (0.27)	0.39 (0.4)
cold	sunny	0.06	0.24 (0.28)	0.14 (0.3)	0.21 (0.32)	0.31 (0.29)	0.47 (0.44)
sunny	rainy	0.09	0.17 (0.28)	0.13 (0.25)	0.23 (0.24)	0.27 (0.3)	0.4 (0.38)
cold	cloudy	0.09	0.34 (0.27)	0.29 (0.27)	0.23 (0.3)	0.28 (0.28)	0.58 (0.41)
cloudy	rainy	0.09	0.13 (0.26)	0.35 (0.29)	0.43 (0.28)	0.2 (0.26)	0.57 (0.35)
cloudy	icy	0.11*	0.22 (0.3)	0.21 (0.29)	0.22 (0.27)	0.24 (0.27)	0.46 (0.46)

*Marginally significant difference between orderings in an unpaired t-test ($p = 0.06$). All other differences in ordering were unsignificant; all other differences gave evidence in favour of the null hypothesis (no difference between orderings) in a JZS Bayes Factor test for two-sample designs. All values for the identities were positive and significantly different from 0 in a one-sample t-test ($p < .0001$).

this question form could eliminate fallacy responses for everyday repeated events.

Half of participants in each group saw all conjunctions and disjunctions in the ordering $A \wedge B$, $A \vee B$, $A \wedge \neg B$ and $\neg A \wedge B$, the other half saw them in the reverse ordering $B \wedge A$ $B \vee A$, $B \wedge \neg A$ and $\neg B \wedge A$. This randomisation of the order of presentation of events in conjunctions and disjunctions was carried out purely to get a unbiased sample of probability estimates for these conjunctions and disjunctions, unaffected by any systematic influence of word order. Here, however, we use this ordering to estimate the degree of incompatibility of the twelve pairs of events used in the experiment, by considering the degree of difference between estimates given for $P(A \wedge B)$ and estimates for $P(B \wedge A)$ for each pair of events. The larger the difference between these standard and reverse-order estimates, the more incompatible the given pair of events.

Participants were given questions containing all single events and all conjunctive and disjunctive events, with questions presented in random order on a web browser. Responses were on an integer scale from 0 to 100 and were divided by 100 prior to analysis, and so probability estimates where given in units of 0.01.

Results

Two participants from the 'probability format' group were excluded (one because they gave responses of 100 to all but 4 questions and the other because they gave responses of 0 to all but 2 questions), leaving 66 participants in total. There was little difference in responses between the 'frequency format' and 'probability format' forms of question, so for simplicity we collapse the groups together in our analysis.

Each participant gave probability estimates for 42 distinct items (4 estimates $P(A \wedge B)$, $P(A \vee B)$, $P(A \wedge \neg B)$ and $P(\neg A \wedge B)$ for each of the 9 A,B pairs, and 6 estimates for the various constituents $P(A)$ and $P(B)$). As a consistency check we split the participants in into two random groups and calculated the average probability estimate in each group for each one of the 42 presented items. If participants were responding consistently we would expect there to be a reliable correlation between these average estimates across the two groups. The correlation was very high ($r = 0.97$, $p < 0.0001$), indicating consistent responses and showing that participants were not simply responding randomly in the probability estimation task.

For each participant we calculated the values of various identities from Table 1 for each of the nine pairs A,B. We also measured the degree of incompatibility between these event pairs in terms of the absolute difference between the average estimate for a conjunction presented in the order $A \wedge B$ and a conjunction in the order $B \wedge A$: the higher this difference, the more incompatible the events are.

Table 2 shows the average values obtained for the relevant identities for each of the 9 event pairs in the experiment, and also shows the degree of measured incompatibility of those event pairs, in increasing order. The degree of incompatibility was low for almost all pairs. There was no statistically significant difference between conjunctive estimates for the ordering $A \wedge B$ and ordering $B \wedge A$ for any pair: only one pair approached significance. A JZS Bayes Factor test for two-sample designs gave evidence in favour of the null hypothesis (no difference between orderings) for all but one pair. There was no relationship between incompatibility and values of the probability theory identities: identity values were reliably positive even when the difference between $P(A \wedge B)$ and ordering $P(B \wedge A)$ was less than 0.01 (that is, less than 1 point on the 100 point rating scale used in the experiment). This is contrary to the predictions of the quantum model, which would predict values of 0 for compatible pairs.

Values for all identities were reliably positive, as predicted by the probability theory plus noise model. For each identity and each event pair, we carried out a single sample t-test comparing individual values for that identity against 0. Since there are 9 event pairs, and 6 identities, this gives 45 separate t-tests. All these t-tests were significant at at least the $p < 0.001$ level, supporting the probability theory plus noise model and going against the quantum model, which would expect either all or most of these values to be 0, and would expect some to be negative.

Finally, recall that the probability theory plus noise model predicts an average value of d for identities 1 to 4, and a value of $2d$ for identity 5. The results in Table 2 support this prediction: the average value for identity 5 was 2.02 times the average value for identities 1 to 4.

Conclusions

The results seen in this experiment appear to clearly contradict the quantum probability model's predictions about compatibility and values of probability theory identities. That model predicts that probability theory identities should have a value of 0 when events are compatible. That prediction did not hold. Even assuming the events are incompatible, the quantum model predicts a value of 0 for all identities but 2 and 4 (and these two identities are predicted to have opposite signs) or for all identities but 1 and 3 (and these two identities are predicted to have opposite signs). These predictions also did not hold: instead, values for all identities were reliably positive, for all events. The results support the probability theory plus noise account, which predicts reliably positive values for all the probability theory identities in Table 1.

The fundamental idea in the probability theory plus noise account is that people's process for estimating probabilities follows the requirements of probability theory, and that the systematic biases away from probability theory seen in people's judgments are simply the consequence of random error in that process. In other work we've shown that this model can explain biases such as conservatism, subadditivity, and binary complementarity. We've also shown that, for expressions in which this model predicts bias should be cancelled, people's probability estimates agree closely with the requirements of probability theory just as predicted by the model (Costello and Watts, 2014). Here we've shown further experimental evidence that supports this model and goes against a competing formal model based on quantum probability. Taken together, our results give evidence against the popular idea that people estimate probabilities using heuristics that do not follow the normative requirements of probability theory (Ariely, 2009, Gigerenzer and Gaissmaier, 2011, Kahneman, 2011, Shafir and Leboeuf, 2002).

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