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How Does an Expert Use a Graph?
a Model of Visual and Verbal Inferencing in Economics

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Abstract

This research aims to clarify, by constructing and testing a computer simulation, the use of multiple representations in problem solving, focusing on the role of visual representations. We model the behavior of an economics expert as he teaches some economics principles while drawing a graph on a blackboard. Concurrent verbal protocols are used to guide construction of a production system. The model employs representation-specific data structures and rules. The graph on the blackboard is represented by a bit map; the pictorial working memory (WM) and long term memory (LTM) representations are node-link structures of a pictorial nature; the auditory WM and LTM representations are node-link structures of a verbal-semantic nature. Pieces from the different representations are linked together on a sequential and temporary basis to form a reasoning and inferencing chain, using cues from LTM and from the external graph. The expert used two representations so as to exploit the unique advantages of each. The graphical representation served as a place holder during reasoning, as well as a summary. The verbal-semantic representation served to give semantic meaning and causal background. Both could initiate reasoning chains. We compare the expert's behavior with novices' trying to learn the same principles.

Introduction

Experts use multiple representations easily and frequently. Indeed, it would be difficult to find a current mathematics, physics or economics textbook without graphs and diagrams, or a blackboard bare of graph or diagram after a physics or economics lecture. When asked to explain certain economics principles, our expert in economics found it extremely difficult do so without referring to visual elements along with his verbal explanation, and in fact failed to do so on three subsequent trials. However, novices have many difficulties using multiple representations. Physics teachers say they must often force students to draw and use diagrams, the students preferring just to crunch equations. In brief, experts get much mileage out of multiple representations, while such representations often handicap novices, especially when an adequate single-representation method is available, such as using equations in physics.

Experts in physics, economics and other sciences almost universally use multiple representations because different representations of a problem are seldom equivalent computationally, even though they may be equivalent informationally(Larkin & Simon, 1987; Glasgow, 1993). It is easier to explain the concept of truth verbally, and to explain the innards of a machine visually. Moreover, many tasks are complex enough to have parts that can be better explained in one than another representation. By modeling an expert's explanation of an economics problem, we illustrate how the graphical and verbal representations complement one another, exploiting the unique advantages of each.

Multiple Representations

Representations have two components: a *format* for recording and presenting information and *operators* for modifying the information; neither is sufficient, by itself, to define a representation. For example, labels, axes, and lines and their definitions are part of the format of a graph. The actions needed to find the x and y coordinates of a point in the graph are examples of operators.

Equivalence of Representations

Two representations are *informationally equivalent* if any information provided by the one representation can be converted into information in the other, and vice versa. As graphs and equations illustrate, the fact that two representations are informationally equivalent does not imply that they are equally useful or efficient, that is, *computationally equivalent*.. The effectiveness of representations for communicating and instructing depends first and foremost on how much they facilitate computation (Larkin and Simon, 1987, Palmer, 1978), and this depends on the representation, the tasks to which it is applied, and the user's familiarity with it. It is easy to find the equilibrium price and quantity on a supply-demand graph: it is the intersection of the supply and demand line. It is harder to find the equilibrium price and quantity with equations: one must know how to solve simultaneous equations.

External vs. Internal Representations

In studying the representations people use, we must distinguish between external information and information in their heads. To understand a drawing of an "A" above a "B", the external drawing must be transformed into an internal representation, the *mental picture*, in the *mind's eye* , a term now commonly used for a pictorial working memory (WM) store (Kosslyn, 1980).

We assume that information encoded in a WM by perception is represented just as it is when the same information is retrieved from long-term memory (LTM). If a sentence is read, it is stored internally in a verbal-semantic representation; if a drawing is viewed, it is stored in a pictorial representation. Similarly, if a verbal thought is called up from memory, we assume it will continue to be processed in verbal-semantic form in auditory working memory (AWM); while if a picture is called up from memory, it will be processed in the mind's eye (M'sE) in a pictorial form as a mental picture. Thus, each mode of representation has its own data structures and operators. Pictorial operators usually cannot work on verbal data structures and vice versa, just as Prolog operators do not work on LISP data. We call this set of assumptions the Mind's Eye Hypothesis.

Evidence for the Mind's Eye Hypothesis, is of three kinds: behavioral (see Kosslyn, 1980; Finke & Shepard, 1986), neuro-physiological (see Kosslyn & Koenig, 1992), and computational (Kosslyn, 1980; Baylor, 1971; Glasgow, 1993). On the basis of the substantial body of evidence cited in these overviews, we employ the Mind's Eye hypothesis; the conformation of our subject's behavior to our model subjects this hypothesis to further tests.

