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Essays on Market Design and Auction Theory

A dissertation submitted in partial satisfaction

of the requirements for the degree

Doctor of Philosophy in Economics

by

Byeonghyeon Jeong

2019

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ABSTRACT OF THE DISSERTATION

Essays on Market Design and Auction Theory

by

Byeonghyeon Jeong
Doctor of Philosophy in Economics
University of California, Los Angeles, 2019
Professor Marek G Pycia, Co-Chair
Professer Ichiro Obara, Co-Chair

This dissertation studies market design and auction theory. Chapter 1 studies the impact of school choice on segregation. It shows that the popular school choice mechanisms lead to substantially different school and residential segregation, an important and overlooked aspect of choosing among school choice mechanisms. We show that open enrollment policy in public school choice program can decrease diversity of individual schools and increase segregation depending on which student allocation mechanism is used. Without open enrollment, we study the model of location choice and show that segregation is mainly associated with income. In comparing mechanisms, we show that Boston mechanism fosters segregation more than the deferred acceptance. With open enrollment, the difference between BM and DA becomes more drastic. We show that BM can actually intensify segregation when open enrollment policy is adopted, while DA is more resilient to segregation. The deferred acceptance with multi tie breaking creates maximally diverse schools. Chapter 2 considers conventional auctions when the seller can design bid spaces. Any symmetric equilibrium in a second price auction with bid spaces can be replicated with an equilibrium in a first price

auction with bid spaces, but the converse doesn't hold. First price auctions with designed bid spaces revenue dominates second price auction with designed bid spaces, and well-designed first price auction is an optimal selling mechanism. Chapter 3 studies one-to-one matching environment without transfer in the presence of incomplete information on one-side. The existing notions of stability under incomplete information are studied and two alternatives are proposed. Weak Bayesian stability requires that the beliefs of the agents are dervided from a common prior via Bayes' rule and are *internally* consistent with the presumption that the given matching is stable. Strong Bayesian stability refines weak Bayesian stability by requiring the beliefs of agents are also *externally* consistent in the sense that the beliefs are narrowed down only when there is a valid reason.

The dissertation of Byeonghyeon Jeong is approved.

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University of California, Los Angeles

2019

TABLE OF CONTENTS

LIST OF FIGURES

ACKNOWLEDGEMENTS

I would never have been able to complete my dissertation without the guidance of my committee members and support from my family.

I am deeply indebted to my primary advisors Marek Pycia for his invaluable guidance and great inspiration. It has been my genuine pleasure to have him as an advisor and to have the opportunity of working with him. I would also like to thank the other members of the committee, Ichiro Obara, Simon Board, Moritz Meyer-ter-Vehn, Tomasz Sadzik, Sushil Bikhchandani. I have greatly benefited from feedback from participants at the theory proseminar of UCLA.

I thank my classmates and friends, Hyo Sang Kim, KeyYong Park, Jinwook Hur, Sangyup Choi, Changsu Ko, Jaeyoung Jang, Jeonghwan Kim, Omer Ali, Jesper Riis-Vestergaard Soerensen, Renato Giroldo, El-Hadi Caoui, Akina Ikudo, Ksenia Shakhgildyan, Alex Fon, Alex Graupner.

Finally, I am grateful to my family. My deep gratitude is with my parents, my sister, and my little niece and nephew.

VITA

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Chapter 1

School Choice in Context: Can Open Enrollment Cure Segregation?

1.1 Introduction

Over the decades school choice policy has been adopted in many countries to expand the schooling options for students and parents. There are many programs available to provide choices such as charter schools, school vouchers, and tuition tax credit program. Especially, open enrollment program with centralized assignment has been widely adopted in many countries. In US, the majority of states have adopted open enrollment programs in some way.[1] Open enrollment policies allow students to attend to public schools outside of their residential school district. Under the neighborhood school system, students are assigned to neighborhood schools based on where they reside, thus, real estate costs decide who can attend which schools and it leads to segregation between school districts. While open enrollment programs were proposed partially as a solution to mitigate the segregation by delinking residential choice and school choice, there are some evidences suggesting that open enrollment programs decreased diversity of each school. Institute on Metropolitan

[1] See the report by Education Commission of the States (2016) for more detail.

Opportunity (2013) reported that overall open enrollment increased segregation by analyzing school districts in the Twin Cities. Moon (2018) studies the impact of adopting the public school choice program on housing prices and student performance in Seoul. In the paper, the result shows that adopting the program closes the gap of housing prices between different school districts, while the gap of academic performance was not affected by the program. Saporito (2003) analyzes public and magnet schools in Philadelphia and concludes that expanding choices lead to increased segregation. Kotok et al. (2017) show an evidence that students movement between public schools and charter schools are segregative. These paradoxical evidences pose an important question that should be answered in designing school choice program. Are school choice programs inherently segregative?

This paper addresses the question by studying the role of specific student assignment mechanisms in the context of segregation.

There are two widely used student assignment mechanisms in practice; Boston mechanism and deferred acceptance.[2] While Boston mechanism or some variations are used around the world, the mechanism has been criticized for its incentive property and unfairness. On the other hand, Deferred acceptance is strategy-proof and fair in the sense that the mechanism eliminates justified envy.

First, we construct a model of residential choice between two school districts when each district has three different schools under the neighborhood school system. Within each district, either Boston mechanism or deferred acceptance are used to assign students. We show that ex-ante symmetric districts are segregated by income and Boston mechanism creates less diverse schools than the deferred acceptance does.

Second, we study the impact of the open enrollment program on the already segregated districts. We show that the open enrollment program mitigates segregation by income between districts, however, it may create or intensify segregation by characteristics other than income between schools if Boston mechanism is used, while the deferred acceptance algorithm

[2] See Agarwal and Somaini (2018) for a partial list of cities using the mechanisms.

is more resilient to segregation.

Third, we compare the performance of Boston mechanism, deferred acceptance with single tie breaking, and deferred acceptance with multi tie breaking in the context of segregation and school diversity. Our results show that the deferred acceptance mechanisms in general are better than Boston mechanism concerning segregation and diversity, and the deferred acceptance with multi tie breaking performs better than the deferred acceptance with single tie breaking.

The rest of paper is organized as follows. Section 1.1 discusses the related literature. Section 2 provides the baseline model. Section 3 studies the outcomes of different school choice mechanisms without open enrollment. Section 4 studies the impact of adopting open enrollment policy under the different school choice mechanisms. Section 5 provides the concluding remarks. All the proofs are in the appendix A.

1.1.1 Related Literature

This paper is related to two lines of literatures. On the one hand, the baseline model without open enrollment in this paper is inspired by the literature on multi-community model. Tiebout (1956) introduces the multi-community model and local public good. Berglas (1976) adds complementarity into Tiebout (1956) model. Benabou (1993) studies general equilibrium model that combines location, education, and occupation choices. Epple and Romano (2003) examine and compare neighborhood system and open enrollment when each district has only one school and the quality of school is determined by peer quality. Our model adds an element of school choice into the multi-community models by having fixed size schools with heterogeneous agents.

On the other hand, there are papers in mechanism design literature that have focused on student assignment algorithms. Gale and Shapley (1962) study stable matching problem and provide that the outcome of Deferred Acceptance algorithm is the proposing side optimal stable matching. Roth (1982) shows that Deferred Acceptance algorithm is strategy-proof.

Abdulkadirolu and Sönmez (2003) approach the school choice problem as a mechanism design problem and show the flaws of widely used mechanisms including Boston mechanism and propose deferred acceptance and top trading cycle as alternatives. Miralles (2009) and Abdulkadirolu et al. (2011) argue that Boston Mechanism is ex-ante more efficient than Deferred Acceptance. Erdil and Ergin (2008), Kesten and Ünver (2015), and Ashlagi and Nikzad (2016) study the impact of tie-breaking in Deferred acceptance algorithm. Pycia (2017) shows that many standard mechanisms are equivalent in terms of mean invariant outcome statistics. In the light of Pycia (2017), it is more natural to focus on non-variant measures when comparing different mechanisms. Our paper provides a comparison between Boston mechanism and deferred acceptance in the context of segregation and school diversity.

Our paper is not the first attempt to connect school choice and segregation in mechanism design context. Avery and Pathak (2015) develop a model linking residential choice and school choice problem with mostly one-dimensional type. In their model, the qualities of schools are determined by peer effect and students' ordinal preferences over schools are homogeneous. In our paper, the qualities of schools are determined by expenditure and there are fundamentally different types of schools with heterogeneous ordinal preferences of students. Calsamiglia et al. (2017) compare Boston mechanism and deferred acceptance algorithm in the context of sorting between schools in a single district. The qualities of schools are determined by peer effect in their paper and their main model assumes one-dimensional type.

1.2 The baseline model

There is a city with equally sized two separate school districts. The size of each district is 1. In each district, there are three public schools and each school provides different types of education; Type A, type B, and type C. Let us denote the school type θ in district i by θ_i and the size of school type θ in district i by $S_i(\theta)$. There is a continuum of households with mass 2. Each household has one child who will enroll in a school and has two dimensional types. The first dimension is income $m \in [\underline{m}, \bar{m}]$. The second dimension is "student type" $a \in [\underline{a}, \bar{a}]$, which determines the relative preference between school type A and B. Each type profile (m, a) is drawn from a joint cdf Φ with pdf ϕ. We assume that m and a are either independent or affiliated throughout the paper. The utility of a household residing in district i and the child attending school type θ_i is given by

$$U(m, a, \theta, i) = u(a, \theta) + q_i(p_i) + v(m - p_i).$$

The utility of living outside of city is given by a constant u_o. p_i is the cost of housing associated with living in district i and $v(\cdot)$ represents the utility for money. $q_i(p_i)$ is the quality of schools district i and increasing in p_i. It reflects the fact that public schools are funded through local property taxes. $u(a, \theta)$ denotes a type specific utility for different types of schools. While we do not assume peer effect in school quality directly, there is an indirect peer effect via housing prices. We impose the following assumptions on the utility function and the school sizes.

Assumption 1.2.1. $u(a, \theta)$ is continuous in a. $u(a, A)$ is strictly increasing in a and $u(a, B)$ is strictly decreasing in a. There exists $\hat{a}$ such that $u(\hat{a}, A) = u(\hat{a}, B)$.

Assumption 1.2.2. $u(a, C) = 0$, $0 < u(a, A)$, and $0 < u(a, B)$ for all a.

Assumption 1.2.3. $v(\cdot)$ is strictly increasing and strictly concave.

Assumption 1.2.4. $S_i(\theta) = S_j(\theta) = S(\theta)$, for all $\theta = A, B, C$.

Assumption 1.2.5. $\Phi(\bar{m}, \hat{a}) > S(B)$ and $1 - \Phi(\bar{m}, \hat{a}) > S(A)$. Furthermore, We refer the case of $\frac{S(A)}{1-\Phi(\bar{m},\hat{a})} \neq \frac{S(B)}{\Phi(\bar{m},\hat{a})}$ as a generic case.

Assumption 1.2.6.

$$v(m - p) + q(p) = v(m - p') + q(p') \tag{1.1}$$

For the equation (1), there exists a unique solution $m(p,p')$ for any $p' > p$ and $m(p,p')$ is increasing in p'. Moreover,

$$\lim_{p\to\infty} v(m-p) + q(p) = -\infty.$$

Assumption 1.2.7.

$$\lim_{a\to\bar{a}} u(a,A) = \infty,\ \lim_{a\to\underline{a}} u(a,B) = \infty.$$

Assumption 2.1 implies that students with higher a have relatively stronger preference for A type schools. Assumption 2.2 implies that school type C is unanimously worst school regardless of student types, and we normalize the utility of attending C to 0. Assumption 2.3 is a usual assumption for utility of money and implies that households with higher income are less sensitive to the differences in housing prices. We assume that two districts are ex-ante symmetric to emphasize in assumption 2.4. Assumption 2.5 implies that both A and B type schools are over-demanded. Assumption 2.6 prevents the housing prices from exploding.

Throughout the paper, we will compare how different school choice mechanisms affect the composition of each schools. There are two dimension in types of households and we define a partial order for diversity regarding "student type" dimension as follows.

Definition 1. A school s_1 is more diverse than s_2 if $\{a|(a,m) \text{ attends } s_2\} \subset \{a|(a,m) \text{ attends } s_1\}$. A school s is *maximally diverse* if $\{a|(a,m) \text{ attends } s\} = [\underline{a},\bar{a}]$. [3]

Definition 2. A school s_1 and a school s_2 are *segregated by income* if $\sup\{m|(a,m) \text{ attends } s_1\} \leqslant \inf\{m|(a,m) \text{ attends } s_2\}$. A school s_1 and a school s_2 are *segregated by student type* if $\sup\{a|(a,m) \text{ attends } s_1\} \leqslant \inf\{a|(a,m) \text{ attends } s_2\}$.

Since there are multiple schools in each district, a centralized allocation mechanism is needed for each district even without open enrollment. Given a mechanism M, each household submits a ranking over schools. We consider a game between households that each

[3]The diversity defined in this way is related to richness index that is widely used in biology. If a school A is more diverse than B, A has a greater richness index than B. For additional information, Delang and Li (2013).

household simultaneously (i) chooses where to live and (ii) submits a ranking over schools. A strategy of a household is

$$\sigma(m, a) = (i, R(m, a)),$$

where i denotes the location choice and $R(\cdot)$ denotes ranking strategy given a mechanism M.

Definition 3. An *equilibrium* consists of housing prices (p_1, p_2) and a strategy $\sigma(m, a)$ such that

a. $\sigma(m, a)$ is optimal given (p_1, p_2), and

b. p_1 and p_2 clear the housing market.

This paper focuses on stable equilibria as in Benabou (1993) and Calsamiglia et al. (2017).

Definition 4. An equilibrium (σ, p_1, p_2) is *stable* if for any converging sequence (p_1^n, p_2^n) there is a sequence of strategy profiles σ^n such that $\sigma^n \to \sigma$ and each σ^n is optimal given (p_1^n, p_2^n).

1.3 Without Open Enrollment

In this section, we examine the implication of different student assignment mechanisms on the segregation of schools and districts without an open enrollment policy. Parents can send their kids only to the schools in the district they reside and three different mechanisms can be used to assign students within each district; The Boston mechanism, the deferred acceptance with single tie breaking, and the deferred acceptance with multi tie breaking. The next theorem shows that the deferred acceptance with multi tie breaking performs better than the others and the Boston mechanism performs worse than the others in terms of mitigating segregation between schools and fostering diversity in each school.

Theorem 1.3.1. *In the unique stable equilibrium with $p_2 > p_1$,*

(a) *schools in the same districts are not segregated under the deferred acceptance in generic cases,*

(b) A_i *and* B_i *are segregated under the Boston mechanism if* $S(C)$ *is large enough,*

(c) *all the schools are maximally diverse under the deferred acceptance with multi tie breaking,*

(d) *either* A_i *or* B_i *are maximally diverse under the deferred acceptance with single tie breaking in generic cases,*

(e) *each school is more diverse under the deferred acceptance with single tie breaking than under the Boston mechanism, if* m *and* a *are independent.*

The rest of the section explains how each mechanism runs and analyzes the equilibrium outcome of the game between households under each student assignment mechanism without open enrollment.

1.3.1 Boston Mechanism Without Open Enrollment

First, we investigate equilibria when the Boston mechanism is used without open enrollment. The Boston mechanism runs as follows.

1. Each household report a ranking of available schools.

2. In the first round, each school admits students that list the school as the first choice until there are no seats left or there are no students left who listed it as the first choice. Any acceptance is final. If a school is over-demanded, some students are rejected based on priority.

3. In round k, consider only the remaining student not yet accepted and k-th choices of students. Each school with available seats admits students until there are no seats left or there are no students left who listed it as the k-th choice.

We assume that schools have no pre-determined priorities.

Lemma 1.3.2. Under the Boston Mechanism, the equilibrium ranking strategy follows cutoff rule with cutoff $\hat{a}_i^{BM}$, where $a > \hat{a}^{BM}$ reports A_i as the most preferred school and $a < \hat{a}_i^{BM}$ reports B_i as the most preferred school.

