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ON EXISTENCE CRITERIA FOR FLUID INTERFACES IN THE ABSENCE OF GRAVITY

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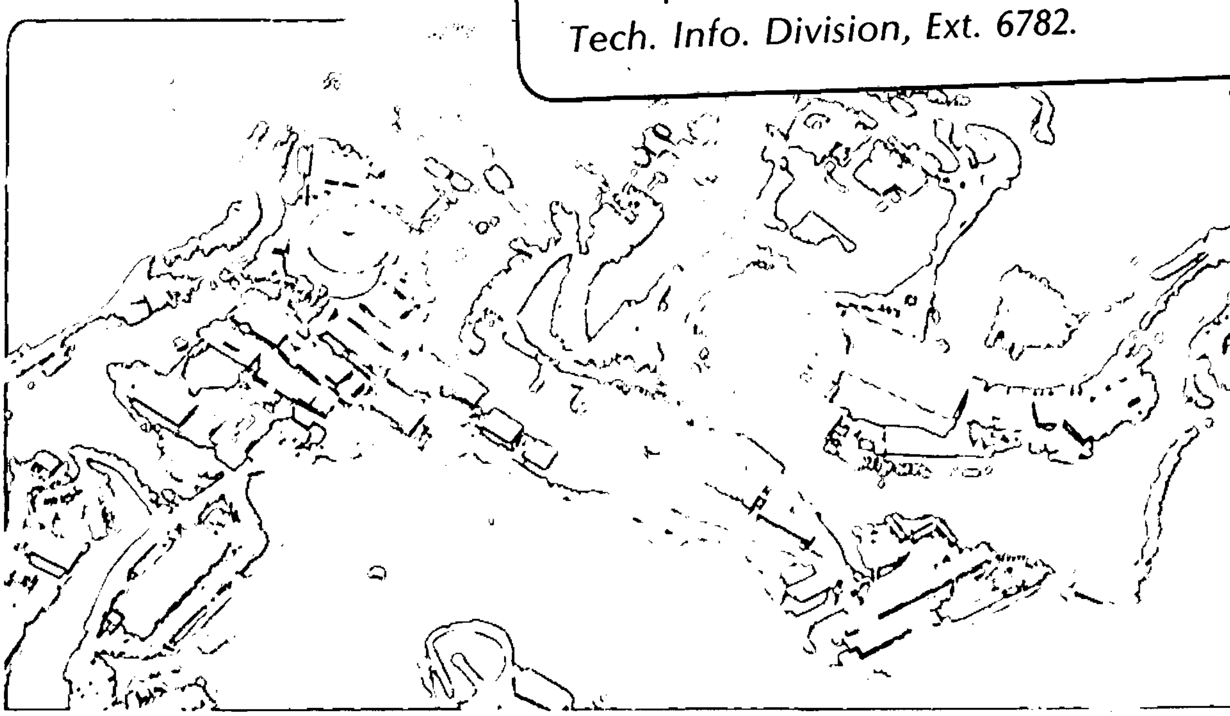

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ON EXISTENCE CRITERIA FOR FLUID INTERFACES IN THE ABSENCE OF GRAVITY[1]

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1. INTRODUCTION

We consider here equilibrium interfaces between two fluids, such as the liquid-gas free surface of a liquid that partly fills its container. The shape of the equilibrium free surface is determined by the interaction of surface and gravitational forces, in such a manner that the mechanical energy of the configuration is stationary with respect to displacements that satisfy prescribed constraints. Of principal interest are stable equilibria, for which the energy is minimized.

Attention is focussed on cylindrical containers of general section. A number of interesting — in some cases striking — results have been uncovered in recent years, most pronounced for the situation in which gravity is absent.

We consider a cylindrical container with axis oriented vertically and assume that the equilibrium free surface of a liquid partly filling the container projects simply onto a section Ω of the cylinder (Fig. 1). There is assumed to be sufficient liquid in the container to cover the base entirely. We denote by $u(x,y)$ the height of the free surface above a horizontal reference plane and by Σ the boundary of Ω. The gravitational field, if present, is taken to be uniform and is positive when directed vertically downward.