Novice Students' Difficulties in Using Multiple Representations

In economics, equations, tables and graphs are widely used to enhance, enrich, and illustrate verbal explanations. Two of our experiments (Tabachneck, 1992) illustrate novices' difficulties in achieving integration of verbal with pictorial representations.

Experiment 1: Pictorial and Verbal Information Are Not Integrated Easily.

Novices in economics read a natural language tutorial explaining basic principles of supply and demand. Data for subsequent problems were presented in informationally (but not computationally) equivalent forms (line graphs, algebraic equations, or tables), in a between-subjects design. The problems were the same for all groups. By tracing the subjects' thought processes with talk-aloud protocols (Ericsson & Simon, 1993), we determined that they were indeed working with the representation we had given them

Subjects who used line graphs did better than subjects who worked with equations or tables, but only on quantitative, precise-number questions, not on explanatory questions requiring them to justify their answers. Apparently, these novice subjects did not attempt to integrate the information in the verbal Tutorial (despite having constant access) with the information in the data sets on the screen. Instead, they relied on a simple, mechanical strategy to answer the quantitative questions: find, and act on, those parts of the display that have changed. In the line graphs, this amounted to reading off the number where supply and demand lines cross, and the strategy met with success. With the equations and tables, changes in the display required computation before a correct answer could be obtained, hence the strategy caused incorrect answers. Subjects did not understand their answers deeply, and thus had little success in explaining them.

Because of the lack of integration with the verbal information, inferences from the data displays, if any, were very shallow, and often wrong. Simple perception could have yielded the answers, *but only to someone who had learned to notice the relevant features of the graph, possessed the appropriate inference operators, and could translate back and forth between the operators and their economic interpretations* (see also Tabachneck & Simon, forthcoming). As we shall see later, experts in economics integrate information in several modes readily.

Experiment 2: Generation of Graphs from Verbal Information: More Non-integration

In a second experiment, we explored whether novice subjects could *generate* graphs from a verbal tutorial similar to that in Experiment 1. Four students saw relevant graphs along with the tutorial ("graph" group), and four saw only the tutorial ("text" group). First, we presented them with a task similar to that taught in the tutorial. Two of the graph subjects used graphs in their replies; the other two did not. Although the graphs drawn by the graph group were very good, the verbal reasoning was not tied well to them and was no better than that of the verbal group.

Subjects were next given the surprise task of drawing the graph for the tutored problem, with full access to the text. Of the four subjects in the text group, only one drew a reasonable graph. They had difficulties assigning variables to axes and drawing the lines within the graphs. Three of the four subjects were unable to represent the causal connection between surplus and shortage on the graph, even when asked to do so. More surprising, in the graph group, only two subjects (those who had used graphs in their previous replies) drew the graph correctly. The other two, like the verbal group, were unable to represent the causal mechanism, though the former had seen it represented explicitly several times. Translating between two representations is definitely difficult, and it is also difficult for subjects to tie elements of two representations together.

Experts' Integration of Pictorial and Verbal Representations

We asked an expert in economics to explain to us the material from the tutorials in the above experiments, as if he were teaching undergraduates, while we taped a concurrent verbal protocol. Graphical and verbal representations were tied closely together: the expert was unable to give a similar explanation without either using or referring to visual elements in three consecutive trials.

Thus, experts can and do use combinations of representations -- in fact, may be dependent on multiple representations. Three good reasons can be found in Larkin & Simon (1987) paper, which suggested three possible advantages of pictorial over verbal representations for certain tasks: (1) <u>less search</u> for information; (2) <u>easier recognition</u> of relevant information; and (3) <u>simpler inference</u> processes.

Here, we are particularly interested in the uses of pictorial representations for storage and retrieval. Larkin and her colleagues showed how familiar patterns could serve as an access route or index, both to factual knowledge, and to information about actions and strategies (Larkin, McDermott, Simon & Simon, 1980). In the model described below, we show how the graphical representation serves both as a place holder and to initiate reasoning, and finally holds, in effect, a summary of the expert's reasoning.