Lemma 1.3.3. In any stable equilibria, $p_1 \neq p_2$.

Intuitively, any small perturbation of prices creates a quality difference of schools between districts and expensive neighborhood draws household with higher income. As a result, any symmetric equilibria unravel. With out a loss of generality we only consider equilibria with $p_2 > p_1$ as two districts are ex-ante symmetric.

Lemma 1.3.4. Let us denote the probability of being admitted to school θ_i by placing θ_i at the top in ranking strategy by $\alpha_{\theta_i,1}$ and the probability of being admitted to school θ_i by placing θ_i at the second by $\alpha_{\theta_i,2}$

(a) Placing C other than at the bottom in ranking strategy is a dominated strategy.

(b) If m and a are independent, $\alpha_{\theta_1,j} = \alpha_{\theta_2,j}$, for $j = 1, 2$, in any equilibria.

(c) If m and a are affiliated, $\alpha_{A_1,1} > \alpha_{A_2,1}$ and $\alpha_{B_1,1} < \alpha_{B_2,1}$ in any equilibria.

Lemma 3.4 (a) is a direct implication of the assumption that C is unanimously the worst school. In order to see why Lemma 3.4 (b) is true, consider a case where $\alpha_{A_1,1} < \alpha_{A_2,1}$. The only reason for $\alpha_{A_1,1} < \alpha_{A_2,1}$ is more students place A at the top in district 1 than in district 2 and it implies that more students demand A in district 1 than in district 2. However, the chance of being admitted to A type school is higher in district 2, thus, for any given income, district 2 attracts students with relatively higher a. On the other hand, $\alpha_{B_1,1} > \alpha_{B_2,1}$ must be true and district 1 attracts students with relatively low a as a result. If m and a are independent, as a result, district 2 becomes populated with more students with higher a and district 1 becomes populated with more students with lower a, which is a contradict to that

A type school in more demanded in district 1. The actual proof is more involved and in appendix A.

Proposition 1.3.5. *Denote the median income by $m_{1/2}$. If m and a are independent, in the unique stable equilibrium with $p_2 > p_1$,*

(a) $\sigma(m, a) = (2, R(m, a))$ *if and only if* $m \geqslant m_{1/2}$*,*

(b) $R(m, a) = (A_i, B_i, C_i)$ *if* $a > \hat{a}^{BM}$*,*

(c) $R(m, a) = (B_i, A_i, C_i)$ *if* $a < \hat{a}^{BM}$*, and*

(d) $\hat{a}^{BM} > \hat{a}$ *if and only if*

$$\frac{S(B)}{\Phi(\bar{m}, \hat{a})} > \frac{S(A)}{1 - \Phi(\bar{m}, \hat{a})}.$$

Under the independence of m and a, equilibrium outcome depicts complete segregation by income. This is a direct implication of Lemma 3.4. If there is no difference in the chance of being accepted to preferred type of school, the only difference between district comes from the quality difference. Since $v(\cdot)$ is concave, households with higher income are more willing to pay for expensive housing to buy an access to better schools.

Proposition 1.3.6. *If m and a are affiliated, in the unique stable equilibrium with $p_2 > p_1$,*

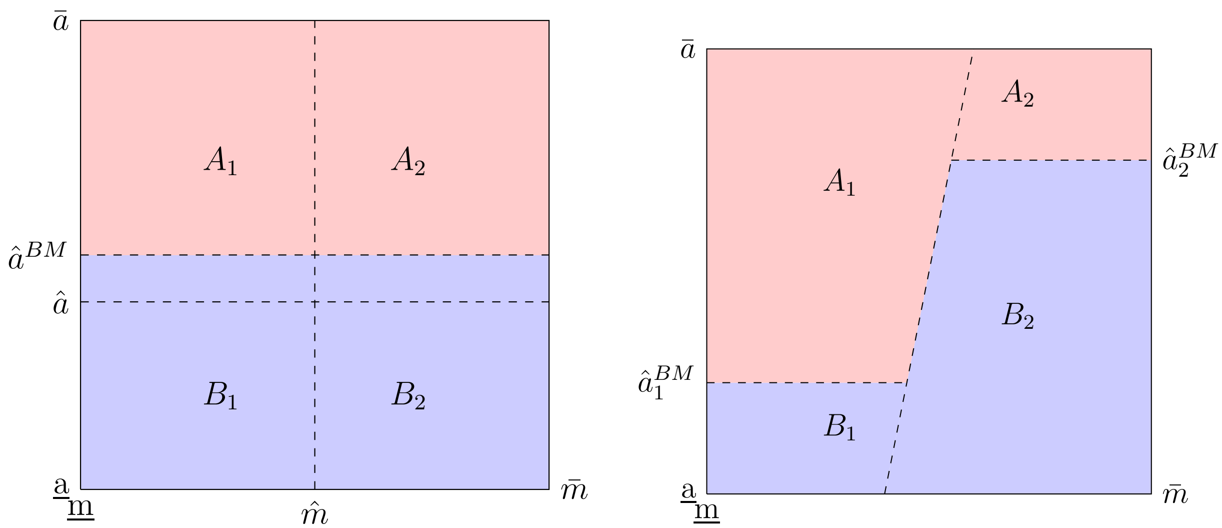
(a) $\sigma(m, a) = (2, R(m, a))$ *if and only if* $m \geqslant m(a)$*,*

(b) $m(a)$ *is increasing in* a*,*

(c) $R(m, a) = (A_i, B_i, C_i)$ *if* $a > \hat{a}_i^{BM}$*,*

(d) $R(m, a) = (B_i, A_i, C_i)$ *if* $a < \hat{a}_i^{BM}$*, and*

(e) $\hat{a}_2^{BM} > \hat{a}_1^{BM}$



(a) Independence

(b) Affiliation

Figure 1.1: Boston Mechanism

By Lemma 3.4, students have lower chance of being accepted to A in district 2 than in district 1. As a result, district 2 becomes less attractive to students with high a. At the same time, district 2 is more attractive to high income households as $p_2 > p_1$. While this leads to a contradiction if a and m are independent, the affiliation between a and m balances the scale so that there are enough students with high a and high m in district 2. Figure 1 depicts the equilibrium strategies for independence and affiliation cases.

1.3.2 Deferred Acceptance Without Open Enrollment

The deferred acceptance runs as follows.

1. Each household report a ranking of available schools.

2. Each round consists of two steps:

   - Applying: Each student applies to his most preferred school that did not reject him yet.

   - Rejections: Each school tentatively accept students as long as capacity allows. If there are no available seats left, rejects students based on priority.

3. The algorithm ends when no rejections are made in a round.

We assume that schools have no pre-determined priorities, thus, all the priorities are determined by lotteries. Under the single tie breaking, each student draws a lottery number independently and uniformly at random from $[0, 1]$ and a student with a smaller number has higher priority than a student with a larger number for all schools. Under the multi tie breaking, each student draws a separate lottery for each school independently and uniformly at random from $[0, 1]$ and priorities for each school are determined by the lotteries for the school.

Because deferred acceptance algorithm is strategy-proof for households, we can restrict our attention to truthful ranking strategy. The results in this section hold regardless of tie breaking rules. The lemma 3.2, 3,3, and 3.4 hold under the Deferred acceptance algorithm as well and the equilibrium strategies are characterized by the following propositions.

Proposition 1.3.7. *If m and a are independent, in the unique stable equilibrium with $p_2 > p_1$,*

(a) $\sigma(m, a) = (2, R(m, a))$ *if and only if* $m \geqslant m_{1/2}$*,*

(b) $R(m, a) = (A_i, B_i, C_i)$ *if* $a > \hat{a}$*,*

(c) $R(m, a) = (B_i, A_i, C_i)$ *if* $a < \hat{a}$*.*

Proposition 1.3.8. *If m and a are affiliated, in the unique stable equilibrium with $p_2 > p_1$,*

(a) $\sigma(m, a) = (2, R(m, a))$ *if and only if* $m \geqslant m(a)$*,*

(b) $m(a)$ *is increasing in* a*,*

(c) $R(m, a) = (A_i, B_i, C_i)$ *if* $a > \hat{a}$*,*

(d) $R(m, a) = (B_i, A_i, C_i)$ *if* $a < \hat{a}$*.*

Figure 2 depicts the equilibrium strategies for independence and affiliation cases.

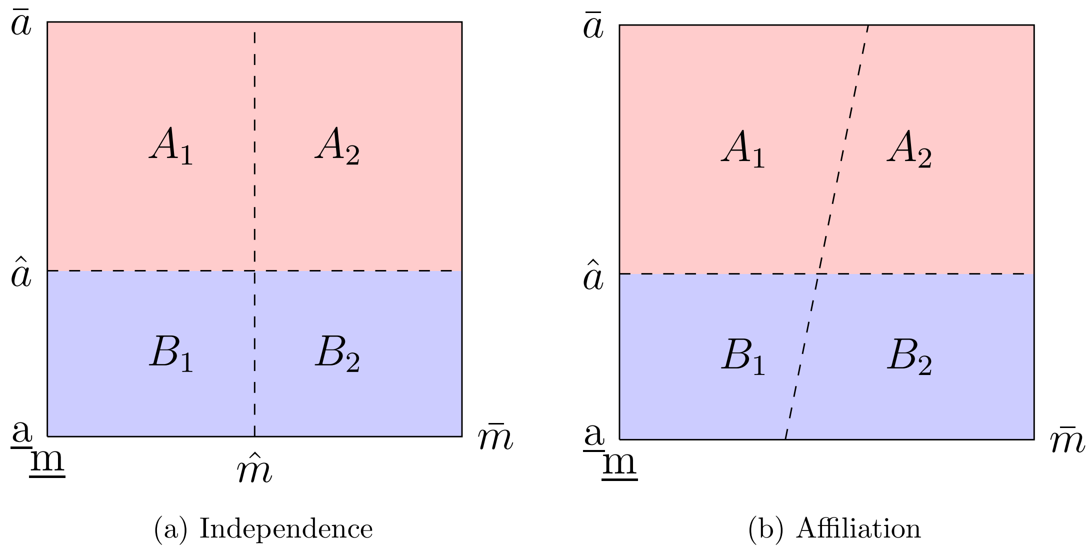


Figure 1.2: Deferred Acceptance

1.4 Open Enrollment

In this section, we do not study the game of locational choice. Rather than, we study the effect of open enrollment by taking the locational choice in section 2 as given, and we will focus on the short-run impact of introduction of the open enrollment program to the city where the neighborhood system has been used originally. Specifically, we will take q_2 and q_1 as given.

Under the open enrollment program, students can apply to schools in other districts, thus advantage of living in certain neighborhood disappears if the rule does not give priority for neighborhood students. The next theorem states that adopting open enrollment has no impact on the outcome if neighborhood priorities are used. While it is possible to apply to schools in the other district, seats are assigned to the students of the households residing in the district if the schools are over-demanded if neighborhood priorities are used.

Theorem 1.4.1. *Open enrollment with neighborhood priority (either Boston mechanism or deferred acceptance) does not change the outcome of the equilibrium without open enrollment.*

For the same type of schools, a school in district 2 is better than a school in district 2. Deferred acceptance algorithm does not allow any violation of priorities, thus, guarantees seats in district 2 to students in the neighborhood. Under the Boston mechanism, as students in district 2 have priorities over schools in the neighborhood, simply reporting the same rankings in the equilibrium without open enrollment is still an equilibrium. Students in district 1 do not have any chance of being admitted to schools in district 2, thus, place the same school at the top is still an equilibrium strategy.

Let us deviate from neighborhood priority to no priority case. Since we take the location choice as given, a strategy of each household is ranking strategy. Under the Boston Mechanism there are many equilibria because bottom of reported ranking usually does not affect the outcome due to the nature of the mechanism, thus we partially characterize the equilibrium strategies.

Proposition 1.4.2. *Under Boston Mechanism, any equilibria follow the following cutoff strategies.*

(a) cutoffs: $a^1 \geqslant a^2 \geqslant a^3 \geqslant a^4$,

(b) if $a > a^1$, *places* A_1 *at the top,*

(c) if $a^2 < a < a^1$, *places* A_2 *at the top,*

(d) if $a^2 < a < a^3$, *places* C_2 *at the top,*

(e) if $a^3 < a < a^4$, *places* B_2 *at the top,*

(f) if $a < a^4$, *places* B_1 *at the top.*

If $q_2 - q_1$ *is small enough,* $a_2 = a_3$.

The main driving force behind the cutoff equilibrium is the self-selection based on the relative intensity of preference toward the preferred type of school. Students with extreme a care more about the chance of being accepted to the preferred type of school A or B rather than the quality of schools. Students with a in the middle care more about the quality of schools than being accepted to the preferred type of school. Figure 3 depicts the intuition

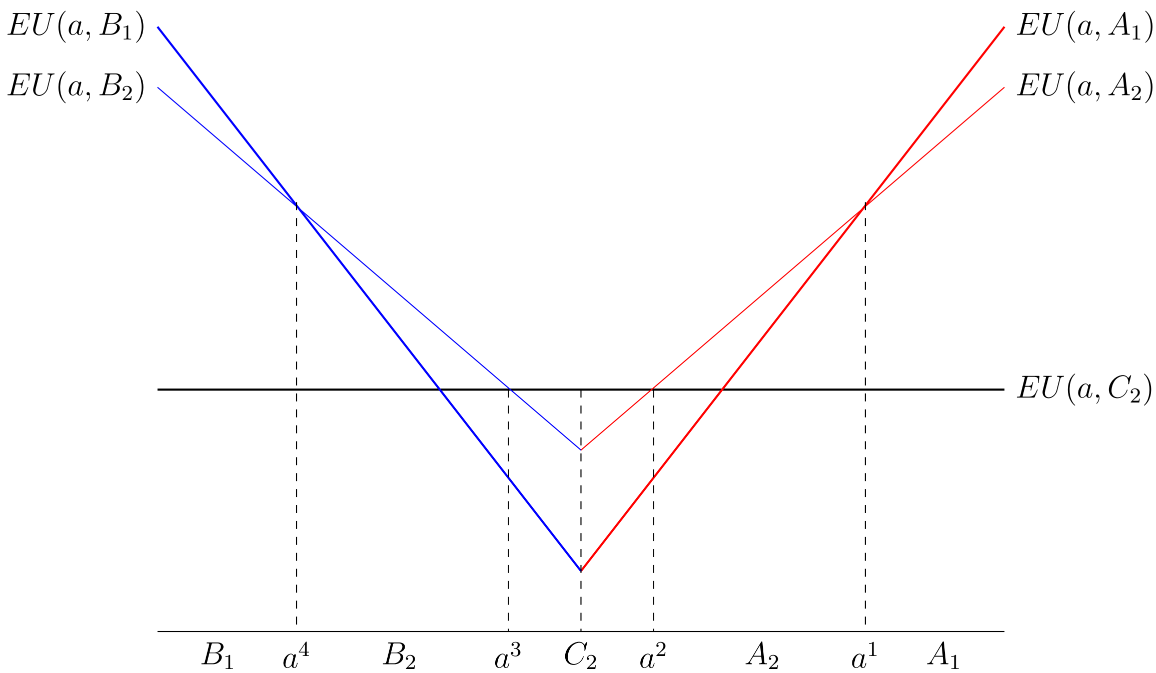


Figure 1.3: Boston Mechanism

behind Proposition 3.3. In the figure, $EU(a, \theta_i)$ denotes the expected utility in equilibrium by placing θ_i at the top. In equilibrium, A_2 and B_2 are more over-demanded than A_1 and B_1 as $q_2 > q_1$. On the other hand, since applying A_1 or B_1 in the first round give a higher chance of being accepted in the first round than A_2 or B_2. More extreme types care more about being admitted to the specific types of school, they are willing to sacrifice q_2 and place A_1 or B_1 at the top.