The condition, that the surface plus gravitational energy be stationary subject to the constraint of fixed liquid volume, leads to the Laplace-Young equation

$$\operatorname{div} Tu = \kappa u + 2H \quad \text{in } \Omega , \tag{1}$$

where

$$Tu = \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} ,$$

and

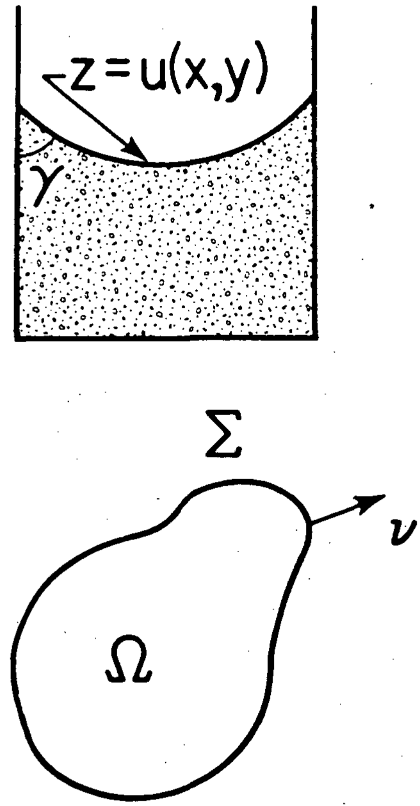


Fig. 1. *A partly filled cylinder and its section* Ω *with boundary* Σ *and exterior normal* ν.

$$\nu \cdot Tu = \cos\gamma \qquad \text{on } \Sigma . \tag{2}$$

These equations are based on surface energy being proportional to area. In (1), $\kappa = \rho g / \sigma$ is the capillary constant, with ρ the difference in densities between the gas and liquid phases, g the acceleration due to gravity, and σ the gas-liquid surface tension; $2H$ is the Lagrange multiplier for the volume constraint and is determined in general by the shape of the cylinder section, the volume of liquid present, and the contact angle γ at which the free surface meets the cylinder wall. The contact angle, which is taken to be constant, is a physically determined quantity that depends on the material properties of the liquid, gas, and container.

The boundary Σ is assumed to be smooth except possibly at a finite number of corners. The contact-angle boundary condition (2), where ν is the unit exterior normal, need not be prescribed at the corners.

Since the quantity on the left of (1) is twice the mean curvature of the free surface, a solution of (1),(2) represents a surface whose mean curvature varies linearly with height and that intersects the cylinder with prescribed angle γ. If $\kappa = 0$ then a solution surface has constant mean curvature, a situation that occurs in the absence of gravity and is taken often as an idealization of those terrestrial situations for which $\kappa\Omega \ll 1$, i.e., for which surface forces dominate gravitational ones. (In this paper Ω, Σ, ... are used interchangeably to denote both a set and its measure). We focus attention in what follows on the case $\kappa = 0$. For simplicity only wetting liquids $0 \le \gamma < \pi/2$ are discussed, as the supplementary case $\pi/2 < \gamma \le \pi$ can be obtained immediately by means of a simple transformation. The case $\gamma = \pi/2$ corresponds to the trivial solution $u \equiv \text{const}$.

In this paper we highlight some of the advances made on this problem during approximately the past decade and indicate some current investigations. The bulk of the author's work on capillary surfaces has been carried out jointly with Robert Finn, whose mathematical and physical insight have played a crucial role in obtaining the results discussed here.

2. SPECIALIZED DOMAINS —NECESSARY CONDITIONS

If $\kappa = 0$, then the constant mean curvature H can be determined from the geometry and the boundary data by integrating (1) over Ω. One obtains, after integration by parts and using (2), the relationship $H = \dfrac{\Sigma}{2\,\Omega\cos\gamma}$. Thus (1) becomes

$$\operatorname{div} Tu = \frac{\Sigma}{\Omega}\cos\gamma \qquad \text{in } \Omega\,. \tag{3}$$

The problem then is to solve (2),(3) in the given domain for the prescribed value of γ. The additive constant, to which a solution of (2),(3) is determined, can be fixed by specifying, for example, the volume of liquid in the cylinder.

