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PLAQUETTE FORMULATION AND THE BIANCHI IDENTITY
FOR LATTICE GAUGE THEORIES*

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ABSTRACT

We extend Halpern's field-strength formulation and dual potentials (for continuum gauge theories) to Abelian and non-Abelian lattice gauge theories. New results include 1) Plaquette formulation of all lattice gauge theories, 2) the strong coupling expansion is seen as (a) a perturbation in dual links or (b) a gradual restoration of the lattice Bianchi identity. To leading order in the strong coupling expansion the lattice Bianchi identity is completely ignored. Geometrical interpretation of the lattice Bianchi identity is presented along with a discussion of the "Abelianization" of the non-Abelian identity and its connection with gauge invariant variables. For Abelian theories we also show that the dual potential is Fourier conjugate to the Bianchi identity and that the Coulomb gas representation of these theories is easily obtained in this formulation.

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I. INTRODUCTION

Some time ago, Halpern¹ proposed two related new formulations of continuum gauge theories: A field strength formulation and a dual potential formulation. It is the purpose of this paper to extend these formulations onto the lattice and study the connections with other lattice formalisms.

Following Halpern's steps, we achieve then a plaquette variable formulation for all lattice gauge theories. For Abelian theories we also show how the dual potential and the Coulomb gas are related to the plaquette formulation.

As in the continuum, the central step in the formulation is the lattice Bianchi identity among the plaquettes. For Abelian lattice gauge theories, the Bianchi identity forms the pivot point from which springs easily the three major known tools for analyzing these theories (Coulomb gas representation, duality transformation, and strong coupling expansion). 1) The lattice Bianchi identity contains the monopole currents explicitly and independent of the action. 2) The dual links are conjugate to the Bianchi identity. 3) The strong coupling expansion is seen to be an expansion towards restoring the lattice Bianchi identity: In leading order of the strong coupling expansion the Bianchi identity is ignored. Higher order terms in the strong coupling expansion correspond to restoring the Bianchi identity in a systematic fashion. Point (3) above, is also shown for non-Abelian theories. For these theories, we also discuss "Abelianization" of the lattice Bianchi identity, i.e. putting it in the form of the product of the six gauge invariant plaquettes forming the surface of a cube.

The techniques and results presented in this paper will be useful for a gauge invariant mean-plaquette formulation of lattice gauge theories to be presented in another paper.

The paper is organized as follows, in Sections II and III we present respectively the plaquette formulation of Abelian and non-Abelian lattice gauge theories. In Section IV we discuss the Abelianization of the non-Abelian lattice Bianchi identity and its geometrical interpretation. Duality transformations and Coulomb gas representations for Abelian theories are presented in Section V. Sections VI and VII discuss the strong coupling expansion for Abelian and non-Abelian theories respectively. Conclusions and some comments are in Section VIII.

II. PLAQUETTE FORMULATION OF ABELIAN LATTICE GAUGE THEORIES

In this section we express lattice QED₄ in terms of plaquette variables. This same derivation holds for other Abelian theories with any action but for illustration we will use the Wilson action.

The partition function is

$$Z = \int \prod U_\mu(r) e^{\frac{1}{4g^2} \sum [P_{\mu\nu}(r) + P_{\mu\nu}^\dagger(r)]} \quad (\text{II.1})$$

where $U_\mu(r)$ is the U(1) link variable and $P_{\mu\nu}(r)$ is a plaquette

$$P_{\mu\nu}(r) \equiv U_\mu(r) U_\nu(r + \hat{\mu}) U_\mu^\dagger(r + \hat{\nu}) U_\nu^\dagger(r). \quad (\text{II.2})$$

In order to express the link variables, $U_\mu(r)$, in terms of the plaquette variables, $P_{\mu\nu}(r)$, we use the completely fixed "path gauge". A path gauge is defined as follows. Choose an origin $(x_0 y_0 z_0 t_0)$ and construct a path to an arbitrary point $(xyz t)$. For example, we start along the t-axis to the point $(x_0 y_0 z_0 t)$, then parallel to the y-axis to $(x_0 yz_0 t)$, then parallel to the x-axis to $(xyz_0 t)$ and finally to $(xyz t)$. All links lying on this path are gauge fixed to unity. This is done with paths to all sites on the lattice. The dotted lines in Figure 1(a) show such paths (in 3 dimensions). The above path gauge is therefore

$$U_3(txyz) = U_1(txyz_0) = U_2(tx_0yz_0) = U_0(tx_0y_0z_0) = 1 \quad (\text{II.3})$$

it is then easy to express links in terms of plaquettes

$$U_1(txyz) = \prod_{z'=z}^{z_0-1} P_{13}(txyz') \quad (II.4a)$$

$$U_2(txyz) = \prod_{z'=z}^{z_0-1} P_{23}(txyz') \prod_{x'=x}^{x_0-1} P_{21}(tx'yz_0) \quad (II.4b)$$

$$U_0(txyz) = \prod_{z'=z}^{z_0-1} P_{03}(txyz') \prod_{x'=x}^{x_0-1} P_{01}(tx'yz_0) \prod_{y'=y}^{y_0-1} P_{02}(tx_0y'z_0) \quad (II.4c)$$

These equations have a very simple geometrical interpretation in the above gauge, each link is a closed Wilson loop, formed by the path from $(x_0y_0z_0t_0)$, along the gauge lines to and through the link and back to $(x_0y_0z_0t_0)$ along the gauge lines. This is shown for $U_2(xyz)$ (in 3 dimensions) in Figure 1(a). Dotted lines denote gauge lines. Each of these Wilson loops may be filled with plaquettes as is also shown in the figure. This is the geometrical content of (II. 4a,b,c).

To change from link to plaquette variables in (II. 1), we fix the gauge by inserting

$$\delta(\text{CGF}) = \delta[U_3(txyz)-1]\delta[U_1(txyz_0)-1]\delta[U_2(tx_0yz_0)-1]\delta[U_0(tx_0y_0z_0)-1] \quad (II.5)$$

where CGF stands for complete gauge fixing. Then insert in (II.1)

$$1 = \int dP_{\mu\nu}(r) \delta[P_{\mu\nu}(r)(U_\mu(r)U_\nu(r+\mu)U_\mu^\dagger(r+\nu)U_\nu^\dagger(r))^\dagger - 1] \quad (II.6)$$

for every plaquette and do the $U_\mu(r)$ integrals. The crucial result is

$$\begin{aligned} & \int DU_\mu(r) \delta(\text{CGF}) \prod_{\mu,\nu} \delta[P_{\mu\nu}(r)(U_\mu(r)U_\nu(r+\mu)U_\mu^\dagger(r+\nu)U_\nu^\dagger(r))^\dagger - 1] \\ & = \prod_{r,\nu} \delta[e^{i\Delta_\mu \tilde{\theta}_{\mu\nu}(r)} - 1] \end{aligned} \quad (II.7)$$

where

$$P_{\mu\nu}(r) \equiv e^{i\theta_{\mu\nu}(r)} \quad (II.8a)$$

and

$$\tilde{\theta}_{\mu\nu}(r) \equiv \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \theta_{\rho\sigma}(r). \quad (II.8b)$$

The proof of (II.7) is left for an appendix. So, the lattice Bianchi identity is given by

$$e^{i\Delta_\mu \tilde{\theta}_{\mu\nu}(r)} = 1 \quad (II.9)$$

and is the product of the six plaquettes forming the six faces of a three dimensional cube.

