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# Stochastic Volatility and Options Valuation: A Pricing-Density Approach

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## ABSTRACT

We develop a simple closed-form valuation model for options when the volatility of the underlying asset is stochastic. Our approach differs from previous research in that we model the pricing density directly. We show that implied volatility estimates from the Black-Scholes model can be very misleading, even when at-the-money options are used in the estimation. We also illustrate that the smile effect in index option prices can be explained by allowing changes in volatility to be correlated with index returns.

## 1. INTRODUCTION

One of the most-important issues in option pricing theory is the question of how options should be valued when the volatility of the underlying asset is stochastic. This issue is relevant to both practitioners and academic researchers given the extensive evidence that volatility varies significantly over time in many financial markets.

In this paper, we develop a simple framework for valuing options when volatility is stochastic. This approach differs from previous research by Wiggins (1987), Hull and White (1987), Scott (1987), Johnson and Shanno (1987), Stein and Stein (1991), Amin and Ng (1993), Heston (1993), Duan (1994) and others in that we model the pricing density directly. The advantage of this approach is that it allows us to derive simple closed-form solutions for option prices under a variety of assumptions about the term structure of volatility. In fact, for some parameter values, the stochastic volatility option pricing model is easier to evaluate than the Black-Scholes option pricing formula. This approach complements recent work by Shimko (1993), Longstaff (1994), Rubinstein (1994), Ait-Sahalia (1994), and Hutchinson, Lo, and Poggio (1994) in which the pricing density is also modelled directly.

The results provide a number of important insights into the pricing of options when volatility is stochastic. For example, we show that implied volatility estimates obtained by inverting the Black-Scholes model can be very misleading when volatility is stochastic—even for at-the-money options. In addition, we show that the majority of the 'smile' effect for S & P 500 index options can be explained by allowing volatility to be correlated with index returns.

The paper is organized as follows. Section 2 presents the basic stochastic volatility valuation model. Section 3 examines the implications of the model for option prices. Section 4 contrasts the stochastic volatility model with the Black-Scholes model. Section 5 shows how alternative volatility specifications can be incorporated into the model. Section 6 summarizes the results and presents concluding remarks.

## 2. THE VALUATION MODEL

In this section, we develop the basic valuation model for pricing options when the

volatility of the underlying asset is stochastic. To make the intuition of the model clear, we use the simplest possible framework in deriving the valuation model. The extension of the model to more general specifications is illustrated in Section 5.

When state variables are hedgable, Cox and Ross (1976) and Harrison and Kreps (1979) show that prices of derivative claims are given by taking the expectation of their payoff with respect to a specific density, and then discounting the expectation at the riskless rate. This is often known as risk-neutral valuation. When there are non-hedgable state variables such as volatility, derivative claims must be priced in an equilibrium rather than no-arbitrage framework. As shown by Cox, Ingersoll, and Ross (1985) and others, however, the equilibrium valuation operator has the same form as the risk-neutral valuation operator. The only difference is that the pricing density may incorporate preference-dependent market price of risk terms. In either case, however, we only need to know the density function to be able to value derivative securities.

Because the density function is ultimately the primitive in any valuation framework, we adopt the more intuitive approach of specifying the pricing density directly. In doing this, of course, we are implicitly assuming that some underlying no-arbitrage or general equilibrium framework could be specified which would be consistent with the pricing density. Specifically, let $S$ denote the current or time-zero value of a traded asset, and let $S_T$ denote its value at time $T$. Furthermore, let $F(S_T)$ be a function of $S_T$ with the property that the expected value of $F^2(S_T)$ is finite and let $r$ denote the constant riskless interest rate.

**Assumption 1.** *We assume the existence of a risk-neutral pricing density such that the present value $PV$ of any claim with payoff function $F(S_T)$ at time $T$ can be represented as the discounted expected value or certainty equivalent,*

$$PV = e^{-rT} E[\, F(S_T)\,]. \tag{1}$$

Here we use the term risk-neutral density in a broad sense since the pricing density may incorporate a market price of volatility risk. All probabilities and expectations are taken with respect to this risk-neutral density.

An immediate implication of this assumption is that the risk-neutral density

must satisfy the no-arbitrage condition given by applying the pricing operator to the underlying asset,

$$S = e^{-rT} E[\ S_T\ ]. \tag{2}$$

This no-arbitrage condition will play a central role in this valuation framework since it places an identification restriction on the mean of the risk-neutral pricing density.

Let $I$ denote the current instantaneous variance of returns on the underlying asset. Let $V_T$ denote the total variance of the return on the underlying asset over the valuation horizon, where

$$V_T = \int_0^T I_t\ dt, \tag{3}$$

is a random variable. In the option pricing models developed by Merton (1973), Hull and White (1987), Scott (1987), and others, the risk-neutral density depends on the instantaneous volatility only through the average variance $A_T = V_T/T$. This is intuitive since it is the average amount of uncertainty during the life of an option which should affect its value. This intuition underlies the next assumption about the conditional distribution of returns.

**Assumption 2.** *Conditional on the realization of* $V_T$, $\ln S_T$ *is normally distributed with mean and variance*

$$E[\ \ln S_T \mid V_T\ ] = \mu_T + \gamma V_T, \tag{4}$$

$$Var[\ \ln S_T \mid V_T\ ] = V_T. \tag{5}$$

For the present, we leave $\mu_T$ unspecified; the value of $\mu_T$ is determined later by requiring the no-arbitrage condition (2) to be satisfied. The parameter $\gamma$ allows for the possibility that realizations of $\ln S_T$ and $V_T$ are correlated. Positive values of $\gamma$ imply positive correlations between $\ln S_T$ and $V_T$, and vice versa. As an illustration, recall that in the Black-Scholes model, $I$ is constant, $V_T = \sigma^2 T$, and $\ln S_T$ is normally distributed with mean $\ln S + rT - \sigma^2 T/2$ and variance $\sigma^2 T$. Thus, the Black-Scholes model satisfies Assumption 2 with $\mu_T = \ln S + rT$ and $\gamma = -1/2$.

**Assumption 3.** *The total variance $V_T$ is gamma distributed with density function*

$$\frac{1}{\Gamma(\alpha^2/\beta^2)V_T}\ \exp\left(\frac{-\alpha V_T}{\beta^2}\right)\left(\frac{\alpha V_T}{\beta^2}\right)^{\alpha^2/\beta^2}, \tag{6}$$

*where $\alpha$ and $\beta^2$ are the mean and variance of $V_T$ respectively.*

This assumption implies that $V_T \geq 0$. Note that a gamma variate is simply the product of a standard chi-square variate and a positive constant. Thus, this assumption is consistent with the implications of many standard econometric models in which the chi-square distribution arises naturally as the distribution of sums of squared normals. In addition, ARCH and GARCH models as well as many continuous-time specifications imply distributions for $V_T$ that converge to, or are closely approximated by this distribution.