2.1 *A Closed-Form Solution*

If Ω is the disc $x^2 + y^2 \le a^2$, then it is well known that the unique solution of (2),(3) (up to an additive constant) is the portion of the lower spherical surface

$$u = -[a^2-(x^2+y^2)\cos^2\gamma]^{1/2}\sec\gamma\ . \tag{4}$$

Less well known is that (4) also is a solution if Σ is a polygon circumscribing the circle $x^2 + y^2 = a^2$, so long as no vertex lies exterior to the concentric circle of radius $a/\cos\gamma$.

What if the polygon does extend beyond this circle? The answer can be obtained from the following result.

2.2 *A Necessary Condition for Existence*

Let a subregion Ω^* and sub-boundary Σ^* be cut from Ω and Σ, respectively, by a curve (or systems of curves) Γ in Ω (Fig. 2). There holds [1]:

A necessary condition for existence of a solution of (2),(3) in Ω is that for every Γ

$$\varphi(\Gamma) \equiv \Gamma - (\Sigma^* - \frac{\Sigma}{\Omega}\Omega^*)\cos\gamma > 0\ . \tag{5}$$

This result can be obtained by integrating (3) over the subdomain Ω^*, integrating by parts, and using (2) and the property that $|Tu| < 1$ for any differentiable u.

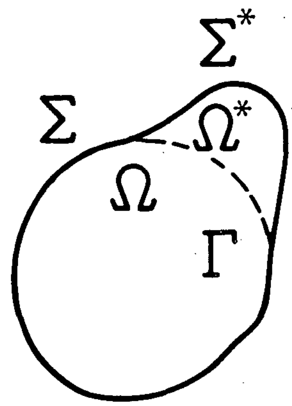


Fig. 2. *The domain* Ω *subdivided by the curve* Γ.

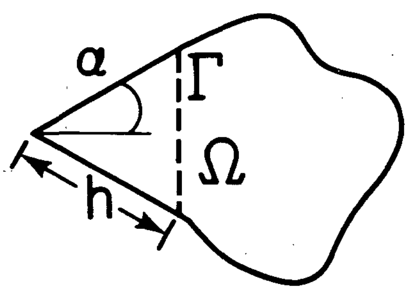


Fig. 3. *Domain with a corner.*

2.3 Corner Phenomenon

By applying the above necessary condition to the domain containing a corner shown in Fig. 3, one can obtain the answer to the question of Sec. 2.1. If the corner has interior angle 2α, and Γ is the straight line segment shown in Fig. 3, then one obtains for this configuration $\varphi(\Gamma) = 2h(\sin\alpha - \cos\gamma) + O(h^2)$. Letting $h \to 0$ gives the following result [1].

If

$$\alpha + \gamma < \pi/2 \tag{6}$$

there is no solution of (2),(3) in Ω.

This non-existence result is sharp, since for polygons circumscribing the disc, the lower spherical cap (4) gives a solution for $\alpha + \gamma \geq \pi/2$, where 2α is the smallest interior angle of the polygon. Thus the solution depends discontinuously on the contact angle γ. As γ decreases from $\pi/2$ to $(\pi/2) - \alpha$, the solution exists and is uniformly bounded and analytic in Ω, but if γ is less than $(\pi/2) - \alpha$ no solution exists.

This remarkable behavior was tested experimentally by W. Masica at the NASA Lewis Research Center Zero-Gravity Facility. Liquid interior to a regular hexagonal cylinder, measuring approximately 4 cm. on a side and made of acrylic plastic, was photographed during free fall. The liquids shown in Fig. 4 make a contact angle $\gamma \approx 48°$ (20% ethanol solution) for case (a) and $\gamma \approx 25°$ (30% ethanol solution) for case (b). Each liquid was initially at rest, and the configurations are shown in Fig. 4 after slightly more than 5 seconds of free fall, at which time free-surface equilibrium had substantially been achieved.

In Fig. 4a the free surface is essentially that given by (4), as would be expected, since $\gamma \geq 30°$. In Fig. 4b, for which $\gamma < 30°$, the fluid has flowed up into the container corners and, after hitting the lid, filled in the corners between the walls and the lid. One would expect basically the same result no matter how tall the cylinder.

The above mathematical results are not spurious ones that depend on the corner being perfectly sharp. They can be obtained as limiting cases of the following result for smooth containers [1].