Equation (II.1) thus becomes

$$Z = \int \prod_{\mu,\nu} d\left(\frac{\theta_{\mu\nu}(r)}{2\pi}\right) \left\{ \prod_{r,\nu} \delta[e^{i\Delta_\mu \tilde{\theta}_{\mu\nu}(r)} - 1] \right\} e^{\frac{1}{2g^2} \sum \cos \theta_{\mu\nu}(r)} \quad (II.10)$$

We have, therefore, obtained a result similar to the continuum:

The integral over plaquettes is constrained only by the lattice Bianchi identity.

The same derivation is applicable to other gauge theories like $Z(N)$ in any number of dimensions. The number of Bianchi identities will depend on the dimensionality of the theory: it is the number of different types of three dimensional cubes one can construct in the space. For example, in two dimensional theories there are no

cubes (and therefore no Bianchi identity), in three dimensional (xyz) theories, there is only one kind of cube: xyz (and thus one Bianchi identity). In general, for an N dimensional theory there are $N(N-1)(N-2)/6$ different types of three dimensional cubes and thus the same number of Bianchi identities.

III. PLAQUETTE FORMULATION OF LATTICE QCD

For illustration, we will use QCD_3 . QCD_4 has four types of cubes (and therefore Bianchi identities), each of which is handled the same way.

The partition function for lattice QCD_3 is given by

$$Z = \int \prod_{\mu} DU_{\mu} e^{\frac{\beta}{4} \sum_{\mu < \nu} \text{Tr}[P_{\mu\nu}(r) + P_{\mu\nu}^+(r)]} \quad (\text{III.1})$$

where $P_{\mu\nu}(r)$ is given by (II.2).

We go through exactly the same steps as in Section II to change the variables of integration in Z from links to plaquettes. The result is

$$Z = \int \prod_{\mu\nu} DP_{\mu\nu} [\prod_C \delta(P_C - 1)] e^{\frac{\beta}{4} \sum_{\mu < \nu} \text{Tr}[P_{\mu\nu}(r) + P_{\mu\nu}^+(r)]} \quad (\text{III.2})$$

where

$$\begin{aligned} \delta(P_C - 1) = & \delta[P_{12}(xyz)P_{23}(xyz)U_2(xyz+1)P_{13}(xy+1z)U_1(xy+1z+1) \\ & \times U_2^+(x+1yz+1)P_{23}^+(x+1yz)U_1^+(xyz+1)P_{13}^+(xyz)U_1^+(xyz+1) \\ & \times U_2(x+1yz+1)U_1^+(xy+1z+1)U_2^+(xyz+1)P_{12}^+(xyz+1)-1]. \quad (\text{III.3}) \end{aligned}$$

The links U_1 and U_2 are given in terms of the plaquettes by (II.4a-b). $\delta(P_C - 1)$ is a δ function of the non-Abelian lattice Bianchi identity associated with a given cube C. This is the lattice version of Halpern's result in the continuum.¹ $\prod_C$ denotes a product over all cubes of the lattice.

Notice that for every U_i that appears in the non-Abelian lattice Bianchi identity (III.3), there also appears its conjugate. However, because the group is non-Abelian, U_i and U_i^+ cannot be made to cancel. In the Abelian case, the U_i terms cancel out and we are left with the product of the six plaquettes forming a cube.

We can also express the partition function in terms of unconstrained plaquette variable by solving the lattice Bianchi identities as Halpern did in the continuum.¹ Using equations (II.4a-b) we get

$$P_{12}(xyz) = U_1(xyz)U_2(x+lyz)U_1^+(xy+lz)U_2^+(xyz)$$

$$= \left[\prod_{z'=z}^{z_0-1} P_{13}(xyz') \right] \left[\prod_{z'=z}^{z_0-1} P_{23}(x+lyz') \prod_{x'=x+1}^{x_0-1} P_{12}^+(x'yz_0) \right]$$

$$\left[\prod_{z'=z}^{z_0-1} P_{13}(xy+lz') \right]^+ \left[\prod_{z'=z}^{z_0-1} P_{23}(xyz') \prod_{x'=x}^{x_0-1} P_{12}^+(x'yz_0) \right]^+ \quad (\text{III.4})$$

It is trivial to check that this is the solution to the lattice Bianchi identity. Thus

$$Z = \int \prod_{13} \prod_{23} \prod_{12}(xyz_0) e^{\frac{\beta}{4} \sum_{i=1,2} \text{Tr}[P_{13}(r) + P_{13}^+(r)] + \frac{\beta}{4} \sum \text{Tr}[P_{12}(r) + P_{12}^+(r)]} \quad (\text{III.5})$$

where $P_{12}(r)$ is given by equation (III.4). This form of the partition function, and others like it (from different gauges), involve no plaquette constraints, but a complicated effective action. We mention that these remaining plaquettes can be thought of as gauge invariant plaquettes (in analogy with Mandelstam's gauge-invariant field strengths²), with their tails along the gauge lines. This is shown in Figure 1(b).

Finally we mention that if the point Z_0 is chosen at infinity. The unconstrained partition function (III.5) may be formally considered as over only $\prod_{13} \prod_{23}$.

IV. ABELIANIZATION OF THE NON-ABELIAN LATTICE BIANCHI IDENTITY

We have noticed that under the following change of variables

$$P'_{23}(x+lyz) = U_1(xyz+1)P_{23}(x+lyz)U_1^+(xyz+1) \quad (\text{IV.1a})$$

$$P'_{13}(xy+lz) = U_2(xyz+1)P_{13}(xy+lz)U_2^+(xyz+1) \quad (\text{IV.1b})$$

the lattice Bianchi identity (III.3), for a given cube becomes

$$\delta [P_{12}(xyz)P_{23}(xyz)P'_{13}(xy+lz)P_{12}^+(xyz+1)P_{23}^+(x+lyz)P'_{13}(xyz)-1] \quad (\text{IV.2})$$

which we call "Abelianization" because it has the form of the lattice Bianchi identity for an Abelian theory. This simplification apparently corresponds to Mandelstam's² "Abelianized" non-Abelian Bianchi identity

$$\nabla \cdot \underline{B}(x, P) = 0 \quad (\text{IV.3})$$

satisfied by his path dependent magnetic fields.

The important question arises, whether this Abelianization can be globally implemented, and variables changed from $P_{\mu\nu}$ to $P'_{\mu\nu}$ (as in the above example) such that $P_{\mu\nu}$ and $P'_{\mu\nu}$ are in one to one correspondence. If possible, this would result in a "totally Abelianized" or local QCD, integrated over the new variables $P'_{\mu\nu}$.

As it stands, we have failed to do this. We have however, found variable changes over large but not complete subspaces of the lattice, for example a two by two by infinite sublattice, where this can be accomplished.

After the completion of this algebraic approach, J. Kiskis³ informed us of the geometrical interpretation of identities such as (IV.2), and it is easiest to explain the more complicated variable changes that we found in that geometrical language.