To close the model, we need to specify the conditional mean and variance of $V_T$. The following specification is perhaps the simplest possible. More general specifications are considered later.

**Assumption 4.** *The mean $\alpha$ and variance $\beta^2$ of $V_T$ are*

$$\alpha = IT, \tag{7}$$

$$\beta^2 = \eta^2 I^2 T^2. \tag{8}$$

Recall that in the Black-Scholes model, $\alpha = I$ and $\beta^2 = 0$. Thus, the Black-Scholes model can be nested within this specification by imposing the restriction $\eta^2 = 0$.

The joint density of $\ln S_T$ and $V_T$ is given by multiplying the conditional density of $\ln S_T$ by the marginal density for $V_T$,

$$\frac{1}{\sqrt{2\pi V_T^3}\ \Gamma(\alpha^2/\beta^2)}\ \exp\left(-\frac{(\ln S_T - \mu_T - \gamma V_T)^2 + 2\alpha V_T^2/\beta^2}{2V_T}\right)\left(\frac{\alpha V_T}{\beta^2}\right)^{\alpha^2/\beta^2}. \tag{9}$$

To value continent claims using (1), we need to solve for the marginal density of $\ln S_T$. This marginal density is obtained by integrating out the value of $V_T$ in (9). Let $z = \ln S_T - \mu$. From Gradshteyn and Ryzhik (1970) 3.471.9,

$$P(z;a,b,c) = \frac{|\,1-c^2\,|^{a+1/2}\;|\,z\,|^a}{\sqrt{\pi}\;2^a\;b^{a+1}\;\Gamma(a+1/2)}\;\exp(-cz/b)\;K_a(|\,z/b\,|), \tag{10}$$

where $P(z;a,b,c)$ is the density of $z$ and

$$a = \frac{2\alpha^2-\beta^2}{2\beta^2}$$

$$b = \frac{1}{\sqrt{\gamma^2+2\alpha/\beta^2}},$$

$$c = \frac{-\gamma}{\sqrt{\gamma^2+2\alpha/\beta^2}},$$

and where $K_a(\,\cdot\,)$ is the modified Bessel function of order $a$.[1]

This expression is the density function for the well-known Bessel distribution and is described in detail in Chapter 12 of Johnson and Kotz (1970). Intuitively, this distribution can be obtained as the distribution of the difference between two independent chi-square variates. Examples of densities implied by this family of distributions are graphed in Fig. 1. As shown, virtually any pattern of skewness and kurtosis for the risk-neutral density is possible by varying the parameters.

To identify $\mu_T$, we require that the model be arbitrage free in the sense of satisfying the no-arbitrage condition in (2). Recalling the definition of $z$, the no-arbitrage condition can be expressed as

$$S = e^{-rT}e^{\mu_T}E[\;e^z\;]. \tag{11}$$

Using the moment generating function of the Bessel density given in Johnson and Kotz (1970) results in

[1]The properties of this Bessel function are described in Chapter 9 of Abramowitz and Stegun (1970).

$$S = e^{-rT} e^{\mu_T} \left( \frac{1 - c^2}{1 - (c - b)^2} \right)^{a+1/2}, \tag{12}$$

provided that $(c - b)^2 < 1$. Inverting this expression and solving for $\mu_T$ gives,

$$\mu_T = \ln S + rT - (a + 1/2) \ln \left( \frac{1 - c^2}{1 - (c - b)^2} \right) \tag{13}$$

Thus, the no-arbitrage condition determines the $\mu_T$ parameter. This means that the risk-neutral pricing operator is fully specified by $\alpha$, $\beta$, and $\gamma$, or equivalently, by $a$, $b$, and $c$. In order to introduce alternative specifications for the mean and variance later, we will derive the option pricing expressions in terms of $a$, $b$, and $c$ rather than explicitly substituting in the specification given in Assumption 4.

## 3. OPTION VALUATION

Let $C(S, X, T)$ denote the value of a European call option on the underlying asset with strike price $X$ and time to expiration of $T$. From (1), the value of this call option can be expressed as

$$C(S, X, T) = e^{-rT} E[\ \max(0, S_T - X)\ ], \tag{14}$$

where the expectation is taken with respect to the risk-neutral density given in (10). Evaluating this expectation leads to the following closed-form expression for the value of a call option

$$C(S, X, T) = S\ B(q; a, b, c - b) - X e^{-rT}\ B(q; a, b, c). \tag{15}$$

where

$$q = \ln \left( \frac{X e^{-rT}}{S} \left( \frac{1 - c^2}{1 - (c - b)^2} \right)^{a+1/2} \right),$$

and where $B(q; a, b, c)$ is the complementary Bessel distribution function

$$B(q;a,b,c) = \int_q^\infty P(z;a,b,c)dz.$$

Structurally, this valuation expression is similar to the Black-Scholes model. In fact, the only substantive difference is that the Bessel distribution function appears in (15) rather than the normal distribution. In this model, the value of a call option is a function of $S$, $r$, $X$, $T$ and the parameters $\alpha$, $\beta$, and $\gamma$. In turn, given the specification in Assumption 4, $\alpha$ and $\beta$ can be expressed in terms of $I$, $T$, and the parameter $\eta$. Thus, option prices depend on $S$, $X$, $I$, $T$, $r$, and the two parameters $\gamma$ and $\eta$.

To illustrate the option pricing function, Fig. 2 graphs the value of a call for different values of the correlation parameter $\gamma$. Recall that in the Black-Scholes model, $\gamma = -1/2$ and the time value of an option is essentially symmetric around the strike price. Fig. 2 shows that when $\gamma$ is very different from zero, the time value of the option can be far from symmetric. Intuitively, this is because the correlation parameter $\gamma$ introduces an asymmetry into the shape of the Bessel density as shown in Fig. 1.

To contrast this model with the Black-Scholes model, we first compute option prices from (15) using the same values of $S$ and $I$ for all options, but for varying values of $X$. We then solve for the 'implied volatility' of each option by inverting the Black-Scholes formula and graph them in Fig. 3. As shown, the resulting implied volatility patterns can be very similar to the smile pattern typically observed in the market. In particular, negative values of $\gamma$ result in the generally downward pattern that has persisted in many index options markets during recent years.

Fig. 3 also illustrates that the implied value of $I$ obtained by inverting the Black-Scholes model is generally very different from the actual value of $I$. In particular, the instantaneous standard deviation using in Fig. 3 is .10. As shown, however, the implied volatility estimates obtained from the Black-Scholes model are quite different from .10 even when the calls are at the money. This result is important since it is frequently claimed in the literature that estimates of $I$ implied from at-the-money options using the Black-Scholes model are unbiased even if volatility is actually stochastic.

