Title
Can If Be Formally Represented?

Journal
Proceedings of the Annual Meeting of the Cognitive Science Society, 3(0)

Author
Bree, D. S.

Publication Date
1981

Peer reviewed

CAN *IF* BE FORMALLY REPRESENTED?

D.S. Brée

Any attempt to arrive at a formal semantics for natural language must at least provide a mapping of the function words into the chosen formal representation. One of the function words that has given considerable problems to logicians is *if*. Even today there is a debate concerning whether or not *if* is equivalent to material implication, $\supset$, in the propositional logic; see for instance the journal Analysis. It has long been recognised that equating *if* with $\supset$ leads to difficulties. I will look at two proposals to cope with this within the confines of traditional propositional logic: one is an older logician's approach, Reichenbach's (1947) connective interpretation; the other is a recent proposal for a 'natural' logic put forward by the psychologist Braine (1978). As neither is satisfactory I will next look at two approaches involving modal logic: the well known 'strict implication' of Lewis and Langford (1932) and an improvement due to Stalnaker (1968), both of which are also unsatisfactory. This will bring me to a consideration of the 'possible worlds' logic which appears to be particularly suitable for conditional propositions. Regrettably, it once again turns out that a second proposal by Stalnaker (1975), namely that *if p,q* may reasonably be inferred from $\tilde{p}$ *or q* and vice-versa, is not proven. At this point one is tempted to abandon the attempt to capture the natural use of *if* in a formalism and to agree with Grice's highly convincing notion that the difference between *if* and material implication can be accounted for by certain conversational implicatures to which all discourse is bound. However, I find that a rather simple interpretation of *if* in the first order predicate calculus, supplemented with a convention to differentiate asserted from presupposed propositions, appears to meet all the standard objections.

The well known problems that arise by equating *if p,q* with $p \supset q$ are:

a. *Affirmation through denial of the antecedent* is permissable for $\supset$, i.e. $\tilde{p} \rightarrow p \supset q$, but not for *if*, e.g.
   A: If God exists we are free to do what we want.
   B: How do you come to that conclusion?
   A: By knowing that God doesn't exist.

b. *Affirmation through assertion of the consequent* is [illegible] for $\supset$, i.e
   A: If the [illegible] there will be rioting in the streets.
   B: How do you know?
   A: Because there are always street riots here at this time of year.

c. *Denying* material implication permits only one true state, i.e. $\sim(p \supset q) \rightarrow p \,\&\, \tilde{q}$, but denying a conditional may be something else, e.g.
   A: It's not so that if God exists we are free to do what we want.
   B: Are you claiming that God exists and we aren't free to do what we want?

Similar problems arise when *or* is equated with $\vee$, a situation which led Reichenbach (1947) to propose that a 'connective' interpretation be given to $\vee$ in order to make it equivalent to *or*. This interpretation requires that all possibilities must remain open. Applying this idea to $\supset$ would require that it must not be possible to do away with any of the three residual statements for the truth of $p \supset q$, i.e. $(p\&q)$, $(\tilde{p}\&q)$ and $(\tilde{p}\&\tilde{q})$, in order for $\supset$ to be equated with *if*. Now both denying the antecedent, $\tilde{p}$, and asserting the consequent, q, rule out two of these three and so no *if* statement may be used. Moreover, denying an *if p,q* may be denying that all three residual statements are open. So Reichenbach's proposal copes with the three well known problems. However, it is open to criticism as it fails to distinguish the relative importance of the three residual statements for an *if* statement:

d. *Homogeneity:* $\tilde{p}\&q \rightarrow p \supset q$ but not *if p,q* as seen in an example of Conditional Perfection (Geis & Zwicky, 1971)
   A: If you mow the lawn, I'll give you 5 dollars.
   B: (Returning after a minute) May I have the 5 dollars now?
   A: But you can't have mowed the lawn already!
   B: I haven't, but you said there was a possibility of my getting the 5 dollars anyway ($\tilde{p}$ and q).

It is not only logicians who have tried to adapt propositional logic so that $\supset$ can be equated with *if*, psychologists also are interested. For example let us consider a recent attempt by Braine (1978) to set up a 'natural' logic based on 18 'natural schemata' which he claims are available for reasoning. These include for example modus ponens but not modus tollens, which is thus not available as [illegible] tive argument but only follows from several steps. Braine equates *if* with /, which is equivalent to an inference in his system. But it can be shown that / suffers from all three of the traditional problems of ⊃, see Figure 1. Note in particular the ease with which the truth of the consequent *q*, can be used to derive *p/q*. I believe that we should conclude from both the logicians' and psychologists' attempts to adapt propositional logic so that *if* can be equated with ⊃ that this is not the way to go.