The lattice Bianchi identity (IV.2) is equivalent to Figure 2 in terms of the indicated gauge invariant plaquettes. The change of variables (IV.1a-b) is precisely the change to these gauge invariant plaquettes in the $U_3 = 1$ gauge. In addition, Equations (III.3) and (IV.2) and Figure 2 are unitarily equivalent to Figure 3 and to many others one can draw. So, Abelianization is expressing the lattice Bianchi identity for a given cube as a product of the six gauge invariant plaquettes forming that cube. Furthermore, the question of the "totally Abelianized QCD" is the question whether each geometrical plaquette corresponds to just one gauge-invariant plaquette.

Now notice that if, in Figure 4, we use Figure 2 for cubes 1, 3, 5, and 7 and Figure 3 for cubes 2, 4, 6 and 8, all the gauge invariant plaquettes shared by these cubes are the same. This can be continued to infinity in the z direction. This is the geometrical interpretation of the algebraic Abelianization mentioned above for large but incomplete regions of the lattice.

It is also easy to see that this Abelianization works for the $2 \times 2 \times$ infinite lattice by reflecting the figures associated with cubes 1 to 7 in the yz plane at x (Fig. 4). To see what is meant by reflection, imagine glueing the cube in Figure 3 to the left xz face

of the cube in Figure 2. The gauge invariant plaquettes of cubes 2 and 3 are reflections of each other (except for the sense of the arrows) in the common xz plaquette. The sense of the arrows can be reversed by taking the conjugate of the Bianchi identity.

We can also show that we can Abelianize the lattice Bianchi identity for cubes 9 and 10 (Fig. 4) and so on in the y direction (and for many other paths). However, in the presence of cubes 9 and 10 we have not managed to Abelianize the identity for cube 11. We found that it will involve plaquettes from nearby cubes, and thus not have an Abelianized (local) form.

The overall pattern seems to be the following. Space can be filled with cubic (i.e. Abelianized) lattice Bianchi identities sharing the same gauge invariant plaquettes, except that cavities will develop in which the identities are not cubic - as for cube 11 in Figure 4. This seems to correspond to the types of regions over which Mandelstam's path dependent phase² can be defined by parallel transport in such a way as to satisfy (IV.3).

The fact that the lattice Bianchi identity can be Abelianized for some regions of the lattice, as described above, provides a simplification both in the strong coupling expansion described in Section VII, and in a new gauge-invariant mean-plaquette formulation of lattice gauge theories which we will describe elsewhere.

V. DUALITY TRANSFORMATION AND COULOMB GAS REPRESENTATION FOR ABELIAN THEORIES

We will show in this section how the duality transformation and the Coulomb gas representation arise from two different ways of writing the δ function of the lattice Bianchi identity. The form that gives the Coulomb gas is

$$\begin{aligned} \prod_{r,v} \delta[e^{\frac{i\Delta_\mu \tilde{\theta}_{\mu\nu}(r)}{g}} - 1] &= \sum_{\{m_\nu(\rho)=-\infty\}}^{+\infty} \prod_{r,v} 2\pi\delta[\Delta_\mu \tilde{\theta}_{\mu\nu}(r) - 2\pi m_\nu(\rho)] \\ &= \sum_{\{m_\nu(\rho)=-\infty\}}^{+\infty} [\prod \delta(\Delta_\nu m_\nu(\rho))] \int_{-\infty}^{+\infty} D\chi_\nu e^{i\sum \chi_\nu(\rho) (\Delta_\mu \tilde{\theta}_{\mu\nu}(r) + 2\pi m_\nu(\rho))} \end{aligned} \quad (V.1)$$

It is clear from (V.1) that the Coulomb gas arises from the lattice Bianchi identity independent of the action being used. It is easy to see that substituting (V.1) in the lattice QED_4 partition function with Villain action⁴ and doing the plaquette and χ_ν integrals immediately yields the well known Coulomb gas representation

$$Z = \sum_{\{m_\nu(\rho)=-\infty\}}^{+\infty} \prod_{\rho} \delta(\Delta_{-\mu} m_\mu(\rho)) e^{\frac{-2\pi^2}{g^2} \sum_{\nu} m_\nu(\rho) G(\rho-\rho') m_\nu(\rho')} \quad (V.2)$$

where

$$\Delta_\mu \Delta_{-\mu} G(\rho-\rho') = -\delta_{\rho\rho'} \quad (V.3a)$$

$$\Delta_{-\mu} f(r) \equiv f(r) - f(r - \hat{\mu}). \quad (V.3b)$$

For the duality transformation, we use the Fourier series expansion of the periodic δ function

$$\prod_{r,v} \delta[e^{\frac{i\Delta_\mu \tilde{\theta}_{\mu\nu}(r)}{g}} - 1] = \sum_{\{n_\mu(\rho)=-\infty\}}^{\infty} e^{i\sum n_\nu(\rho) \Delta_\mu \tilde{\theta}_{\mu\nu}(\rho)} \quad (V.4)$$

$n_\nu(\rho)$ and $\chi_\nu(\rho)$ are both lattice versions of Halpern's continuum dual potential $\tilde{A}_\mu$, and both lie on the link dual to the cube formed by the six plaquettes in $\exp i\Delta_\mu \tilde{\theta}_{\mu\nu}(r)$. In general, in D dimensions the dual potential has as many components as there are Bianchi identities.

Substituting (V.4) in (II.10) and doing the plaquette integrals gives the dual to lattice QED_4 with a Wilson action

$$Z = \sum_{\{n_\mu(\rho)=-\infty\}}^{+\infty} \left[\prod_{\rho} \prod_{\nu} I_{\tilde{n}_{\mu\nu}(\rho)}(g^{-2}) \right] \quad (V.5)$$

where $I_n(\beta)$ is a modified Bessel function and

$$\tilde{n}_{\mu\nu}(\rho) = \frac{1}{2\epsilon_{\mu\nu\alpha\beta}} [\Delta_\alpha n_\beta(\rho) - \Delta_\beta n_\alpha(\rho)]. \quad (V.6)$$

This method of doing the duality transformation shows that the dual potential $n_\mu(\rho)$ is the Fourier conjugate to the lattice Bianchi identity. This is the same as Halpern's result in the continuum.

VI. STRONG COUPLING EXPANSION AS A RESTORATION OF THE BIANCHI IDENTITY: ABELIAN CASE

From equation (V.5) it is clear that the strong coupling expansion is a perturbation expansion in small $n_{\mu\nu}$. This means that in such an expansion, we keep only some of the terms in the expansion of the lattice Bianchi identity (V.4).

This leads to the interpretation of the strong coupling expansion as an expansion towards restoring the lattice Bianchi identity: To leading order we ignore it completely. The higher the order of the terms we keep, the closer the partition function is to accepting contributions only from plaquette configurations that satisfy the lattice Bianchi identity.

It should be clear that the strong coupling expansion obtained in this formalism gives the same results as those obtained by, say, cluster expansions because both methods give the same dual.

To illustrate these points, we will calculate the first two terms in the strong coupling expansion of the string tension in lattice QED₄.