Figure 4 depicts the equilibrium strategy. While segregation pattern by income disappears, other pattern of sorting by a appears compared to the equilibrium without open enrollment.

The next theorem provides one possible reason why the open enrollment programs may fail to mitigate segregation.

Theorem 1.4.3. *Suppose that open enrollment is adopted to a city that has been using the Boston mechanism. Then, (a) A_2 and B_2 are segregated by student type. (b) If $q_2 - q_1$ is small enough, all schools but C_1 and C_2 are segregated by student type.*

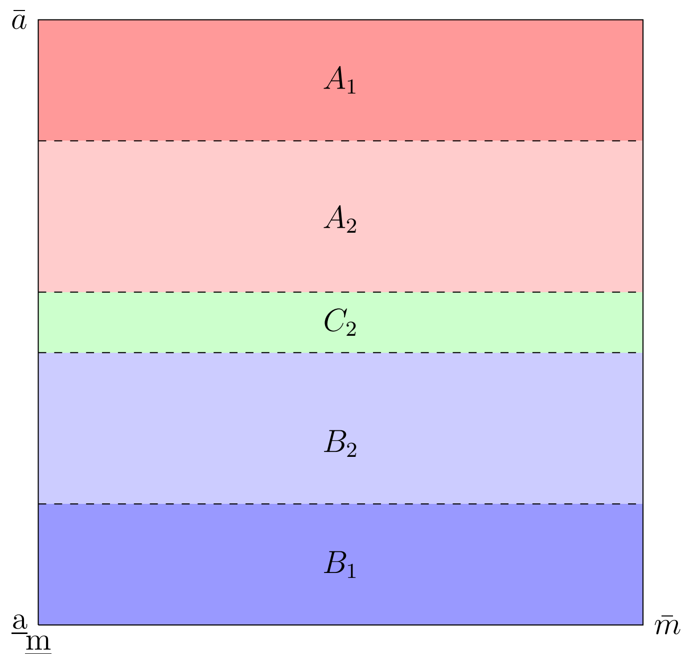


Figure 1.4: Boston Mechanism

Theorem 4.3 states that another form of segregation appears after open enrollment. Moreover, a pair of schools that were not segregated before the open enrollment can be segregated after the introduction of the open enrollment if Boston mechanism is used. Notice that A_1 and B_1 were not segregated without the open enrollment. Even though Theorem 4.3 analyzes the pattern of segregation in the short-run after the introduction of the open enrollment, we want to emphasize (b) of Theorem 3.3 has an implication in the long run. In the long run, as housing prices become adjusted the difference between $q(p_2)$ and $q(p_1)$ becomes smaller. Theorem 4.3 (b) implies that the pattern of segregation might not disappear if the difference in housing prices become arbitrarily small. As far as we know, there is no empirical work on studying the impact of open enrollment combining different mechanisms. However, school districts in Twin cities experienced an increase in segregation after the adoption of open enrollment and they use the Boston mechanism.[4]

Now, we investigate the impact of open enrollment under deferred acceptance algorithm. Truthful reporting is a dominant strategy under the deferred acceptance algorithm regardless

[4]See Institute on Metropolitan Opportunity (2013).

of tie breaking rule, thus, students with $a > \hat{a}$ will place A_2 at the top and students with $a < \hat{a}$ will place B_2 at the top in ranking strategy.

Proposition 1.4.4. *Under the deferred acceptance with single tie breaking and open enrollment, the equilibrium outcome is as follows.*

(a) If $\frac{S(B)}{\Phi(\bar{m},\hat{a})} > \frac{S(A)}{1-\Phi(\bar{m},\hat{a})}$, *then*

- A_2 *is composed of only students with a higher than* $\hat{a}$, *(i.e.,* $\{a|(m,a) \text{ attends } A_2\} = [\hat{a}, \bar{a}]$*) and*
- B_1 *and* B_2 *are maximally diverse*

(b) If $\frac{S(B)}{\Phi(\bar{m},\hat{a})} < \frac{S(A)}{1-\Phi(\bar{m},\hat{a})}$, *then*

- B_2 *is composed of only students with a lower than* $\hat{a}$, *(i.e.,* $\{a|(m,a) \text{ attends } B_2\} = [\underline{a}, \hat{a}]$*) and*
- A_1 *and* A_2 *are maximally diverse.*

The intuition behind the proposition is as follows. Single tie breaking favors applicants accepted in the earlier round as the those applicants pick relatively better lotteries and has higher chance of securing the seats as the applicants competing those seats later rounds likely picked bad draws. For instance, if A_2 is relatively more demanded than B_2, there is no chance of being accepted to A_2 after the first round. On the other hand, under the multi tie breaking rule, there is any students accepted to a school in the first round can be rejected down the road if the school becomes over-demanded in some round of the algorithm.

The next theorem contrasts the deferred acceptance algorithm with the Boston mechanism in that the deferred acceptance is more resilient to segregation than the Boston mechanism when the open enrollment is adopted.

Theorem 1.4.5. *(a) The deferred Acceptance with multi tie breaking under open enrollment results in maximally diverse schools.*

(b) No schools are segregated under the deferred acceptance with any tie breaking.

1.5 Concluding Remarks

This paper has studied the comparison between different school choice systems and mechanisms by connecting multi-community model and mechanism design problem. On the one hand, there are empirical evidences suggesting that open enrollment program decreased diversity of each school and this paper provide one possible reason by focusing on the role of specific student assignment mechanism and how they affect the outcome differently. On the other hand, there have been many studies comparing the Boston mechanism and the deferred acceptance mechanism in market design literature. However, there are very few research about how they compare combined with location choices or under specific environments with regard to segregation and diversity of schools and this paper provides one more aspect to consider in comparing school choice mechanisms.

We show that the Boston mechanism can create less diverse schools than the deferred acceptance mechanism even under the neighborhood school system. Combined with open enrollment, the difference between two mechanisms become more drastic and open enrollment program actually worsens segregation if it is combined with Boston mechanism. The deferred acceptance with multi tie breaking makes all schools maximally diverse.

1.6 Appendix

1.6.1 Proofs

Lemma 3.2

Consider a district i. Notice that reporting C_i as the most or second preferred school is a dominated strategy. Denote the mass of students that report A as the most preferred school by β and denote the expected utility of reporting θ_i as the most preferred school by $EU(a, \theta, \beta)$. Since $EU(a, A, \beta)$ is increasing in a and $EU(a, B, \beta)$ is decreasing in a, there exists a cutoff $a^{BM}(\beta)$ such that $EU(a, A, \beta) \geqslant EU(a, \theta, \beta)$. Now, we show the existence of the unique equilibrium. Denote the cutoff type a by $a(\beta)$ given β. For $S(A) < \beta < 1 - S(B)$,

$$EU(a, A, \beta) = \frac{S(A)}{\beta} u(a, A),\ EU(a, B, \beta) = \frac{S(B)}{1 - \beta} u(a, B),$$

because both A and B are filled in the first round of the Boston mechanism. Then,

$$\frac{S(A)}{\beta} u(a(\beta), A) = \frac{S(B)}{1 - \beta} u(a(\beta), B).$$

Note that $a(\beta)$ is increasing in β. Now suppose that $\beta < S(A)$. Then, only B is filled in the first round of the Boston mechanism and

$$EU(a, A, \beta) = u(a, A),\ EU(a, B, \beta) = \frac{S(B)}{1 - \beta} u(a, B) + \frac{S(A) - \beta}{1 - \beta} u(a, A),$$

and

$$EU(a, A, \beta) = EU(a, B, \beta) \iff (1 - S(A)) u(a, A) = S(B) u(a, B).$$

Therefore, $\alpha(\beta) = \tilde{a}$ for some $\tilde{a} < \hat{a}$. The inequality $\tilde{a} < \hat{a}$ comes from $1 - S(A) > S(B)$. Similarly, if $\beta > 1 - S(B)$, the cutoff $a(\beta)$ is another constant $\dot{a} > \hat{a}$. To show the existence, it is enough to show that there exists $\beta \in [0,1]$ such that

$$\Gamma(\beta) = \beta - \int_{\underline{a}}^{a(\beta)} \phi_i(a)da = 0.$$

Note that $\Gamma(0) = -\int_{\underline{a}}^{\tilde{a}} \phi_i(a)da < 0$ and $\Gamma(1) = 1 - \int_{\underline{a}}^{\tilde{a}} \phi_i(a)da > 0$. By the intermediate value theorem, there exists β such that $\Gamma(\beta) = 0$ and such β is unique since $\alpha(\beta)$ is monotone.

Lemma 3.3

Consider an equilibrium where $p_1 = p_2 = p$ and a sequence of prices $p_1^n < p_2^n$ converging to (p,p). By Lemma 3.4, the equilibrium residential choice follows a cutoff form, where (m,a) chooses district 1 if and only if $m \leqslant m^n(a)$.[5] Now, pick a different sequence of prices, where $p_1^n > p_2^n$. The residential choice follows a cutoff form, where (m,a) chooses district 1 if and only if $m > m^n(a)$. The limit of two different residential choice cannot coincide, thus, an equilibrium with $p_1 = p_2$ cannot be stable.

Lemma 3.4

(b): Denote the mass of students that place school A at the top in district i by β_i. Then,

$$\begin{aligned}
\alpha_{A_i,1} &= \min\Big\{1, \frac{S(A)}{\beta_i}\Big\}, \\
\alpha_{B_i,1} &= \min\Big\{1, \frac{S(B)}{1-\beta_i}\Big\}, \\
\alpha_{A_i,2} &= \Big(1 - \min\Big\{1, \frac{S(B)}{1-\beta_i}\Big\}\Big)\Big(\frac{\max\{0, S(A)-\beta_i)\}}{(1-\beta_i) - S(B)}\Big), \\
\alpha_{B_i,2} &= \Big(1 - \min\Big\{1, \frac{S(A)}{\beta_i}\Big\}\Big)\Big(\frac{\max\{0, S(B)-(1-\beta_i)\}}{\beta_i - S(A)}\Big).
\end{aligned}$$

Notice that at least one of $\alpha_{A_i,1}$ or $\alpha_{B_i,1}$ should be less one since $S(A) + S(B) < 1$, for each i. We will show that $\beta_1 = \beta_2$ in any equilibria.

Suppose that $\beta_1 > \beta_2$ in an equilibrium. Then $\alpha_{A_1,1} \leqslant \alpha_{A_2,1}$ with equality only if both

[5] This part of Lemma 3.4 does not rely on lemma 3.2.

are 1. $\alpha_{A_1,1} = \alpha_{A_2,1} = 1$ implies that there is a student with $a > \hat{a}$ that applies B first than A by Assumption 2.5, which leads to a contradiction to equilibrium condition since such a student can apply to A and be accepted for sure. By the same reason, $\alpha_{B_1,1} = \alpha_{B_2,1} = 1$ does not happen in any equilibria and we can conclude that $\alpha_{A_1,1} < \alpha_{A_2,1}$ and $\alpha_{B_1,1} > \alpha_{B_2,1}$. This implies that $\alpha_{B_2,2} = 0$ since students can be accepted to the second choice only if there is a remaining seat in the school after the first round under Boston mechanism. As we assume that $\beta_1 > \beta_2$, $\alpha_{A_1,2} = \alpha_{B_2,2} = 0$. We can write the expected utility of a student in district i to applying to school θ_i first as follows.

$$EU(m, a, A_1) := \alpha_{A_1,1} u(a, A) + \alpha_{B_1,2} u(a, B) + q(p_1) + v(m - p_1),$$

$$EU(m, a, B_1) := \alpha_{B_1,1} u(a, B) + q(p_1) + v(m - p_1),$$

$$EU(m, a, A_2) := \alpha_{A_2,1} u(a, A) + q(p_2) + v(m - p_2),$$

$$EU(m, a, B_2) := \alpha_{A_2,2} u(a, A) + \alpha_{B_2,1} u(a, B) + q(p_2) + v(m - p_2).$$

Since $\beta_1 > 0$, either there exists a cutoff $\hat{a}_1$ such that $EU(m, \hat{a}_1, A_1) = EU(m, \hat{a}_1, B_1)$ and any students above $\hat{a}_1$ in district 1 apply A first in the equilibrium or every student in the district applies to A first. In district 2, either there exists a cutoff $\hat{a}_2$ such that $EU(m, \hat{a}_2, A_2) = EU(m, \hat{a}_2, B_2)$ and any students above $\hat{a}_2$ apply A first in the equilibrium or every student in the district applies to B first. If such cutoffs $\hat{a}_1$ and $\hat{a}_2$ exist, the cutoffs solve the following equations.

$$\frac{\alpha_{A_1,1}}{(\alpha_{B_1,1} - \alpha_{B_1,2})} u(\hat{a}_1, A) = u(\hat{a}_1, B), \tag{1}$$

$$\frac{(\alpha_{A_2,1} - \alpha_{A_2,2})}{\alpha_{B_2,1}} u(\hat{a}_2, A) = u(\hat{a}_2, B). \tag{2}$$

We argue that $\hat{a}_2 < \hat{a}_1$.

Consider a case where both $\alpha_{A_2,2} = \alpha_{B_1,2} = 0$. In other words, both types of schools are over-demanded in the first round of Boston mechanism. Then,

$$\frac{\alpha_{A_1,1}}{(\alpha_{B_1,1}-\alpha_{B_1,2})}=\frac{1-\beta_1}{\beta_1}\frac{S(A)}{S(B)},\ \frac{(\alpha_{A_2,1}-\alpha_{A_2,2})}{\alpha_{B_2,1}}=\frac{1-\beta_2}{\beta_2}\frac{S(A)}{S(B)}.$$

Since we assume that $\beta_2<\beta_1$, then $\frac{\alpha_{A_1,1}}{(\alpha_{B_1,1}-\alpha_{B_1,2})}<\frac{(\alpha_{A_2,1}-\alpha_{A_2,2})}{\alpha_{B_2,1}}$ and $\hat{a}_2<\hat{a}_1$.

Consider a case where both $\alpha_{A_2,2}$ and $\alpha_{B_1,2}$ are postive. This happens only if A_2 and B_1 are under-demanded in the first round of Boston mechanism, thus, $\alpha_{A_2,1}=\alpha_{B_1,1}=1$ and

$$\frac{\alpha_{A_1,1}}{(\alpha_{B_1,1}-\alpha_{B_1,2})}=\frac{S(A)}{1-S(B)},\ \frac{(\alpha_{A_2,1}-\alpha_{A_2,2})}{\alpha_{B_2,1}}=\frac{1-S(A)}{S(B)}.$$

Notice that $S(A)+S(B)<1$ implies that $\frac{\alpha_{A_1,1}}{(\alpha_{B_1,1}-\alpha_{B_1,2})}<\frac{(\alpha_{A_2,1}-\alpha_{A_2,2})}{\alpha_{B_2,1}}$, thus, $\hat{a}_2<\hat{a}_1$.

Consider a case where $\alpha_{A_2,2}>0$ and $\alpha_{B_1,2}=0$. In this case, A_2 is under-demanded in the first round. Then,

$$\frac{\alpha_{A_1,1}}{(\alpha_{B_1,1}-\alpha_{B_1,2})}=\frac{1-\beta_1}{\beta_1}\frac{S(A)}{S(B)},\ \frac{(\alpha_{A_2,1}-\alpha_{A_2,2})}{\alpha_{B_2,1}}=\frac{1-S(A)}{S(B)}.$$

Since $\alpha_{A_1,1}<1$, $S(A)<\beta_1$. This implies $\frac{\alpha_{A_1,1}}{(\alpha_{B_1,1}-\alpha_{B_1,2})}<\frac{(\alpha_{A_2,1}-\alpha_{A_2,2})}{\alpha_{B_2,1}}$ and $\hat{a}_2<\hat{a}_1$.

