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There are n boys and m girls. The boys are all expected utility maximizers and agree about the desirability of the prospect of dating the girls. Boys assign utility v_i to a date with girl i, where $v_1 > v_2 > \ldots v_n > 0$ and a utility of 0 to having no date. Girls don't care which boy they go out with and they prefer having a date to not having a date.

Each boy is allowed to ask one girl for a date. If a girl gets more than one offer, she chooses one of her offers at random. If she gets no offer, she stays home. Boys make their offers without knowing the actions of others. We seek a symmetric Nash equilibrium in mixed strategies for the boys.

In a symmetric equilibrium, each boy makes an offer to girl i with probability p_i. Let π_i be the conditional probability that girl i will go out with boy i if he asks her. In equilibrium the boys must be indifferent between asking any of the girls who have a positive probability of being asked out. This implies that for some $k > 0$,

$$\pi_i v_i = k \tag{1}$$

for every girl i who is asked out with positive probability. If nobody else asks a girl out, one could have a date with her for sure, just by asking. Therefore it must also be that in equilibium, $v_i \leq k$ for any girl who is not asked with positive probability.

The conditional probability π_i is determined by the probability p_i that each boy asks girl i. In particular, we note that the probability that girl i will get a date if and only if at least one boy asks her out. The probability that this happens is $1-(1-p_i)^n$. In equilibrium, all boys have the same chance of getting a date with girl i. Therefore the probability that any specified boy gets a date with girl i is

$$\hat{P}_i = \frac{1-(1-p_i)^n}{n}. \tag{2}$$

Since p_i is the probability that a boy asks girl i and π_i is the conditional probability that she goes out with him if he asks, it must be that $p_i\pi_i = \hat{P}_i$. It follows that where

$$\pi_i = \frac{\hat{P}_i}{p_i} = \frac{1-(1-p_i)^n}{p_i n}. \tag{3}$$

Let us define

$$F(p) = \frac{1-(1-p)^n}{pn} \tag{4}$$

Therefore we know that if all m girls are asked out with positive probability, then the probability distribution $(p_1, \ldots, p_m)$ will be a symmetric Nash equilibrium if and only if for $i = 1, \ldots, m$, $F(p_i) = k/v_i$ for some $k > 0$.

The key to doing comparative statics and to showing uniqueness of equilibrium is to show that the function $F(p)$ is strictly monotone decreasing. We show this in the appendix below. Since $v_i F(p_i) = k$ for all i and since F is a strictly decreasing function of p_i it follows that in equilibrium, p_i must be an increasing function of v_i and consequently the probability that a girl gets a date is an increasing function of her desirability. This is no great surprise. No paradoxes lurk here. Though perhaps some will find it comforting that even the most beautiful girl faces a positive probability of a lonely Saturday night.

Now let us prove uniqueness and existence. Let us first show uniqueness conditional on the assumption that there exists an equilibrium where the g best girls are asked with positive probability and no other girls are available. Note that since F is strictly decreasing, the function F^{-1} is well-defined and strictly decreasing. In equilibrium, we need $p_i = F^{-1}(k/v_i)$ and

$$\sum_{i=1}^{g} p_i = \sum_{i=1}^{g} F^{-1}(k/v_i) = 1. \tag{5}$$

Since F^{-1} is monotone decreasing, there can be at most one solution for k. Given k, each p_i is uniquely determined by $p_i = F(k/v_i)$.

Here I will just sketch an argument by which one can prove existence and determine the number of girls who have positive probabilities of getting offers by an inductive argument. There is a unique equilibrium if the only girl available is the best girl. Every boy makes her an offer with probability 1. Notice that if getting a date with the second best girl for sure is better than a chance of 1/n of getting the first best girl, then there will be a unique mixed strategy equilibrium in which only the two best girls are available and both of them have positive probabilities of being proposed to. With a little work one can show how to work down the list of girls, adding the next best girl so long as the prospect of a sure date with her is better than the equilibrium utility when the set of available girls excludes her.

Appendix

To see that $F'(p) < 0$, we can simplify $F(p)$ as follows. Let $q(p) = 1 - p$. Then

$$F(p) = \frac{1 - q(p)^n}{n\,(1 - q(p))} = \frac{1}{n}\left(1 + q(p) + q(p)^2 + \ldots + q(p)^{n-1}\right).$$

Differentiating and applying the chain rule, we have

$$F'(p) = -\frac{1}{n}\left(1 + 2q(p) + 3q(p)^2 + \ldots + (n-1)q(p)^{n-2}\right) < 0$$

for all $p \in [0, 1]$.