Consider a Wilson loop of minimal surface area A and lying in the xy plane. The loop can be expressed in terms of the plaquettes forming the minimal surface giving for the expectation value

$$\langle W[C] \rangle = Z^{-1} \sum_{\{n_{\mu\nu}(\rho)=-\infty\}}^{+\infty} \int_{-\pi}^{+\pi} D\theta_{\mu\nu} e^{i \sum_{\nu} n_{\nu}(\rho) \Delta_{\mu} \tilde{\theta}_{\mu\nu}(r)} \times \frac{1}{e^{2g^2}} \sum \cos \theta_{\mu\nu}(r) + i \sum_{A_{xy}} \theta_{xy}(r) \quad (VI.1)$$

here $\exp i \sum_{A_{xy}} \theta_{xy}(r)$ is the Wilson loop expressed in terms of its minimal plaquettes. Note that $\exp i n_{\nu}(\rho) \Delta_{\mu} \tilde{\theta}_{\mu\nu}(r)$ is the product of six plaquettes forming a cube raised to the power $n_{\nu}(\rho)$. Thus it is clear that keeping such terms with nonzero n_{ν} amounts to including contributions from nonminimal surfaces.

To obtain the first two terms in the strong coupling expansion, we ignore the Bianchi identity everywhere except at cubes in contact with the minimal surface of the loop. In other words, the expansion (V.4) of the δ function of the Bianchi identity is replaced by 1 everywhere except at the cubes mentioned above. At these cubes, we ignore most of the Bianchi identity by replacing (V.4) by the $n_{\nu} = 0, \pm 1$ terms of the expansion. With this approximation to the Bianchi identity, Eq. (VI.1) becomes

$$\langle W[C] \rangle = Z^{-1} \int_{-\pi}^{+\pi} D\theta_{\mu\nu} \prod'_{\substack{r' \\ v=t,z}} \left[1 + e^{\pm i \Delta_{\mu} \tilde{\theta}_{\mu\nu}(r')} \right] \times \frac{1}{e^{2g^2}} \sum \cos \theta_{\mu\nu}(r) + i \sum_{A_{xy}} \theta_{xy}(r) \quad (VI.2)$$

Π' is defined to mean that we take the product over all r' such that $\exp \pm i \Delta_{\mu} \tilde{\theta}_{\mu\nu}(r')$ has a face in common with the minimal surface, and the sign of the exponent is chosen so that this common face cancels. ν is restricted to be t or z because when $\nu = t$ the cube formed by the exponential is an xyz cube and when $\nu = z$ it is an xyt cube, and these are the only cubes that have a face in the xy plane.

The integrals are easy to do giving

$$\langle W[C] \rangle = \left[\frac{I_1(g^{-2})}{I_0(g^{-2})} \right]^{\frac{A}{a^2}} + \frac{4A}{a^2} \left[\frac{I_1(g^{-2})}{I_0(g^{-2})} \right]^{\frac{A}{a^2} + 4} + 0 \left[\left(\frac{I_1(g^{-2})}{I_0(g^{-2})} \right)^{\frac{A}{a^2} + 6} \right] \quad (\text{VI.3})$$

where a is the lattice spacing.

The first term in (VI.3) results from the 1 (Bianchi identity gone) in the product in (VI.2), and by itself gives the familiar leading term in the strong coupling string tension

$$\sigma = \frac{1}{a^2} \ln \left(\frac{I_0(g^{-2})}{I_1(g^{-2})} \right) \quad (\text{VI.4})$$

Therefore, we see that ignoring the Bianchi identity everywhere gives the correct leading term for the string tension. This of course amounts to ignoring all correlations among the plaquettes because, as is clear from (II.10), the lattice Bianchi identity is the only source of such correlations. Moreover, this contribution of the $n_v = 0$ term is the contribution of the minimal surface of the Wilson loop.

The second term in (VI.3) arises from the cross terms in the product in (VI.2) that contain only one exponential. This says that the first correction to the leading term in (VI.3) comes from putting all dual links (n_v) except one equal to zero. The nonzero link (whose value and sign were discussed above) is dual to a cube in contact with the minimal surface. The $4A/a^2$ factor counts the number of allowed locations for such a cube. Clearly this is the contribution of the smallest nonminimal surface.

Thus, the first step towards restoring the Bianchi identity, Eq. VI.3, gives the string tension

$$\sigma = \frac{1}{a^2} \left[\ln \left(\frac{I_0(g^{-2})}{I_1(g^{-2})} \right) - 4 \left(\frac{I_1(g^{-2})}{I_0(g^{-2})} \right) \right] \quad (\text{VI.5})$$

which agrees with cluster expansion results.

The procedure for calculating higher order corrections (i.e. contributions from more non minimal surfaces) to (VI.3) is now clear: We keep more Fourier components of the δ function of the lattice Bianchi identity. Clearly, the more Fourier components we keep, the closer we get to restoring the Bianchi identity.

We therefore reach quite a striking picture of strong coupling confinement. The strong coupling limit corresponds to totally ignoring the Bianchi identity, and is thus a state of maximal disorder among the plaquettes. The strong coupling expansion is a gradual restoration of the Bianchi identity, i.e. a gradual restoration of a certain degree of order among the plaquettes.

VII. STRONG COUPLING EXPANSION AS A RESTORATION OF THE BIANCHI IDENTITY: NON-ABELIAN CASE

In this section we will show that the strong coupling expansion for QCD₃, as for Abelian theories, is an expansion towards restoring the non-Abelian lattice Bianchi identity. The same method applies to QCD₄ and other non-Abelian theories.

Consider a rectangular Wilson loop $\frac{1}{2} \text{Tr} W$, where W is the ordered product of the links forming the boundary of the loop. We need to express $\text{Tr} W$ in terms of plaquettes, and since it is gauge invariant, we are free to choose the gauge in such a way as to make this expression as simple as possible. Following Halpern,¹ we choose the Wilson loop in the xy gauge plane at z_0 . This makes both of its sides that are parallel to the x -axis equal to unity. Furthermore, we can choose one of the y sides along the gauge line $x = x_0, z = z_0$ (the y -axis in Fig. 1). This reduces the Wilson loop to

$$\begin{aligned} \frac{1}{2} \text{Tr} W &= \frac{1}{2} \text{Tr} \prod_{y'=y_1}^{y_2-1} U_2(x_1 y' z_0) \\ &= \frac{1}{2} \text{Tr} \prod_{y'=y_1}^{y_2-1} \left[\prod_{x'=x_1}^{x_0-1} P_{12}^+(x' y' z_0) \right] \end{aligned} \quad (\text{VII.1})$$

where $(x_1 y_1 z_0)$ and $(x_1 y_2 z_0)$ are, respectively, the beginning and end of the remaining side of the Wilson loop. This is Halpern's¹ "Abelianized" Wilson loop (involving only the plaquettes of the minimal area).