The Expert Model

The model is partially inspired by Jeff Shrager's model of gas laser physics, based on representations that are specific to different modalities, but easily associate with each another. (Shrager, 1990). The model also resembles ISAAC (Novak, 1977), which can solve physics problems by constructing and using drawings. Previously learned information was integrated from several separate schemas to draw a picture either on the computer screen or internally. Like our model, ISAAC's WM gradually assembled the information to solve a problem on an as-needed and temporary basis, and both systems could inference from this integrated drawing.

The reasoning employed by the model is called *immediate reasoning*, as that term is used in situated action, where the environment is an integral part of the reasoning process (Agre, 1988 Suchman, 1987). Another system having this feature is Vera, Lewis and Lerch's (1993) model that learns how to use an automated teller machine -- the various buttons and labels cueing the behavior. As in our model, the memory their system built up consists of small substructures and is heavily recognitional, using the external cues to reconstruct the actions needed to operate the machine. Neither model needs a structured plan. Vera et al.'s model provides a more detailed account of verbal processes, but a less detailed account of visual processes.

Memory Contents and Formats

We have modeled the visual part of our expert's protocol in detail. We also modeled the parts of the verbal reasoning that supply needed verbal-semantic labels for the visual structures being drawn on the blackboard (which we call EM for External Memory), and the parts that resolve *impasses*. An impasse is created when a reasoning chain ends and no external cues or WM cues are available to start another one. For each impasse, we made an LTM element corresponding to the piece of previously known information that would have caused the next observed behavior in the expert's protocol. This element then initiated a new reasoning chain, generally by drawing something on the EM. The content of these LTM knowledge elements came solely from the problem statement and the expert's statements previous to the impasse; no other information was needed. Sentences that described drawing and writing processes were not modeled (please compare the protocol excerpt in appendix 1 with the model excerpt in appendix 2).

As seen in Figure 1, the representations of the pictorial and verbal parts of the memories are different. The M'sE and pictorial LTM are pictorial-based node-link structures. The Auditory Working Memory (AWM) and verbal LTM are verbal-semantic based node-link structures. The external (visual) display (EM), which contains no semantics, is represented by a raster of pixels. The LTM structures are subdivided into knowledge that both a novice and an expert would have (e.g., graphs have axes) and knowledge only an expert would have (e.g., the effects of surpluses and shortages). The model was constructed by assigning the expert's utterances and actions to visual and verbal modes, and then deriving chains of reasoning from the protocol of the expert. Appendix 1 shows the first part of the expert's reasoning of why the equilibrium price stays where it is. Notice the interaction between the two reasoning modes, as pictured on the right hand side of Appendix 1. Notice also that the verbal-semantic parts supply causal reasoning which drives the drawing of some elements on the blackboard; the interaction of such elements with the already existing pictorial structure supply recognitional elements to further the reasoning chain.

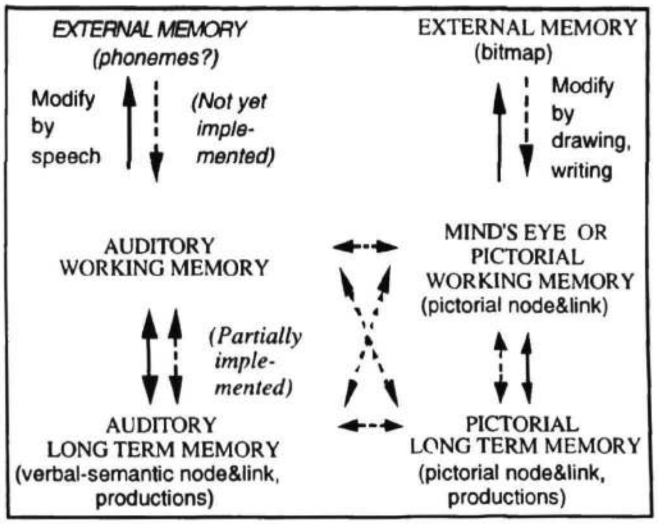

Figure 1: Model Overview.
See text for explanation.

Memory Interactions

In our model, information in the M'sE is derived from small pictorial and auditory data structures (providing lines and labels, respectively) stored in LTM. These are activated by rules stored in LTM on recognition of objects presented to the eyes and ears, or by internal associations of thoughts.