Figure 1

Proofs that Braine's system of Natural Logic suffers from the traditional problems

Affirmation from p̃				Affirmation from q				Denial to p & q̃			
#	from	by	gives	#	from	by	gives	#	from	by	gives
A1		N16	p̃	B1:		N16	p&q	C1		N16	F(p/q)
A2	1	N 5	F(p&q̃)	B2:	1	N 7	q				
A3	1,2		p̃/F(p&q̃)	B3:	1,2		(p&q)/q	C2		N16	q
								C3	2	B 6	p/q
								C4	1,3	N 1	(p/q)&F(p/q)
A4		N16	p	B4:		N16	q	C5	2,4		q/((p/q)&F(p/
A5		N16	F(p&q̃)	B5	3,4	N18	p/q	C6	5	N 7	q̃
A6	4,5	N12	F(q̃)	B6:	4,5		q/(p/q)				
A7	6	N 8	q					C7		N16	p̃
A9	4,5,7		(p&F(p&q̃))/q					C8	7	A13	p/q
								C9		N 1	(p/q)&F(p/q)
A10		N16	F(p&q̃)					C10	7,9		p̃/((p/q)&F(p/
A11	9,10	N18	p/q					C11	10	N 7	p
A12	10,11		F(p&q̃)/(p/q)								
								C12	5,11	N 1	p&q̃
A13	3,12	N17	p/(p/q)					C13	1,12		~(p/q)/(p&q)

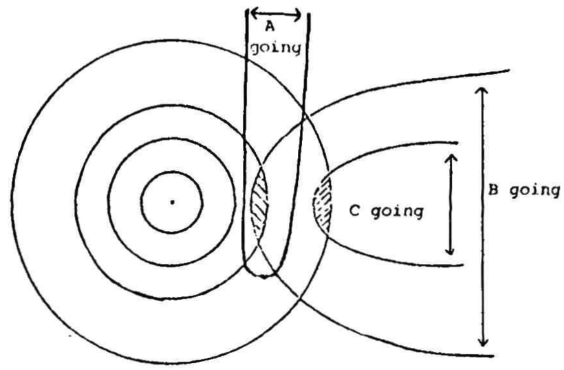
N1 to N20 are Braine's natural schemas, most of which are obvious: N16 is 'Assuming'.

Attempts to equate *if* with a modal operator have also met with difficulties. For instance the well known proposal that *if* might be equivalent to strict implication put forward by Lewis and Langford (1932) partially overcomes all four of the problems of material implication. Strict implication is defined as asserting the impossibility of *p&q*, i.e. $\sim\Diamond(p\tilde{q})$. Strict implication is thus a stronger concept than material implication; the latter can be deduced from the former. However, it still suffers from stronger versions of the two problems of affirmation as $\sim\Diamond p \rightarrow \sim\Diamond(p\tilde{q})$ and $\Box q \rightarrow \sim\Diamond\tilde{q} \rightarrow \sim\Diamond(p\tilde{q})$ but it is not so that from either the impossibility of the antecedent or from the necessity of the consequent that a conditional *if* statement may be deduced, as can be seen by appropriately modifying the examples given for the affirmation problems above. At one point Stalnaker (1968) attempted to improve on the Lewis and Langford proposal by equating *if p,q* with $p>q$; his corner, >, entailed material implication and is entailed by strict implication, i.e. $p>q \rightarrow p \supset q$ and $\sim\Diamond(p\tilde{q}) \rightarrow \Box(p \supset q) \rightarrow p>q$. Unfortunately this leaves it open to the same criticism as strict implication, for instance the impossibility of the antecedent is sufficient for the corner, i.e. $\sim\Diamond p \rightarrow \Box(p \supset q) \rightarrow p>q$. Modal logic has not offered us a formal representation of *if*, although it has come closer than simple propositional logic.

It is then with high expectations that we turn to the logic of 'possible worlds' which seems designed to cope with conditionals. In brief this postulates a universe of possible worlds arranged in an ordered series of sets such that each set contains all the worlds of the previous set and some more worlds besides. The initial set contains one world, which is usually interpreted as our world as it is. Worlds added in going from one set, K, to the next set, K+1, are further removed from our world than all worlds in set K. These sets are the contexts in which sentences are interpreted. For a conditional sentence the appropriate context is the first set which contains a world in which the proposition underlying the antecedent is true. A nice consequence of this is that *if* is no longer a transitive relation as can be seen by examining Figure 2. Even 'if B goes to the party, A will go' and 'if C goes, B will go' it does not follow that 'if C goes, A will go'; A may well want to go to meet B but not in the unlikely circumstances of C's going as well.