The expectation value of this loop is given by

$$\left\langle \frac{\text{Tr} W}{2} \right\rangle = Z^{-1} \int \mathcal{D}P_{\mu\nu} \left[\prod_c \delta(P_c - 1) \right] \text{Tr} \frac{W}{2} e^{\frac{\beta}{4} \sum_{\mu < \nu} \text{Tr} [P_{\mu\nu}(r) + P_{\mu\nu}^+(r)]} \quad (\text{VII.2})$$

where $\text{Tr} \frac{W}{2}$ is given by (VII.1). The δ function of the Bianchi identity has the form $\delta(P_c - 1)$ where P_c is an $SU(2)$ matrix. $\delta(P_c - 1)$ is invariant under similarity transformations and may therefore be character expanded

$$\begin{aligned} \delta(P_c - 1) &= \sum_{J_c} (2J_c + 1) \chi_{J_c}(P_c) \\ J_c &= 0, \frac{1}{2}, 1, \frac{3}{2}, \dots \end{aligned} \quad (\text{VII.3})$$

The subscript c refers, as before, to the cube which is associated with the Bianchi identity we are considering. $\chi_{J_c}(P_c)$ is the trace of P_c in the $(2J_c + 1)$ dimensional representation.

J_c is the dual potential for lattice QCD. This is the non-Abelian generalization of the Abelian case where the dual potential was shown to be the Fourier conjugate to the lattice Bianchi identity.

Substituting (VII.3) in (VII.2) gives

$$\left\langle \frac{\text{Tr} W}{2} \right\rangle = Z^{-1} \int \mathcal{D}P_{\mu\nu} \left[\prod_c \sum_{J_c} (2J_c + 1) \chi_{J_c}(P_c) \right] \text{Tr} \frac{W}{2} e^{\frac{\beta}{4} \sum_{\mu < \nu} \text{Tr} [P_{\mu\nu}(r) + P_{\mu\nu}^+(r)]} \quad (\text{VII.4})$$

For $\beta \ll 1$, the dominant contribution to $\frac{1}{2} \langle \text{Tr} W \rangle$ comes from all $J_c = 0$, i.e. we ignore the Bianchi identity, which means we have plaquette disorder. This contribution is the familiar leading term⁵

$$\left\langle \frac{\text{Tr} W}{2} \right\rangle = \left[\frac{I_2(\beta)}{I_1(\beta)} \right]^{A/a^2} \quad (\text{VII.5})$$

where A is the minimal area of the loop and a the lattice spacing.

The next contribution comes from putting $J_c = 0$ for all cubes except for one cube in contact with the minimal surface of the loop. In other words, the next contribution comes from ignoring the Bianchi identity at all cubes except one which is in contact with the minimal surface. At this cube we take $J_c = \frac{1}{2}$, i.e. at this cube we still ignore most of the lattice Bianchi identity by ignoring most of the characters in its expansion. The term that we keep from this expansion has the form

$$2\text{Tr}[P_{12}(xyz_0)P_{23}(xyz_0)U_2(xyz_0+1)P_{13}(xy+1z_0)U_1(xy+1z_0+1)U_2^+(x+1yz_0+1)P_{23}^+(x+1yz_0)U_1^+(xyz_0+1)P_{13}^+(xyz_0)U_1(xyz_0+1)U_2(x+1yz_0+1)U_1^+(xy+1z_0+1)U_2^+(xyz_0+1)P_{12}^+(xyz_0+1)]. \quad (\text{VII.6})$$

Applying the change of variable (IV.1a-b) Abelianizes it into the form

$$2\text{Tr}[P_{12}(xyz_0)P_{23}(xyz_0)P_{13}'(xy+1z_0)P_{12}^+(xyz_0+1)P_{23}^+(x+1yz_0)P_{13}^+(xyz_0)]. \quad (\text{VII.7})$$

Now, by applying the Gross and Witten trick,⁶ we change the trace in (VII.7), and $\text{Tr}W$, into a product of the traces of single plaquettes. It is simplest to apply this trick to all plaquettes appearing in (VII.7) and $\text{Tr}W$ except $P_{12}(xyz_0)$ because it is the only one that appears in both. The plaquette integrals are then easy to do, and up to this order we get

$$\langle \frac{\text{Tr}W}{2} \rangle = \left[\frac{I_2(\beta)}{I_1(\beta)} \right]^{A/a^2} \left[1 + \frac{2A}{a} \left(\frac{I_2(\beta)}{I_1(\beta)} \right)^4 + 0 \left(\frac{I_2(\beta)}{I_1(\beta)} \right)^6 \right] \quad (\text{VII.8})$$

In fact it would have been just as simple, in this case, to use the Gross and Witten trick directly on the plaquettes $P_{23}(xyz_0)$, $P_{13}(xy+1z_0)$, $P_{23}^+(x+1yz_0)$, $P_{13}^+(xyz_0)$ and $P_{12}^+(xyz_0+1)$ in (VII.6) in

which case, the rest of the U_i variables cancel. However, we wanted to demonstrate the use of Abelianization because for higher order terms (such as two cubes sharing a plaquette and lying on the surface of the Wilson loop, to be discussed below), we found it much simpler to Abelianize before using the Gross and Witten trick.

The next contribution comes from $J_c = 0$ for all cubes except one cube not touching the minimal surface. For this cube $J_c = \frac{1}{2}$. The contribution after this comes from $J_c = 0$ for all cubes except two that lie on the minimal surface and at the same time share a face. For each of these two cubes $J_c = \frac{1}{2}$. The string tension up to this order is

$$\sigma = -\ln U - 2U^4 + 4U^6 - 6U^4V$$

$$U \equiv \frac{I_2(\beta)}{I_1(\beta)}, \quad V \equiv \frac{I_3(\beta)}{I_1(\beta)} \quad (\text{VII.9})$$

This result agrees with the result from cluster expansions.⁵

We therefore see that, as in the Abelian case, the strong coupling expansion is an expansion towards restoring the lattice Bianchi identity.

Finally a comment about the use of Abelianization in this calculation. Since we carried the strong coupling expansion only to low order, we only needed to consider the contributions from a few Bianchi identities. This enabled us to Abelianize them as discussed in Section

IV and thus simplify the calculation. However, going to higher orders, one will encounter situations where not all Bianchi identities can be Abelianized. Moreover, the lower the temperature, the more of these non-Abelianized Bianchi identities will contribute. So, it seems that the high temperature region is adequately described by Abelianized Bianchi identities while in the low temperature region non-Abelianized identities will play an important role.

VIII CONCLUSIONS

We have accomplished in this paper a lattice analogue of Halpern's field strength and dual potentials for gauge theories. As in the continuum, the lattice Bianchi identity is the pivot point for the structure of these theories.

One of our most interesting results is the interpretation of the strong coupling expansion as a restoration of the Bianchi identity. In a gauge theory, the Bianchi identity is the only source of plaquette correlations, and it is ignored in the strong coupling limit. This corresponds to maximal plaquette disorder. As the strong coupling expansion proceeds, the Bianchi identity, and thus plaquette order, are gradually restored.

We can, therefore, say that the confining phase in lattice gauge theories (Abelian and non-Abelian) corresponds to disordered plaquettes. This is seen very clearly in two dimensional theories. In such theories, there are no Bianchi identities and thus no correlations among the plaquettes; the plaquettes are disordered at all couplings, and therefore these theories always confine. Another way of saying what we have seen here is then that the only kind of confinement we have so far seen in models is "plaquette disorder confinement"-the common ingredient in all gauge theories in any number of dimensions.

Another very interesting result is the "Abelianization" of the non-Abelian lattice Bianchi identity and the connection of this procedure to gauge invariant plaquettes. This is interpreted as the lattice version of Mandelstam's² Abelian Bianchi identity satisfied by gauge invariant non-Abelian field-strengths.