Consider a case where $\alpha_{A_2,2}=0$ and $\alpha_{B_1,2}>0$. In this case, B_1 is under-demanded in the first round, $\alpha_{B_1,1}=1$, and $\alpha_{B_2,1}<1$.

$$\frac{\alpha_{A_1,1}}{(\alpha_{B_1,1}-\alpha_{B_1,2})}=\frac{S(A)}{1-S(B)},\ \frac{(\alpha_{A_2,1}-\alpha_{A_2,2})}{\alpha_{B_2,1}}=\frac{1-\beta_2}{\beta_2}\frac{S(A)}{S(B)}.$$

In this case, $\frac{S(B)}{1-\beta_2}<1$ implies $\frac{\alpha_{A_1,1}}{(\alpha_{B_1,1}-\alpha_{B_1,2})}<\frac{(\alpha_{A_2,1}-\alpha_{A_2,2})}{\alpha_{B_2,1}}$ and $\hat{a}_2<\hat{a}_1$.

By Assumption 2.7, for any given a, there exist a cutoff income $m(a)$ where (m,a) is indifferent between two districts. We argue that $m(a)$ is decreasing in a. As we established that $\hat{a}_2<\hat{a}_1$, there are three disjoint intervals in $[\underline{a},\bar{a}]$ we need to consider.

For $a \in [\hat{a}_1, \bar{a}]$, $m(a)$ solves the following equation.

$$(\alpha_{A_1,1} - \alpha_{A_2,1})u(a, A) + \alpha_{B_1,2}u(a, B) = q(p_2) - q(p_1) - (v(m - p_1) - v(m - p_2)).$$

Notice that the left hand side is decreasing in a and the right hand side is increasing in m, thus, $m(a)$ is decreasing in a.

For $a \in [\hat{a}_2, \hat{a}_1]$, $m(a)$ solves the following equation.

$$\alpha_{B_1,1}u(a, B) - \alpha_{A_2,1}u(a, A) = q(p_2) - q(p_1) - (v(m - p_1) - v(m - p_2)).$$

The left hand side is decreasing in a and the right hand side is increasing in m, thus, $m(a)$ is decreasing in a.

For $a \in [\bar{a}, \hat{a}_2]$, $m(a)$ solves the following equation.

$$(\alpha_{B_1,1} - \alpha_{B_2,1})u(a, B) - \alpha_{A_2,2}u(a, A) = q(p_2) - q(p_1) - (v(m - p_1) - v(m - p_2)).$$

the left hand side is decreasing in a and the right hand side is increasing in m, thus, $m(a)$ is decreasing in a.

Now, we show that β_2 cannot be smaller than β_1.

$$\begin{aligned}
\beta_1 &= 2\int_{\hat{a}_1}^{\bar{a}} \int_{\underline{m}}^{m(a)} \phi_m(m)\phi_a(a)dmda = 2\int_{\hat{a}_1}^{\bar{a}} \Phi_m(m(a))\phi_a(a)da, \\
\beta_2 &= 2\int_{\hat{a}_2}^{\bar{a}} \int_{m(a)}^{\bar{m}} \phi_m(m)\phi_a(a)dmda = 2\int_{\hat{a}_2}^{\bar{a}} (1 - \Phi_m(m(a)))\phi_a(a)da, \\
1 - \beta_1 &= 2\int_{\underline{a}}^{\hat{a}_1} \int_{\underline{m}}^{m(a)} \phi_m(m)\phi_a(a)dmda = 2\int_{\underline{a}}^{\hat{a}_1} \Phi_m(m(a))\phi_a(a)da, \\
1 - \beta_2 &= 2\int_{\underline{a}}^{\hat{a}_2} \int_{m(a)}^{\bar{m}} \phi_m(m)\phi_a(a)dmda = 2\int_{\underline{a}}^{\hat{a}_2} (1 - \Phi_m(m(a)))\phi_a(a)da.
\end{aligned}$$

Then, $\beta_2 + 1 - \beta_1 > \beta_1 + 1 - \beta_2$ for the following reason.

$$\beta_2 + 1 - \beta_1 = 2\int_{\hat{a}_1}^{\bar{a}} (1 - \Phi_m(m(a)))\phi_a(a)da + 2\int_{\underline{a}}^{\hat{a}_2} (\Phi_m(m(a)))\phi_a(a)da + 2(\Phi_a(\hat{a}_1) - \Phi_a(\hat{a}_2)),$$
$$\beta_1 + 1 - \beta_2 = 2\int_{\hat{a}_1}^{\bar{a}} \Phi_m(m(a))\phi_a(a)da + 2\int_{\underline{a}}^{\hat{a}_2} (1 - \Phi_m(m(a)))\phi_a(a)da.$$

Notice that $\Phi_m(m(a))$ and $1 - \Phi_m(m(a))$ cross exactly once and

$$\int_{\underline{a}}^{\bar{a}} \Phi_m(m(a))\phi_a(a)da = \frac{1}{2}.$$

Moreover, $\Phi_m(m(a))$ is decreasing in a. This implies that, for any $x \in [\underline{a}, \bar{a}]$,

$$\int_{x}^{\bar{a}} (1 - \Phi_m(m(a)))\phi_a(a)da \geqslant \int_{x}^{\bar{a}} \Phi_m(m(a))\phi_a(a)da,$$
$$\int_{\underline{a}}^{x} \Phi_m(m(a))\phi_a(a)da \geqslant \int_{\underline{a}}^{x} (1 - \Phi_m(m(a)))\phi_a(a)da.$$

Therefore, $\beta_2 + 1 - \beta_1 > \beta_1 + 1 - \beta_2$ and $\beta_2 > \beta_1$, which is a contradiction to the assumption $\beta_2 < \beta_1$.

Now, suppose $\beta_1 < \beta_2$. Then, $m(a)$ is increasing in a and it leads to the same contraction.

(c): Notice that the proof of (b) does not rely on independence until the last step. Suppose that $\beta_2 < \beta_1$. Then, $m(a)$ in decreasing in a and $\hat{a}_1 > \hat{a}_2$.

$$\beta_2 + 1 - \beta_1 = 2\int_{\hat{a}_1}^{\bar{a}} (1 - \Phi(m(a)|a))\phi_a(a)da + 2\int_{\underline{a}}^{\hat{a}_2} (\Phi(m(a)|a))\phi_a(a)da + 2(\Phi_a(\hat{a}_1) - \Phi_a(\hat{a}_2)),$$
$$\beta_1 + 1 - \beta_2 = 2\int_{\hat{a}_1}^{\bar{a}} \Phi(m(a)|a)\phi_a(a)da + 2\int_{\underline{a}}^{\hat{a}_2} (1 - \Phi(m(a)|a))\phi_a(a)da.$$

Note that $\Phi(m(a)|a))$ is decreasing in a due to the stochastic dominance and decreasing $m(a)$.

Suppose that $\beta_1 = \beta_2$. Then, $m(a)$ does not change with a and $\hat{a}_1 = \hat{a}_2 = \hat{a}$. Denote the median income by $m_{1/2}$.

$$\beta_1 = 2\int_{\hat{a}}^{\bar{a}} \Phi(m_{1/2}|a)\phi_a(a)dmda,\ \beta_2 = 2\int_{\hat{a}}^{\bar{a}} (1-\Phi(m_{1/2}|a))\phi_a(a)dmda.$$

Stochastic dominance implies that $\beta_2 > \beta_1$, which is a contradiction.

Proposition 3.5

(a): By Lemma 3.4, we know that $\beta_1 = \beta_2$. Given (p_1, p_2) there is a cutoff income $m(p_1, p_2)$ such that

$$v(m-p_1) + q(p_1) = v(m-p_2) + q(p_2).$$

(b), (c): Ranking strategy in any equilibria follows a cutoff strategy as established in Lemma 3.4. $\hat{a}_1 = \hat{a}_2$ is implied by $\beta_1 = \beta_2$.

(d): For $\beta_1 = \beta_2 = \beta$, both types of schools are over-demanded in the first round and cutoff strategy is characterized by the equation

$$\frac{S(A)}{\beta}u(\hat{a}^{BM}, A) = \frac{S(B)}{1-\beta}u(\hat{a}^{BM}, B).$$

If $\frac{S(A)}{\beta} \geqslant \frac{S(B)}{1-\beta}$, then $\hat{a}^{BM} \leqslant \hat{a}$. In equilibrium,

$$\beta = 1 - \Phi(\bar{m}, \hat{a}^{BM}).$$

Then, the following inequalities hold.

$$\frac{S(B)}{\Phi(\bar{m}, \hat{a})} \leqslant \frac{S(B)}{\Phi(\bar{m}, \hat{a}^{BM})} \leqslant \frac{S(A)}{1-\Phi(\bar{m}, \hat{a}^{BM})} \leqslant \frac{S(A)}{1-\Phi(\bar{m}, \hat{a})}.$$

This is a contradiction to the assumption that $\frac{S(A)}{1-\Phi(\bar{m},\hat{a})} < \frac{S(B)}{\Phi(\bar{m},\hat{a})}$.

To establish the uniqueness of the equilibrium, notice that p_1 is determined by the indifference condition of the household with minimum expected utility in district 1 so that the household is indifferent between living in district 1 and living outside of the city. The household's type with minimum expected utility is $(\underline{m}, \hat{a}^{BM}$. The equilibrium p_1 solves

$$\frac{S(B)}{\Phi(\bar{m}, \hat{a}^{BM})} u(\hat{a}^{BM}, B) + q_1(p_1) + v(m - p_1)$$

Existence of the solution is guaranteed by Assumption 2.6. There can be more than one prices that solve the equation, and the maximum solution p_1 is the equilibrium price in that case. Given the p_1, the market clearing price p_2 solves

$$v(m_{1/2} - p_2) + q(p_2) = v(m_{1/2} - p_1) + q(p_1).$$

Proposition 3.6

If an equilibrium exists, (a),(b),(c),(d), and (e) are already established in the proof of Lemma 3.4. Suppose that β_i mass of students apply to A_i and $1 - \beta_i$ mass of students apply to B_i for each i. Then the cutoff $\hat{a}_i$ is increasing in β_i and denote the relationship by $\hat{a}_i = \hat{a}^(\beta_i)$.

To show this proposition, we will fix an arbitrary p_1 and show the existence of the equilibrium strategy profile and p_2 that clears the housing market by ignoring outside option. As p_1 can be arbitrary, it shows the existence of the equilibrium.

Given $\beta_2 > \beta_1$, the cutoff income m solves,

$$EU(m, a, A_1) = EU(m, a, A_2), \ \forall \ a \in [\hat{a}(\beta_2), \bar{a}],$$

$$EU(m, a, A_1) = EU(m, a, B_2), \ \forall \ a \in [\hat{a}(\beta_1), \hat{a}(\beta_2)],$$

$$EU(m, a, B_1) = EU(m, a, B_2), \ \forall \ a \in [\underline{a}, \hat{a}(\beta_1)].$$

Denote the solution m by $m = m(a, p_2, \beta_1, \beta_2)$. Then, m is increasing in β_2 and decreasing in β_1 for $a \in [\hat{a}(\beta_2), \bar{a}]$ since the higher β_2 makes district 2 less attractive and increasing β_1 makes district 2 more attractive for the students who will place A_2 at the top. For $a \in [\underline{a}, \hat{a}(\beta_1)]$, m is decreasing in β_2 and increasing in β_1 for the opposite reason. For $a \in [\hat{a}(\beta_1), \hat{a}(\beta_2)]$, m is decreasing in each of both β_1 and β_2 since the students with those types will place B_2 at the top in district 2 and place A_1 at the top in district 1.

The equilibrium is characterized by the following three equations.

$$2\int_{\underline{a}}^{\bar{a}}\int_{m(a,p_2,\beta_1,\beta_2)}^{\bar{m}} \phi(m,a)dmda = 1, \tag{2}$$

$$2\int_{\underline{a}}^{\hat{a}(\beta_1)}\int_{\underline{m}}^{m(a,p_2,\beta_1,\beta_2)} \phi(m,a)dmda = 1-\beta_1, \tag{3}$$

$$2\int_{\hat{a}(\beta_2)}^{\bar{a}}\int_{m(a,p_2,\beta_1,\beta_2)}^{\bar{m}} \phi(m,a)dmda = \beta_2. \tag{4}$$

The first equation is the housing market clearing condition, and the last two are consistency condition. Denote the solution of (3) by $\beta_1 = \beta_1(p_2, \beta_2)$ and the solution of (4) by $\beta_2 = \beta_2(p_2, \beta_1)$. The existence and uniqueness of the solutions $\beta_1(\cdot,\cdot)$ and $\beta_2(\cdot,\cdot)$ is established in Lemma 2.8. From (3), $\beta_1(p_2, \beta_2)$ is increasing in β_2 because $m(a, \beta_1, \beta_2, p_2)$ is decreasing in β_2. From (4), $\beta_2(p_2, \beta_1)$ is increasing in β_1 because $m(a, \beta_1, \beta_2, p_2)$ is decreasing in β_1. Also, notice that $a(\beta_1)$ and $a(\beta_2)$ are bounded below and above by some constants $\tilde{a}$ and $\dot{a}$.[6] This implies that

$$\lim_{\beta_2 \to 1} \beta_1(p_2, \beta_2) < 1, \ \lim_{\beta_1 \to 1} \beta_2(p_2, \beta_1) < 1, \tag{5}$$

$$\lim_{\beta_2 \to 0} \beta_1(p_2, \beta_2) > 0, \ \lim_{\beta_1 \to 0} \beta_2(p_2, \beta_1) > 0, \tag{6}$$

Combining the monotonicity of $\beta_1(\beta_2, p_2)$ and $\beta_2(\beta_1, p_2)$, (5), and (6), there is a unique fixed

[6]This is also established in Lemma 2.8.

point $(\beta_1^*(p_2),\, \beta_2^*(p_2))$ such that

$$\beta_2(\beta_1^*(p_2), p_2) = \beta_2^*(p_2),\ \beta_1^*(p_2) = \beta_1(\beta_2^*(p_2), p_2). \tag{7}$$

By Assumption 2.6, $m(a, p_2, \beta_1, \beta_2)$ is decreasing in p_2, thus, there exists unique p_2^* that solves the equation (2). So far, we show that there exist a unique solution to the system of equations (2), (3), and (4). One needs to verify the solution $\beta_2^* > \beta_1^*$. Fix the equilibrium by β_1^*, β_2^*, and p_2^* and denote the cutoff income in the equilibrium by $m^*(a)$.

Claim 1. $\lim_{\beta_1\to\beta,\beta_2\to\beta} m(a, p_2^*, \beta_1, \beta_2) = m^*(\hat{a}(\beta))$

Proof. This follows from that how $m(a, \beta_1, \beta_2)$ rotates about $\hat{a}(\beta_i)$ as β_j changes.

Consider the equation (4) given p_2^*. We look for a fixed point of a mapping $\beta_2(\beta_1, p_2^*) = \beta_1$. Denote the fixed point $\tilde{\beta}_2$. Then,

$$2\int_{\hat{a}(\tilde{\beta}_2)}^{\bar{a}} 1 - \Phi(m^*(\hat{a}(\tilde{\beta}_2)|a)\phi_a(a)da = \tilde{\beta}_2.$$

Similarly, let us denote the fixed point from the equation (3) by $\tilde{\beta}_1$, then

$$2\int_{\underline{a}}^{\hat{a}(\tilde{\beta}_1)} \Phi(m^*(\hat{a}(\tilde{\beta}_1)|a)\phi_a(a)da = 1 - \tilde{\beta}_1.$$

The existence and uniques of $\tilde{\beta}_2$ and $\tilde{\beta}_1$ follow from the intermediate value theorem.