The different memories interact as follows (figure 2). Anything visible on the EM is available for inspection by the M'sE. Within a modality, reasoning employs that modality's rules and data structures. In interaction between modalities, auditory elements can activate (point to) corresponding pictorial elements and vice versa. Information moving from LTM to EM (e.g., drawing a line) must go through WM. Similarly, EM cannot directly activate LTM structures; to be recognized, they must first be represented in the WM. For example, a line in LTM must first be added to the M'sE before it can be drawn. Intersections in the drawing are then found by the M'sE (e.g., intersection at 208, 333), after which they are represented on the M'sE by rough translation into coarse axial coordinates (intersection at [20, 30]). The M'sE then seeks semantic information about the intersecting lines (demand and supply lines). Then, matching structures can be found in LTM by comparing semantics (recognize intersection of demand and supply lines as equilibrium).

Reasoning and Inference

All reasoning and inference involves collaboration between the M'sE and LTM. Each element of the expert's reasoning is linked to the next by LTM and EM cues and inference rules operating on them The dynamically created representation in the EM (the graph on the blackboard) is involved in the reasoning process in three ways: as a depository for pieces of reasoning as they become available (the expert's drawing and labeling), as a source for information missing from the M'sE, and to provide cues for reasoning processes. An example of the latter use is recognition of a shortage or surplus (the surplus reasoning is illustrated in appendix 1, the verbal protocol of the expert, and appendix 2, the model's run. See also fig. 4 and 5 below appendix 2). These constructs are cued by drawing a line parallel to the x-axis at the presumed selling price, which generates two intersections, one with the supply and one with the demand line. The segment of this line that lies between the supply and demand lines represents a surplus or a shortage. This segment then cues the appropriate verbal term, which activates verbal reasoning to continue the explanation. This is an example of immediate reasoning.

The EM is used also as a summary of the past reasoning. Because of this latter property, the system is interruptable: cues in the EM can, when recognized, also initiate a reasoning chain. For instance, as soon as supply and demand lines are drawn, their intersection cues LTM, via the M'sE, to begin to reason about equilibrium. If the expert is interrupted, he can refer back to the drawing and pick up this cue. Thus, the reasoning can be continued as long as this summary is available (which might not be possible if the graph were created only in the M'sE). On the other hand, the expert might miss the intersection cue, or think that the response to the cue had already been provided, and proceed to the next step. Larkin (1989) found the same phenomenon in research on sequential behavior in coffee making. After an interruption, people would assume that the filter and grounds had already been put in the basket if they saw the basket in place, even though they had not yet filled the basket. The external state, summarizing the act-in-progress, cued them to what they mistakenly supposed was the next step. The familiar result: hot water instead of coffee.

Predictions and Extensions

The structure of our expert model yields several predictions. It predicts that students taught with multiple representations may learn the material in short reasoning chains that are initiated by the EM. Since visual perceptions offer powerful cues, one would predict that they learn these perceptual-initiated chains first. The verbal material, on the other hand, may be learned as one long causal story. A problem then arises of integrating the visual recognitional chains with the verbal causal knowledge. We intend to gain insight into these issues by removing from the model the expert's domain-specific LTM data, recognitional material, and the productions designed specifically for handling these LTM data, thereby simulating a novice, and then comparing the novice model with novices' concurrent verbal protocols. In the present model, working memory limits are not yet implemented, but provision was made for their implementation by dividing the connections between elements in the M'sE into two types: temporary and memorized. The temporary connections are stored only in the M'sE, and can be lost when one image is replaced by another. WM can be restricted by limiting the number of temporary connections.

The auditory representation is less complete than the pictorial, primarily because the expert received few external auditory cues (only the problem statement) and did not need to do any learning. The situation will be different for novices, however, for learning will require auditory as well as pictorial input.

Conclusion

We model our expert as, after long experience, having distilled a model that exploits computationally efficient aspects of both graphical and verbal representations. The graphical representation served as a place holder, as a summary, and to initiate reasoning, while the verbal representation gave meaning and causal background. The memory of the expert is recognitional and fine-grained, and could thereby be used flexibly without an overall plan (i.e., in working-forward mode). The external pictorial representation, here a blackboard, provides cues for the next steps in reasoning, thus resulting in "immediate reasoning". These cues become available by recognition as they are constructed. This is especially clear with equilibrium, provided by the diagram in the form of line intersections, and shortage/surplus cues, represented in the diagram with line segments.

By storing the reasoning-in-progress in the external diagram, the expert can continue after an interruption. Some of the reasoning that cued off dynamic creation of elements of the representation might be skipped if these elements were created before the interruption, but the cue had not been picked up yet.