Figure 2

Possible worlds representation of if, p, q is not transitive



Suppose: 'If B's going to the party, A's going'
[illegible], B's going'
But not: 'If C's going to the party, A's going'

Figure 3

A formal version of Stalnaker's proof

<u>To Proove</u>: 'P̃ *or* Q therefore *if* P, Q' is a reasonable inference

<u>Proof</u>: Suppose 'P̃ *or* Q' is appropriate in K.
S2: Suppose 'P̃ *or* Q' is accepted in K.
S3: So P&Q is appropriate in K.
S4: So P is appropriate in K.
S5: If X is appropriate and Y is accepted in K, '*if* X, Y' is accepted in K.
S6: So '*if* P, then P̃ *or* Q' is accepted in K.
S7: So '*if* P, then Q' is accepted in K.
S8: So 'P̃ *or* Q therefore *if* P, Q' is a reasonable inference.

It is within this formalism that Stalnaker (1975) set out to show that, although *if p,q* may not be truth functionally equivalent to $\tilde{p}$ *or q* the one is a reasonable inference from the other and vice-versa. A reasonable inference occurs when for all contexts, K, in which the premise is appropriate and acceptable the conclusion is also. A proposition, P, is appropriate in K if there exists at least one world in K in which P holds; P is acceptable if it holds in all worlds in K. A formal version of Stalnaker's proof that '$\sim$P or Q therefore if P,Q' is a reasonable inference is reproduced in Figure 3. Unfortunately it is flawed as it assumes what it is setting out to prove, namely that *if* P,Q is equivalent to $P \supset Q$ in step 5. The result is not surprising as even the 'impossible worlds' formalism of *if* accepts any appropriate conditional in which the consequent is necessarily true, the second of the traditional problems of affirmation.

Those of you who never believed in the logical basis of natural language will by now be thinking 'I told you so' and those convinced in the formal program will be busy finding a new formalism. Perhaps we should follow a middle road, for example the path marked out by Grice (1967) in his William James lectures. He virtually divided the problem into two parts: retain a simple formal representation for natural language connectives, including *if*, and account for deviations between the formalism and the normal usage by postulating a set of Indirect Conditions, IC. Then $if \equiv \supset + IC$. The IC's are the cancellable part of the meaning of *if*, e.g. part of the IC is that the speaker doesn't know the truth values of either the antecedent nor the consequent but these are cancellable as in

> I know where Smith is and what he is doing; all I'll say is that if he's in London he's attending the meeting.

Although they are cancellable they are not detachable, i.e. it isn't possible to find another formulation which is equivalent to *if*-IC, e.g. both

> Either Smith isn't in London or he's attending the meeting.
>
> It isn't the case that Smith's in London and not attending the meeting.

contain the same IC. Such IC's Grice called Conversational Implicatures, CI's, which hold for conversation in general not only for *if*. His first CI is the maxim of quantity: a speaker does not say less than he knows. Thus a speaker will not use *if p,q* when he knows $\tilde{p}$ *or q* for certain, which avoids the two affirmation problems. His third CI is more specific: *if p,q* has an implicature 'supposing *p*, then *q*'. This CI is also present with $\tilde{p}$ *or q*. It enables the problems of denial and homogeneity to be avoided. However, it doesn't have the generality that one would like from a CI. Would it be too uncharitable to say that it ducks the issue?

With a certain trepidation I would like to propose a formalism for *if p,q* in the first order predicate calculus

$$F1: \forall w(P(w) \supset Q(w)) \ \& \ [\exists(x,y)(P(x) \& \sim Q(y))]$$

in which the proposition P is the proposition derived from *p* in the context K, and P(w) is true if P is the case in world w. The universal, $\forall$, and existential, $\exists$, operators operate over all worlds in the context set K, following the 'possible worlds' formalism. The necessity for differentiating *p* and P is not only that the proposition underlying a sentence depends on the context in which that sentence is interpreted, but also because the protasis and apodosis of a conditional are not necessarily equivalent to sentences, they sometimes cannot stand alone, e.g.

> If anyone has a malignant cancer of the backbone, they'll be dead within 6 months.
>
> If Alexander was afraid, I didn't notice it.

As Ryle (1950) pointed out *if* sentences contain statement indents, not statements. They can be used for making inferences, but are not in themselves inferences.