The analytical expression for $C(S, X, T)$ can be differentiated to obtain hedge ratios and comparative statics. For example, the derivative of the call price with respect to $S$ gives the delta of the option

$$C_S = B(q; a, b, c - b), \tag{16}$$

which is positive and always between zero and one. Fig. 4 illustrates that the delta of a call option can differ significantly from the delta implied by the Black-Scholes model.

A number of other derivatives can be obtained in closed form. In particular,

$$C_{SS} = \frac{P(q; a, b, c - b)}{S} > 0, \tag{17}$$

$$C_X = -e^{-rT} B(q; a, b, c) < 0, \tag{18}$$

$$C_\tau = X e^{-rT} T \; B(q; a, b, c) > 0. \tag{19}$$

An increase in the current value of $I$ increases both the mean and the variance of the total volatility $V_T$. An analysis of the pricing expression in (15) shows that the call price is an increasing function of $I$. This is intuitive, since an increase in $I$ implies that the average volatility over the life of the option will be higher. Furthermore, as $I$ increases, the increase in the variability of the average variance contributes to the total amount of uncertainty about the option payoff, which tends to increase the value of the option. The relation between $I$ and the value of a call option is illustrated in Fig. 5.

An increase in $T$ has a similar effect on the value of an option as it does in the Black-Scholes model. As the time until expiration $T$ increases, the flexibility provided by the option is more valuable (assuming that there are no dividends or similar stock price changes). Thus, an increase in $T$ increases the value of a call option. This is illustrated in Fig. 6 which graphs the value of a call option as a function of $T$.

The pricing expression for the value of a European put option can be obtained from (15) using the standard put-call parity relation. The basic comparative statics for puts parallel those of call options described above.

Although computing the Bessel distribution function is straightforward, the Bessel distribution function can be expressed entirely in terms of elementary functions when the ratio $\alpha^2/\beta^2 = 1/\eta^2$ is an integer. In this situation, $a = N + 1/2$, where $N$ is also an integer. From Gradshteyn and Ryzhik (1970) 8.468,

$$K_{N+1/2}(\mid z/b \mid) = \sqrt{\frac{\pi b}{2 \mid z \mid}}\ e^{-\mid z/b \mid} \sum_{i=0}^{N} \frac{(N+i)!\ b^i}{i!\ (N-i)!\ 2^i \mid z \mid^i}. \tag{20}$$

Substituting this expression into (10) and integrating allows the Bessel distribution function to be expressed entirely in terms of simple algebraic and exponential terms.

As an example, consider the empirically relevant case where the mean and standard deviation of $V_T$ are equal. In this case, the Bessel distribution function has the simple form

$$B(q; 1/2, b, c) = \frac{(1-c)}{2}\ e^{-(1+c)q/b}, \tag{21}$$

for $q \geq 0$. When $q < 0$, the distribution function can be evaluated using the identity

$$B(q; a, b, c) = 1 - B(\ -q\ ; a, b, -c), \tag{22}$$

which is true for general $a$, $b$, and $c$. Similarly, in the case where $\alpha^2/\beta^2 = 2$, the Bessel distribution function has the form

$$B(q; 3/2, b, c) = \frac{(1-c)^2}{4}\left(2 + c + \frac{(1+c)q}{b}\right)\ e^{-(1+c)q/b}. \tag{23}$$

for $q \geq 0$. Note that both (21) and (23) are much easier to evaluate than the cumulative normal distribution function that appears in the Black-Scholes model. In the general case where $\alpha^2/\beta^2 = N + 1$, where $N$ is a positive integer, the Bessel distribution function is given by the sum

$$B(q; N+1/2, b, c) =$$

$$\sum_{i=0}^{N} \sum_{j=0}^{N-i} \frac{(1-c^2)^{N+1}\,(N+i)!}{2^{N+1+i}\,b^j\,N!\,i!\,j!\,(1+c)^{N+1-i-j}}\; q^j\; e^{-(1+c)q/b}. \qquad (24)$$

for $q \geq 0$.

## 4. IMPLIED VOLATILITY ESTIMATES

To illustrate further how the stochastic volatility option pricing model differs from the Black-Scholes model, we use both models to imply estimates of the instantaneous volatility $I$ from a common set of S & P 500 index option prices. We then contrast the estimates from the Black-Scholes model with those from the stochastic volatility model.

The S & P 500 index option prices are obtained by taking the closing call and put prices listed in the November 22, 1994 *Wall Street Journal* for the next two expiration months of December and January. This procedure resulted in a set of 8 call prices for both the December and January expirations, and 11 put prices for both the December and January expirations. Since S & P 500 index options are European options, the Black-Scholes and the stochastic volatility models are directly applicable. The closing stock index value and the value of the riskless rate are also obtained from the *Wall Street Journal*. We assume that the dividend rate on the index is .03 per annum. In inverting the stochastic volatility model, we also assume that $\gamma = -20$ and that $\eta = 1$. Although realistic, these parameter values are illustrative only. By specifying both $\gamma$ and $\eta$, however, the stochastic volatility model can now be inverted to obtain an individual estimate of $I$ from each option. The implied volatility estimates for the December and January calls are shown in Fig. 7 and Fig. 8. The implied volatility estimates for the December and January puts are shown in Fig. 9 and Fig. 10.

If the Black-Scholes model were correctly specified, then the estimates of $I$ would be similar across strike prices, expiration dates, and type of option. As shown, however, the implied volatility estimates from the Black-Scholes model vary dramatically across strike prices (illustrating the smile effect), expiration dates, and

whether the options are calls or puts. In contrast, the estimates of $I$ implied from the stochastic volatility model are much more stable. In particular, the estimates of $I$ are generally within 100 basis points across all strike prices, expiration dates, and option type.

This simple illustration suggests that the stochastic volatility option pricing model can potentially explain the persistent smile effect characterizing many option markets. Furthermore, this illustration suggests that the term structure of volatility obtained by implying estimates of $I$ from options with different maturities may be largely an artifact of the constant volatility assumption underlying the Black-Scholes model.

## 5. ALTERNATIVE VOLATILITY SPECIFICATIONS

In assumption 4, we specified the functional form of the mean $\alpha$ and the variance $\beta^2$ of the total variance $V_T$. This specification, however, is only one of many possible. To illustrate how alternative specifications can be incorporated within this option pricing framework, we provide several other specifications of $\alpha$ and $\beta^2$ in this section and show how closed-form expressions can be obtained.

One strategy for specifying the mean and variance parameters is to assume that the instantaneous volatility $I$ follows a continuous-time process and then set $\alpha$ and $\beta^2$ equal to the mean and variance of the corresponding value of $V_T$.