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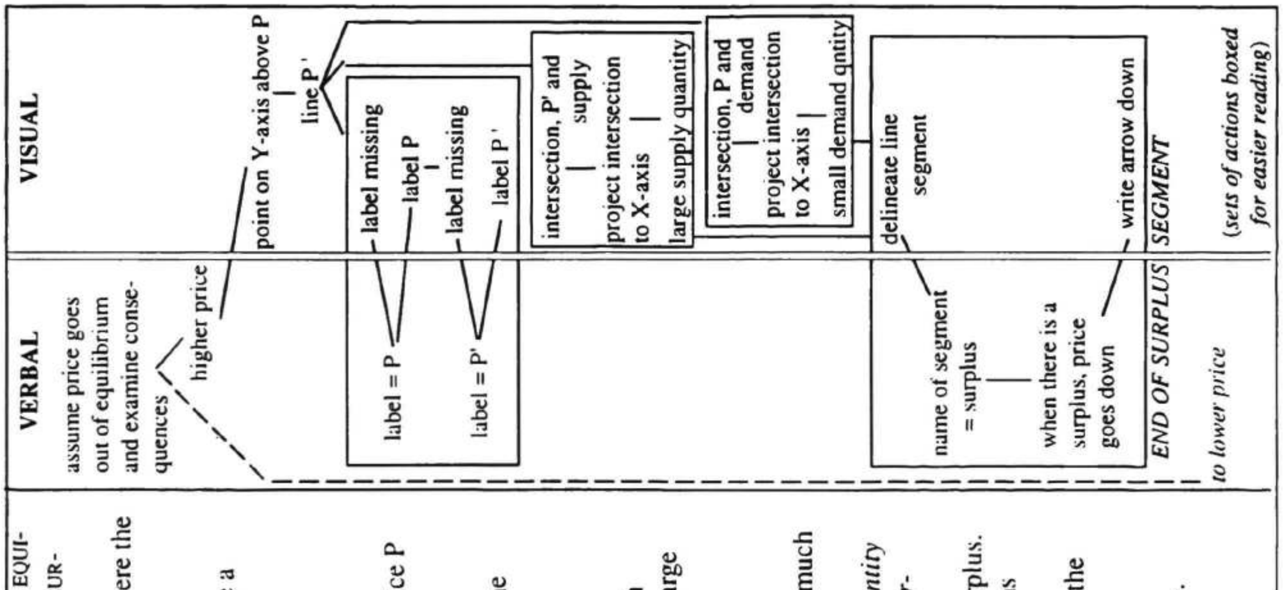

Figure 2: Visual and verbal processes, an excerpt from the expert's protocol.

Appendix 2: Run of model for the surplus reasoning section.

The numbers in front of the name of the productions refer to corresponding verbal remarks or drawings, as shown on Appendix 1. The production's name is in bold capitals. The first part of the name shows where it is situated in the model. For instance, MEYE <-> VERBAL means this is a production that communicates between the mind's eye and the verbal-semantic working memories. Capitalized words and all numbers are added by the productions to the print statements in small letters. Further comments by the authors (in italics) are interspersed with the productions.

43)* MEYE<->VERBAL!HYPOTHESIS-GENERATOR

The graph as in fig. 3 has been drawn, and the expert is explaining why the price remains at the equilibrium point.

Why is the equilibrium point the equilibrium point? Consider the relation between the quantity of the product supplied and demanded when the price of the product is larger or smaller than that of the equilbrium price.

MEYE!DRAW-LINE

This line corresponds to a higher price than the equilibrium.

drawing HORIZONTAL-LINE with slope 0 and intercept 37 into mind's eye representation.

MEYE<->VERBAL!LABEL-OBJECT

The label to this line is recognized as missing and is filled in.

labeling mind's eye node [for HORIZONTAL-LINE] with verbal label P2.

(44) MEYE<->EM!DRAW-LINE

the line is transferred from the mind's eye to the blackboard

drawing a line with with slope 0 and intercept 60 onto external memory. *(note: the P1 label has been added earlier)*

(46) MEYE<->EM!LABEL-OBJECT

labeling external memory structure [for horizontal-line] as P'.