Finally, we mention that everything discussed in this paper can be applied to spin systems. There one changes from site to link variables and the Bianchi identity is a plaquette.

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APPENDIX

Appendix: Proving Equation (II.7).

Written out in full, the left hand side of (II.7) is

$$\begin{aligned}
 & \int DU_1 DU_2 DU_0 \\
 & \delta[U_1(txyz)_0 - 1] \delta[U_2(tx_0 y z_0) - 1] \delta[U_0(tx_0 y_0 z_0) - 1] \\
 & \times \delta[P_{01}(txyz) U_1(txyz) U_0(tx+lyz) U_1^+(t+lx yz) U_0^+(txyz) - 1] \\
 & \times \delta[P_{02}(txyz) U_2(txyz) U_0(txy+lyz) U_2^+(t+lx yz) U_0^+(txyz) - 1] \\
 & \times \delta[P_{03}(txyz) U_0(txyz+lyz) U_0^+(txyz) - 1] \\
 & \times \delta[P_{12}(txyz) U_2(txyz) U_1(txy+lyz) U_2^+(tx+lyz) U_1^+(txyz) - 1] \\
 & \times \delta[P_{13}(txyz) U_1(txyz+lyz) U_1^+(txyz) - 1] \delta[P_{23}(txyz) U_2(txy+lyz) \\
 & \times U_2^+(txyz) - 1]
 \end{aligned} \tag{A.1}$$

where we have already performed the trivial integration over $U_3(r)$. The $U_1(r)$ integral can be easily done using the first and eighth δ functions. It gives Equation (II.4a) for U_1 in terms of P_{13} .

To do the U_2 integral, I change variables from U_2 to $V_{21}(txyz)$ according to

$$U_2(txyz) = \prod_{z'=z}^{z_0-1} P_{23}(txyz') V_{21}(txyz) \tag{A.2}$$

The second δ function simply becomes $\delta(V_{21}(tx_0 y z_0) - 1)$. Making this substitution (A.1) becomes

$$\begin{aligned}
 & \int DV_{21}(txyz) DU_0(txyz) \delta[V_{21}(tx_0 y z_0) - 1] \delta[U_0(tx_0 y_0 z_0) - 1] \\
 & \times \delta[P_{01}(txyz) \prod_{z'=z}^{z_0-1} P_{13}(txyz') U_0(tx+lyz) \prod_{z''=z}^{z_0-1} P_{13}^+(t+lx yz'') U_0^+(txyz) - 1] \\
 & \times \delta[P_{02}(txyz) \prod_{z'=z}^{z_0-1} P_{23}(txyz') V_{21}(txyz) U_0(txy+lyz) \prod_{z''=z}^{z_0-1} P_{23}^+(t+lx yz'') \\
 & \times V_{21}^+(t+lx yz) U_0^+(txyz) - 1] \\
 & \times \delta[P_{03}(txyz) U_0(txyz+lyz) U_0^+(txyz) - 1] \delta[V_{21}(txyz+lyz) V_{21}^+(txyz) - 1] \\
 & \times \delta[P_{12}(txyz) \prod_{z'=z}^{z_0-1} P_{23}(txyz') V_{21}(txyz) \prod_{z_1=z}^{z_0-1} P_{13}(txy+lyz_1) \\
 & \times \prod_{z''=z}^{z_0-1} P_{23}^+(tx+lyz'') V_{21}^+(tx+lyz) \prod_{z_2=z}^{z_0-1} P_{13}^+(txyz_2) - 1]
 \end{aligned} \tag{A.3}$$

Calling the product of variables in the last δ function $A(txyz)$, we can rewrite the last δ as

$$\delta(A(txyz) - 1) = \delta[A(txyz_0) - 1] \delta[A(txyz+lyz) A^+(txyz) - 1] \tag{A.4}$$

i.e. we split it into a δ function of the initial condition at $z = z_0$ and a relation between A and A^+ at two neighboring z . It is trivial to see that the R.H.S. of (A.4) is the same as the L.H.S. Working out $A(txyz_0)$ and $A(txyz+lyz) A^+(txyz)$ (A.4) becomes

$$\delta(A(txyz) - 1) = \delta[P_{12}(txyz_0) V_{21}(txyz_0) V_{21}^+(tx+lyz_0) - 1] \delta[e^{i\Delta_{\mu\bar{\theta}} \bar{\mu}_0(r)} - 1] \tag{A.5}$$

where I put $P(r) = e^{i\theta_{\mu\nu}(r)}$ and I used $V_{21}(txyz+1) = V_{21}(txyz)$ as demanded by one of the δ functions. So, replacing (A.5) in (A.3) and doing the V_{21} integral gives

$$V_{21}(txyz_0) = \prod_{x'=x}^{x_0-1} P_{21}(tx'yz_0) \quad (A.6)$$

and (A.3) becomes

$$\begin{aligned} & \int DU_0(txyz) \delta[U_0(tx_0y_0z_0)-1] \delta[P_{01}(txyz) \prod_{z'=z}^{z_0-1} P_{13}(txyz') U_0(tx+lyz) \\ & \times \prod_{z''=z}^{z_0-1} P_{13}^+(t+lxz'') U_0^+(txyz)-1] \\ & \times \delta[P_{02}(txyz) \prod_{z'=z}^{z_0-1} P_{23}(txyz') \prod_{x'=x}^{x_0-1} P_{21}(tx'yz_0) U_0(txy+lyz) \prod_{z''=z}^{z_0-1} P_{23}^+(t+lxz'') \\ & \times \prod_{x''=x}^{x_0-1} P_{21}^+(t+lx''yz_0) U_0^+(txyz)-1] \\ & \times \delta[P_{03}(txyz) U_0(txyz+1) U_0^+(txyz)-1] \times \delta[e^{i\Delta_{\mu\tilde{\theta}}\mu_0}(r)-1] \quad (A.7) \end{aligned}$$

To do the U_0 integral we again change variables to $V_0(txyz)$

$$U_0(txyz) = \prod_{z'=z}^{z_0-1} P_{03}(txyz') \prod_{x'=x}^{x_0-1} P_{01}(tx'yz_0) \prod_{y'=y}^{y_0-1} P_{02}(tx_0y'z_0) V_0(txyz) \quad (A.8)$$

