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Essays in Auction Theory and Financial Economics

A dissertation submitted in partial satisfaction of the
requirements for the degree Doctor of Philosophy
in Economics

by

Paulo Bráulio de Souza Coutinho

2013

ABSTRACT OF THE DISSERTATION

Essays in Auction Theory and Financial Economics

by

Paulo Braulio de Souza Coutinho

Doctor of Philosophy in Economics

University of California, Los Angeles, 2013

Professor Marek Pycia, Chair

The primary focus of this dissertation is to study the connections between the auction mechanism used to sell Treasuries securities and the secondary market activity, when auction participants can trade the auctioned securities among themselves. I use a game theoretically framework where participants understand the market rules and behave strategically so as to maximize their returns. The model is capable of explaining puzzling empirical patterns from the U.S. Treasury market and provides policy implications. In the third chapter of the dissertation I use tools developed by the auction theory to analyze corporate organizations capital structure decisions. For a firms perspective, using debt to finance new investments could be cheaper than

using equity. Nevertheless, a higher debt level increases the risks the investment will be liquidated before it yields the promised dividends. The chapter studies firms' optimal decisions when facing the trade-off between debt and equity and its implications on equilibrium prices and interest rates on the financial market

The dissertation of Paulo Braulio de Souza Coutinho is approved.

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To my parents and my beloved wife, for all their support.

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CHAPTER 1

Introduction

Auctions provide the principal means of financing federal deficits and refinancing maturing debts in the US and many other countries around the world. On a weekly basis, the US Treasury uses an auction structure to determine the interest rate it will pay on its Bills, Notes and Bonds. The typical auction value exceeds ten billion dollars. A single basis point (.01%) increase or decrease on the resulted interest rate represents, respectively, an extra cost or savings of millions of dollars for taxpayers. Treasury securities are not traded only during the official Treasury auctions. In fact, dealers actively buy and sell securities before (using forward contracts) and after the auction takes place. Dealers' positions and strategies in other market stages have a direct impact on how they formulate their bids during the auction stage. The literature addressing investors' strategic interactions on all three market stages (auction, pre and post-markets), and its implications on prices and trading patterns, is fairly incipient. My dissertation is an attempt to fill this gap. Several empirical studies have documented that the price of Treasury securities is lower in the official auctions than in the forward market preceding it, which is called the when-issued market. Nevertheless, the volume of when-issued securities being traded in the inter-dealer market is remarkably large. In other words, some of the dealers are willing to acquire a substantially large quantity of securities at this stage despite the price gap between the when-issued market and the auction. The first chapter of my dissertation, titled When-Issued Securities and Treasury Auctions, builds up a dynamic model to show how the foreseen strategic interaction of dealers in the auction stage gives dealers incentives to trade when-issued securities and how the price gap arises in equilibrium. The model also implies that previous empirical papers

which used observed prices of Treasury securities from the when-issued market as proxies for their true market value may have reached into biased results. Finally, from a social welfare perspective, I show that when-issued markets improve how the auction procedure distributes Treasury securities to the public. The possibility that a dealer can resell the security has a direct impact on her bidding behavior during the auction. Although there exists a relatively new literature studying the effects of the possibility of resale on auctions of a single unit, not much has been done to understand its effect when multiple units of the same good are being auctioned at the same time, which is essentially the case in Treasury auctions. The second chapter of my dissertation, titled Treasury Auctions with Resale, is an attempt to shed light on this problem. I investigate how the existence of a resale market affects the behavior of participants in Treasury auctions. I am able to conclude that increasing the number of participants in the auction can actually decrease the auction's revenue, which goes against the common intuition that more competition in the auction is always good for the Treasury. Moreover, the paper clarifies important distinctions between attracting additional final investors and attracting additional speculators in the equilibrium outcome of the market. Finally, the third chapter of my dissertation is a joint paper with Professor Antonio Bernardo from the Finance department at the Anderson School of Management. We use the theoretical background developed by auction theorists to analyze an old corporate finance problem, namely how do firms choose their capital structure. In many real-world situations, corporate firms may prefer debt over equity as the way to finance new or continuing projects. For instance, under the U.S. law, interests paid in debt are not subject to taxation whereas the same is not true about dividends paid on equity. However, a higher debt level increases the leverage of a firm. This means that firms could go underwater with small variations on its market value, which increases the chance it is forced to liquidate the project before it yields returns. Early liquidation is undesirable since it can be associated to termination of socially desirable project. Therefore, institutional rules which incentivize firms to artificially

increase their debt levels (as the tax exemption for interest paid on debt), can deepen the negative effects of financial crises. Besides explaining observed firm's decision, the framework developed in this chapter can be used to policy analysis. Indeed, it provides insights on the relationship between the institutional environment firms are facing and their debt level decisions.

CHAPTER 2

When-issued markets and Treasury auctions

2.1. Introduction

Investors set up transactions involving new Treasury Securities before they are effectively offered to the public. A commonly used financial instrument for these type of transactions are When-Issued (WI) securities. A when-issued security is simply a forward contract with a specific settlement day: the issuance date of the underlying Treasury security. The when-issued market is particularly active. [Fabozzi and Fleming \[2004\]](#) and [Barclay et al. \[2006\]](#) documented that it accounts roughly for six percent of the entire volume involving Treasury securities in the inter-dealer market in U.S.¹ This figure is even more remarkable when one takes into account that when-issued securities can be traded only during the small window between the Treasury's official announcement and the effective issuance dates, usually consisting of five trading days.

Several studies have observed that the price of securities is relatively higher in the when-issued market than in the official Treasury auction.² The price gap, called *underpricing*, is observable on the same day and even within minutes to when the auction takes place. Surprisingly, some of these studies used data from inter-dealer markets. This means that some dealers, who actively participate in both Treasury auctions and when-issued markets, are willing to acquire a substantial amount at the when-issued market in spite of the price gap.

¹[Barclay et al. \[2006\]](#) do not explicitly document this percentage in their article. However, they argued that "Approximately 93% of the trades in our sample have one-day settlement. The majority of the trades with nonstandard (longer than 1 day) settlement occur during the when-issued trading period".

²e.g, [Goldreich \[2007\]](#); [Bikhchandani et al. \[2000\]](#); [Simon \[1994\]](#).

At first glance, these two empirical observations might seem puzzling. Why do dealers buy in the when-issued market instead of waiting a few minutes for the auction and paying a lower price for the same securities? In this paper, I show that the structure of the market can give rise to the observed patterns in prices and trading activity. Specifically, the way the Treasury auction is organized - as uniform price auction - gives dealers incentives to trade when-issued securities even in the presence of a price premium. Moreover, I show that underpricing arises as an outcome of equilibrium when the same set of dealers interact strategically in both Treasury auctions and when-issued markets.

I consider a model in the lines of [Wilson \[1979\]](#) auction of shares.³ The mechanism is similar to a Walrasian auction. The Treasury uses a uniform price auction to sell perfectly divisible securities to a finite number of dealers. The dealers simultaneously submit complete bid schedules determining the amount of securities they are willing to acquire for each possible price. The Treasury, then, aggregates the bids and determines the price that clears the market - the *stop-out price*. This price determines how much dealers pay for the bids and the amount of securities they get.

The existence of equilibria where auction participants strategically "shade" their bids is a well know property of this class of models. In order to artificially reduce the auction stop-out price, dealers submit bid schedules that understate their true valuation for the securities. The intuition is the same as in a simple monopsonist problem: dealers face the trade-off between buying larger quantities of the security at the expense of increasing the auction price. However, the price does not increase only for the additional securities, but also for the ones the dealer was already acquiring. In the end, dealers prefer to acquire a lower amount of securities and pay a lower price for them.

³Examples of other papers using this type of model are [Kyle \[1989\]](#), [Back and Zender \[1993\]](#), [Ausubel and Cramton \[2002\]](#), [Wang and Zender \[2002\]](#), [Pycia et al. \[2010\]](#) and [Coutinho \[2012a\]](#).

The difference between a dealer’s true valuation and the bid he submits is often called *bid shading*. I show how *bid shading* can play a central role in giving dealers incentives to trade when-issued securities. When dealers are heterogeneous, the magnitude of their bid shadings will be different at equilibrium. As a consequence, the auction mechanism fails to distribute securities efficiently. Dealers acquiring relatively larger amounts will be more sensitive to price variations and as a consequence will have a higher incentive to shade their bids. In the end, larger/smaller dealers acquires less/more than they would get if the securities were distributed efficiently.⁴

Dealers have incentives to find additional means to improve the way the auction is allocating securities. This paper focus on how they can use when-issued securities to improve the allocation. I extend the benchmark model to allow dealers to trade securities on a when-issued basis. I model the when-issued market as a uniform price auction, in a similar fashion as the auction stage. Dealers submit complete bid schedules determining the amount of when-issued securities they want to buy/sell to a central inter-dealer broker.⁵ The broker determines the when-issued price, which will be the the price that clears the market, and distributes securities accordingly.