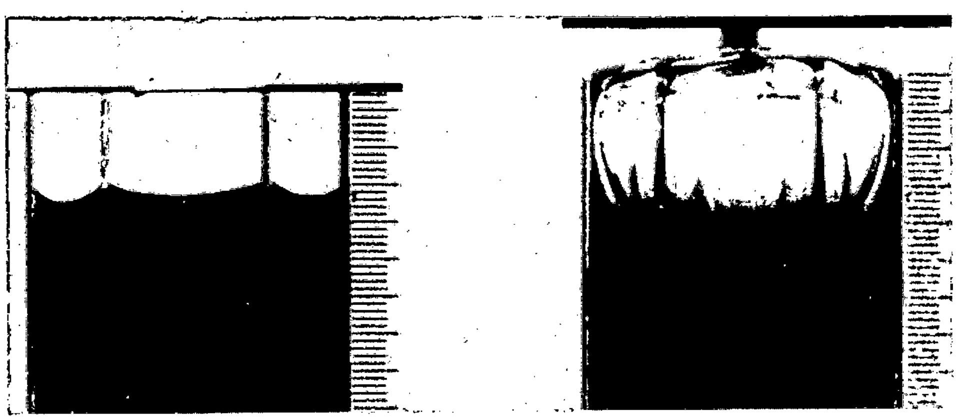
Suppose there is a point on Σ *at which the curvature exceeds* Σ/Ω. *Then there exists* γ_{cr}, $0 < \gamma_{cr} \leq \pi/2$ *such that there is no solution whenever* $0 \leq \gamma < \gamma_{cr}$.

This result for smooth domains is sharp, as well, since explicit solutions can be given for $\gamma = 0$ for certain configurations in which the boundary curvature achieves the value Σ/Ω.

Discontinuous behavior can occur for domains with corners even if gravity is not zero [2]. If $\kappa > 0$, the solution height at a corner with interior angle 2α can change abruptly from a bounded value to infinity as γ traverses the critical value $(\pi/2) - \alpha$.

3. SUFFICIENT CONDITIONS FOR EXISTENCE — MORE GENERAL DOMAINS

An important result was obtained by Giusti, who gave sufficient conditions for there to exist a solution of (2),(3). With Γ, Ω^*, Σ^*, and $\varphi(\Gamma)$ as in Sec. 2.2, he proved in effect [7]:



(a) (b)

Fig. 4. *Equilibrium free surface in a hexagonal cylinder at zero gravity. (a)* $\gamma \approx 48°$, *(b)* $\gamma \approx 25°$.

If $\varphi(\Gamma) > 0$ holds for every Γ then a solution of (2),(3) exists.

A slight modification may be required for the case in which boundary corners are present and $\alpha + \gamma = \pi/2$. Giusti's result permits study of domains for which closed-form solutions are not available. Based on it, the following result, which does not require the testing of every Γ, was obtained in [4].

A solution exists if and only if there is a vector field $W(\mathbf{x})$ in $\bar{\Omega}$, with $\operatorname{div} W = \Sigma/\Omega$, $\nu \cdot W = 1$ *on* Σ, *and* $|W| < 1/\cos\gamma$ *in* Ω.

The proof, which uses integration by parts over Ω^*, permits neglecting any set of Hausdorff measure zero on Σ on which the normal is not defined.

3.1 Parallelogrammatical Domains

For the parallelogram, an explicit vector field $W(\mathbf{x})$ can be constructed that has the required properties if $\alpha + \gamma \geq \pi/2$, where 2α is the smaller interior angle [5]. Denote $W = (u,v)$ and $\mathbf{x} = (x,y)$. Then for the parallelogram shown in Fig. 5 this vector field is

$$u = \frac{1}{a \sin 2\alpha}\left[x + \left(\frac{1}{b} - \frac{1}{a}\right) y \cot 2\alpha\right]$$

$$v = \frac{1}{b \sin 2\alpha} y \; .$$

Thus one finds that the nonexistence result (6) is sharp for parallelograms, as well as for polygons circumscribing a circle.