**Example 1.** Assume that the instantaneous variance $I$ follows the following geometric Brownian motion process

$$dI = \sigma dZ, \tag{25}$$

where $Z$ is a standard Wiener process. These dynamics imply the following mean and variance for $V_T$

$$\alpha = IT, \tag{26}$$

$$\beta^2 = I^2\left(\frac{2}{\eta^4}\left(e^{\eta^2 T} - 1\right) - T^2 - \frac{2T}{\eta^2}\right). \tag{27}$$

This specification is very similar to that given in Assumption 4.

**Example 2.** Assume that the instantaneous variance $I$ follows the square root process

$$dI = (\theta - \kappa I)dt + \sigma\sqrt{I}dZ. \tag{28}$$

These dynamics imply the following mean and variance for $V_T$

$$\alpha = \frac{\theta}{\kappa}T + \left(\frac{I}{\kappa} - \frac{\theta}{\kappa^2}\right)(1 - e^{-\kappa T}), \tag{29}$$

$$\begin{aligned}\beta^2 = {} & \frac{\sigma^2}{\kappa^2}\left(\frac{\theta}{\kappa} - 2\left(I - \frac{\theta}{\kappa}\right)e^{-\kappa T}\right)T \\ & - \frac{2\theta\sigma^2}{\kappa^4}(1 - e^{-\kappa T}) \\ & + \left(\frac{\theta\sigma^2}{2\kappa^4} + \frac{\sigma^2}{\kappa^3}\left(I - \frac{\theta}{\kappa}\right)\right)(1 - e^{-2\kappa T}).\end{aligned} \tag{30}$$

Similarly, many other possible specifications can be obtained by assuming that $I$ follows an ARCH or GARCH type of process, integrating $I$ to obtain $V_T$, and then taking expectations and setting the first two moments equal to $\alpha$ and $\beta^2$.

Since the option pricing formula is expressed in terms of the general parameters $\alpha$, $\beta$, and $\gamma$ (or $a$, $b$, and $c$), these alternative specifications can simply be substituted into (15) provide alternative closed-form solutions. Thus, this framework allow us significant flexibility in deriving closed-form stochastic volatility option pricing models under a variety of assumptions about the properties of the average volatility.

## 6. CONCLUSION

We have presented a risk-neutral-density-based framework for valuing options when the volatility of the underlying asset is stochastic. An important advantage of this approach is that it allows us to derive simple closed-form stochastic volatility option pricing expressions and to study their implications for the behavior of option prices.

We find that the implied volatility estimates obtained from the Black-Scholes model can be very biased when volatility is stochastic. This result is important because of the widespread use of the Black-Scholes model in obtaining volatility estimates. In addition, we find that allowing for stochastic volatility that is correlated with the returns on the underlying asset has the potential to explain many of the observed biases of the Black-Scholes model.

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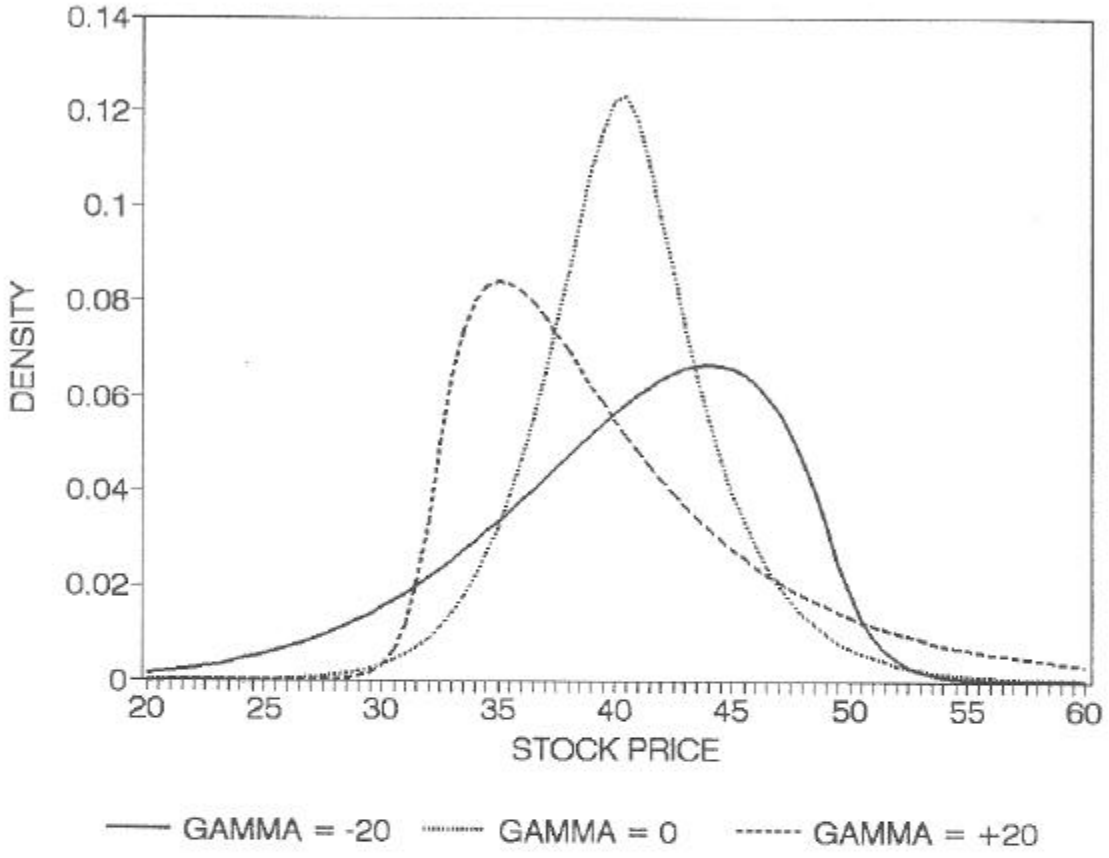


Fig. 1. Examples of the Bessel density function. The parameter gamma governs the correlation between volatility and returns on the underlying asset.