Underpricing will arise endogenously, as an outcome of the strategic interaction of dealers in both when-issued and auction stages. Specifically, larger dealers need to induce smaller dealers to take short positions in the when-issued market. The only way the smaller dealers would be willing to sell a when-issued security is if the price they get for it were larger than his valuation for the security. However, their valuation is larger than the auction price since dealers submit bids with *shading* in the auction.

The analysis has further implications on interpretation of empirical exercises. For instance, when-issued prices are frequently used to determine how much the Treasury is losing,

⁴For a detailed discussion on the inefficiency of uniform price auctions, see [Ausubel and Cramton \[2002\]](#).

⁵Complete schedules in central markets can be attained by a combination of limit and stop orders.

in terms of revenue, due to the auction mechanism. The idea is that the when-issued price resembles the true market value of the underlying security, thus the price gap would give the loss, *per security*, for the Treasury.⁶ I show that, although the price of a security in the when-issued market is larger than in the auction stage, it still falls below the security's true market value. Therefore, the magnitude of the underpricing is a lower bound for Treasury's loss of revenue.

When-issued prices are also used in empirical papers as a benchmark to compare relative performances of different auction mechanisms. The idea is to compare the magnitude of the resulting underpricing when the auction follows different pricing and allocation rules (e.g. uniform vs. discriminatory auctions). While this paper does not consider other mechanisms besides uniform price auctions, the analysis below highlights the fact that the prices in the when-issued and auction stages are jointly determined in equilibrium. Indeed, dealers' behavior in the when-issued market depends on how they will affect the outcome of the auction. However, the latter depends directly on the specific mechanism being used. That said, there is no straightforward reason why the equilibrium gap between the price in the two market stages is independent of the choice of the auction mechanism. It is also not straightforward that a higher underpricing implies lower revenues for the Treasury. Therefore, just comparing the magnitude of empirically observed underpricing might not be very informative about relative revenue and efficient performances across auction mechanisms.

Related literature: Despite the importance of the when-issued market for Treasury securities, the literature considering its relationship with auction outcomes is fairly incipient. From a theoretical perspective, I am aware of only two studies analyzing the strategic behavior of agents participating in a pre-market before an auction, and both follow an approach substantially different from the one considered in this paper. The first one, [Chatterjea and Jarrow \[1998\]](#), restricts market participants to submit a single bid for the entire quantity

⁶By "true market value", I mean the price that would arise in equilibrium under perfect competition.

being supplied in the auction. Therefore, they do not analyze how the uniform price mechanism, per se, can give incentives to participants to trade in a when-issued market before the auction takes place. The second one, [Nyborg and Strebulaev \[2004\]](#), analyzes equilibrium of multiple unit auctions when dealers arrive with an *exogenous* set of positions from the when-issued market.⁷ In the present paper, however, I allow market participants to *endogenously* choose these positions, which pins down an equilibrium price for when-issued securities.

Outside the contexts of multiple unit auctions and the Treasury securities market, a number of papers have considered the interaction between spot and forward markets in the presence of market power. The pioneering work in this area is [Allaz and Vila \[1993\]](#). They considered an environment where two duopolists can sell their products to final consumers in a cournot spot market and in a forward market. In the context of energy markets, [Powell \[1993\]](#), [Green \[1999\]](#) considered an environment where generators can supply electricity in a forward and in a spot market. These studies focus on how firms can commit to behave more aggressively in the spot market by selling part of their supply to *final* consumers in forward markets. This paper adds to this literature by considering competition in bid schedules in both market stages, by highlighting why agents have incentives to trade securities *among* themselves before the auction, which generates underpricing, and by considering the market for Treasury securities.

This paper is also related to literature on competition markets for financial securities where agents have market power. Following the seminal work of [Kyle \[1989\]](#), competition in demand (supply) schedules has been a commonly used framework to model markets where dealers recognize that their orders have an impact on the price of the underlying security. For instance, [Vayanos \[1999\]](#), [Rostek and Weretka \[2011\]](#) extended the benchmark model to a dynamic setting closely related to the one considered in this paper. The former uses a

⁷Their focus is how the possibility of being squeezed in an after-auction market affects their strategy in the auction stage.

dynamic double uniform price auction mechanism to analyze how strategic traders adjust their holdings of a risk asset when they are subject to random endowment shocks in each period. [Rostek and Weretka \[2011\]](#), on the other hand, consider an environment similar to [Vayanos \[1999\]](#), but focus on the implications of strategic behavior of trader in the efficiency and arbitrage properties of equilibrium. This paper differs from their analysis by explicitly considering the Treasury auction, as well as considering different environments for the zero-sum pre-auction forward market (“when-issued market”), and by providing further comparative statics for the analysis of the Treasury securities market.

This paper points out an alternative reason bidders can benefit from participating in a pre-auction market. Pre-auction markets can be used as a mechanism to ensure efficient collusion among a subset of auction participants. In the context of single-unit auctions, [McAfee and McMillan \[1992\]](#) showed how participants (or a subset of participants) use the pre-auction stage to decide *ex ante* which one of them will compete for the security in the auction. The reduced competition decreases the price of the good in the auction stage and participants can share the surplus extracted from the auctioneer among themselves. In the environment considered in the current paper, however, dealers are not trading on a pre-auction market (when-issued) to reduce the auction’s price. Indeed, here, the price of securities in the auction are invariant to their actions in the when-issued market. Even so, dealers can still benefit from participating in a pre-auction market as this market stage improves the final allocation of securities, and they can share the efficiency gains among themselves.

2.2. Example: Two Dealers, Two Units

In this section, I present a simple example that illustrates how the strategic behavior from dealers is consistent with underpricing. Suppose there are only two risk neutral dealers, $N = 2$, competing for two identical units of an indivisible Treasury security through a

uniform price auction. For simplicity, I assume that their valuation for securities, defined as $v_i > 0, i = 1, 2$, is constant for both units, with (i) $v_1 > v_2$ and (ii) $2v_2 > v_1$. We can express dealer i 's payoff if he acquires q units in exchange for a monetary payment m as:

$$U_i(q, p) = qv_i - m.$$

In the auction stage, dealer $i \in \mathcal{I}$ submits a bid $\beta_i = (\beta_i^1, \beta_i^2) \in \mathbb{R}_+^2$, representing how much he is willing to pay for the first security, β_i^1 , and for the second, β_i^2 . Without loss of generality, I assume that $\beta_i^1 \geq \beta_i^2$, i.e., the bid function should be weakly decreasing. The securities are allocated to the two highest bids. Dealers pay the stop-out price, $p^{so} \geq 0$, defined as the highest rejected bid. In case of a tie, the security goes to dealer 1.

For exposition motives, I focus on a specific equilibrium for the auction stage. One can use a standard argument to show that it is weakly dominant for dealers to bid their true valuation for the first security, i.e., $\beta_i^1 = v_i$. Within the class of equilibria of the auction in which dealers are submitting their true valuation for the first unit, I will focus on the one which maximizes dealers' payoffs. All other equilibria are Pareto dominated for both dealers.⁸ We have the following result:

CLAIM 1. *There is an equilibrium of the auction with dealer $i \in \mathcal{I}$ submitting $\beta_i^0 = (v_i, 0)$. The equilibrium stop out price is $p^{so} = 0$ and i 's payoffs is $U_i^0 = v_i$.*

It is easy to see that dealers do not have incentives to deviate from the equilibrium. Both end up acquiring one unit of the security and pay $p^{so} = 0$ for it. Since the stop-out price is already in its lower bound, the only possible way a dealer could increase his payoff is by increasing the number of securities he gets. Suppose that i decided to deviate and acquire both securities in the auction. The stop-out price would increase to v_{-i} . His payoff, in this

⁸This equilibrium selection criteria is used, for instance, in [Pagnozzi \[2010\]](#).

case, would be given by $\hat{U}_i = 2(v_i - v_{-i})$ which is strictly less than U_i^0 by the assumptions on dealers' valuations.

Note that there is bid shading in the equilibrium described above. i submits $\beta_i^2 = 0$ even though his true valuation for the second security is v_i . Dealers' strategic behavior in the auction leads to an inefficient final allocation: since $v_1 > v_2$, dealer 1 should be getting the two securities instead of only one. However, she knows that her bid for the second unit will directly affect the price she pays for the first unit. In this case, she does better by giving up the second security in order to reduce the cost of acquiring the first one.

Is it possible that both dealers benefit from trading a when-issued security before the auction takes place? The answer is yes. Suppose they enter into an agreement establishing that dealer 2 will deliver one unit of the security to dealer 1 in exchange for pre-established price $p^{wi} \in (v_2, v_1)$ after the auction takes place. This means that dealer 2 should acquire at least one of the two securities being auctioned in order to fulfill his contract with dealer 1.⁹ I will make the hypothesis that there is an exogenous monetary penalty, $\pi \geq v_2$, to dealer 2 if he does not deliver the promised security to dealer 1. This assumption rules out cases where B optimally chooses to default in equilibrium. If dealers arrive in the auction with this contract, we have the following result:

CLAIM 2. *There is an equilibrium of the auction where dealers submit $\beta_1^{wi} = (v_1, 0)$ and $\beta_2^{wi} = (\pi, 0)$. The equilibrium stop out price is $p^{so} = 0$ and payoffs are given by:*

$$U_1^{wi} = 2v_1 - p^{wi}.$$

$$U_2^{wi} = p^{wi}.$$

⁹I am not allowing trade to occur after the auction takes place. This assumption is not necessary to get the qualitative results in this section (and in the following ones).