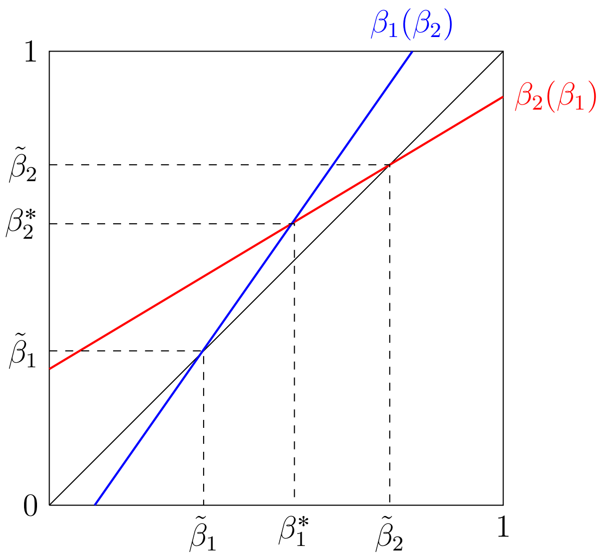
Claim 2. $\tilde{\beta}_2 > \tilde{\beta}_1$.

Proof. Suppose that $\tilde{\beta}_2 \leqslant \tilde{\beta}_1$ and denote $\hat{a}(\tilde{\beta}_1) = \tilde{a}_1$, $\hat{(}\tilde{\beta}_2) = \tilde{a}_2$, $m^*(\hat{a}(\tilde{\beta}_1 = m_1^*$, and $m^*(\hat{a}(\tilde{\beta}_2 = m_2^*$. Then, $\tilde{a}_2 \leqslant \tilde{a}_1$ and $m_2^* \leqslant m_1^*$. Notice that

$$2\int_{\tilde{a}_2}^{\bar{a}}(1-\Phi(m_2^*|a))\phi_a(a)da = \tilde{\beta}_2,$$
$$2\int_{\underline{a}}^{\tilde{a}_1}\Phi(m_1^*|a)\phi_a(a)da = 1-\tilde{\beta}_1.$$

The same trick used in the proof of Lemma 3.4 can be used here to show $\tilde{\beta}_2 + 1 - \tilde{\beta}_1 > \tilde{\beta}_1 + 1 - \tilde{\beta}_2$, which is a contradiction.

So far, we have shown that the fixed point of $\beta_2(\beta_1, p_2^*)$ is greater than the fixed point of $\beta_1(\beta_2, p_2^*)$ and the fixed points are unique. The following diagram shows that $\beta_2^* > \beta_1^*$.



Proposition 3.7

Since the deferred acceptance is strategy-proof, in any non-dominated equilibrium, households report truthful ranking and (b) and (c) directly follow. To establish (a), Suppose that $\beta_2 > \beta_1$, which implies that $\alpha_{A_2,1} < \alpha_{A_1,1}$ and $\alpha_{B_2,2} > \alpha_{B_1,2}$ Then, cutoff income $m(a)$ is increasing and this leads to a contradiction as follows.

$$\beta_2 = 2\int_{\hat{a}}^{\bar{a}} (1 - \Phi_m(m(a)))\phi_a(a)da,$$
$$\beta_1 = 2\int_{\hat{a}}^{\bar{a}} \Phi_m(m(a))\phi_a(a)da.$$

$\Phi_m(m)$ crosses $1 - \Phi_m(m)$ exactly once from below and

$$\int_{\underline{a}}^{\bar{a}} \Phi_m(m(a))\phi_a(a)da = \frac{1}{2}.$$

Therefore,

$$\int_{x}^{\bar{a}} \Phi_m(m(a))\phi_a(a)da \geqslant \int_{x}^{\bar{a}} (1 - \Phi_m(m(a)))\phi_a(a)da, \ \forall\ x.$$

If $\beta_2 < \beta_1$, $m(a)$ is decreasing and it leads to a similar contradiction. If $\beta_1 = \beta_2$ in an equilibrium, the optimal strategy is to reside in district 2 if and only if the income m exceeds some threshold. Market clearing condition requires that such a threshold is the median income.

Proposition 3.8

(a),(b):

If $\beta_1 > \beta_2$, $m(a)$ is $m(a)$ is decreasing in (a). Then

$$\beta_2 = 2\int_{\hat{a}}^{\bar{a}} (1 - \Phi(m(a)|a))\phi_a(a)da,$$
$$\beta_1 = 2\int_{\hat{a}}^{\bar{a}} \Phi(m|a)\phi_a(a)da,$$
$$\int_{\underline{a}}^{\bar{a}} \Phi(m|a)\phi_a(a)da = \frac{1}{2}.$$

Then, $1 - \Phi(m(a)|a)$ is increasing in a and $1 - \Phi(m(a)|a)$ crosses $\Phi(m|a)$ exactly once from below. Therefore,

$$\beta_2 = \int_{\hat{a}}^{\bar{a}} (1 - \Phi(m(a)|a))\phi_a(a)da > \int_{\hat{a}}^{\bar{a}} \Phi(m|a)\phi_a(a)da = \beta_1,$$

which is a contradiction. Even if $\beta_1 = \beta_2$ and $m(a)$ is constant, $1 - \Phi(m|a)$ is still increasing in a due to affiliation, and it leads to the same contradiction. This establishes (a) and (b) since $\beta_2 > \beta_1$ in any equilibria if an equilibrium exists. Under DA algorithm, the minimum utility in district 1 is given by

$$EU(\underline{m}, \hat{a}, A) = [S(A) + S(B)]u(\hat{a}, A),$$

and p_1 in the equilibrium solves

$$[S(A) + S(B)]u(\hat{a}, A) + v(\underline{m} - p_1) + q(p_1) = u_o.$$

Given $(a, p_2, \beta_1, \beta_2)$, denote the cutoff income by $m(a, p_2, \beta_1, \beta_2)$. Then, $m(\cdot)$ is increasing in β_1 if and only if $a < \hat{a}$, increasing in β_2 if and only if $a > \hat{a}$. The equilibrium is characterized by the following equations.

$$2\int_{\underline{a}}^{\bar{a}} \Phi(m(a, p_2, \beta_1, \beta_2))|a)\phi_a(a)da = 1, \tag{8}$$

$$2\int_{\underline{a}}^{\hat{a}} \Phi(m(a, p_2, \beta_1, \beta_2))|a)\phi_a(a)da = 1 - \beta_1, \tag{9}$$

$$2\int_{\hat{a}}^{\bar{a}} (1 - \Phi(m(a, p_2, \beta_1, \beta_2))|a))\phi_a(a)da = \beta_2.. \tag{10}$$

Denote the solution β_1 of (9) by $\beta_1 = \beta_1(p_2, \beta_2)$. By (9), $\beta_1(p_2, \beta_2)$ is increasing in β_2. Similarly, the solution of (10), $\beta_2(p_2, \beta_1)$ is increasing in β_1. The remaining proof is similar to the proof of Proposition 2.12.

Theorem 3.1

(b): Based on the equilibrium characterized in Proposition 3.5, if $S(C)$ is large enough, A_i and B_i are filled in the first round in the equilibrium. Since the equilibrium strategy takes a form of a cutoff strategy, A_i and B_i are segregated by student type.

(c): The outcome of DA-MTB is characterized by cutoffs $c^j_{\theta_i} \in [0,1]$ for each student j and each school θ_i and each student j in district i draw 3 lotteries $l^j_{A_i}$, $l^j_{B_i}$, and $l^j_{C_i}$ independently from a uniform distribution on $[0,1]$. A student has a *right* to attend a school θ_i if and only if $c^j_{\theta_i} > l^j_{\theta_i}$. Thus, a student j that reports $A_i \succ B_i$ draw lotteries such that $l^j_{A_i} > c^j_{A_i}$ and $l^j_{B_i} < c^j_{B_i}$, then the student will be assigned to B_i. The same logic applies to a student j that reports $B_i \succ A_i$. Therefore, all the schools are maximally diverse under DA-MTB.

(d): The outcome of DA-MTB is characterized by cutoffs $c_{\theta_i} \in [0,1]$ for each school θ_i and each student draws a lottery from a uniform distribution on $[0,1]$. If $\frac{S(B)}{\Phi_i(\hat{a})} > \frac{S(A)}{1-\Phi_i(\hat{a})}$, school A is relatively over-demanded in district i, and the cutoff C_{A_i} is less than C_{B_i}. Thus, any students rejected from B_i in the first round will not be accepted to A_i the following rounds and the school A_i does not admit any students with type below $\hat{a}$. On the other hand, there are positive mass of students rejected from A_i in the first round and some of those students will be accepted to B_i in the second round, thus, B_i is maximally diverse. If $\frac{S(B)}{\Phi_i(\hat{a})} < \frac{S(A)}{1-\Phi_i(\hat{a})}$, the same proof still holds.

(e): Without loss of generality, suppose that $\frac{S(B)}{\Phi_i(\hat{a})} > \frac{S(A)}{1-\Phi_i(\hat{a})}$. By Proposition 3.5, $a^{BM} > \hat{a}$ and the school A_i is filled in the first round under the Boston mechanism. Thus, A_i is less diverse under the Boston mechanism than the deferred acceptance with single tie breaking. B_i is maximally diverse under the deferred acceptance with single tie breaking because there are student rejected from A_i in the first round but accepted to B_i in the second round. (a): Established in the proof for (c) and (d).

Theorem 4.1

Consider the Boston mechanism. As the students in each district have priories for the schools in the district, the same ranking strategy followed by any order of schools from other district consists an equilibrium. For example, if it was an equilibrium strategy of a student in district in i to report $A_i \succ B_i \succ C_i$, then, $A_i \succ B_i \succ C_i \succ A_j \succ B_j \succ C_j$ is an equilibrium strategy under open enrollment. Consider the deferred acceptance mechanism. Since truthful reporting is an equilibrium strategy, all the students place θ_2 higher than θ_1 for each θ. As students in district 2 have higher priority, no students from district 1 can be accepted to schools in district 2 at the end of the deferred acceptance algorithm.

Proposition 4.2

Denote the mass of students that place θ_i at the top by β_i^θ.

Claim 1. $\beta_2^B > S(A)$ and $\beta_2^B > S(B)$ in any equilibria.

Proof. Suppose not, the probability of being accepted to A_2 or B_2 is 1 and every student that has not placed A_2 or B_2 at the top can profitably deviate by doing so. □

Claim 2. If $\beta_2^\theta > \beta_1^\theta$, there exist cutoffs a^1 and a^4 such that any students with $a > a^1$ prefer to place A_1 at the top rather than A_2 and any students with $a < a^4$ prefer to place B_1 at the top rather than B_2

Proof. Since $\beta_2^\theta > \beta_1^\theta$, placing θ_1 at the top gives a higher chance of being accepted to θ type school than placing θ_2 at the top, while placing θ_2 at the top gives a higher chance of enjoying $q_2 > q_1$. As $u(a, A) \to \infty$ as $a \to \bar{a}$, $u(a, B) \to \infty$ as $a \to \underline{a}$, and u is continuous, there exist cutoffs a^1 and a^4. □

Claim 3. If $q_2 - q_1$ is large enough, there exist cutoffs α^2 and α^3 such that $a \in [\alpha^3, \alpha^2]$ places C_2 at the top. If $q_2 - q_1$ is small enough, then no students place C_2 at the top in equilibrium.

Proof. First of all, in any equilibrium, $\beta_2^C < \beta_2^B$ and $\beta_2^C < \beta_2^A$. Otherwise, the students placed C_2 at the top will deviate by placing A_2 or B_2 at the top. Given that $\beta_2^C < \beta_2^A$, denote the probability of being accepted to C_2 by placing it at the top by $\alpha_{C_2,1}$ and denote the probability of being accepted to A_2 by placing it at the top by $\alpha_{A_2,1}$. For a student that places C_2 at the top, placing A_2 or B_2 at the second is dominated strategy as A_2 and B_2 are filled in the first round in any equilibrium. Thus, the expected utility of placing C_2 at the top is

$$\alpha_{C_2,1} q_2 + (1 - \alpha_{C_2,1}) f(q_1),$$

where f is increasing in q_1. f depends on which school the student places at the second in ranking and we know that it is not a school in district 2. The expected utility of placing A_2 at the top is

$$\alpha_{A_2,1} u(a, A) + \alpha_{A_2,1} q_2 + (1 - \alpha_{A_2,1}) f(q_1).$$

As $\alpha_{A_2,1} < \alpha_{A_2,1}$, there exist students that prefers to place C_2 at the top rather than A_2 if $q_2 - q_1$ is large enough. To show the latter part of the claim, note that the expected payoff by placing C_2 at the top is bounded from above by q_2 if it is an equilibrium strategy. Suppose that there is a student with type $a \geqslant \hat{a}$ that places C_2 at the top. If the student places A_2 at the top instead of C_2, the expected payoff is bounded from below by $\frac{S(A)}{2}(u(\hat{a}, A) + q_2) + (1 - \frac{S(A)}{2}) q_1$ and such deviation if profitable if

$$q_2 < \frac{S(A)}{2}(u(\hat{a}, A) + q_2) + (1 - \frac{S(A)}{2}) q_1 \iff (1 - \frac{S(A)}{2})(q_2 - q_1) < \frac{S(A)}{2} u(\hat{a}, A).$$

The inequality holds if $q_2 - q_1$ is small enough. □

Suppose that $q_2 - q_1$ is large enough, so that there are students placing C_2 at the top in an potential equilibrium. Given $\beta_1^A < \beta_2^A$, let $a^1(\beta_1^A, \beta_2^A)$ be the student type that is indifferent between applying A_1 first or A_2 first. Then, the equation

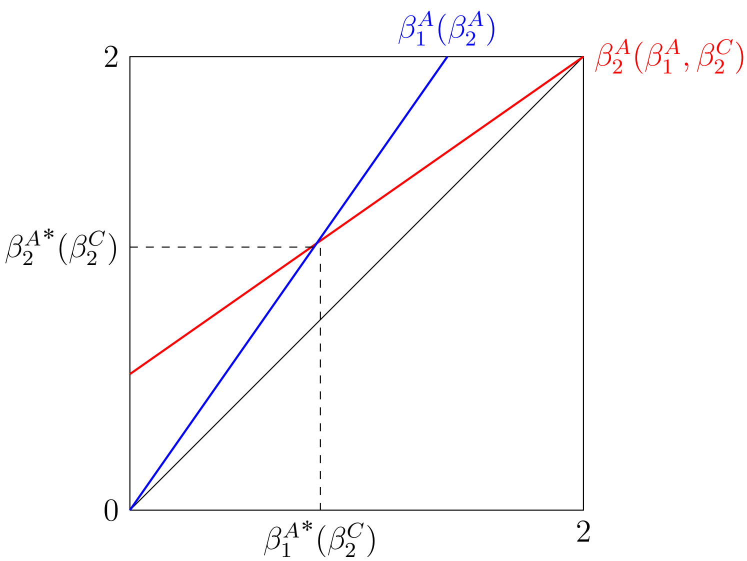
$$2[1 - \Phi_a(a(\beta_1^A, \beta_2^A))] = \beta_1^A$$

specifies a mapping $\beta_1^A = \beta_1^A(\beta_2^A)$. Since $a^1(\beta_1^A, \beta_2^A)$ is decreasing in β_2^A and increasing in β_1^A, $\beta_1^A(\beta_2^A)$ is increasing in β_2^A. As $\underline{a} < a(\beta_1^A, 2)$, $\beta_1(2) < 2$ and $\beta_1(0) = 0$.

Consider the indifferent type between A_2 and C_2. Then, $\beta_2^A(\beta_1^A, \beta_2^C)$ solves the equation

$$2[\Phi_a(a^1(\beta_1^A, \beta_2^A)) - \Phi_a(a^2(\beta_2^A, \beta_2^C))] = \beta_2^A.$$

$a(\beta_2^A, \beta_2^C)$ is increasing in β_2^A and is decreasing in β_2^C. Therefore, $\beta_2^A(\beta_1^A, \beta_2^C)$ is increasing in β_1^A and β_2^C. Moreover, since $\underline{a} < a^1(0, \beta_2^A)$, $0 < \beta_2^A(0, \beta_2^C)$. Thus, for any given β_2^C, there exists a fixed point $(\beta_1^{A*}(\beta_2^C), \beta_2^{A*}(\beta_2^C))$ and $\beta_2^{A*}(\beta_2^C) > \beta_1^{A*}(\beta_2^C)$. Moreover, $\beta_1^{A*}(\beta_2^C)$ and $\beta_2^{A*}(\beta_2^C)$ are increasing in β_2^C.