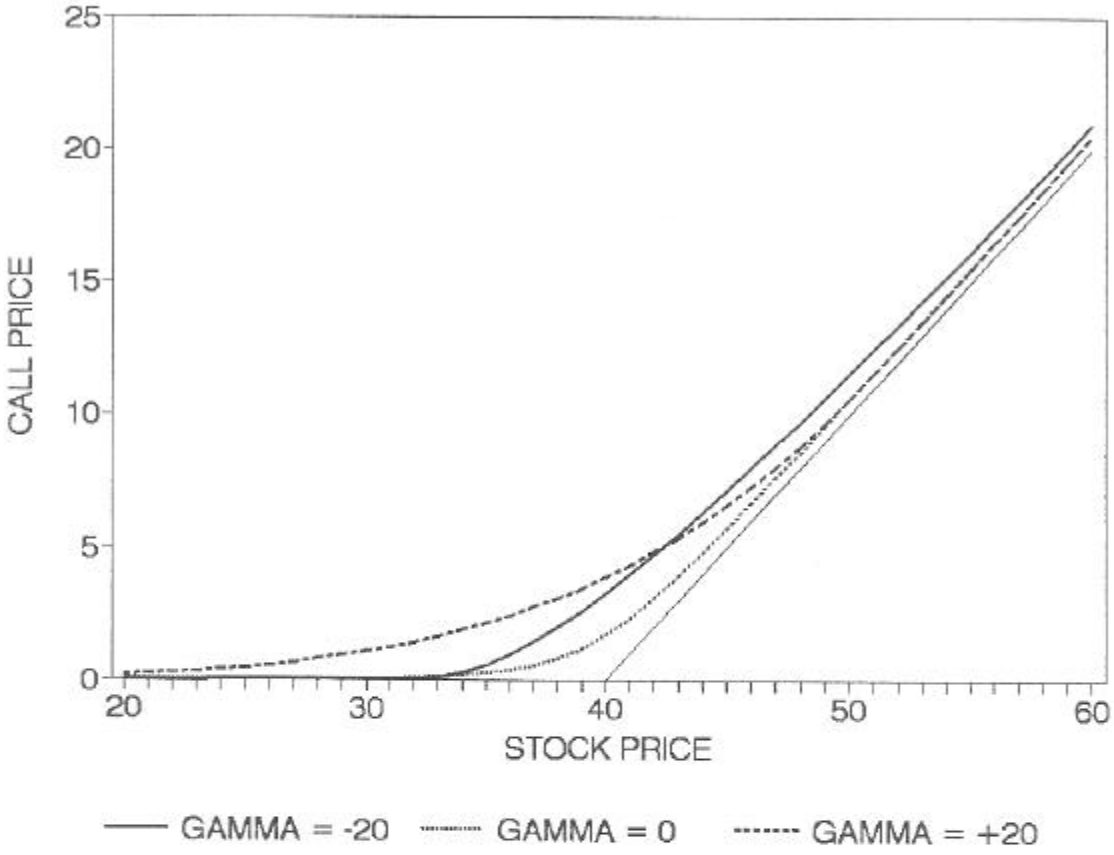


Fig. 2. Examples of call prices implied by the model. The model inputs used are $r = .05$, $T = .25$, $X = 40$, $I = .04$, and $\eta = 1$. The parameter gamma governs the correlation between volatility and returns on the underlying asset.

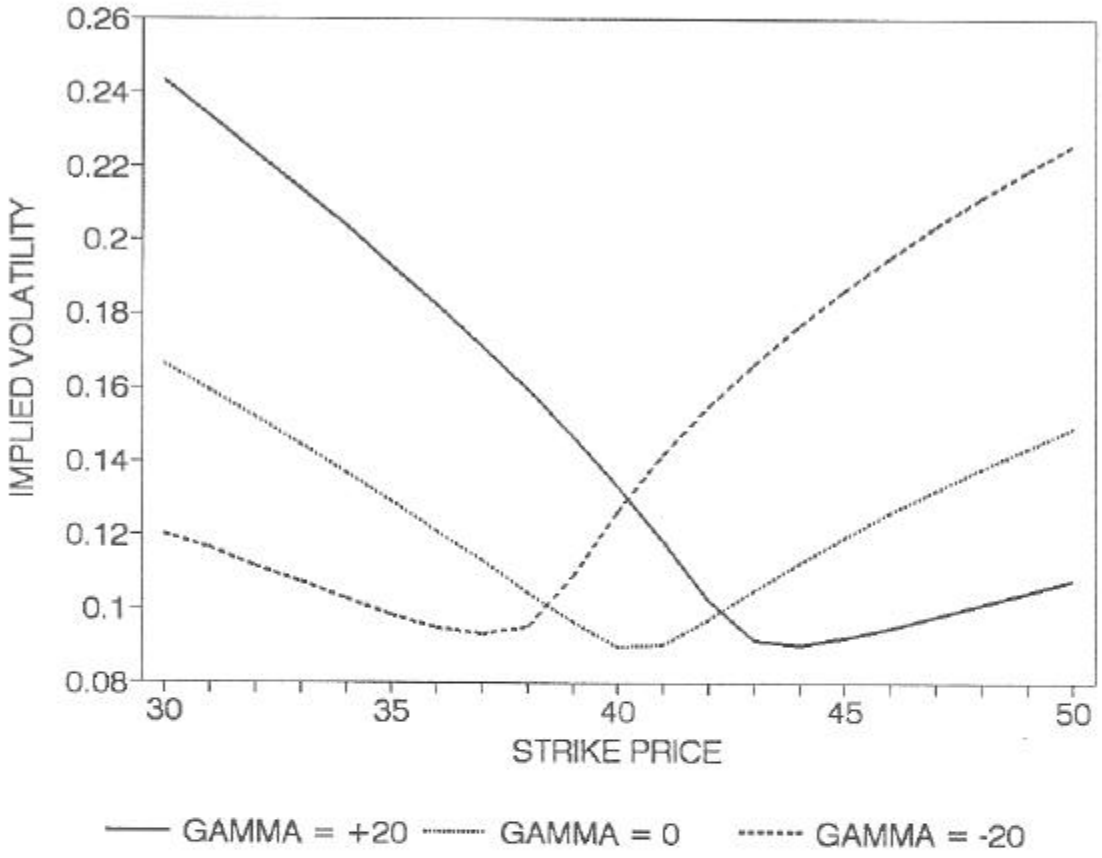


Fig. 3. Implied volatilities obtained by inverting the Black-Scholes model when the underlying option prices are generated by the stochastic volatility option pricing model. The actual instantaneous volatility is .10.