As before, each dealer is acquiring one unit of the security in the auction stage and paying $p^{so} = 0$ for them. However, now dealer 2 will have to transfer the acquired security in the auction to dealer 1 in exchange for p^{wi} as established in the when-issued transaction.

The simple environment considered in this section illustrates how strategic behavior of dealers in both the auction and the when-issued market can generate the pattern observed in the data. First, it illustrates that dealers can benefit by trading when-issued securities prior to the auction. Both dealers are better off after we allow them to trade the when-issued security compared to the case where they were not allowed.

Second, the equilibrium is consistent with *underpricing*. The price of a when-issued security must satisfy $p^{wi} \in (v_2, v_1)$, otherwise the trade would be blocked by one of the two dealers.¹⁰ Therefore, we have $p^{wi} > p^{so}$, i.e., the price dealers trade the when-issued security is higher than they pay for the securities in the auction. Appendix B shows that, if dealers compete for the when-issued security through a double auction, the equilibrium price satisfies $p^{wi} \in (v_2, v_1)$.

Third, dealers have incentives to acquire securities in the when-issued market even in the presence of the price premium. Dealer 1 has the incentive to acquire a security in the when-issued market even though the price she is paying is larger than the auction's stop-out price. The argument is more subtle: if dealer 1, instead of acquiring the when-issued security, decided to acquire the second unit in the auction, she would actually pay a lower price for it. In this case, the stop-out price in the auction would rise to $\hat{p}^{so} = v_2$, which is still strictly lower than p^{wi} . However, in choosing to do so, she would affect the price she pays for the *first* unit she is acquiring in the same auction. On the other hand, if she decides to buy the additional security in the when-issued market, the auction stop-out price wouldn't be affected. The overall cost for dealer 1 is lower when she adopts the latter strategy.

¹⁰I am assuming that they prefer not to trade a when-issued security if they are indifferent to do so.

2.3. Main Framework

In this section I describe the basic model used in the rest of the paper. Treasury securities are simultaneously offered to dealers through a divisible good uniform price auction. Unlike auctions of a single indivisible good, in divisible good auctions, participants compete for shares of a positive quantity of the good being auctioned. A bid submitted by the participants of the auction is an entire schedule describing how many securities they are willing to acquire for all possible prices.

There are N dealers in the economy with preferences over two goods: Treasury securities, q , and money m . The preference of a dealer $i \in \mathcal{I} = \{1, \dots, N\}$ is represented by a quasi-linear utility function $U_i : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ such that $U_i(q, m) = u_i(q) + m$. For tractability, dealers' marginal utility in respect to securities is assumed to be linear, $u_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfies:

ASSUMPTION 1. $\frac{\partial u_i(q)}{\partial q} = v_i - \rho q$.

Dealers have complete information, so v_i is common knowledge.

The Treasury uses a uniform price auction to sell an exogenous quantity, Q , of a perfectly divisible security. Dealer $i \in \mathcal{I}$ submits a bid schedule, $q_i^A(p, h)$, $q_i^A : \mathbb{R}_+ \times \mathcal{H} \rightarrow \mathbb{R}_+$, specifying the quantity demanded for each possible price and when-issued history $h \in \mathcal{H}$, which will be described later. The bid $q_i^A(p, h)$ is left continuous and weakly decreasing in the first argument. After collecting all individual bids, the Treasury determines the *stop-out price*, p^{so} . The stop-out price is defined as the highest value of $p \geq 0$ such that the aggregate bid is larger than or equal to the realized supply, i.e., $\sum_{j=1}^N q_j^A(p^{so}, h) \geq Q$ ($p^{so} \equiv 0$ if $\sum_{j=1}^N q_j^A(0, h) < Q$). Dealer i gets $\psi_i^A \equiv q_i^A(p^{so}, h)$ units of the good, i.e., the amount he bidded for at p^{so} and pays the stop-out price for all units he acquires, $p^{so} \times \psi_i^A$.

A history is defined as vector $h = (p^{wi}, \{\theta_i^{wi}\}_{i \in \mathcal{I}})$ specifying the price of when-issued securities, p^{wi} , and a set of positions $\{\theta_i^{wi}\}_{i \in \mathcal{I}}$ dealers bring from the when-issued market. I

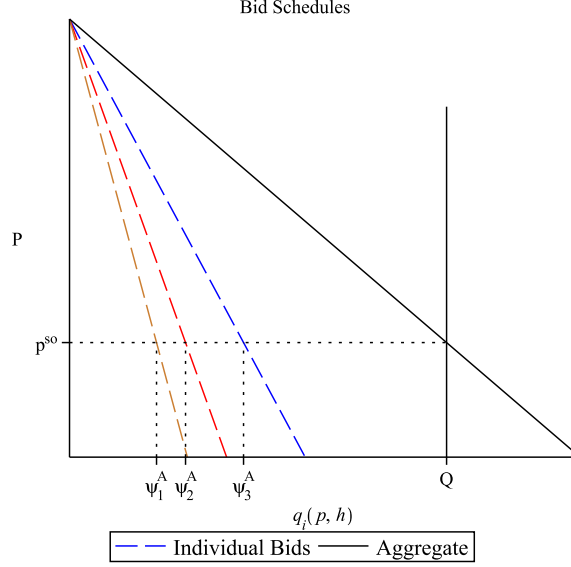


FIGURE 2.3.1. Payment and allocation rules on a uniform price auction. Individual bid schedules are represented by dashed lines. Aggregate demand and supply are represented by solid lines. For a given realization of Q , the stop-out price is defined as the one that clears the market, and each dealer gets the amount of securities submitted at the stop-out price.

am assuming a complete information environment, so all dealers know h when submitting their bids in the auction.

Even though agents submit strictly decreasing bids in all equilibria described in this paper, we need to specify a tie-breaking rule for the cases where the aggregate demand is greater than the supply at the stop-out price, i.e. when we have $\sum_j q_j^A(p^{so}, h) > Q$. In this case, there is no alteration in the way the securities in the segment of the aggregate demand curve strictly above the stop-out price are allocated. The remaining units, given by $Q - \lim_{p \rightarrow p_+^{so}} \sum_j q_j^A(p, h)$ will be distributed to the bidders on a pro-rata basis.¹¹

The total supply being auctioned, Q , is stochastic with a support $[\underline{Q}, \overline{Q}]$. As discussed in [Back and Zender \[1993\]](#), a noisy supply in the auction considerably reduces the set of equilibria for uniform price auctions. From a dealer's perspective, a noisy supply is realistic

¹¹[Kremer and Nyborg \[2004\]](#) compare equilibria of uniform auctions with different tie-breaking rules.

in the context of the Treasury auctions. For instance, in these auctions, there are two ways participants can submit their bids: *competitively* or *non-competitively*. *Competitive* bids take the form of weakly decreasing schedules like the ones described in the paragraph above. A *Non-competitive* bid is simply a quantity a bidder wishes to acquire independently of the value of the stop-out price. It can be interpreted as an agent submitting a perfectly inelastic bid schedule. The Treasury first subtracts *non-competitive* bids from total supply before allocating the remaining securities to *competitive* bidders. Dealers may only submit *competitive* bids. Since they do not know *ex ante* the total amount of *non-competitive* bids, they perceive the supply in the auction as a stochastic variable.¹²

Let $\gamma \equiv \frac{N-2}{N-1}$, and $\bar{v} \equiv \frac{\sum_j v_j}{N}$. I will make following restrictions on the support of Q ;

ASSUMPTION 2. *The bounds of the support of Q satisfy:*

- $\bar{Q} \leq \frac{\gamma}{\rho} \bar{v}$.
- $\frac{Q}{N} \geq -\frac{\gamma}{\rho} \min_i \{v_i - \bar{v}\}$.

The assumption in the upper bound guarantees that the equilibrium stop-out price in the linear equilibrium described below is non-negative. The restriction on the lower bound is made in order to assure that all dealers are acquiring a positive quantity of the securities. In other words $\psi_i^A \geq 0$ for all $i \in \mathcal{I}$ in equilibrium.

When-Issued Market: Prior to the auction, dealers can trade when-issued securities amongst each other. A when-issued security is promise to buy/sell a Treasury security after the auction for a pre-established price. For instance, if dealer i sold q^{wi} when-issued securities to j for a price p^{wi} , it means that i will have to deliver q^{wi} units of Treasury securities after the auction takes place and j will pay $p^{wi} \times q^{wi}$ for them. The contract is established in the

¹²*Indirect bidders*, who purchase roughly 22% of the entire supply of securities being auctioned (Fleming [2007]), also do not appear to be playing strategically (submitting strictly decreasing bid schedules). These type of bidders are mainly formed by foreign institutions which might demand a specific reserve of US securities not too sensitive on their price. Therefore, their bids also contribute to stochasticity of the total supply perceived by direct bidders.

when-issued market, but the actual exchange will materialize only after the auction takes place and dealers have received the securities from the Treasury. There are no physical transfers, either of securities or money, at the time the contracts are negotiated.