EM!FIND-INTERSECTION

As soon as lines are drawn, the external memory drives the search for intersections, since these are perceptually salient. It notes ALL the intersections, and the following set of intersection - productions interact with the expert's recognition about which ones are interesting. Four are found, two discarded.

searching external memory for intersection along line [0 , 60]; new intersection at ((75 60) (15 60) (60 60) (90 60))

MEYE!INTERSECTION (fires 4 times)

representing pixel intersection point {[90 , 60] [60 , 60] [15 , 60] [75 , 60]} in mind's eye.

MEYE<->EM!DETERMINE-INTERSECTION-LINES (fires 3 times)

lines: NO-INTERESTING-INTERSECTIONS contain the points [15 , 60] [75 , 60]}

MEYE<->EM!DETERMINE-INTERSECTION-LINES

The model finds the lines that make up the firstinteresting intersection, and draws and labels the missing x-coordinate on the mind's eye and the blackboard.

lines: ((0 60) (-1 150)) contain the point [90, 60]

(47) MEYE<->EM!IDENTIFY-INTERSECTION

intersection between SUPPLY-LINE and horizontal-line at point: [90 , 60]

MEYE!DRAW-LINE

drawing VERTICAL-LINE with slope INFINITE and intercept 37 into mind's eye representation.

MEYE<->VERBAL!LABEL-OBJECT

labeling mind's eye node [for vertical-line] with verbal label QUANTITY3.

(48) MEYE<->EM!DRAW-LINE

drawing a line with slope INFINITE and intercept 90 onto external memory.

(49) MEYE<->EM!LABEL-OBJECT

labeling external memory structure [for vertical-line] as QUANTITY3.

EM!FIND-INTERSECTION

See explanation on previous instance; same set of productions (omitted) fires as when the intersection with the supply line was found. The model now finds the intersection with the demand line, and represents the x-coordinate on the mind's eye and the blackboard as before.

searching external memory for intersection along line [

MEYE!DRAW-LINE

drawing VERTICAL-LINE with slope INFINITE and intercept 13 into mind's eye representation.

(50) MEYE<->EM!DRAW-LINE

drawing line with slope INFINITE and intercept 60 onto external memory.

(50) MEYE<->EM!LABEL-OBJECT

labeling external memory structure [for vertical-line] as QUANTITY2.

(51 - 55) MEYE<->VERBAL!DIS-EQUILIBRIUM-RECOGNIZER

The pattern on the blackboard is now recognized by the expert; verbal-semantic memory supplies the meaning of the pattern and some causal reasoning.

if the HORIZONTAL-LINE intersects both DEMAND-LINE and SUPPLY-LINE at y = 37
and the VERTICAL-LINE intersects DEMAND-LINE at x = 13;
and the VERTICAL-LINE intersects SUPPLY-LINE at x = 37,
then,
there is a state of SURPLUS.[since the quantity of the SUPPLY-LINE (x = 37) is GREATER than the quantity of the DEMAND-LINE (x = 13)]
and when there is a surplus, the sellers want to lower the price in order to get rid of the excess of goods.

Note: Fig. 4 shows the graph. as now drawn on the "blackboard"

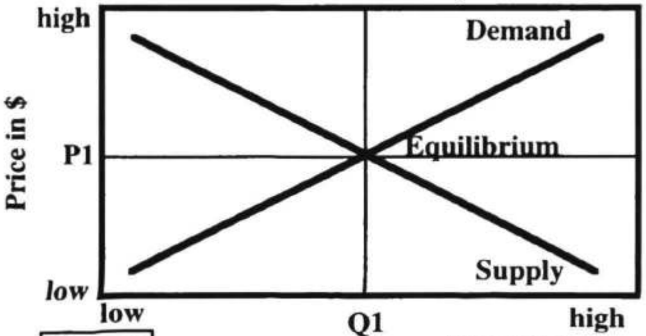

Figure 3

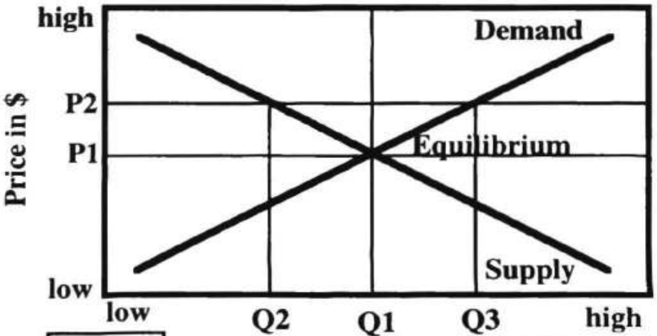

Figure 4