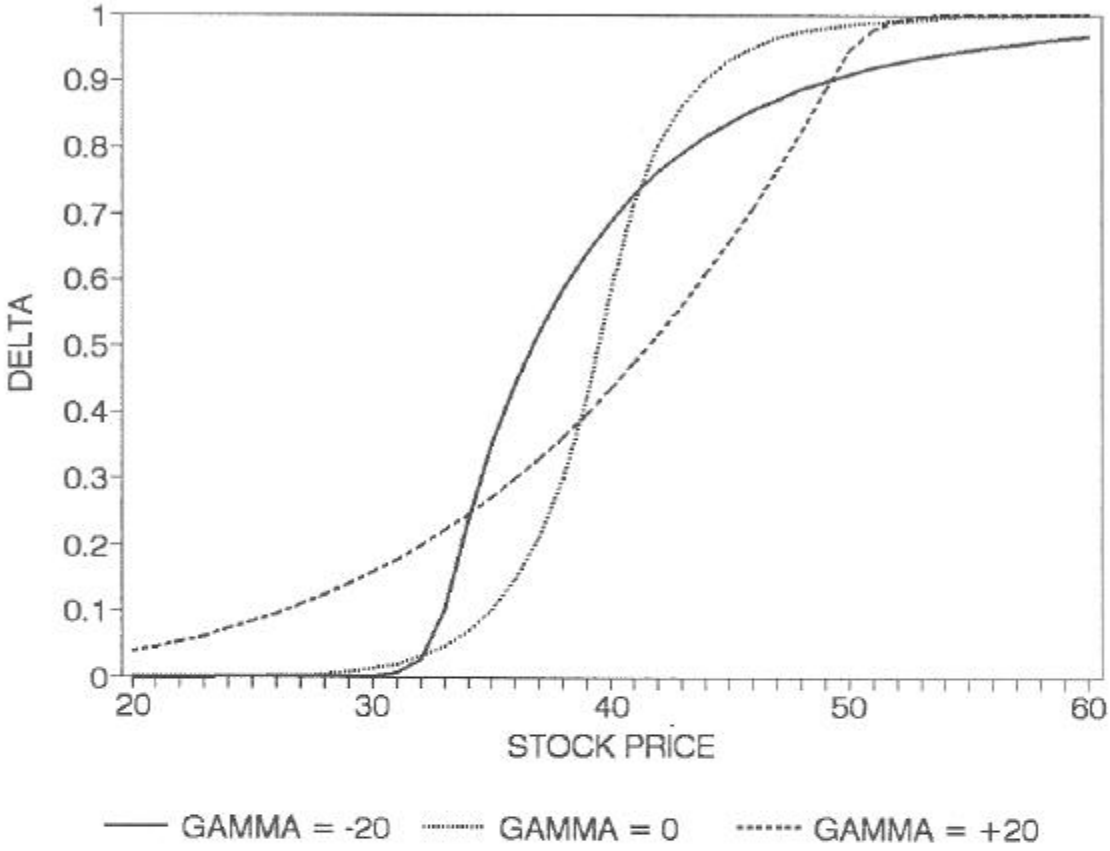


Fig. 4. Call option deltas implied by the model. The model inputs used are $r = .05$, $T = .25$, $X = 40$, $I = .04$, and $\eta = 1$. The parameter gamma governs the correlation between volatility and returns on the underlying asset.

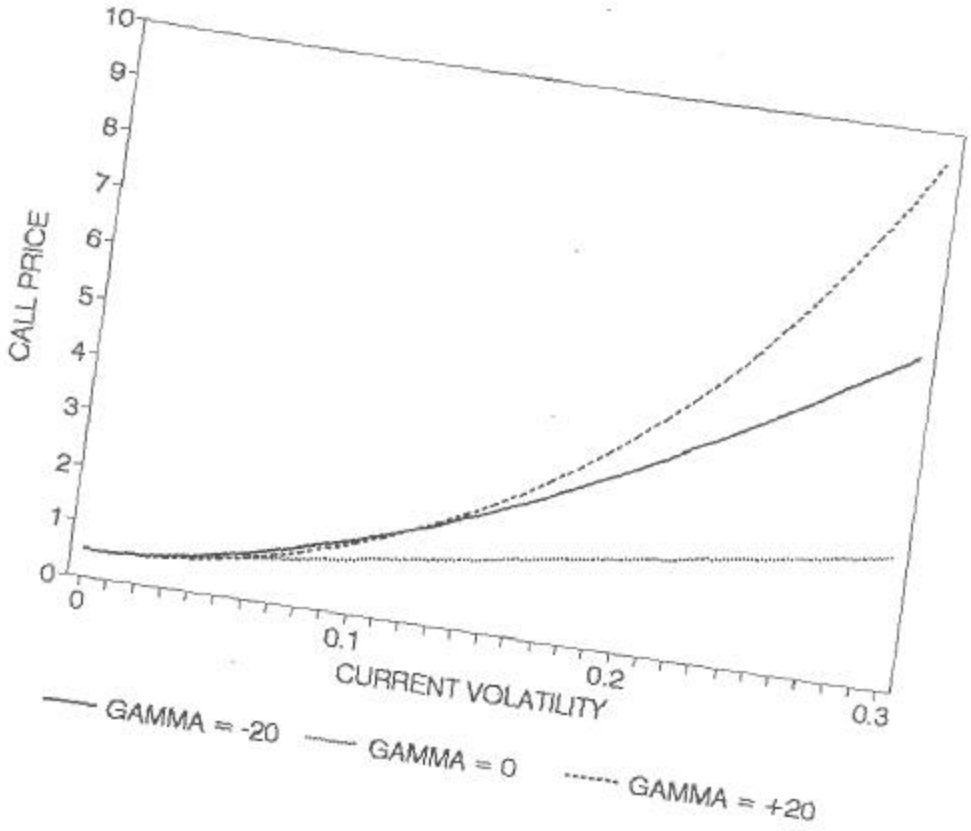


Fig. 5. Call option prices graphed as a function of the instantaneous standard deviation. The model inputs used are $r = .05$, $T = .25$, $X = 40$, $S = 40$, and $\eta = 1$. The parameter gamma governs the correlation between volatility and returns on the underlying asset.