I consider two different structures for the when-issued market. In the first, a perfect competitive market, dealers take the price of when-issued securities as given and simply choose how many when-issued securities they wish to acquire (sell). In the second environment, I model the when-issued market as a uniform price auction as well. In latter environment, dealers submit demand schedules representing their desire to acquire (or sell) when-issued securities in a similar fashion to how they submit bid schedules in the auction stage. Dealers submit left continuous, weakly decreasing demand schedules $q_i^{wi}(p) : \mathbb{R} \rightarrow \mathbb{R}$, $i \in \mathcal{I}$ specifying the quantity of when-issued securities they wish to buy (sell) for all possible prices. A central *inter dealer* broker collects all individual schedules and forms the aggregate bid, $\sum_{j=1}^N q_j^{wi,t}(p)$. Since the when-issued market is a zero-supply market, the stop-out price p^{wi} is defined as the highest value of $p \geq 0$ such that the aggregate bid is larger or equal to zero, i.e., $\sum_{j=1}^N q_j^{wi,t}(p) \geq 0$. As in the auction stage, the excess demand will be split in a pro-rata base if the inequality is strict.

The outcome for Dealer $i \in \mathcal{I}$ from the when-issued market is a forward contract establishing that he will receive $\theta_i^{wi} \equiv q_i^{wi}(p^{wi})$ units of the security in exchange to a payment of $p^{wi} \times \theta_i^{wi}$ as soon as the auction takes place. He is said to be taking a long(short) position at this round if $\theta_i^{wi} > (<) 0$. There is a per-unit monetary penalty, π , in case a dealer fails to deliver the security. The penalty is assumed to be big enough in order to avoid dealing with cases where dealers optimally chooses to default in equilibrium.

For tractability, I will restrict the maximum position a dealer can take in the when-issued market. In all periods t , we have that:

ASSUMPTION 3. $\theta_i^{wi} \leq \bar{\theta}_i$ for $i \in \mathcal{I}$, where $\bar{\theta}_i \leq \frac{Q}{N} + \frac{1}{\rho} [v_i - \bar{v}]$.

Assumption 3 guarantees that all dealers acquire a positive amount of securities in the linear equilibrium of the auction described in the next sections.¹³

A dealer's final holdings of securities is given by the sum of what he acquires in the when-issued and auction stages. For example, if i arrives with θ_i^{wi} from the when-issued and acquires additional ψ_i^A in the auction, he will end up with $\psi_i^F \equiv \psi_i^A + \theta_i^{wi}$.

2.4. Assumptions

I make three main assumptions in order to get the results of this paper. First, dealers must have market power in the auction stage. Bid shading, which plays a central role on the analysis made here, will arise in equilibrium only if dealers take into account that their bid choices affect the auction price. Second, dealers should have heterogeneous demands for Treasury securities. With this assumption, dealers will end up acquiring different amounts of securities in the auction stage. Together with bid shading, this implies that the auction mechanism will fail to distribute securities efficiently, which gives dealers incentives to trade when-issued securities. The third assumption of the paper is the absence of asymmetric information among dealers, which will be assumed for tractability reasons.

There are a couple of facts corroborating the market power assumption. First, Primary Dealers, which are the only institutions allowed to trade directly with the Federal Reserve System in the secondary market, absorbs roughly 70% of all securities being auctioned by the Treasury [Fleming, 2007]. In fact, even among the Primary Dealers, the market is fairly concentrated. For instance, in the first quarter of 2012, five dealers were responsible, roughly, for 50% of the total outright volume of transactions involving Treasury securities by the

¹³According to the "Administration of Relationships with Primary Dealers": "(...)the New York Fed will expect a primary dealer to bid in every auction, for, at minimum, an amount of securities representing its pro rata share, based on the number of primary dealers at the time of the auction, of the offered amount, its bid prices should be reasonable when compared to the range of rates trading in the when-issued market (...)".

This assumption could be dropped if we allowed dealers to acquire negative positions, i.e. sell securities, in the auction stage.

Primary Dealers.¹⁴ Second, in order to get the status of a Primary Dealer, the Fed requires that a financial institution participate in all auctions for Treasury Securities.¹⁵ This costly requirement illustrates the extent to which the Fed is worried about fomenting competition in the auctions.

There are many reasons why dealers have heterogeneous demand for a specific Treasury security. For instance, dealers might differ in their endowments of a correlated security, inventory costs, outside investment opportunity costs and on their financial constraints. Moreover, primary dealers often submit bids on behalf of their clients.¹⁶ Idiosyncrasies from these network of clients will also be reflected in their demand for the Treasury security. These characteristics are specific to each dealer and are not necessarily related to the intrinsic fundamental value of the security.

In the specific case of the Treasury market, asymmetric information is not as relevant as for other types of securities. The type of information that affects the fundamental value of a Treasury security mainly takes the form of public macroeconomic announcements.¹⁷ Many authors have suggested that differences in the ability to interpret public information could actually be a source of asymmetric information among dealers ([Fleming and Remolona, 1999](#), [Green, 2005](#)). However, the impact of such announcements in the market are almost completely absorbed a few hours after they are released and do not endure until the auction period.

¹⁴See [Primary Dealers Statistical Releases](#), fed of New York.

¹⁵See ["Administration of Relationships with Primary Dealers"](#).

¹⁶For example, China, one of the largest holder of U.S. Treasury securities, submitted their bids through dealers until 2011. Other big players, as Japan, still uses the primary dealers to do it. See, for example, [Reuters](#).

¹⁷CPI, PPI, etc. See [Fleming and Remolona \[1999\]](#).

2.5. No Market Power

This section considers the benchmark Walrasian equilibrium of the environment described above. Here, dealers take the prices of securities as given in both market stages. We have the following result:

PROPOSITION 1. *If dealers have no market power, they submit their true valuation function as a bid in the auction:*

$$(2.5.1) \quad q_i^W(p, h) = \frac{1}{\rho} (v_i - p) - \theta_i^{wi} \quad i \in \mathcal{I}.$$

For any history $h = (p^{wi}, \{\theta_i^{wi}\}_{\mathcal{I}})$ and realization $\tilde{Q}$, the equilibrium stop-out price in the auction is:

$$(2.5.2) \quad \tilde{p}^W = \bar{v} - \rho \frac{\tilde{Q}}{N}.$$

Moreover, the equilibrium price in the when-issued market is given by:

$$(2.5.3) \quad p^{wi} = E[p^W].$$

Dealers are indifferent between their holdings of when-issued securities which implies that any set $\{\theta_i^{wi}\}_{\mathcal{I}}$ satisfying Assumption 3 and $\sum_{\mathcal{I}} \theta_i^{wi} = 0$ could hold in equilibrium.

A dealer's optimal strategy is to equate his marginal valuation to the security's price. Although p^W is not known by the time dealers place their bids, they are able to do so for all realizations of p^W by submitting their true marginal valuation functions as their bid schedules. To see that, we can invert the bid schedule (2.5.1) above to get:

$$(2.5.4) \quad \beta_i^W(q, h) = u'_i(q + \theta_i^{wi}).$$

Dealers' final allocations are efficient and independent of the specific history from the when-issued market. It is clear from the proposition above that only the positions $\{\theta_i^{wi}\}_{\mathcal{I}}$ affect the auction outcome. However, it is also clear that these positions affect the way securities are distributed in the auction, but not their final allocation. An increase in θ_i^{wi} is completely canceled out by a decrease in the amount dealer i acquires in the auction, as can be seen in (2.5.1). In the end, this dealer ends up holding the same amount of securities he was holding before the increase in his when-issued position.

Since ψ_i^W is independent of h , the price of securities in the when-issued market should be equal to the equilibrium price in the auction, in expected terms. If $p^{wi} < E[p^W]$, all dealers would take upper bound position $\bar{\theta}_i$ and the market would not clear. Conversely, if $p^{wi} > E[p^W]$, all dealers would have incentives to take strictly negative positions. Therefore, when dealers do not have market power, the equilibrium price in the when-issued market must be the same as in the auction in expected terms. In other words, there is no *underpricing*.

Since the price of when-issued securities satisfies (2.5.3), dealer i is indifferent between any position $\theta_i^{wi} \in \mathfrak{R}$. This implies that any set $\{\theta_i^{wi}\}_{\mathcal{I}}$ satisfying Assumption 3 and $\sum_{\mathcal{I}} \theta_i^{wi} = 0$ can be sustained as an equilibrium for the when-issued market. A special set satisfying these conditions has $\theta_i^{wi} = 0$, for $i \in \mathcal{I}$, the case which the when-issued market is shut down. In fact, these would be the unique equilibrium positions if we made the additional assumption that there are arbitrarily small transactions costs in the when-issued market.