Similarly, there exists a fixed point $(\beta_1^{B*}(\beta_2^C), \beta_2^{B*}(\beta_2^C))$ such that $\beta_2^{B*}(\beta_2^C) > \beta_1^{B*}(\beta_2^C)$.

The equilibrium β_2^{C*} solves

$$\beta_1^{A*}(\beta_2^C) + \beta_2^{A*}(\beta_2^C) + \beta_1^{B*}(\beta_2^C) + \beta_2^{B*}(\beta_2^C) + \beta_2^C = 2.$$

The uniqueness and existence of β_2^{C*} is followed by the intermediate value theorem and the monotonicity of $\beta_i^{\theta*}(\beta_2^C)$.

If $\beta_2^C = 0$ in a potential equilibrium, we have three mappings $\beta_1^A(\beta_2^A)$, $\beta_2^A(\beta_1^A, \beta_2^B)$, and $\beta_1^B(\beta_2^B)$ instead of the four and the remaining proof is the same.

Theorem 4.3

(a): A_2 and B_2 are segregated since both schools are filled in the first round of the Boston mechanism and the students follows cutoff strategy in the equilibrium.

(b): If $q_2 - q_1$ is small enough, no students place C_2 at the top and A_1 and B_1 are filled in the first round as well as A_2 and B_2. [7] Since all schools but C_1 and C_2 are filled in the first round, A_1, A_2, B_1, B_2 are all segregated by student type.

Proposition 4.4

(a): The deferred acceptance is characterized by cutoffs. $\frac{S(B)}{\Phi(\bar{m},\hat{a})} > \frac{S(A)}{1-\Phi(\bar{m},\hat{a})}$, then the cutoff C_{A_i} for A_i is lower than C_{B_i} and the cutoff C_{A_2} is the minimum among cutoffs. Any acceptances to A_2 in the first round are final since any students rejected in the first round drew higher lottery than C_{A_2}. B_2 becomes maximally diverse because there are students with the true preference $A_2 \succ B_2 \succ \ldots$ and those students draw a lottery number in $[C_{A_2}, C_{B_2}]$ with positive probability. There are students with the true preference $A_2 \succ B_2 \succ A_1 \succ B_1$ and if those students draw a lottery in $[C_{A_1}, C_{B_1}]$, then the students are assigned to B_1, thus, B_1 becomes maximally diverse. A_1 may or may not be maximally diverse depending on whether $C_{A_1} > C_{B_2}$ or not. The same proof applies to (b).

[7] See the proof of Proposition 3.2

Theorem 4.5

(a): The proof for Theorem 3.3 (a) applies here.

(b): Corollary of Proposition 4.4.

Chapter 2

Auctions with Designed Bid Spaces

2.1 Introduction

In different auction formats, there are some restrictions implicitly or explicitly on bids level that each bidder can choose. For example, eBay uses different minimum bid increment for different current maximum bid levels. Sometimes each bidder has constrained budget, so there can be endogenous restriction on bid space. In this paper, we establish the revenue ranking between conventional auction formats when the restriction on bid spaces is a part of auction design. Moreover, we show that any Bayesian incentive compatible mechanism is a convex combination of first price with designed bid spaces. Section 2 discusses the revenue comparison between first price auctions and second price auctions when the seller can design the bid spaces. Section 3 shows that the optimal selling mechanism can be implemented by first price auctions with well-designed bid spaces. Section 4 provides a characterization of any symmetric Bayesian incentive compatible mechanisms.

The revenue equivalence principle was established by Riley and Samuelson (1981) and Myerson (1981) for continuous type space. Maskin and Riley (2000) considered conventional auctions when the type space is discrete, but the revenue raking is ambiguous. Che and Gale (2006) and Chung and Olszewski (2007) considered the revenue equivalence on general type space. Our work considers finite type space with restriction on bid spaces. We establish the revenue raking by showing that any outcome in second price auctions can be replicated with an equilibrium in first price auction, but the converse doesn't hold. Furthermore, we also show that *well-designed* first price auction is actually an optimal selling mechanism. Bergemann and Pesendorfer (2007) characterized the optimal direct selling mechanism for finite type space, and our well-designed auction can implement the same outcome of the optimal mechanism.

2.2 Revenue Rankings

There are n bidders indexed by $i \in \{1, \ldots, N\}$. Each bidder i's value $\theta_i \in \Theta \subset [0, \infty)$ is distributed i.i.d. according to a distribution F, and utility is linear in monetary transfer. Only bids from finite set of bids $\mathcal{B}$ are allowed; same bid levels for all bidders.

We study conventional auctions in which the highest bidder wins, in case of a tie the winner is chosen uniformly at random from among the highest bidders. We will also assume that only the winning bidder pays. Furthermore, we restrict attention to the finite type space.

Proposition 2.2.1. *For any bid space $\mathcal{B}_S$, there is a symmetric equilibrium $\sigma_S : \Theta \to \triangle(\mathcal{B}_S)$ under second price auction with the bid spaces $\mathcal{B}_S$. Moreover, there is a symmetric equilibrium σ_F under first price auction with some bid space $\mathcal{B}_F$ that generates same buyer payoffs for each type, same allocation, hence, same revenue as σ_S.*

The existence of a symmetric equilibrium is a direct implication of Nash (1951). For the remaining part of the theorem, denote the induced distribution over the bid spaces by $G(\cdot, \sigma) = G(\cdot)$. Let $\pi_G(B)$ and $T_G(B)$ denote the winning probability and the expected payment from bidding B, given $G(\cdot, \sigma)$. When a bidder with type θ bids B, the interim utility $U(\theta, B, G)$ is given by $\theta\pi_G(B) - T_G(B)$.

Proof. Denote $\{B \in \mathcal{B}_S \mid B \in \text{supp}(\sigma_S(\theta)), \text{ for some } \theta \in \Theta\}$ by $\{B_1^S, \ldots, B_K^S\}$, where $B_k^S >$

B^S_{k-1}, for all $k = 2, \ldots, K$. Construct the bid spaces $\mathcal{B}_F$ and an equilibrium strategy σ_F in first price auction as follows.

$$B^F_k := \frac{T_G(B^S_k)}{\pi_G(B^S_k)}, \text{ for all } k = 1, \ldots, K.$$
$$\sigma_F(\theta)(B^F_k) := \sigma_S(\theta)(B^S_k), \text{ for all } \theta \in \Theta \text{ and } k = 1, \ldots, K.$$

In order to show that σ_F and σ_S achieve the same allocation, it is enough to show that B^F_k is strictly increasing in k because the allocation only depends on the order of submitted bids. Since B^S_k is chosen with positive probability in equilibrium σ_S, there is some type θ such that

$$\pi_G(B^S_k)\theta - T_G(B^S_k) \geqslant \pi_G(B^S_{k+1})\theta - T_G(B^S_{k+1}).$$

This implies that

$$\Big(\theta - \frac{T_G(B^S_k)}{\pi_G(B^S_k)}\Big)\frac{\pi_G(B^S_k)}{\pi_G(B^S_{k+1})} \geqslant \theta - \frac{T_G(B^S_{k+1})}{\pi_G(B^S_{k+1})} \iff B^F_{k+1} > B^F_k.$$

Note that $\pi_G(B^S_k) < \pi_G(B^S_{k+1})$ since the both are played with positive probability. Therefore, σ_F and σ_S generate the same distribution over the allocation. Denote the induced bidding distribution from σ_F over $\{B^F_1, \ldots, B^F_K\}$ by G_F. The utility of bidding B^F_k under first price auction can be written as

$$\pi_{G_F}(B^F_k)(\theta - B^F_k) = \pi_G(B^S_k)(\theta - \frac{T_G(B^S_k)}{\pi_G(B^S_k)}) = \pi_G(B^S_k)\theta - T_G(B^S_k),$$

which is the same payoff under σ_S. We established (1) allocation equivalence, (2) payoff equivalence, hence, (3) revenue equivalence. The remaining step is to show that σ_F is a Bayesian Nash equilibrium. Since σ_S is a Bayesian Nash equilibrium in the original second price auction with $\mathcal{B}_S$,

$$\pi_G(\sigma_S(\theta))\theta - T_G(\sigma_S(\theta)) \geqslant \pi_G(B)\theta - T_G(B),\ \forall\theta,\ \forall B.$$

By the construction, $\pi_G(\sigma_S(\theta)) = \pi_{G_F}(\sigma_F(\theta))$ and $T_G(\sigma_S(\theta)) = T_{G_F}(\sigma_F(\theta))$. Therefore, σ_F is an equilibrium in the first price auction with the bid spaces $\mathcal{B}_F$. □

According to the proposition, the designer can get any desired outcome in a first price auction with designed bids if the outcome is achievable in some second price auction with designed bids. However, our following example shows that the converse doesn't hold.

Example 2.2.2. *There are 2 bidders, and each bidder i's value v_i is drawn independently following*

$$v_i = \begin{cases} 1 & \textit{with probability } 0.6, \\ 11/4 & \textit{with probability } 0.2, \\ 13/2 & \textit{with probability } 0.2. \end{cases}$$

Suppose $\mathcal{B}_F = \{1, 2, 3\}$. Then

$$\sigma_F(v_i) = \begin{cases} 1 & \textit{if } v_i = 1, \\ 2 & \textit{if } v_i = 11/4, \\ 3 & \textit{if } v_i = 13/2. \end{cases}$$

is a symmetric equilibrium in pure strategies of first price auction with $\mathcal{B}_\mathcal{F}$ and

$$G_F(B) = \begin{cases} 0.6 & \textit{if } B = 1, \\ 0.8 & \textit{if } B = 2, \\ 1 & \textit{if } B = 3. \end{cases}$$

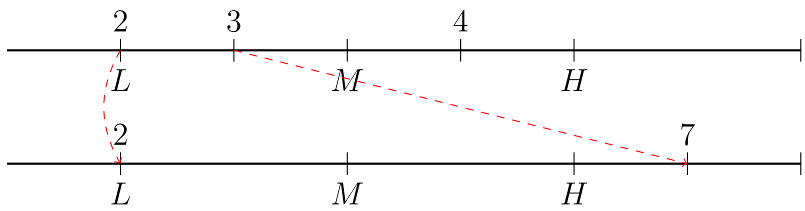


Figure 2.1: Overshooting

The expected payment for each type (denote the type by L, M, H in the order of the lowest to the highest) is

$$T(L) = 0.3,\ T(M) = 1.4,\ T(H) = 2.7.$$

The payoff equivalent bid level for L type is $B_L = 1$. *Given that, the payoff equivalent bid level for M type is 8, which is greater than* $H = 13/2$. *Notice that if* $B_H > B_M > H$ *then, H type never bids* B_H *in any symmetric equilibrium.*

The problem in the example is that the bid level for middle type can *overshoot* the value of the high type as depicted in figure 1. We can characterize the condition that prevents the overshooting. Denote $\{B \in \mathcal{B}_F \mid B = \sigma_F(\theta),\ \text{for some } \theta \in \Theta\}$ by $\{B_1^F, \dots, B_K^F\}$, where $B_k^S > B_{k-1}^S$, for all $k = 2, \dots, K$. Denote the bidding distribution over $\{B_1^F, \dots, B_K^F\}$ in the equilibrium σ_F by G_F. $\pi_{G_F}(\cdot)$ and $T_{G_F}(\cdot)$ denote the winning probability and expected payment, given G_F. We can define $\mathcal{B}_S$ recursively as follows.

$$B_1^S = B_1^F,$$

$$B_k^S = \frac{\pi_{G_F}(B_{k-1}^F) - G_F(B_{k-1}^F)^{N-1}}{\pi_{G_F}(B_k^F) - G_F(B_{k-1}^F)^{N-1}} B_{k-1}^S + \frac{\pi_{G_F}(B_k^F)B_k^F - \pi_{G_F}(B_{k-1}^F)B_{k-1}^F}{\pi_{G_F}(B_k^F) - G_F(B_{k-1}^F)^{N-1}},\ \text{for } k \geqslant 2. \qquad (*)$$

B_k^S is strictly increasing if and only if

$$\frac{\pi_{G_F}(B_k^F)B_k^F - \pi_{G_F}(B_{k-1}^F)B_{k-1}^F}{\pi_{G_F}(B_k^F) - \pi_{G_F}(B_{k-1}^F)} > B_{k-1}^S,\ \text{for all } k \geqslant 2. \qquad (**)$$

Proposition 2.2.3. *For any symmetric equilibrium that* $(**)$ *holds in* pure *strategies* $\sigma_F : \Theta \to \mathcal{B}_F$ *under first price auction with some bid space* $\mathcal{B}_F$*, there is a symmetric equilibrium* σ_S *under second price auction with some bid space* $\mathcal{B}_S$ *that generates same buyer payoffs for each type, same allocation, hence, same revenue as* σ_F*.*

Proof. Set

$$\sigma_S(\theta)(B_k^S) := \sigma_F(\theta)(B_k^F), \text{ for all } \theta \in \Theta \text{ and } k = 1, \ldots, K.$$

Since B_k^S is strictly increasing, σ_S and σ_F generate the same allocation. The remaining step is to show that $T_{G_S}(B_k^S) = T_{G_F}(B_k^F)$. Since $T_{G_S}(B_1^S) = T_{G_F}(B_1^F)$, it is enough to show that $T_{G_S}(B_k^S) - T_{G_S}(B_{k-1}^S) = T_{G_F}(B_k^F) - T_{G_F}(B_{k-1}^F)$, for all $k \geqslant 2$.

$$T_{G_S}(B_k^S) = (\pi_{G_S}(B_k^S) - G_S(B_{k-1}^S)^{N-1})B_k^S + \sum_{j=1}^{k-1}(G_S(B_j^S)^{N-1} - G_S(B_{j-1}^S)^{N-1})B_j^S.$$

$$\begin{aligned}
\Rightarrow & T_{G_S}(B_k^S) - T_{G_S}(B_{k-1}^S) \\
& = (\pi_{G_F}(B_k^F) - G_F(B_{k-1}^F)^{N-1})B_k^S - (\pi_{G_F}(B_{k-1}^F) - G_F(B_{k-1}^F)^{N-1})B_{k-1}^S \\
& = T_{G_F}(B_k^F) - T_{G_F}(B_{k-1}^F).
\end{aligned}$$

The last equality holds by construction. σ_S and σ_F achieve the same allocation and the same interim payment for each type, therefore the revenue is the same for the both. ∎

Even though we show that an outcome of some equilibrium in first price auction with designed bid spaces cannot be achieved with an equilibrium in second price auction with designed bid spaces, it may be possible to achieve the same revenue. Furthermore, the revenue of an auction depends on equilibrium strategies played and designed bid spaces. In order to consider revenue ranking it makes sense to compare the revenue of each auction formats when the bid spacess are designed in a way to achieve the maximal revenue. We define a first (second) price auction is *well-designed* if the bid spaces is designed to maximize the revenue of the seller. The next example shows that *well-designed* first price auction can revenue dominates *well-designed* second price auction.