3.2 Trapezoidal Domains

The sharpness of the nonexistence result (6) based on the corner phenomenon does not carry over to more general polygonal domains. The type of behavior that can occur is illustrated by containers with trapezoidal section. Departure from a parallelogram need not be substantial for a significant change in critical contact angle. Consider the trapezoid shown in Fig. 6 with bases

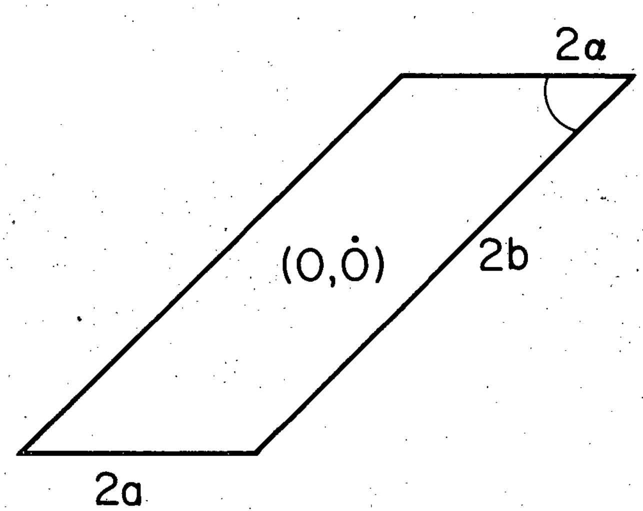


Fig. 5. *Parallelogrammatical domain.*

$b > a$ and acute angle 2α. The following result is proved in [5].

For any $\gamma < \pi/2$ *and any* $\varepsilon > 0$ *there is a trapezoid with* $|\alpha - \pi/4| < \varepsilon$, $|b/a - 1| < \varepsilon$ *for which there is no solution of* (2),(3).

Thus a trapezoid can be found arbitrarily close in the above sense to a rectangle, for which the critical contact angle can depart from the rectangle's $\pi/4$ to as close as desired to $\pi/2$.

The above result is obtained through a careful choice of the curve Γ that cuts off the subdomain Ω^*. The optimal choice $\Gamma = \Gamma_{cr}$, which is a circular arc of curvature $\frac{\Sigma}{\Omega}\cos\gamma$ that meets Σ with angle γ, is shown in Fig. 6. It will be discussed further in Sec. 4.

A different type of discontinuous dependence than discussed previously occurs for the trapezoid [5].

For fixed α *and all sufficiently large* h, *there exists a unique* $\gamma = \gamma_{cr}$. *A solution exists if and only if* $\gamma > \gamma_{cr}$. *As* γ *approaches* γ_{cr} *from above,* $u \to \infty$ *in* Ω^* *and* $|\nabla u| \to \infty$ *on* Γ_{cr}. *The solution surface in* $\Omega \setminus \Omega^*$ *approaches the vertical circular cylinder over* Γ_{cr}, *as* Γ_{cr} *is approached from within* $\Omega \setminus \Omega^*$.

To observe the manner in which the limiting behavior occurs, numerical calculations were carried out using the finite element method for a sequence of trapezoidal domains [8]. For $b = 2$, $h = 25$, $\gamma = 58°$, and the four values $a = 2$, 1.5, 1.4, 1.3, the problem (2),(3) was solved numerically using reduced biquadratic elements on a 7×75 mesh. For $a = 1.3$ the critical value of γ for the trapezoid is $\gamma_{cr} \approx 57.6°$, which is substantially larger than the one that would be yielded by the corner condition $(\pi/2) - \alpha \approx 45.4°$. In Fig. 7 are shown the numerically obtained values for $u(0,y)$ as a function y for the four trapezoids.