2.6. Equilibrium with Market Power

In this section, I will characterize an equilibrium where dealers take into account that the auction's stop-out price is a function of the specific bid schedule they submit. As will be shown below, dealers will strategically submit bid schedules strictly below their true marginal valuation, i.e., they will "shade" their bids. For tractability, I focus on equilibrium where

investors submit linear bid schedules in both market stages, which restricts dealers to have strictly decreasing valuation functions, i.e. $\rho > 0$.

2.6.1. Auction. Suppose dealers arrive in the auction stage after a given history $h \in \mathcal{H}$ from the when-issued market satisfying Assumption 3. An equilibrium of the auction subgame is a set of bid schedules $\{q_i(p, h)\}_{\mathcal{I}}$ and a random variable $\tilde{p}^{so}$ such that $q_i^A(p, h)$, $i \in \mathcal{I}$ solves:

$$(2.6.1) \quad \max_{q_i(\cdot)} E \left[u_i \left(q_i(p^{so}, h) + \theta_i^{wi} \right) - p^{so} q_i(p^{so}, h) \right].$$

given the other dealers are submitting $\{q_j(p, h)\}_{\mathcal{I}_{-i}}$ and, for each realization of the supply, the stop-out price satisfies the market clearing condition $\sum_{\mathcal{I}} q_j(\tilde{p}^{so}, h) = \tilde{Q}$.

Suppose, for a moment, that dealers know the realization of the supply of securities in the auction, $\tilde{Q}$. Let $y_i(p, h) \equiv \tilde{Q} - \sum_{\mathcal{I}_{-i}} q_j^A(p, h)$ denote the residual supply faced by dealer i . Given the other dealers' bid schedules, $y_i(p, h)$ gives the amount of securities i receives as a function of the auction price. Dealer i 's optimization problem is reduced to one of choosing the auction stop-out price that solves:

$$\max_{p^{so}} u_i \left(y_i(p^{so}, h) + \theta_i^{wi} \right) - p^{so} y_i(p^{so}, h).$$

Suppose also that each $j \in \mathcal{I}_{-i}$ submits differentiable and strictly decreasing bid schedules. The first order condition of the problem above is:

$$(2.6.2) \quad u_i' \left(y_i(p^{so}, h) + \theta_i^{wi} \right) = p^{so} + \frac{1}{\frac{\partial}{\partial p} y_i(p^{so}, h)} y_i(p^{so}, h).$$

Dealer i 's problem is similar to one of a monopsonist facing a positive sloped residual supply. He faces a trade-off between increasing the quantity of securities he acquires in the auction at the cost of increasing the price he pays, not only for the additional security, but

for all the ones he was already acquiring.¹⁸ At optimum, the dealer equalizes the marginal expenditure to the marginal benefit he gets from acquiring an additional security. The marginal expenditure, the term in the right hand side of the equation above, has two components: (i) the payment for the additional security, p^{so} , and (ii) an increase in the payment of all infra-marginal units due to an increase in the stop-out price.

The equilibrium described in the proposition below is found by making the above optimality condition hold for each realization of the supply $\tilde{Q}$. This equilibrium is *ex post* efficient in the sense that dealers are acquiring the optimal quantity of securities for all realizations of the stochastic supply.

PROPOSITION 2. *In the unique linear equilibrium of the uniform price auction, dealers submit the following bid schedules:*

$$(2.6.3) \quad q_i^A(p, h) = \frac{\gamma}{\rho} (v_i - p) - \gamma \theta_i^{wi}, \quad i \in \mathcal{I}.$$

where $\gamma = \frac{N-2}{N-1}$. For a given realization $\tilde{Q}$, the equilibrium stop-out price is:

$$(2.6.4) \quad \tilde{p}^{so} = \tilde{p}^W - \frac{1-\gamma}{\gamma} \rho \frac{\tilde{Q}}{N}.$$

Let's take a closer look at how we arrive at the above expression for the equilibrium bid schedule. If $j \in \mathcal{I}_{-i}$ are submitting bid schedules like (2.6.3), the residual supply $y_i(p, h)$ has a constant slope in its first argument given by $\frac{\gamma}{1-\gamma} \frac{1}{\rho}$. We can rewrite the first order condition in (2.6.2) as:

$$v_i - \rho \left(y_i(p^{so}, h) + \theta_i^{wi} \right) - p^{so} - \rho \frac{1-\gamma}{\gamma} y_i(p^{so}, h) = 0.$$

Solving for $y_i(p^{so}, h)$ gives exactly the quantity on the bid schedule 2.6.3 and the price on 2.6.4 for a given realization of the supply $\tilde{Q}$.

¹⁸See [Milgrom \[2004\]](#) for a detailed discussion.

In equilibrium, dealers are optimally "shading" their bids. The magnitude of the bid shading can be seen if we invert the equilibrium bid schedule (2.6.3) of dealer i :

$$\beta_i^A(q, h) = u'_i(q + \theta_i^{wi}) - \rho \frac{1 - \gamma}{\gamma} q$$

The inverse of the bid schedule is equal to i 's marginal valuation minus the bid shading term. As long as dealer i acquires a positive amount q of securities in the auction, the bid schedule he submits will lie strictly below his marginal valuation curve.

2.6.1.1. *(In)efficiency*: The magnitude of bid shading is strictly increasing in the quantity a dealer is acquiring in the auction. This is intuitive. If the auction price increases by a given amount, the impact on dealer i 's total payment is higher if he is acquiring a relatively larger amount of securities. Consequently, dealers will bid less aggressively for larger amounts of securities. The left panel of Figure 2.6.1 depicts the marginal valuation function and the bid schedules from two different dealers when they arrive in the auction with a null position from the when-issued market. For both dealers, the marginal valuation curve and the bid schedule coincides for the first unit of the security since the price impact component is zero. However, as dealers bid for more securities, the gap between the two curves increases.

The fact that bid shading is increasing in quantity has implications on how the auction allocates securities among dealers. Given the heterogeneity in demand intercepts and when-issued positions, the amount of securities acquired in the auction will not be the same across dealers. This implies that dealers with higher demands, who end up acquiring larger amounts, end up with higher marginal valuations when evaluated at the final allocation of securities. This property of the equilibrium is illustrated on Figure 2.6.1. For a given realization of $\tilde{p}^{so}$, the dealer with the highest demand acquires more securities in the auction and ends up with a higher marginal valuation than the smaller one. The following corollary summarizes this property.

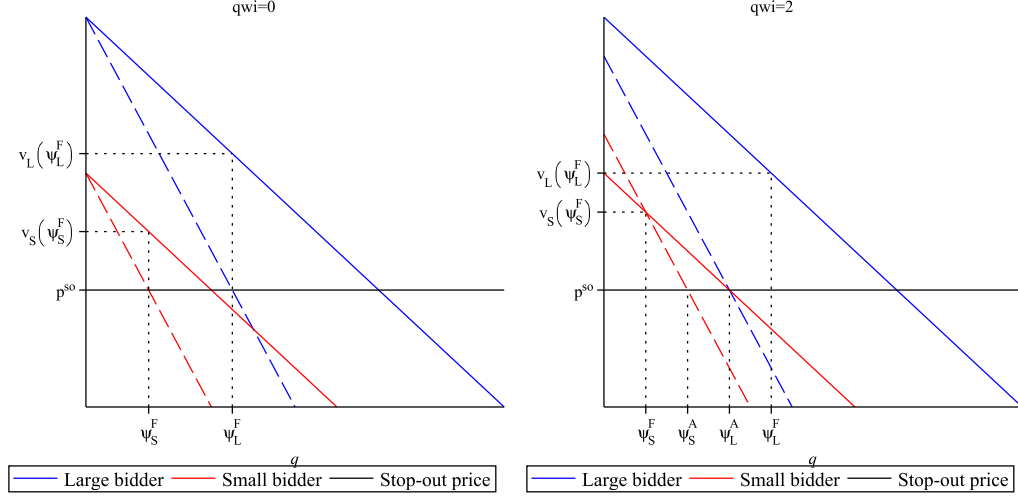


FIGURE 2.6.1. Left panel: The solid lines represent dealers' demand for securities. The dashed lines represent dealers' equilibrium bid schedules when both dealers arrive with a null position from the when-issued market. Right: Dashed lines represent the equilibrium bid schedules when dealers arrive in the auction with net positions $\theta_L = 2 = -\theta_S$ from the when-issued market.

COROLLARY 1. *In the linear equilibrium of the auction we have that $v_i - \rho\theta_i^{wi} > v_j - \rho\theta_j^{wi} \iff u'_i(\tilde{\psi}_i^F) > u'_j(\tilde{\psi}_j^F)$ for all dealers $i, j \in \mathcal{I}$ and realization of the supply $\tilde{Q}$.*

The auction does not distribute securities in an efficient way. The larger a dealer's demand for securities, adjusted for his when-issued position, the higher will be his marginal valuation evaluated at the final allocation $\{\psi_i^F\}_{\mathcal{I}}$. Even though dealers with high demand are getting larger amounts of securities, they are getting less than if securities were distributed efficiently.