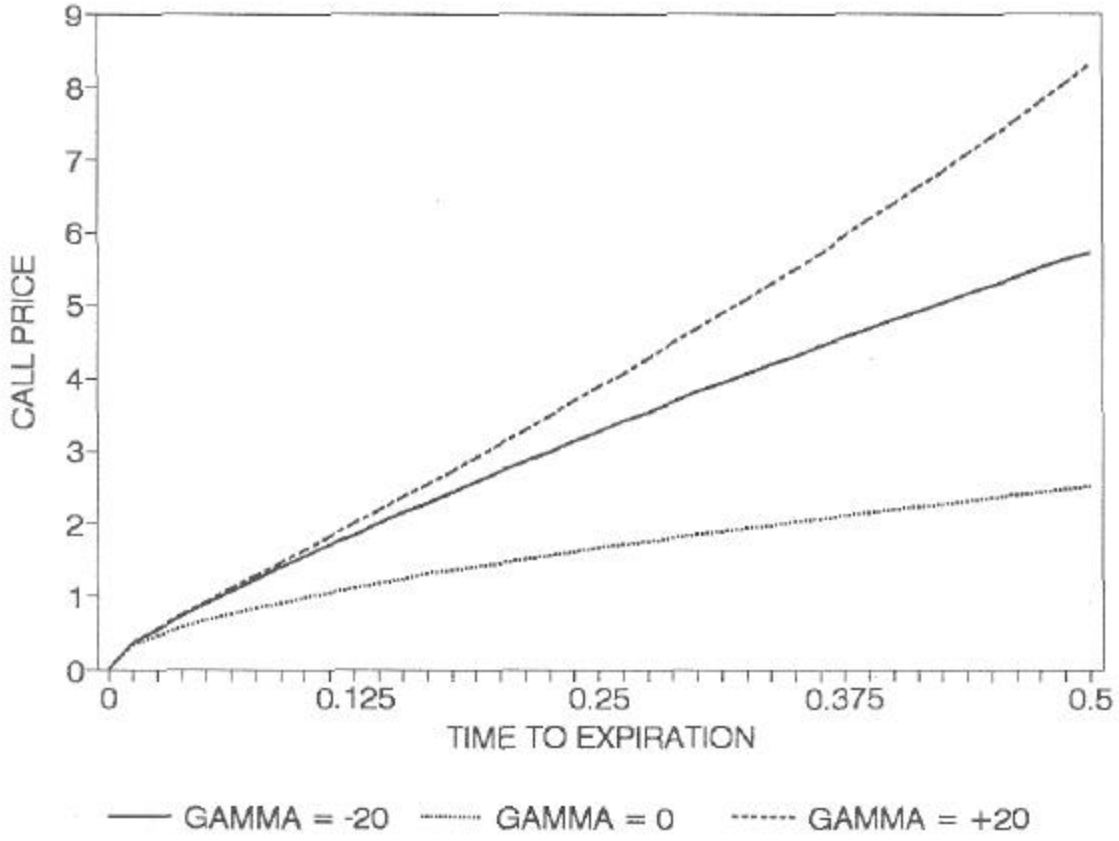


Fig. 6. Call option prices graphed as a function of the time until expiration. The model inputs used are $r = .05$, $I = .04$, $X = 40$, $S = 40$, and $\eta = 1$. The parameter gamma governs the correlation between volatility and returns on the underlying asset.

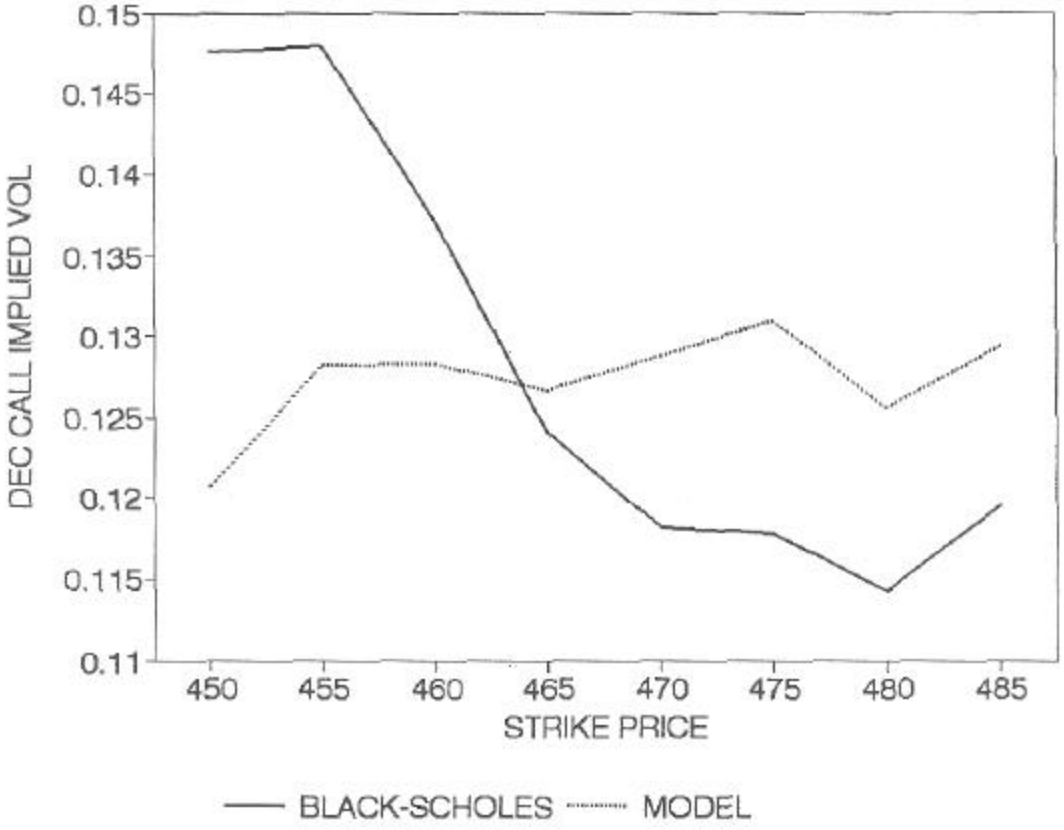


Fig. 7. Implied volatility estimates for December 1994 S & P 500 index call options.

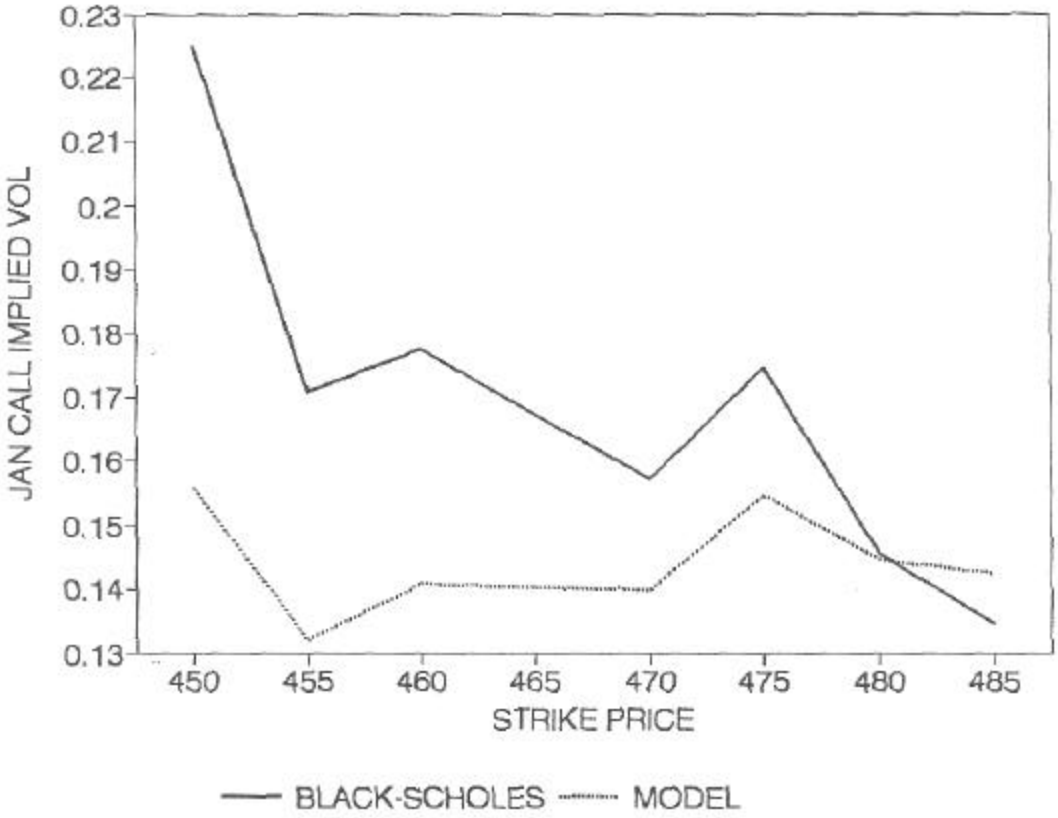


Fig. 8. Implied volatility estimates for January 1995 S & P 500 index call options.