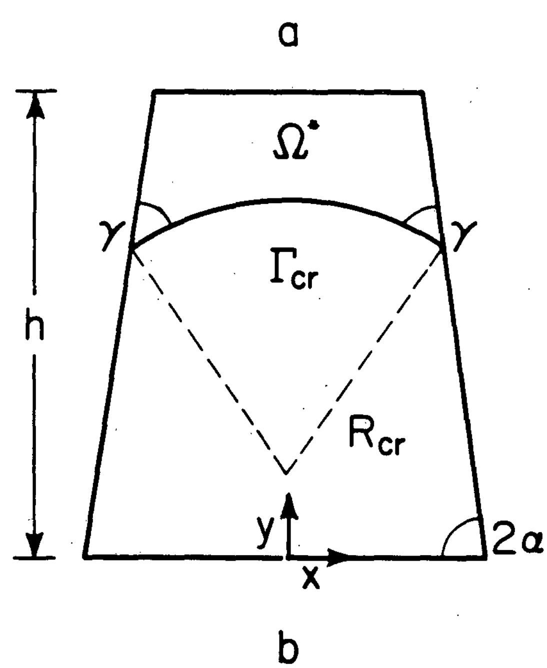


Fig. 6. *Trapezoidal domain with extremal arc* Γ_{cr}

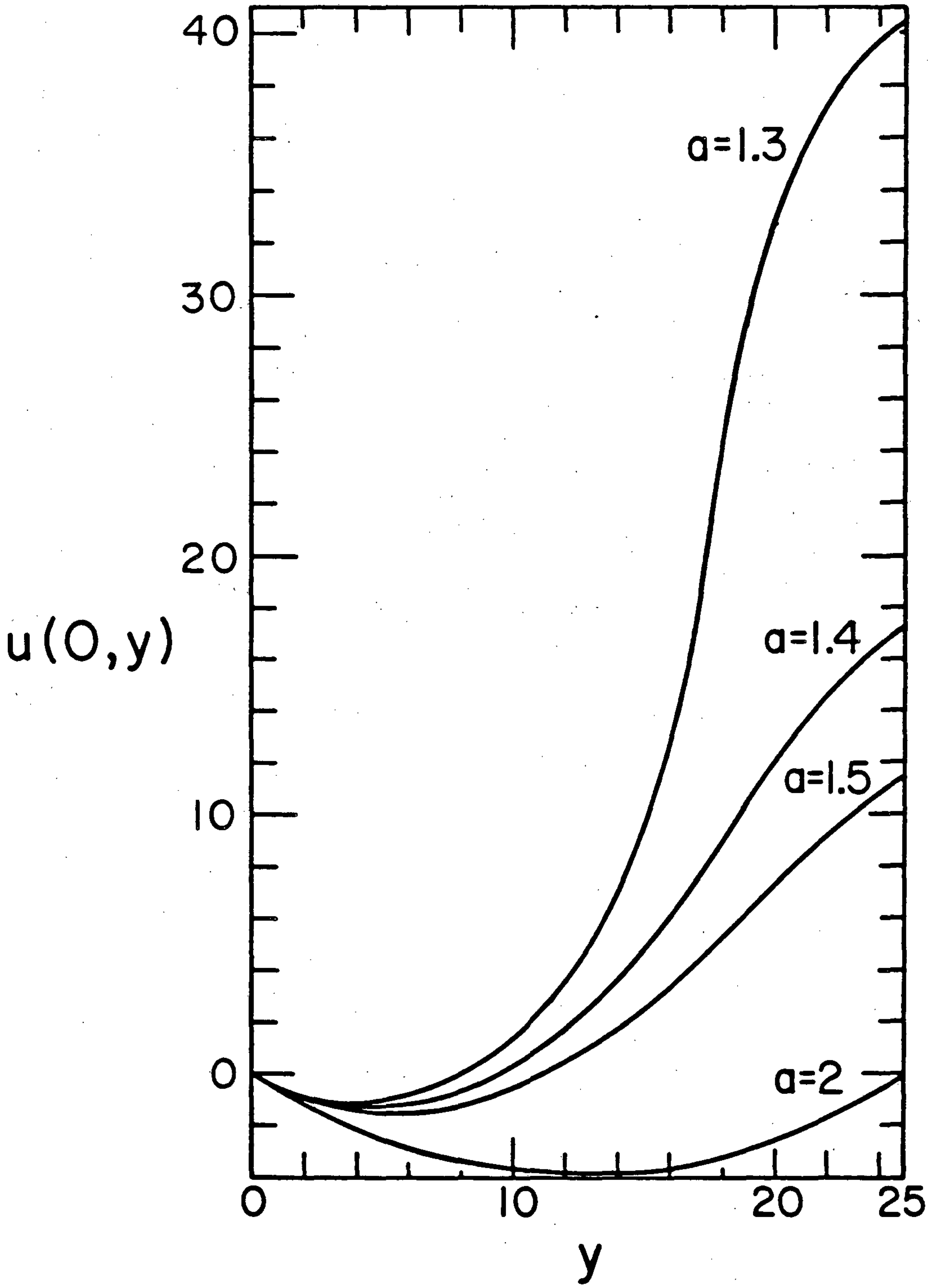


Fig. 7. *Free surface height $u(0,y)$ vs. y for four trapezoids.*

The additive constants are chosen so that $u(0,0) = 0$. The differing behavior between the near-critical configuration ($a = 1.3$) and the rectangular domain ($a = 2$) is easily observed. The near verticality of the numerically obtained curve for $a = 1.3$ is seen to be maximal at a value of y that is in good accord with the value $y \approx 17.6$ that Γ_{cr} assumes at $x = 0$ for this configuration. Experimental investigation of these properties aboard a NASA Spacelab flight is currently being planned in collaboration with D. Coles, R. Finn, and L. Hesselink, in conjunction with the NASA Lewis Research Center.

4. SUBSIDIARY VARIATIONAL PROBLEM

For general domains, determination of whether the sufficient condition for existence holds, that $\varphi(\Gamma) > 0$ for every Γ, can be approached as a subsidiary variational problem [6]. One seeks the minimum of $\varphi(\Gamma) \equiv \Gamma - \Sigma^* \cos\gamma + \left(\frac{\Sigma}{\Omega}\cos\gamma\right)\Omega^*$ over all admissible Γ. The sign of this minimum, if one exists, would then indicate existence or nonexistence of a solution to the original capillary free surface problem.

Of significance is the feature that the variational problem for minimizing $\varphi(\Gamma)$ is essentially the same as the one for the original capillary surface problem, but in one dimension lower. The capillary surface problem (2),(3) corresponds to seeking a minimum of the total surface energy subject to fixed fluid volume V, that is, to minimizing $\sigma[S - S^* \cos\gamma + (\frac{\Sigma}{\Omega}\cos\gamma)\, V]$, where S is the capillary free surface area and S^* is the wetted surface area of the container. Thus, as a minimizing surface S must be a surface of constant mean curvature $\frac{\Sigma}{\Omega}\cos\gamma$ intersecting the bounding cylinder walls with angle γ, so must Γ be a curve (or system of curves) of constant curvature $\frac{\Sigma}{\Omega}\cos\gamma$ intersecting Σ with angle γ. One need investigate only these specific circular arcs in seeking a minimum for $\varphi(\Gamma)$.