This formalism avoids the three traditional problems of equating *if* with $\supset$. Let W be our world. Then neither $\sim$P(W) nor Q(W) is sufficient to affirm *if p,q* unless W is the only world in K. Nor can it be confirmed by $\tilde{p}$ nor by *q* as these two extremes are ruled out by the 'presupposition' of *if*, given in F1 between square brackets. The problem of denial is also avoided, as denying *if* denies the assertion $\forall w(P(w) \supset Q(w))$ which is equivalent to $\exists w(P(w) \ \& \ \sim Q(w))$, but the world that satisfies this denial is not necessarily our world. However, this formalism fails to avoid the homogeneity problem: there is no difference between a world for which PQ is true and one for which $\tilde{P}Q$ is true. This problem can be avoided if we are prepared to use strict implication rather than material implication. We then define *if p,q* to be represented by

$$F2: \forall w(P(w) \rightarrow Q(w)) \ \& \ [\exists(x,y)(P(x) \ \& \ \sim Q(y))]$$

in which the same conventions hold as above, and $\rightarrow$ is strict implication. Specifically strict implication has a known truth value only when the antecedent is true. Thys $P \rightarrow Q$ is true for PQ and false for $P\tilde{Q}$, which avoids the homogeneity problem. It remains to be seen if there are other problems which this formalism is not capable of handling.

At first glance this formulation for *if* seems to cope with problems that arise for 'other' interpretations of *if* than the standard. Most notorious of these is the counterfactual in which the premise is claimed to be false of the actual world, but from a false proposition anything follows! As Lewis (1973) has shown the 'possible worlds' approach, which I have here adapted, can cope with counterfactuals. There is no claim in F2 that P(W) is false (or true), e.g.

If Paul had stuck to his plan, he'd (still) have been famous.

(Examples are adapted from the Brown corpus of American English, Kučera and Francis, 1967). F2 also copes with factuals in which both P(W) and Q(W) hold, e.g.

If Wilhelm Reich is the Moses who has led them out of the Egypt of sexual slavery, Dylan Thomas is the poet who offers them the Dionysian dealectic of justification for their indulgence in liquor.

As P(W) is true Q(W) must hold. But contrary to the possible worlds formulation, the context, K, must not be confined to W as then $\sim\exists w(\sim Q(w))$. Nor is there any problem with Austin's (1961) stipulative use of *if*, e.g.

There are some biscuits on the table, if you want some.

as again the truth status of Q(W) is not necessarily open but may be true. And I think it will handle cases of doubtful presupposition as in

It made him conspicuous to the enemy, if it was the enemy.

Here the interpretation of *q* is problematic unless P is true, but since F2 uses strict implication this does not matter.

I do not claim that the use of F2 for *if* can decide which of these 'interpretations' is actually the case. Rather that decision should not rest upon *if*, but must be made using other aspects of the sentence, e.g. what the listener knows that the speaker knows. This is particularly the case for the use of *if* within the scope of performative verb, e.g.

He promised vengence on V.L. if ever the chance came his way.

What I do say is that F2 *permits* these different interpretations, with the exception of the performative.

REFERENCES

AUSTIN, J.L. (1961)
Ifs and cans. In J.O. Urmson and G.J. Warnock, eds., *Philosophical papers of J.L. Austin*. London; Oxford University Press.

BRAINE, M.D.S. (1979)
On the relation between the natural logic of reasoning and standard logic. *Psychological Review, 85*, 1-21.

GEISS, M.L. and ZWICKY, A.M. (1971)
On invited inferences. *Linguistic Inquiry, 2*, 561-566.

GRICE, H.P. (1967)
William James Lecturers, Harvard University. Published in part as "Logic and conversation", P. Cole and J.L. Morgan, eds., *Syntax and Semantics, Vol. 3: Speech Acts*. New York; seminar Press, 1975, 41-58.

KUCERA, H. and FRANCIS, W. N. (1967)
Computational analysis of present-day American English. Providence; Brown University Press.

LEWIS, C.I. and LANGFORD, C.H. (1932 and 1959)
Symbolic Logic. 2nd ed. New York; Dover.

LEWIS, D.K. (1973)
Counterfactuals. Cambridge, Mass.; Harvard University Press.

REICHENBACH, H. (1947)
Elements of symbolic Logic. London; Macmillan.

RYLE, G. (1950)
'If', 'so' and 'because'. In M. Black, ed., *Philosophical analysis*. New York; Cornell University Press.

STALNAKER, R.C. (1968)
A theory of conditionals. In: N. Rescher, ed. *Studies in logical Theory*. Oxford; Blackwell. Reprinted in E. Sosa, ed., *Causation and counterfactuals*. Oxford University Press, 1975.

STALNAKER, R.C. (1975)
Indicative conditionals. *Philosophia, 5*, 269-286.