(A.7) becomes

$$\begin{aligned} & \delta[e^{i\Delta_{\mu\tilde{\theta}}\mu_0}(r)-1] \int DV_0 \delta[V_0(tx_0y_0z_0)-1] \delta[P_{01}(txyz) U_1(txyz) U_0(tx+lyz) \\ & \times U_1^+(t+lxz) U_0^+(txyz)-1] \\ & \times \delta[P_{02}(txyz) U_2(txyz) U_0(txy+lyz) U_2^+(t+lxz) U_0^+(txyz)-1] \\ & \times \delta[V_0(txyz+1) V_0^+(txyz)-1] \quad (A.9) \end{aligned}$$

where we wrote variables in terms of U_1 , U_0 and U_2 because the expressions were becoming cumbersome. U_1 , U_2 and U_0 are given by (II.4a), (II.4b) and (A.8) respectively. As we did before, if we call the product of variables in the second δ function of (A.9) $A(txyz)$, we can split this δ as we did in (A.4). The result is

$$\begin{aligned} & \delta[P_{01}(txyz) U_1(txyz) U_0(tx+lyz) U_1^+(t+lxz) U_0^+(txyz)-1] \\ & = \delta[e^{i\Delta_{\mu\tilde{\theta}}\mu_2}(r)-1] \delta[V_0(tx+lyz) V_0^+(txyz)-1] \quad (A.10) \end{aligned}$$

therefore (A.9) becomes

$$\begin{aligned} & \delta[e^{i\Delta_{\mu\tilde{\theta}}\mu_0}(r)-1] \delta[e^{i\Delta_{\mu\tilde{\theta}}\mu_2}(r)-1] \int DV_0 \delta[V_0(tx_0y_0z_0)-1] \delta[V_0(txyz+1) V_0^+(txyz)-1] \\ & \times \delta[V_0^+(txyz_0) V_0(tx+lyz_0)-1] \\ & \times \delta[P_{02}(txyz) U_2(txyz) U_0(txy+lyz) U_2^+(t+lxz) U_0^+(txyz)-1] \quad (A.11) \end{aligned}$$

Again splitting the last δ at Z_0 and X_0 (in the same manner as before) we finally end up with

$$\delta[e_{\mu\mu_0}^{i\Delta\tilde{\theta}}(r) - 1] \delta[e_{\mu\mu_2}^{i\Delta\tilde{\theta}}(r) - 1] \delta[e_{\mu\mu_1}^{i\Delta\tilde{\theta}}(r) - 1] \delta[e_{\mu\mu_3}^{i\Delta\tilde{\theta}}(txyz_0) - 1]$$

$$\times \int DV_0 \delta[V_0(tx_0y_0z_0) - 1] \delta[V_0(tx_0yz_0) - 1] \delta[V_0^+(tx_0yz_0) - 1] \delta[V_0^+(tx_0yz_0) - 1] \\ \times \delta[V_0(tx_0y+1z_0) - 1] \delta[V_0^+(tx_0yz_0) - 1] \quad (A.12)$$

It is trivial to see that the solution to all the δ functions in the integrand is $V_0(tx_0yz_0) = 1$ because the second δ function tells us that $V_0(tx_0yz_0) = V_0(tx_0yz_0)$. The third δ says $V_0(tx_0yz_0) = V_0(tx_0yz_0)$ and the last that: $V_0(tx_0yz_0) = V_0(tx_0yz_0)$. But by the first δ this is 1. Therefore $V_0(tx_0yz_0) = 1$.

Thus, we have shown that (A.1) is equal to:

$$\delta[e_{\mu\mu_0}^{i\Delta\tilde{\theta}}(r) - 1] \delta[e_{\mu\mu_1}^{i\Delta\tilde{\theta}}(r) - 1] \delta[e_{\mu\mu_2}^{i\Delta\tilde{\theta}}(r) - 1] \delta[e_{\mu\mu_3}^{i\Delta\tilde{\theta}}(tx_0yz_0) - 1] \quad (A.13)$$

These are the "3.1" Bianchi identities which can be easily shown to imply the full 4 Bianchi identities.

So, finally, the Jacobian of the variable change from links to plaquettes is

$$\prod_{v,r} \delta[e_{\mu\mu\nu}^{i\Delta\tilde{\theta}}(r) - 1] \quad (A.14)$$

REFERENCES

1. M. B. Halpern, Phys. Rev. D19, 517 (1979); Phys. Lett. 81B, 245 (1979).
2. S. Mandelstam, Phys. Rev. 175 (1968) 1580.
3. J. Kiskis, private communication.
4. J. Villain, J. Phys. (Paris) 36, 581 (1975).
5. G. Münster, Nucl. Phys. B180[FS2], (1981) 23.
6. D. Gross and E. Witten, Phys. Rev. D21, (1980) 446.

FIGURE CAPTIONS

- Figure 1-a. $U_2(xyz)$ as a Wilson loop filled with plaquettes. The dotted lines are gauge lines.
- Figure 1-b. A gauge invariant plaquette. The tail runs along gauge (dotted) lines and is therefore invisible.
- Figure 2. The gauge invariant plaquettes whose product, in the order abcdef, gives the lattice Bianchi identity for the cube.
- Figure 3. The product of these gauge invariant plaquettes, in the order afedcb, gives a lattice Bianchi identity which is unitarily equivalent to that of Fig. 2.
- Figure 4. The lattice Bianchi identity can be Abelianized for cubes 1 through 10 but not 11.