2.6.1.2. *Relationship Between When-issued Positions and the Auction Outcome.* As in the simple example analyzed in section 2.2, dealers can use the when-issued market to improve the auction allocation. Indeed, Proposition 2 implies that a dealer's bidding behavior in the auction is directly affected by the position he carries from the when-issued market. They anticipate this relationship when choosing their strategies in the when-issued stage.

The following corollary gives the exact relationship between the auction and final allocation of securities and the history from the when-issued market.

COROLLARY 2. *For any realization of $\tilde{Q}$, history h , and $i \in \mathcal{I}$,*

- $\frac{d\tilde{\psi}_i^A}{d\theta_i^{wi}} = -\gamma$ and $\frac{d\tilde{\psi}_i^F}{d\theta_i^{wi}} = (1 - \gamma)$.
- $\frac{d\tilde{\psi}_i^A}{d\theta_{-i}^{wi}} = \frac{d\tilde{\psi}_i^F}{d\theta_{-i}^{wi}} = 0$.
- $\frac{d\tilde{p}^{so}}{d\theta_i^{wi}} = 0$.

The first conclusion we can draw from the corollary above is that only a dealer's own position, θ_i^{wi} $i \in \mathcal{I}$, will affect his allocation of securities. The way the remaining when-issued securities are distributed among $j \in \mathcal{I}_{-i}$, or the specific when-issued price p^{wi} will not have any influence on the amount of securities i acquires in the auction and his final position. Second, the relationship between a dealer's allocation and his when-issued position is independent of the specific realization of $\tilde{Q}$ or the history h from the when-issued market. Finally, the last point states that the auction price is independent of the positions dealers carry from the when-issued market.

Suppose, for instance, that the dealer with the higher marginal valuation acquires a when-issued security from the dealer with the lower valuation. Corollaries 1 and 2 above imply that the gap between the dealers' marginal valuations evaluated at the final allocation will be reduced. The price of a when-issued security will determine how these gains are split between dealers trading when-issued securities.

To what extent are dealers able to mitigate the inefficiencies coming from bid shading using when-issued securities? How do dealers share the rents from the allocational improvements? In other words, what will be the equilibrium price of securities in the when-issued market? These questions will be analyzed in the following subsection.

2.6.2. When-Issued Market. I continue the analysis by characterizing the equilibrium in the when-issued market. When choosing their strategies at this market stage, dealers

anticipate the auction outcome will be determined by the equilibrium described in Proposition 2. Therefore, they internalize the relationship between their positions and the auction outcome given by Corollary 2.

I will focus on two particular equilibria for the when-issued market: a perfectly competitive equilibrium and a linear bid schedules equilibrium.¹⁹ In the perfectly competitive equilibrium, dealers take the price they pay for securities in the WI market as given (but not in the auction stage). This environment provides intuitions about how the price of WI securities is determined in equilibrium - thus, how underpricing arises. Moreover, it emphasizes that it is the market power in the *auction stage* alone that gives dealers incentives to trade when-issued securities. In the linear equilibrium, dealers take into account that they can affect the price of securities in the when-issued stage as well. They submit entire demand (supply) schedules for WI securities in the same fashion as they do in the auction stage.

As will be shown below, the outcome of the two equilibria share some characteristics. For instance, the prices of the security, in the auction and in the when-issued stages, are invariant between the two equilibria. Moreover, the final allocation of securities (and money) in the imperfect competitive equilibrium gets arbitrarily close to that in the perfectly competitive equilibrium as we increase the number of rounds in the when-issued market.

2.6.2.1. Competitive When-Issued Market. In this subsection, dealers take the price of when-issued securities as given when choosing their positions in the when-issued market. It is important to emphasize, however, that dealers do take into account how these positions will affect the auction outcome. They anticipate that the equilibrium in the auction will be the one described on Proposition 2. Therefore, they also anticipate the resulting relationship between when-issued positions and auction allocations.

Given a price p^{wi} for the when-issued securities, dealer i 's problem is reduced to:

¹⁹As in the case of the auction sub game, there will be multiple equilibria in demand schedules. Linearity is chosen for tractability.

$$(2.6.5) \quad \max_{\theta_i^1 \leq \theta_i} E \left[U_i \left(\psi_i^F \right) - p^{so} \psi_i^A \right] - p^{wi} \theta_i^{wi}.$$

where ψ_i^A and ψ_i^F are, respectively, the auction and final allocations of securities to dealer i resulting from the auction equilibrium described on Proposition 2. Corollary 2 describes the exact relationship between ψ_i^A and ψ_i^F with i 's when-issued position, θ_i^{wi} .

We have the following result:

PROPOSITION 3. *Suppose that dealers anticipate that the equilibrium in the auction stage is the one described on Proposition 2. If the when-issued market is competitive, the equilibrium price and positions are given by:*

$$(2.6.6) \quad p^{wi,c} = (1 - \gamma) E \left[p^W \right] + \gamma E \left[p^{so} \right].$$

$$(2.6.7) \quad \theta_i^c = \frac{1}{\rho} (v_i - \bar{v}).$$

where p^{so} and p^W are, respectively, the price of securities in the auction stage and the price that would arise if dealers had no market power in the auction stage.

Efficiency: Proposition 3 leads to the following corollary:

COROLLARY 3. *If the when-issued market is competitive, the final allocation of securities is efficient: $\psi_i^F = \psi_i^W$ for all $i \in \mathcal{I}$.*

Two conclusions can be drawn from the corollary above. First, dealers are able to eliminate all inefficiencies associated with the auction process if they take the right set of positions in the when-issued market. Note that this is true for any realization of $\tilde{Q}$. Second, this set of positions, denoted by $\{\theta_j^c\}_{j=1}^I$, is attained in equilibrium under perfect competition in the when-issued market.

If dealers arrive with $\{\theta_j^c\}_{j=1}^N$ from the when-issued market, they will submit symmetric bid schedules in the auction stage. This can be readily seen by plugging back θ_i^c on the equilibrium bid schedule 2.6.3. The resulting bid will not depend directly on i 's marginal valuation intercept, v_i , but only on the average intercept across dealers $\bar{v}$. As a result, all dealers end up with the same demand for securities in the auction. It follows from Corollary 1 from the previous subsection that this symmetrization property implies that all dealers end up with the same marginal valuation for the security evaluated at the final allocation.

Underpricing: The equilibrium of the environment considered in this subsection is consistent with *underpricing*. Indeed, the price of securities in the when-issued market, $p^{wi,c}$, is given by a weighted sum of the expected values of p^{so} and p^W - the price of securities in the auction stage and the price that would arrive in a competitive market, respectively. Since $p^W > p^{so}$ for all realizations of Q , it follows that $p^{wi} > E[p^{so}]$, i.e., the equilibrium price of the when-issued market is greater than the expected auction price.

The price of securities in the when-issued market is lower than the Walrasian price p^W . Dealers anticipate that they will be able to acquire the security for an artificially low price in the auction stage, which drives the price in the when-issued stage down as well. However, the price at the latter stage will not fully go down to the auction's level, otherwise none of the dealers would have incentives to take short positions.

Let's take a closer look at how we arrive at equation (2.6.6) for p^{wi} . Suppose dealer i acquires an additional amount Δ of securities in the when-issued market. Since the price of securities is fixed in the WI market, the cost of the additional Δ will be simply $p^{wi,c} \times \Delta$. On the other hand, the gains are split in two effects - the first comes from an increase in his final allocation and the second comes from a reduction in his payment at the auction stage. Indeed, Corollary 2 implies that ψ_i^F will increase by $(1 - \gamma) \times \Delta$ and ψ_i^A will be reduced by

$\gamma \times \Delta$. The sum of the two components is written as:

$$E \left[u'_i \left(\psi_i^F \right) \right] \times (1 - \gamma) \times \Delta + E \left[p^{so} \right] \times \gamma \times \Delta.$$

However, evaluated at the equilibrium allocation, $u'_i \left(\psi_i^F \right) = u'_i \left(\psi_j^W \right) = p^W$ for all i, j . At the optimal, the marginal benefit from increasing the quantity acquired at this stage should be equal to the cost, which gives the expression (2.6.6) for the WI price.

The next corollary describes how the magnitude of the underpricing depends on the parameters' values in equilibrium.

COROLLARY 4. *The expected magnitude of the underpricing in the equilibrium described above is given by:*

$$p^{wi,c} - E \left[p^{so} \right] = \frac{(1 - \gamma)^2}{\gamma} \rho \frac{E \left[Q \right]}{N}.$$

Not surprisingly, the same set of parameters determining the magnitude of dealers' bid shading in equilibrium will be the same determining the magnitude of the underpricing. This is not surprising since dealers trade when-issued securities in order to exploit the inefficiencies caused by bid shading. We can interpret the when-issued price as the way dealers share the trading gains from the when-issued market.