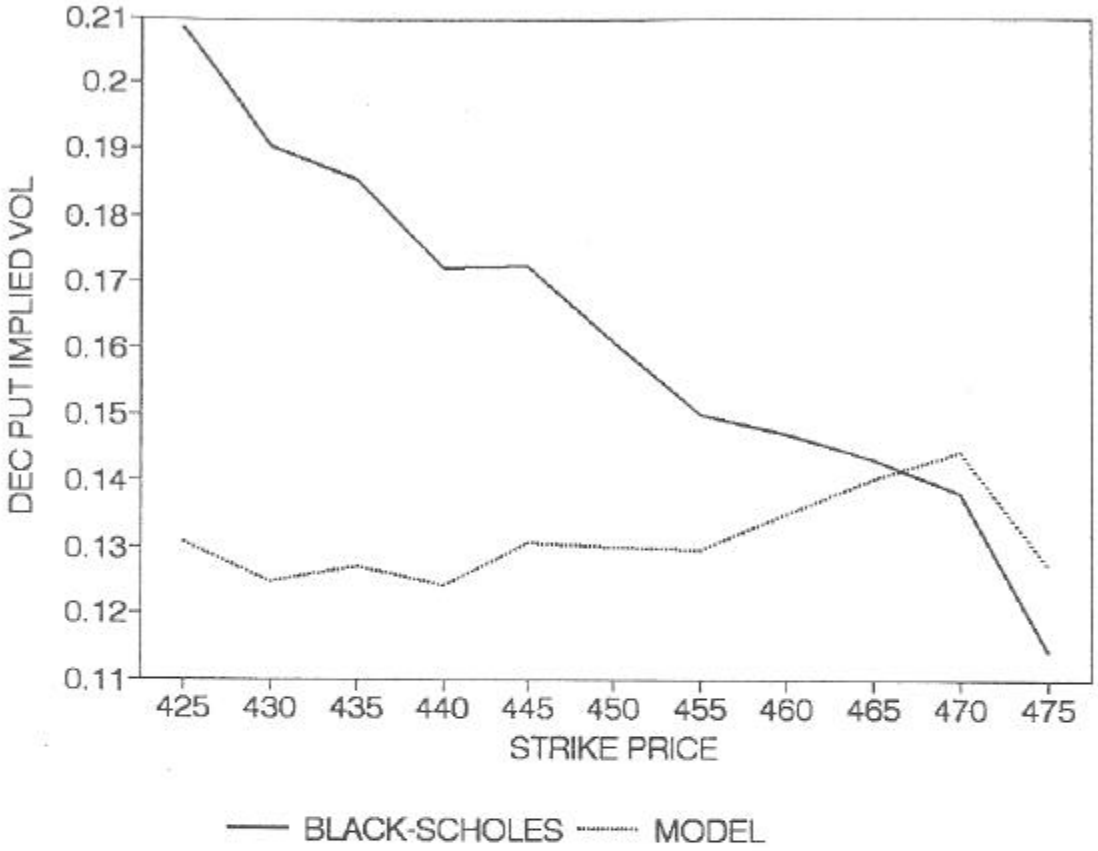


Fig. 9. Implied volatility estimates for December 1994 S & P 500 index put options.

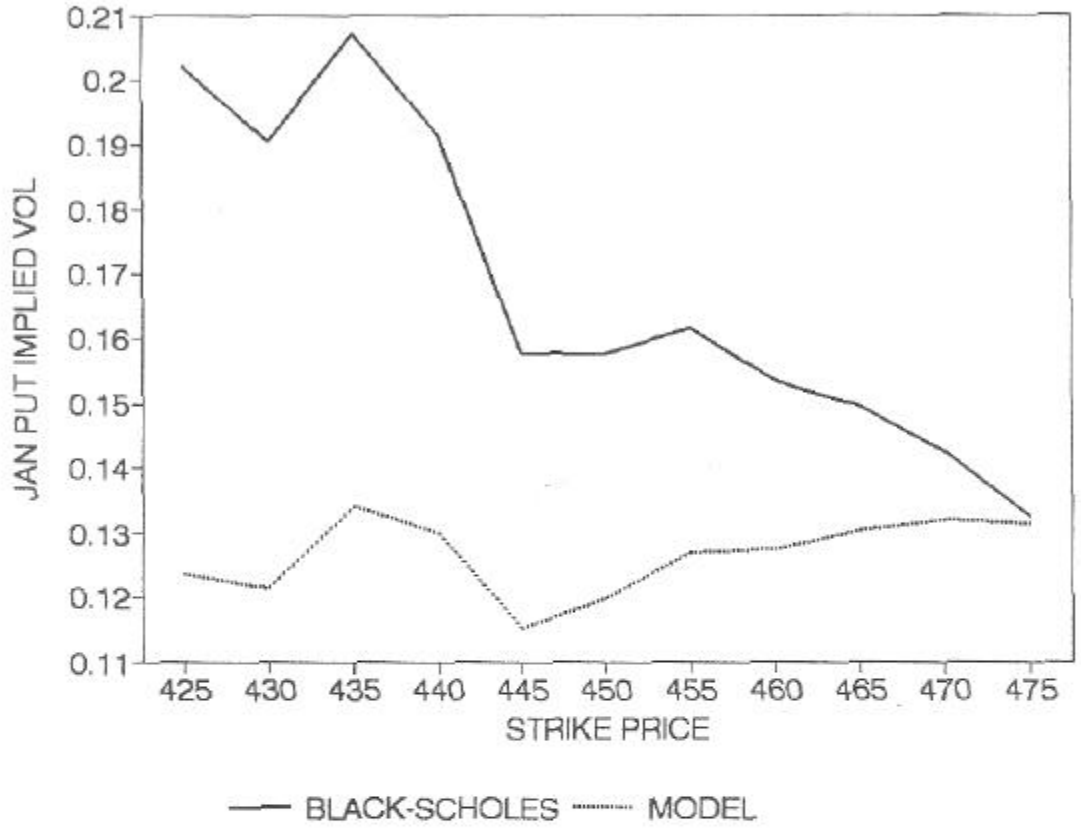


Fig. 10. Implied volatility estimates for January 1995 S & P 500 index put options.