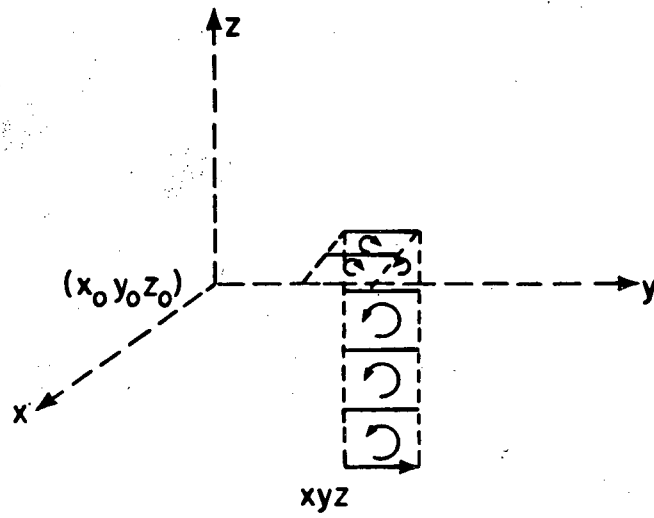
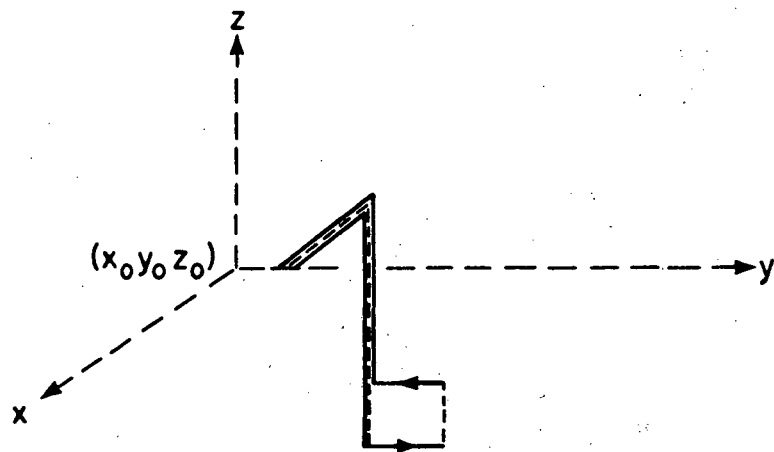
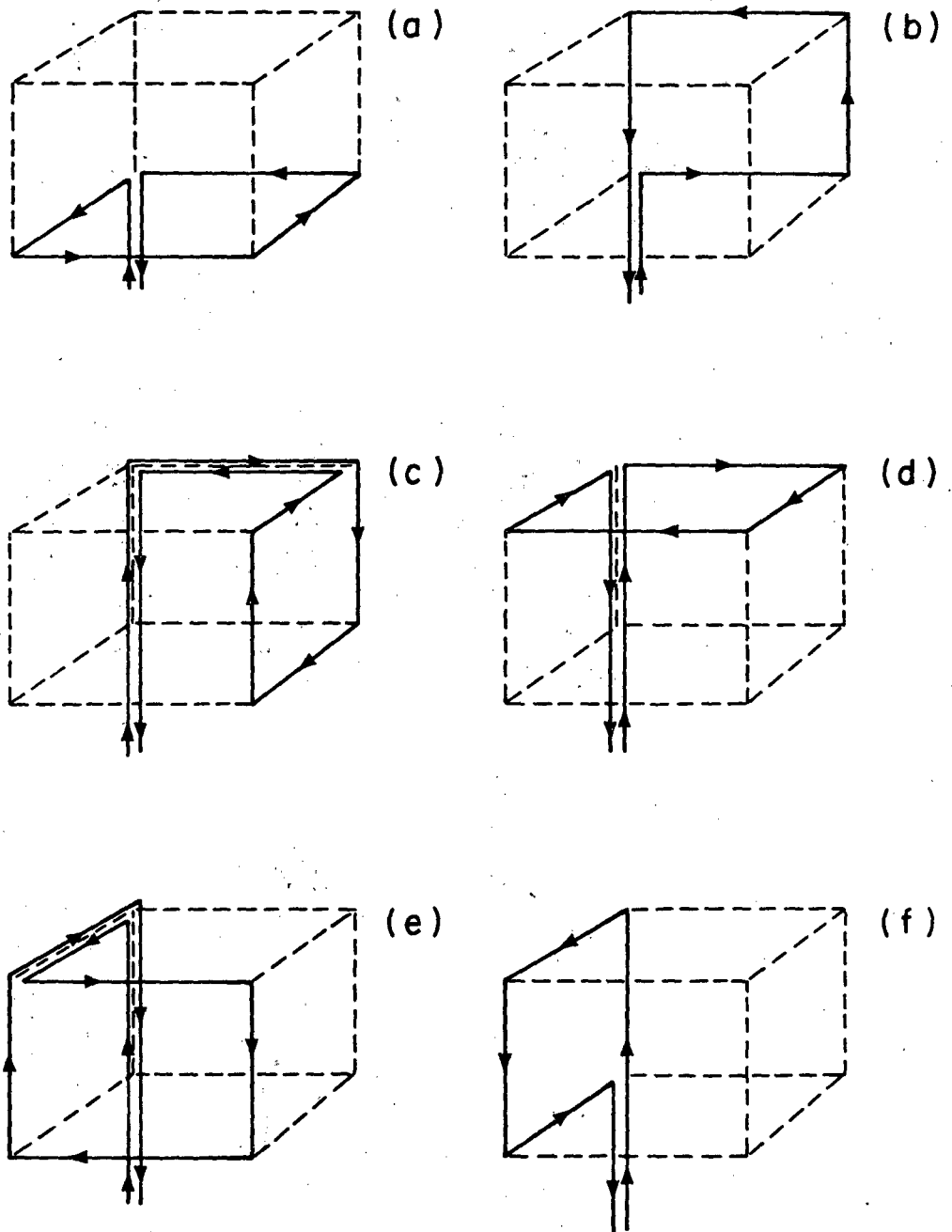


FIGURE 1-A



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FIGURE 1-B



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FIGURE 2

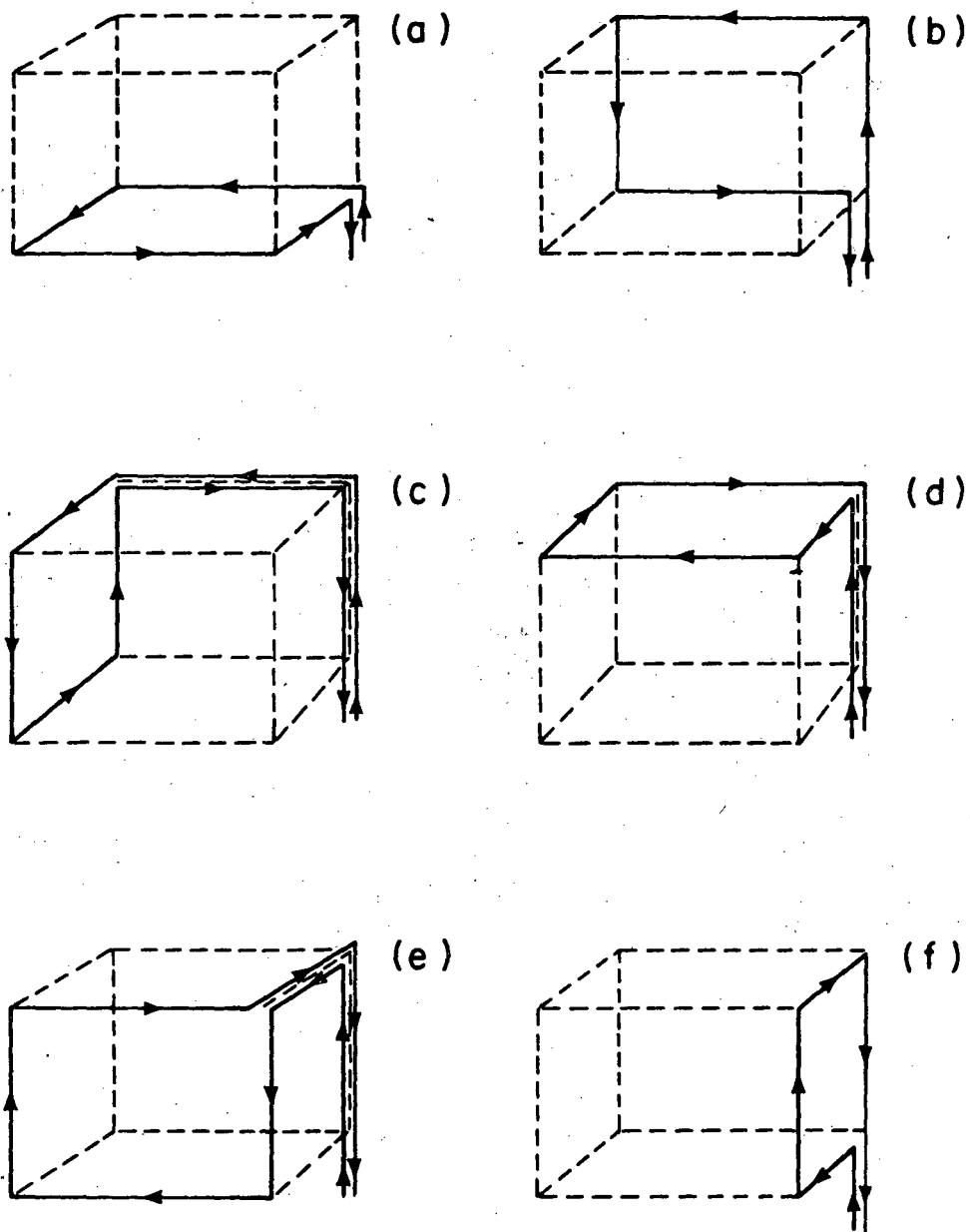


FIGURE 3

XBL 822-4474

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