Example 2.2.4. There are 2 bidders, and each bidder i's value v_i is drawn independently following

$$v_i = \begin{cases} 2 & \text{with probability } 0.5, \\ 11/3 & \text{with probability } 0.25, \\ 13/2 & \text{with probability } 0.25. \end{cases}$$

The well-designed bid space under first price auction is $\mathcal{B}_F = \{2, 3, 4\}$ with an equilibrium

$$\sigma_F(v_i) = \begin{cases} 2 & \text{if } v_i = 2, \\ 3 & \text{if } v_i = 10/3, \\ 4 & \text{if } v_i = 13/2. \end{cases}$$

The expected revenue equals to 51/16. Suppose $\mathcal{B}_S = \{B_L, B_M, B_H\}$. If $B_L < B_M < B_H$, then

$$\pi_{G_S}(B_L) = \frac{1}{4},\ \pi_{G_S}(B_M) = \frac{5}{8},\ \pi_{G_S}(B_H) = \frac{7}{8},$$
$$T_{G_S}(B_L) = \frac{1}{4}B_L,\ T_{G_S}(B_M) = \frac{4}{8}B_L + \frac{1}{8}B_M,\ T_{G_S}(B_H) = \frac{4}{8}B_L + \frac{2}{8}B_M + \frac{1}{8}B_H.$$

In this case, revenue maximization problem becomes

$$\max_{B_L, B_M, B_H} 2 \sum f(v) T_{G_S}(B_v) \text{ subject to}$$

$$13 \geqslant B_H + B_M, \tag{IC_H}$$

$$11 \geqslant B_M + 2B_L,$$

$$2 \geqslant B_L,$$

$$B_H > B_M > B_L \tag{Monotonicity}$$

Not only doesn't the solution exist but also any bid levels satisfying the constraints give less revenue than 50/16, which comes from the solution of the same problem with relaxed monotonicity constraint to weak inequalities. Suppose $\mathcal{B}_S = \{B_L, B_M, B_H\}$. If $B_L = B_M < B_H$, then

$$\pi_{G_S}(B_L) = \pi_{G_S}(B_M) = \frac{3}{8},\ \pi_{G_S}(B_H) = \frac{7}{8},$$
$$T_{G_S}(B_L) = T_{G_S}(B_M) = \frac{3}{8} B_L,\ T_{G_S}(B_H) = \frac{1}{8} B_H + \frac{6}{8} B_L.$$

In this case, revenue maximization problem becomes

$$\max_{B_L, B_H} 2 \sum f(v) T_{G_S}(B_v) \text{ subject to}$$

$$26 \geqslant B_H + 3B_L, \tag{IC_H}$$

$$2 \geqslant B_L.$$

The solution is $B_L = B_M = 2$, $B_H = 20$. The expected revenue is 50/16.

Proposition 2.2.5. *The revenue from well-designed first price auction is greater than or equal to the revenue from well-designed second price auction. In some cases, it is strict.*

Proof. The first part is a corollary of Proposition 1. The second part comes from the previous example. ☐

2.3 Optimal Mechanism

So far, we only consider two auction formats; first price and second price auctions. The next proposition shows that *well-designed* first price auction is an optimal selling mechanism for a single object. Bergemann & Pesendorfer (2007) characterizes the optimal direct mechanism for a sale of single object when the type space is finite. Suppose each bidder i's type θ_i is distributed over Θ following c.d.f. F, for all i. Then the allocation rule of the optimal direct mechanism is described as follows.

1. Each bidder i reports θ_i.

2. Award the good to i with maximum positive *virtual valuation* $\mu(\theta) = \theta - \frac{1-F(\theta)}{f(\theta)}$. If the maximum $\mu(\theta)$ is negative, the seller keeps the good. If there are more than one bidder with maximum $\mu(\theta)$ then the winner is picked uniformly at random. 3. If $\mu(\theta)$ is not monotone, then use *ironed virtual value*.

Proposition 2.3.1. *Well-designed first price auction is an optimal selling mechanism for single unit.*

Proof. Collect the interim payment and interim allocation rule $(T(\theta), \pi(\theta))$ from the direct mechanism, only for the type θ such that $\mu(\theta)$ is positive. Construct the bid spaces and equilibrium strategies in first price auction as follows.

$$B_\theta = \frac{T(\theta)}{\pi(\theta)},\ \forall \theta.$$

$$\sigma(\theta) = B_\theta.$$

Since we only consider θ that wins the item with positive probability, B_θ is well-defined and

B_θ is increasing in θ by the following inequality.

$$\theta\pi(\theta) - T(\theta) \geqslant \theta\pi(\theta') - T(\theta') \iff \frac{\pi(\theta)}{\pi(\theta')}(\theta - B_\theta) \geqslant \theta - B_{\theta'}.$$

According to the allocation rule of the optimal direct mechanism, $\pi(\theta) \geqslant \pi(\theta')$ if and only if $\mu(\theta) \geqslant \mu(\theta')$. Therefore σ_F achieves the same allocation of the optimal mechanism. Furthermore, the interim payoff for each type is the same for the direct mechanism and σ_F. Therefore, σ_F is an equilibrium that achieves the same revenue of the optimal direct mechanism. □

Chapter 3

Weak and Strong Bayesian Stability

3.1 Introduction

In the classical two-sided matching model, the main solution concept is stable matchings; no one can find a better partner who is willing to be matched with him or her. While the classical definition of stability relies on the complete information, where the value of a matching is known to everyone, it is not trivial problem how to define stability if there is incomplete information. A pair of agents (i, j) can *block* the matching if i and j consider each other better than their current partners. If i or j do not know for sure that the potential blocking is mutually beneficial, the assessment of the value of the blocking depends on the beliefs each agent has. Thus, defining a notion of stability under incomplete information must entail how the beliefs of the agents are formed and maintained, thus, there can be different sets of stable outcomes depending on how to discipline beliefs. This paper investigates existing definitions of stable outcomes and proposes an alternative definition when each agent in one side has a private information that might affect the value of potential matchings. The paper is organized as follows. Section 2 introduces the notations and the definition of stability under complete information. Section 3 and 4 explore existing definitions of stable outcomes and propose *weak Bayesian stability*. Section 5 presents a refinement of weak Bayesian stability.

3.1.1 Related Literature

This paper is related to the literature on two-sided matching with incomplete information. Liu et al. (2014) introduce incomplete information stable outcome under transferable utility based on a belief-formation process resembling rationalizability in non cooperative game theory. Bikhchandani (2017) investigates stability under non-transferable utility and proposed a refinement of Liu et al. (2014). Section 3 will discuss these papers in detail. Pomatto (2015) proposes non-cooperative solution concept that is equivalent to incomplete information stable outcome in Liu et al. (2014). Chakraborty et al. (2010) study stability of mechanisms rather than stable outcomes in the context of college admission and show that stable mechanisms exist only if students have identical preferences. Peivandi and Vohra (2017) explore a notion of blocking a centralized mechanism and core under incomplete information.

3.2 Complete Information

There are a finite set I of workers and a finite set J of firms. A *matching* μ is a one-to-one correspondence from $I \cup J$ to $I \cup J$ such that if $\mu(i) \notin I$ then $\mu(i) \in J$, if $\mu(j) \notin J$ then $\mu(j) \in I$, and $\mu^2(x) = x$. Each firm j has a known type $\mathbf{f}(j)$ and a vector of types is denoted by $\mathbf{f}$. Worker i's type is denoted by $\mathbf{w}(i)$ and firm j's type is denoted by $\mathbf{f}(j)$. The utility of each worker and each firm only depends on their own types and their partner's types. Worker i's utility is denoted by $v_i(\mathbf{w}(i), \mathbf{f}(j))$ and firm j's utility is denoted by $u_j(\mathbf{w}(i), \mathbf{f}(j))$, given that i and j are matched. The utility of being unmatched is normalized to 0. I can define $\mathbf{f}(i) = \varnothing$ if $i \in I$ and $\mathbf{w}(j) = \varnothing$ if $j \in J$ and $u_j(\varnothing, \mathbf{f}(j)) = v_i(\mathbf{w}(i), \varnothing) = 0$ for any i, j, $\mathbf{w}$, and $\mathbf{f}$. Now, I can define a stable outcome under *complete information* originated from Gale and Shapley (1962) in the notations above.

Definition 5. An outcome $(\mu, \mathbf{w}, \mathbf{f})$ is *individually rational* if $v_i(\mathbf{w}(i), \mathbf{f}(\mu(i))) \geqslant 0$ and $u_j(\mathbf{w}(\mu(j), \mathbf{f}(j))) \geqslant 0$ for all $i \in \mu(J)$ and $j \in \mu(I)$.

An individually rational outcome guarantees the payoff no worse than being unmatched to each agent.

Definition 6. An individually rational outcome $(\mu, \mathbf{w})$ *complete-information stable* if for any (i, j) such that $v_i(\mathbf{w}(i), \mathbf{f}(j)) > v_i(\mathbf{w}(i), \mathbf{f}(\mu(i)))$,

$$u_j(\mathbf{w}(\mu(j)), \mathbf{f}(j)) \geqslant u_j(\mathbf{w}(i), \mathbf{f}(j)).$$

Under complete information, a matching outcome is not stable if either it is not individually rational or there are worker i and firm j that prefer each other than their current partners.

3.3 Incomplete Information

The assumption of complete information is relaxed and environments where agents are uncertain about the types of others are considered. Specifically, I consider environments where the type of firms $\mathbf{f}$ is publicly known, while each worker i's type $\mathbf{w}(i)$ is only known to the worker i and his or her partner $\mu(i)$. Assume that $\mathbf{w}(i)$ is drawn independently from some set $\Omega \subset \mathbb{R}^I$ following some distribution. Since the type of workers are unknown, firms can form different beliefs after some thought processes and an outcome may or may not be stable depending on the beliefs of each firm. Denote the belief of firm j on $\mathbf{w}(i)$ by G_{ij}.

Definition 7. An individual rational outcome $(\mu, \mathbf{w}, \mathbf{f})$ is *stable with beliefs* $\{G_{ij}\}_{(i,j)\in I\times J}$ if for any (i, j) such that $v_i(\mathbf{w}(i), \mathbf{f}(j)) > v_i(\mathbf{w}(i), \mathbf{f}(\mu(i)))$,

$$u_j(\mathbf{w}(\mu(j)), \mathbf{f}(j)) \geqslant \mathbb{E}[u_j(\mathbf{w}(i), \mathbf{f}(j))|G_{ij}].$$

This definition is similar to the definition of a stable outcome under complete information other than each firm j uses a set of beliefs $\{G_{ij}\}$ to assess the potential blocking partner i. Without disciplining the beliefs further, it is usual that any individually rational outcomes are stable with *some* beliefs $\{G_{ij}\}$. One can derive different concepts of stability by restricting the beliefs firms can have in different ways. As a comparison, I introduce a modified version of the definition of *incomplete-information stable outcomes* in Liu et al. (2014).[1] The following three definitions are adapted from Liu et al. (2014).

Definition 8. A matching outcome $(\mu, \mathbf{w}, \mathbf{f}) \in \Sigma$ is Σ-*stable* if for any (i, j) such that $v_i(\mathbf{w}(i), \mathbf{f}(j)) > v_i(\mathbf{w}(i), \mathbf{f}(\mu(i)))$, there exists $(\mu, \mathbf{w}', \mathbf{f}) \in \Sigma$ satisfying

(a) $u_j(\mathbf{w}'(\mu(j)), \mathbf{f}(j)) \geqslant u_j(\mathbf{w}'(i), \mathbf{f}(j))$,

(b) $\mathbf{w}(\mu(j)) = \mathbf{w}'(\mu(j))$, and

(c) $v_i(\mathbf{w}'(i), f(j)) > v_i(\mathbf{w}'(i), f(\mu(i)))$.

Suppose that worker i applies to firm j. For firm j not to replace its current worker $\mu(j)$, the firm has to be pessimistic enough about i's type. The first and second parts of the definition says that there exists such a type of worker i so that firm j weakly prefers the current worker to i. The last part of the definitions says that such beliefs of firm j must be consistent with the fact that worker i applies to firm j. So, the beliefs that the current matching outcome is in the set Σ are reinforced by observing no pair of agents have formed blocking pairs.

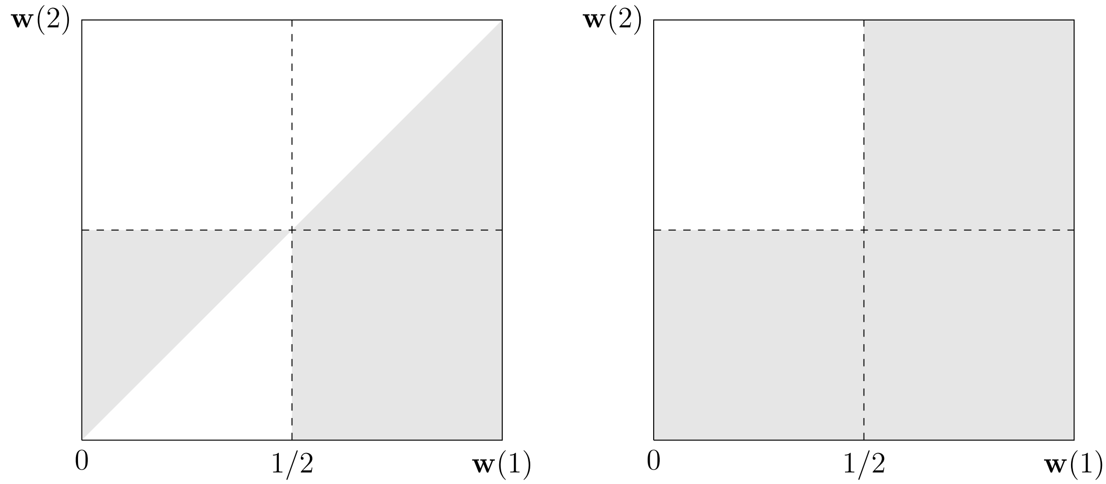
Definition 9. A nonempty set of individually rational matching outcomes Σ is *self-stabilizing* if every $(\mu, \mathbf{w}) \in \Sigma$ is Σ-stable.

Definition 10. The largest self-stabilizing set of outcomes is the set of *incomplete information stable outcome.*[2]

I present an example to show that how the definition actually works.

[1]Liu et al. (2014) consider environments with transferable utilities. The adapted version to non-transferable environments is introduced by Bikhchandani (2017).

[2]The actual definition of *the set of incomplete information stable outcomes* in Liu et al. (2014) is the surviving set through the process of iterated elimination of *blocked* matching outcomes under the most conservative beliefs. One main finding of Liu et al. (2014) is to show that the definition is equivalent to the self-stabilization.



(a) Complete information

(b) Incomplete information

Figure 3.1: Stable outcomes

Example 3.3.1. There are two firms $\{a, b\}$ and two workers $\{1, 2\}$. $\Omega = [0, 1] \times [0, 1]$. Both firms prefer a higher type worker, while worker i's utilities are given by $v_i(w(i), a) = w$ and $v_i(w(i), b) = 1/2$. One can interpret that firm a uses incentive payment system while firm b uses fixed payment system, and $w(i)$ is the productivity parameter. Consider a matching $\mu = \{(1, a), (2, b)\}$, where $w(2) > w(1) > 1/2$. Under complete information, this matching outcome is not stable since $(2, a)$ would be a blocking pair. Under incomplete information, if firm a believes that $w(2)$ is $1/2 + \epsilon$ for some small $\epsilon > 0$, firm a would not replace worker 1 with worker 2. Such a belief is justified as firm 2 would still prefer firm a even if $w(2) = 1/2 + \epsilon$, and this line of logic satisfies the condition (3) in the definition of Σ-stable. One can easily verify that $\{\mu, (0.5 + \epsilon, w(2)), (w(2), 0.5 + \epsilon)\}$ is self-stabilizing. Now, suppose that $w(1) < 1/2 < w(2)$. Consider a potential blocking pair $(2, a)$. In this case, for any type of worker 2 that prefers a to b is bigger than 1/2. Thus, worker 2 is able to convince firm a that his or her productivity is higher than 1/2. Therefore, μ is not incomplete information stable if $w(1) < 1/2 < w(2)$.

In Figure 2(b), the reasoning behind of ruling out the upper left region from the stable outcomes solely relies on the fact that firm a can infer something from the fact that only a certain types of worker actually prefers firm a to b. If the utilities of workers are similar to each other regardless of the types, there would be nothing to infer for firms. The following proposition shows that the incomplete information stability is too permissive.

Proposition 3.3.2. *Suppose that*

(1) $\Omega = [\underline{w}, \bar{w}]^I$*, and any matching is individually rational.*

(2) $v_i(\cdot, f)$ *is a positive monotonic transformation of* $v_{i'}(\cdot, f)$ *with respect to* f *for all* $i, i' \in I$.

(3) $|I| \geqslant |J|$.