Detailed properties of minimizing curves Γ are given in [6]. An important feature is that:

The nonexistence of a minimizing Γ in Ω is a sufficient condition for the existence of an (energy minimizing) solution u of the capillary surface problem.

Work is currently underway to characterize geometric conditions on general domains for a minimizing Γ to exist and for a minimizing Γ to give $\varphi > 0$.

5. CONTINUOUS AND DISCONTINUOUS DISAPPEARANCE OF CAPILLARY SURFACES

As a concluding remark, emphasizing the seemingly anomalous nature of the capillary problem, we comment on the behavior at the critical contact angle γ_{cr}. In general, a solution exists for $\gamma > \gamma_{cr}$ and does not exist for $\gamma < \gamma_{cr}$. If the domain is smooth, then a solution does not exist at the critical value $\gamma = \gamma_{cr}$.

However, if the domain is not smooth and has a corner, then a solution may exist for $\gamma = \gamma_{cr}$. This feature is discussed in [3,6].

REFERENCES

1. Concus, P. and R. Finn, *On capillary free surfaces in the absence of gravity*, Acta Math., 132 (1974), pp. 177-198.
2. ______________, *On capillary free surfaces in a gravitational field*, Acta Math., 132 (1974), pp.207-223.
3. ______________, *Continuous and discontinuous disappearance of capillary surfaces*, preprint LBL-15356, Lawrence Berkeley Lab., University of California, Berkeley, CA, 1983.
4. Finn, R., *Existence and non-existence of capillary surfaces*, Manuscripta Math., 28 (1979), pp. 1-11.
5. ________, *Existence criteria for capillary free surfaces without gravity*, Indiana University Math. J., to appear.
6. ________, *A subsidiary variational problem and existence criteria for capillary surfaces*, preprint, Math. Dept., Stanford University, Stanford, CA, 1982.
7. Giusti, E., *Boundary value problems for non-parametric surfaces of prescribed mean curvature*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 3 (1976), pp. 501-548.
8. Roytburd, V., *On singularities of capillary surfaces on trapezoidal domains*, Internat. J. Math. Math. Sci., to appear.

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