The above corollary implies that underpricing and auction size are positively related. It is clear from expression 2.6.4 that an increase in the total amount of securities *per dealer* being offered by the Treasury implies a larger gap between the Walrasian price, p^W , and the auction price, p^{so} . Since the when-issued price is a convex combination of these two prices, the gap between p^{wi} and p^{so} is also going to be positively related with the size of the auction. Therefore, *caeteris paribus*, we should expect a positive correlation between the magnitude of underpricing and the quantity of securities being auctioned.

An increase in the number of dealers also reduces the magnitude of underpricing. We can split the effects of an additional dealer in two. The first is a reduction in the auction

size - more dealers for the same amount of securities. The second is a decrease in the degree of competition - their ability to influence the price is diminished. Both effects imply a more aggressive bidding behavior by dealers at the auction stage. Their bids will be closer to their true marginal valuation function, which implies a smaller gap between p^W and p^{so} . Consequently, the gap between p^{wi} and p^{so} will also be reduced.

Sell high and buy low: One may argue that the gap between the price of the securities in the two market stages creates an arbitrage opportunity. A dealer could sell securities in the when-issued market, where the price is higher, and buy them back in the auction, where the price is lower. Why can't dealers profit from this simple "sell high and buy low" strategy?

The answer lies on the impact on the auction price such a strategy produces. As discussed in the previous section, there are two effects coming from a marginal increase in the quantity acquired in the auction. The first is simply the cost of this additional unit, given by p^{so} . The second comes from the fact that the auction's price will also increase, which affects the cost of all other securities this dealer was already acquiring. The net profit dealer i gets if he takes this strategy is:

$$p^{wi,c} - \left(E[p^{so}] + \frac{1-\gamma}{\gamma} \rho \times E[\psi_i^A] \right)$$

Substituting the equilibrium values in the expression above, one can show that the profit is given by $-(1-\gamma)\rho \frac{E[Q]}{N}$. Therefore, the dealers' payoff would actually decrease if they decided to take such strategy.

2.6.2.2. Imperfect Competition in the When-Issued Market. This section extends the above analysis to the case where dealers internalize their market power in the when-issued market as well. One may wonder whether equilibrium underpricing depends on the assumption of competition in the when-issued market. As will be shown below, this is not true. Indeed, underpricing will arise in equilibrium when investors do take into account they can

influence the price of when-issued securities. In fact, the gap in the price from the two market stages will be exactly the same as in the previous section. Moreover, it is not straightforward to justify why dealers internalize their market power in the auction stage, but act as price takers in the when-issued market.

Given that all dealers $j \in \mathcal{I}_{-i}$ are submitting $\{\theta_j^{wi}(p)\}_{\mathcal{I}_{-i}}$, dealer i 's maximization problem in the when-issued market can be written as:

$$(2.6.8) \quad \max_{\theta_i^{wi}(\cdot)} E \left[u_i(\psi_i^F) - p^{so} \psi_i^A \right] - p^{wi} \theta_i^{wi}(p).$$

where ψ_i^F and ψ_i^A are related to θ_i^{wi} accordingly Corollary 2, and the when-issued price satisfies the market clearing condition $\sum_{\mathcal{I}} \theta_j^{wi}(p^{wi}) = 0$. We have the following result,

PROPOSITION 4. *Suppose that dealers anticipate that the equilibrium of the auction is the one described on Proposition 2. In the unique linear Sub game Perfect Equilibrium (robust), dealers submit the following bid schedules in the when-issued market:*

$$(2.6.9) \quad \theta_i^{wi}(P) = \frac{\gamma}{(1-\gamma)^2 \rho} \{A_i - P\}, \quad i \in \mathcal{I}.$$

where $A_i \equiv p^{wi,c} + (1-\gamma)^2 \rho \theta_i^c$. The equilibrium price and positions are given by:

$$\begin{aligned} p^{wi} &= p^{wi,c} \\ \theta^{wi} &= \gamma \theta^{wi,c} \end{aligned}$$

The variables $p^{wi,c}$ and θ_i^c were defined on Proposition 3. They represent, respectively, the price and i 's position of equilibrium when the when-issued market was perfectly competitive.

Underpricing: The equilibrium described above is consistent with underpricing. The price of securities in the when-issued market is higher than their expected price in the auction. In fact, the when-issued price when dealers submit complete demand/supply schedule at this

stage takes the exact form as in the competitive when-issued market case. Consequently, the analysis made in the previous subsection about the relationship between the model parameters and the magnitude of the underpricing can be extended to the equilibrium above.

At this point, a natural question arises: if market power is the source of a relatively lower auction price, why doesn't the price of when-issued securities change when we let dealers make use of their market power in the when-issued market? Unlike the auction stage, both sides of the market have market power at the when-issued stage. At the auction, the Treasury is selling an exogenous quantity $\tilde{Q}$ and dealers, who are all on the buy side of the market, make use of their market power to reduce the price they pay for the securities. The when-issued market, in turn, is a zero supply market - dealers are trading securities amongst themselves. Dealers taking long positions will behave strategically to reduce the price of securities they are acquiring, and dealers taking short positions will also behave strategically, but in order to increase the price. The linearity of bid schedules implies that the two effects cancel out in equilibrium.

The intuition above can be readily seen on Figure 2.6.2. The future expected payoff that when-issued securities bring to dealers (their true demands for when-issued securities) are represented by the solid lines. However, dealers submit the dashed lines as their bid schedule in equilibrium. Whenever a dealer is taking a long position, his demand schedule will lie strictly below his valuation. This is the bid shading discussed previously - the dealer reduces the quantity he acquires in order to artificially decrease the price. However, when a dealer is taking a short position, his supply schedules will lie strictly above his true valuation. The shade goes in the other direction - he reduces the quantity he sells in order to artificially *increase* the price.

2.6.2.3. Sequential When-issued Market. In practice, the when-issued market is open for trade over the entire period between the announcement and the auction days. This section extends the model considered above to a multi-period environment in order to capture this

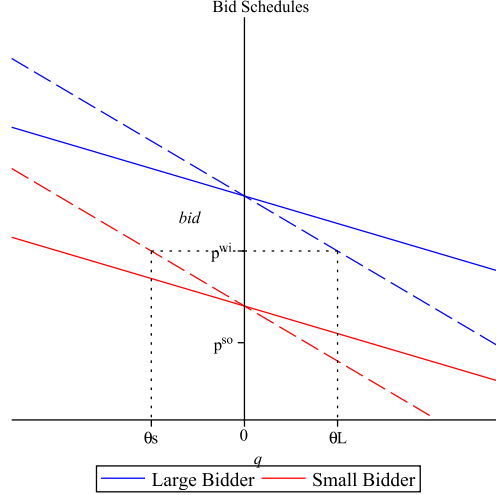


FIGURE 2.6.2. The solid lines represent the future net payoff that buying/selling when-issued securities will bring to dealers. The dashed lines represent dealers' bid schedule submitted in equilibrium.

dynamic characteristic from this market. Instead of a single period, dealers are allowed to trade when-issued securities in a finite number T of rounds before the auction takes place. Each of these rounds works exactly the same as in the previous section. Let $t = 1, \dots, T$ denote the number of rounds *until* the auction stage. Given a history up to period t , dealers submit a demand schedules $q_i^{wi,t}(p, h^{t+1}) : \mathcal{R} \times \mathcal{H}^{t+1} \rightarrow \mathcal{R}$, $i \in \mathcal{I}$ at each round t . The central *inter-dealer* broker determines the when-issued price and allocations for the specific round denoted, respectively, by p^t and $\psi_i^t \equiv q_i^{wi,t}(p^t, h^t)$.

Let $\theta_i^t \equiv \sum_{\tau=t}^T \psi_i^\tau$ denote dealer i 's net position on forward contracts at the end of period t , which is given by the summation of all his positions from the prior $T - (t + 1)$ rounds.²⁰ After all rounds of the when-issued market have occurred, dealer i arrives at the auction stage with a net position θ_i^1 . This means that, in the absence of default, this dealer will get θ_i^1 securities in addition to the ones he ends up acquiring in the auction stage. For instance, if he acquires ψ_i^A units in the auction, he will end up holding $\psi_i^F \equiv \psi_i^A + \theta_i^1$ securities on his portfolio. As will be discussed below, θ_i^1 is sufficient to determine how the outcome of the T

²⁰To simplify the notation, I am assuming $\theta^t \equiv 0$ for all $t > T$

rounds in the when-issued market will influence i 's final allocation. However, since the price of the when-issued security is specific to each round t , his payment will depend specifically on his transactions at each specific round. Indeed, the net payment i promised to make after the when-issued contracts are settled is given by:

$$\sum_{t=1}^T p^t (\theta_i^{t+1} - \theta_i^t).$$

I will assume that dealers have complete information. Let $h^t \equiv \left(\left\{ \theta_j^{\tau+1} \right\}_{j=1}^N, p^{\tau+1} \right)_{\tau=t}^T \in \mathcal{H}^t$ be the history up to period t from the when-issued market. On each round t , the history h^t is common knowledge among dealers. These assumptions are made in order to isolate the relationship between market power in the auction stage and the incentives to trade when-issued securities.