Then a matching outcome $(\mu, \mathbf{w}, \mathbf{f})$ *is incomplete information stable if all firms are matched.*

Condition (1) is imposed to make a clean statement and can be relaxed. Condition (2) says that if worker i with some type w prefers a firm with type f to a firm with type f', any other workers with the same type w prefer f to f' as well.[3]

Proof. It is enough to show that any outcome $(\mu, \mathbf{w}, \mathbf{f})$ is $(\mu, \Omega, \mathbf{f})$-stable, for any $\mathbf{w}$, $\mathbf{f}$, and μ. Consider a matching outcome $(\mu, \mathbf{w}, \mathbf{f})$ and a potential blocking pair (i, j). It is enough to provide a $\mathbf{w}'$ that satisfies the conditions in Definition 4. Set $w'(i) = w'(\mu(j)) = w(\mu(j))$ for given i and j and $w'(i') = w(i')$ for any other $i' \neq i$ or $i' \neq \mu(j)$. The condition (a) holds with equality and (b) holds trivially. Condition (c) holds by the assumption (2). Therefore, there is no blocking pair (i, j) such that j is currently matched with someone else.

3.4 Refinements

One possible refinement of incomplete information stability is to discipline the beliefs of firms so that each firm's posterior should be Bayes-consistent with the common prior. Denote the common prior on $\mathbf{w}$ by F.

[3] There is a similar proposition in Bikhchandani (2017), but one does not imply another. For instance, Bikhchandani (2017) requires that the utilities increase in types.

Assumption 3.4.1. Each $\mathbf{w}(i)$ is drawn independently.

As mentioned in Section 3, to define incomplete information stability, Liu et al. (2014) use the iterated elimination of possible blocking outcome under the pessimistic beliefs. One might apply such elimination procedure using *conditional expectation* rather than the most pessimistic beliefs. I provide an example to illustrate how the procedure works rather than stating the definition of *Bayesian stability* provided by Bikhchandani (2017). For more information, refer to Bikhchandani (2017).

Example 3.4.2. There are two firms $\{a, b\}$ and two workers $\{1, 2\}$. Suppose that firm 1 prefers higher type while firm 2 prefers lower type. Specifically, $u_a(w) = w$ and $u_b(w) = 1-w$. Worker 1 prefers a regardless of $\mathbf{w}(1)$ while worker 2 prefers b regardless of $\mathbf{w}(2)$. Assume that each $\mathbf{w}(i)$ is drawn from uniform $[0, 1]$ independently. Consider a matching $\{(a, 2), (b, 1)\}$. Iterated elimination goes by rounds.

- Round 1: Notice that worker 1 always wants to form a blocking pair with firm a as long as firm a is willing to. Moreover, if $\mathbf{w}(2)$ is low enough, firm a is willing to do so. In other words, $\{(a, 2), (b, 1)\}$ would have been blocked if $\mathbf{E}[u_a(\mathbf{w}(1))] > u_a(\mathbf{w}(2))$, where the expectation is taken with respect to the common prior. The condition is equivalent to $1/2 > \mathbf{w}(2)$. Similarly, one can derive the condition $\mathbf{w}(1) > 1/2$ for firm b. Eliminate $[0, 1/2)$ form Ω_2 and $(1/2, 1]$ from Ω_1.

- Round 2: After the elimination, $\mathbf{E}[u_a(\mathbf{w}(1))] = 0.25$. Notice that for any $\mathbf{w}(2) \in [1/2, 1]$, firm a prefers to stay with 2 rather than blocking with 1. The similar logic holds for firm b. There is no possible types that can block the matching with the condition expectation, no further elimination is necessary. $\mu = \{(a, 2), (b, 1)\}$ is *Bayesian stable* if and only if $\mathbf{w} \in [0, 1/2] \times [1/2, 1]$.

Even though *Bayesian stability* is a natural extension of incomplete information stability, there is a caveat. The thought process of agents should be done in a very coordinated way. In Example 4.1, firm a and firm b should eliminate certain possibility at the same time in each round. In other words, the order or elimination actually matters. This caveat does not exist in Liu et al. (2014). Intuitively, since they rely on the elimination based on the most pessimistic beliefs and the most pessimistic beliefs become less pessimistic as eliminations are iterated, there is *monotonicity* in elimination. However, if one eliminates certain types based on expectation, the monotonicity does not present anymore. In Example 4.1, if firm a rules out $(1/2, 1]$ from Ω_1, then only $[0, 1/4)$ would be eliminated from Ω_2.

An alternative way of imposing Bayesian consistency is to use the concept of *self-stabilization*. Suppose that an outcome is $(\mu, \mathbf{w}, \mathbf{f})$ is *stable with beliefs* $\{G_{ij}\}_{(i,j)\in I\times J}$. How would such beliefs G_{ij} be justified? If for any pair of agents have no incentive to form a blocking pair whenever the realized types of workers are in the support of beliefs, such beliefs are reinforced by observing the given matching, thus, the matching will remain stable. I propose the following definition to capture the idea.

Definition 11. A set of beliefs $\{G_{ij}\}$ is weak Bayes-consistent with a matching μ if

1. $(\mu, \mathbf{w}, \mathbf{f})$ is stable with beliefs $\{G_{ij}\}$ for any $\mathbf{w} \in \prod_{i\in I} supp(G_{ij})$,
2. $G_{ij} = \text{marg}_i F$ conditional on $supp(G_{ij})$.
3. $supp(G_{ij})$ is not measure 0 with respect to the prior.

The first condition imposes the consistency of the beliefs with the observation that no one hasn't blocked the matching, and the second condition requires that such beliefs should be consistent with the common prior.

Definition 12. A matching outcome $(\mu, \mathbf{w}, \mathbf{w})$ is weak Bayesian stable if

1. there exists a set of beliefs $\{G_{ij}\}$ that is weak Bayes-consistent with the matching μ,
2. $\mathbf{w} \in \prod_{i\in I} supp(G_{ij})$, for any $j \in J$.

3.4.1 Examples

In this subsection, I provide an example and a proposition. While *weak Bayesian stability* is a natural extension of *self-stabilization* under the assumption that firms are Bayesian, the definition is still too permissive.

Example 3.4.3. Suppose that there are two firms $\{a, b\}$ and two workers $\{1, 2\}$. Both worker prefers firm a to firm b, and both firms prefer higher type, where types are drawn from $U[0, 1]$ independently. Note that only firm a has a possible blocking opportunity. For a realization of (w_1, w_2), define $S_{\mu(a)}$ and $S_{\mu(b)}$ as follows.

$$S_{\mu(a)} = [w_{\mu(a)}, 1],$$
$$S_{\mu(b)} = [0, \epsilon] \cup \{w_{\mu(b)}\}.$$

By taking ϵ arbitrarily small, we can make firm a arbitrarily pessimistic so that the conditional expectation based on $S_{\mu(b)}$ is less than $w_{\mu(a)}$. This is Bayesian self-stabilizing since for any realization firm b has no blocking opportunity, and firm a is pessimistic.

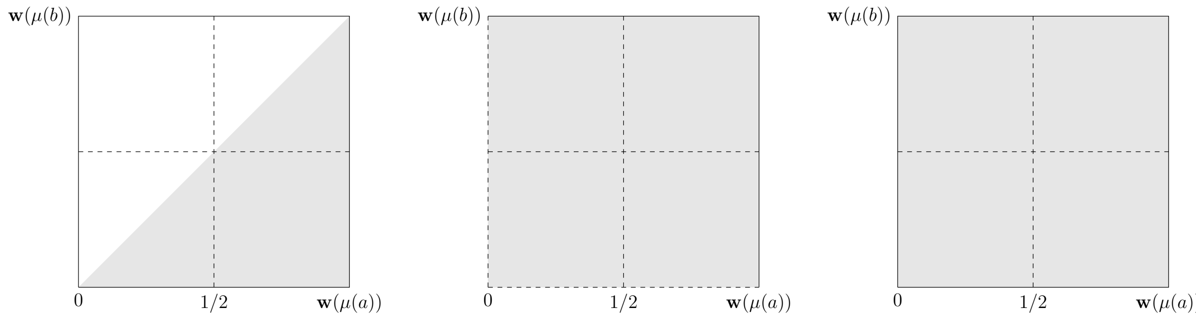
The idea behind of the example is if we do not impose certain restrictions on the types outside of the support of the beliefs, one can always infuse arbitrarily pessimistic beliefs to the firms so that no blocking occurs for any realization and such beliefs can be justified if firms share the same "worst" type of workers. Figure 2 illustrates the set of stable outcomes for different concepts of stability.

Proposition 3.4.4. *Suppose that*

(1) $\Omega = [\underline{w}, \bar{w}]^I$, *and any matching is individually rational.*

(2) Common prior is atom-less.

(3) $v_i(\cdot, f)$ *is a positive monotonic transformation of* $v_{i'}(\cdot, f)$ *with respect to* f *for all* $i, i' \in I$.



(a) Complete information (b) Weak Bayesian (c) Incomplete information

Figure 3.2: Stable outcomes

(4) $u_j(w, \cdot)$ *is increasing in* w *for all* $j \in J$.

(5) Utilities are continuous.

Then an $(\mu, \mathbf{w}, \mathbf{f})$ *is weak Bayesian stable for any* $\mathbf{w} >> (\underline{w}, \ldots \underline{w})$.

Proof. Define a set $S_i = [underlinew, underlinew + \epsilon] \cup \{\mathbf{i}\}$ for each i. By defining $G_{ij} = \text{marg}_i F$ conditional of S_i, $(\mu, \mathbf{w}, \mathbf{f})$ is weak Bayesian stable with the beliefs $\{G_{ij}\}$. □

Even if I drop the condition (4), the proposition still holds if firms share a common worst type $w \in [\underline{w}, \bar{w}]$.

3.5 Strong Bayesian Stability

The main reason behind why imposing Bayesian is not enough to make sharper prediction on stable outcomes is because it only requires the beliefs are *internally* consistent, thus, firms can be arbitrarily pessimistic by ruling out any possibilities that the potential blocking partner has higher type than the worst type. I propose a refinement of weak Bayes-consistency by requiring the beliefs are *externally* consistent as well.

Definition 13. A set of beliefs $\{G_{ij}\}$ is strong Bayes-consistent with a matching μ if

- it is weak Bayes-consistent with μ,

- for any j and $\mathbf{w} \notin \prod_{i \in I} supp(G_{ij})$, there is j' such that

  (1) $u_i(\mathbf{w}(i), \mathbf{f}(j')) > u_i(\mathbf{w}(i), \mathbf{f}(\mu(i)))$,

  (2) $\mathbb{E}[v_j(\mathbf{w}(i), \mathbf{f}(j'))|G_{ij'}] > v_j(\mathbf{w}(\mu(j')), \mathbf{f}(j'))$.

If firm j rules out $\mathbf{w}$ from the realms of possibility, there must be a valid reason for it. Strong Bayesian stability requires that for the firm j to rule out $\mathbf{w}$, it has to be the case that there would been a blocking pair if $\mathbf{w}$ were the actual state of the world. By not observing a blocking pair, firm j can eliminate $\mathbf{w}$ from the possible state of the world.

Definition 14. A matching outcome $(\mu, \mathbf{w}, \mathbf{w})$ is strong Bayesian stable if

1. there exists a set of beliefs $\{G_{ij}\}$ that is strong Bayes-consistent with the matching μ,
2. $\mathbf{w} \in \prod_{i \in I} supp(G_{ij})$, for any $j \in J$.

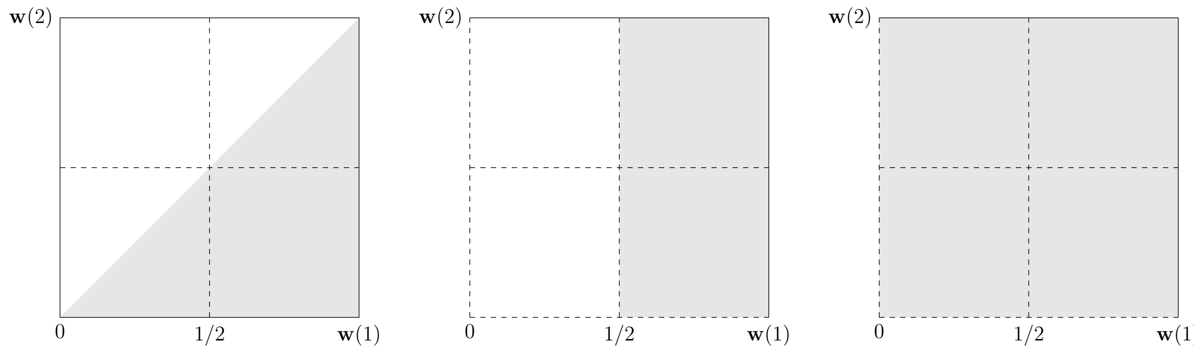
3.5.1 Examples

The next example compares Strong Bayesian stability with Weak Bayesian stability.

Example 3.5.1. Suppose that there are two firms $\{a, b\}$ and two workers $\{1, 2\}$. Both worker prefers firm a to firm b, and both firms prefer higher type, where types are drawn from $U[0, 1]$ independently. The utility of firms are given by $u_i(w) = w$. Note that only firm a has a possible blocking opportunity. Consider a matching $\mu = \{(a, 1), (b, 2)\}$. Since b is unequivocally worse firm than a, b does not have any blocking opportunity, thus, no types are eliminated from Ω_2. For firm a, given that there is no elimination from Ω_2, it would block the matching if $\mathbf{w}(1)$ were less than 1/2. Thus, by defining

$$S_1 = [0.5, 1],$$

$$S_2 = [0, 1],$$



(a) Complete information (b) Strong Bayesian (c) Weak Bayesian

Figure 3.3: Stable outcomes

and with the beliefs $G_{ia} = G_{ib} = F$ conditional on S_i, $(\mu, \mathbf{w}, \mathbf{f})$ is strongly Bayesian stable for any $\mathbf{w} \in [0.5, 1] \times [0, 1]$.

Figure 3 illustrates how the stable outcomes differ for Strong and Weak stable concepts. Since firm a is preferred by every worker, in expectation, firm a will get a better match than firm b under Strong Bayesian stability. In fact, Strong Bayesian stability coincides with Bayesian stability in Bikhchandani (2017) in this example.

The second example illustrates how Strong Bayesian stability is different from Bayesian stability in Bikhchandani (2017).

Example 3.5.2. There are two firms $\{a, b\}$ and two workers $\{1, 2\}$. Suppose that firm 1 prefers higher type while firm 2 prefers lower type. Specifically, $u_a(w) = w$ and $u_b(w) = 1 - w$. Worker 1 prefers a regardless of $\mathbf{w}(1)$ while worker 2 prefers b regardless of $\mathbf{w}(2)$. Assume that each $\mathbf{w}(i)$ is drawn from uniform $[0, 1]$ independently. Consider a matching $\mu = \{(a, 2), (b, 1)\}$. This is Example 4.2. The Bayesian stable types of workers, given the matching μ, was $\mathbf{w} \in [0, 1/2] \times [1/2, 1]$. With Strong Bayesian stability, bigger set of outcomes can be supported as stable outcomes. Suppose that firm a believes that $\mathbf{w}(1) \in [0, 2/3]$ and firm b believes that $\mathbf{w}(2) \in [1/3, 1]$. For any $\mathbf{w}(2) \in [1/3, 1]$, firm a prefers the current partner worker 2 over worker 1 in expectation as $\mathbb{E}[u_a(\mathbf{w}(1))|[0, 2/3]] \geqslant 1/3$. For any $\mathbf{w}(1) \in [0, 2/3]$, firm b prefers the current partner worker 1 over worker 2 in expectation as

$\mathbb{E}[u_b(\mathbf{w}(1))|[1/3, 1]] \geqslant 1/3$. Thus, such beliefs are weak Bayes-consistent. To check strong Bayes-consistent, notice that for any type profiles outside of $[0, 2/3] \times [1/3, 1]$, there would be at least 1 blocking pair.

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