Once more, I focus on the equilibrium where dealers submit linear demand (supply) schedules in all of the T rounds of the when-issued market. We have the following result.

PROPOSITION 5. *In the linear Subgame Perfect Equilibrium for the when-issued market, dealers submit the following bid schedules on periods $t = 1, \dots, T$:*

$$(2.6.10) \quad q_i^{wi,t}(P) = \frac{\gamma}{(1-\gamma)^{2t} \rho} \{A_i^t - P\} - \gamma \theta_i^{t+1}, \quad i \in \mathcal{I}.$$

where $A_i^t \equiv p^{wi,c} + (1-\gamma)^{2t} \rho \theta_i^c$.

Figure 2.6.3 depicts the equilibrium bid schedules for different periods in the when-issued market. As can be readily seen, the slope of the bid schedules decreases as dealers approach the auction. This result is intuitive. With a larger number of periods to go before the auction, dealers have more opportunities to revert any undesired position they might be holding at a specific period. Consequently, their true valuation function for when-issued securities will get more elastic the further they are from the auction.

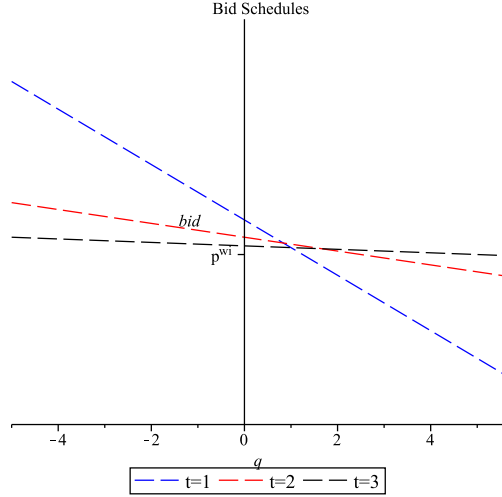


FIGURE 2.6.3. Equilibrium bid schedules in the when-issued market. As dealers approach the auction, their bid schedules get steeper.

Flatter valuation functions, *per se*, already give dealers incentives to submit flatter demand schedules. However, this effect is augmented by the interaction process that takes place in equilibrium. From an individual dealer's perspective, the fact that all his competitors submit flatter demand schedules implies that the price of when-issued securities is less sensitive to the amount of securities he decides to acquire (sell). Therefore, he will have less incentives to try to manipulate the price by increase the shade in his bid at this stage. As a consequence, he will submit bids closer to his true valuation for the security - flatter bid schedules.

Underpricing: Proposition 5 implies on the following corollary.

COROLLARY 5. *The equilibrium price in the when-issued market satisfies $p^{wi,t} = p^{wi,c}$ for all $t = 1, \dots, T$.*

The above corollary states that the equilibrium described in this subsection is consistent with underpricing as well. As a matter of fact, the equilibrium price of securities in the when-issued and auction stages is exactly the same as in the case where the when-issued

market was assumed to be competitive. Therefore, the relationship between the magnitude of underpricing and the underlying parameters from the model is exactly the same as in the previous subsection.

The corollary also highlights the fact that the price of securities does not change over the entire when-issued period. Even though dealers are submitting different demand schedules, the price clearing the market will be the same in all rounds. This implies that, in the absence of any shock on the underlying parameters, we should expect the price of when-issued securities to be constant over the entire when-issued period.

Dealers' Positions Over Time. How do dealers' positions evolve over the when-issued market? Proposition 5 implies that they follow a simple recursive form.

COROLLARY 6. *Suppose that dealers arrive in round t with a set of positions $\{\theta_i^{t+1}\}_{i=1}^N$. The equilibrium dealers' positions by the end of round t from the when-issued market are given by:*

$$(2.6.11) \quad \theta_i^t = \gamma \theta_i^c + (1 - \gamma) \theta_i^{t+1}, \quad i \in \mathcal{I}.$$

Positions are gradually formed over the when-issued market. In the absence of an exogenous shock, dealers taking (strictly) long positions at the first round will keep taking (strictly) long positions in all the remaining periods. Moreover, the sign of a dealer's position depends only on the relative size of their demand in comparison to the demand of other dealers. This comes straightforwardly from the fact that θ_i^t is a multiple of θ_i^c for all periods t . As argued before, dealers use the when-issued market to exploit the inefficiencies created by their strategic behavior in the auction. However, the fact that dealers also have market power in the when-issued market implies that they do not exploit all trading gains on a single round. The remaining gains are exploited on following rounds.

At the end of round t , dealer i will hold a net position given by a weighted average between his position in the previous round and the optimal position. The higher the market power of dealers (the lower γ is) the lower will be his adjustment in round t . In other words, the lower will be the amount of when-issued securities he acquires (sells) in this round. This comes from the fact that dealers bid less aggressively when they have more market power because his demand will have a higher impact on the price.

Corollary 6 also implies on a positive relationship between dealers positions in the when-issued market, $\{\theta_j^t\}_{j=1}^N$, and the amount they acquire in the auction $\{\psi_j^A\}_{j=1}^N$. Dealers taking long positions will be the ones acquiring larger amounts in the auction. This contrasts with the results when the when-issued market was competitive. When this was the case, we've seen that all dealers end up acquiring the same quantity of securities in the auction stage. However, dealers are not able to exploit all gains of trade when they have market power in the when-issued market. This implies that we would still observe dealers with higher intercepts on their valuation functions acquiring larger quantities at the auction stage. Furthermore, these are the same dealers taking the long positions in the when-issued market.

Efficiency: At a given round t in the when-issued market, dealers will exploit the remaining gains of trade not exploited in the previous rounds. The next corollary states that, as the number of rounds gets arbitrarily large, the equilibrium allocation of securities among dealers converges to the efficient allocation.

COROLLARY 7. *When $T \rightarrow \infty$, the equilibrium final allocation of dealer i satisfies $\psi_i^F \rightarrow \psi_i^W$.*

The result come straightforwardly from Corollary 6. Given the simple recursive expression that the positions of dealers evolve, it is not difficult to see that $\{\theta_j^1\}_{j=I}^N$ converges to $\{\theta_j^c\}_{j=I}^N$ as T goes to infinity. However, we saw in the previous section that if dealers arrive in the auction holding $\{\theta_j^c\}_{j=I}^N$ the final allocation will be efficient.

2.7. Further Empirical Implications

2.7.1. Measuring the Cost of the Auction Mechanism. What is the cost for the Treasury, in terms of loss of revenue, due to the strategic behavior of dealers in the auction stage? This question is the subject of many empirical and theoretical papers analyzing Treasury auctions. A benchmark for how much the Treasury is leaving on the table is the difference between the highest price dealers would be willing to pay for the securities - their "true value" - and the price they are actually paying in the auction. The magnitude of the gap between the true value and the auction prices is interpreted as the cost, *per security*, of the auction for the Treasury. Unfortunately, in practice we observe only what dealers are actually paying in the auction, but not what they were willing to.

In order to deal with this problem, many empirical papers use the outcome of the secondary market surrounding auctions to estimate the securities' true value. Specifically, the price at which securities are negotiated at the when-issued market has been frequently used as proxies for their true value. Since dealers trade in the when-issued market during the minutes surrounding the auction, using when-issued prices allow researchers to avoid problems concerning duration mismatches.

The simple environment considered in this paper highlights that using when-issued securities for this purpose may be misleading. As showed in the previous sections, the equilibrium price at the when-issued market will not be equal to dealers' true valuation for the security in the presence of market power. Dealers anticipate that the equilibrium price in the auction will be artificially below the underlying security's true market value. This fact will drive the equilibrium price of when-issued securities down as well. Indeed, in the environment considered above, the highest value for the price such that dealers would be willing to absorb the entire supply - the securities "true value" - is the Walrasian price p^W . However, in

equilibrium, the Walrasian and the when-issued prices satisfy:

$$(2.7.1) \quad E[p^{wi} - p^{so}] = (1 - \gamma) E[p^W - p^{so}].$$

Since $\gamma \in (\frac{1}{2}, 1)$, the gap between the auction and when-issued prices is strictly smaller than the gap between the auction price and the "true value" of the security. This means that using the when-issued prices as proxies would actually underestimate the revenue loss for the Treasury.

Equation (2.7.1) suggests a way to correct to the potential bias related to using WI prices as proxies. Note that the magnitude of the multiple factor $(1 - \gamma)$ depends solely on the number of participants in the market. Therefore, the average gap between the Walrasian and auction's prices could be backed up straightforwardly from the average gap between the when-issued and auction's prices.

One could also be interested in the gap between the two prices for a specific realization. Equation (2.7.1) proposes a way to back up the average revenue loss for the Treasury. Nonetheless, it does not say anything about the realized loss on a given auction. In other words, it tells us what $E[p^W - p^{so}]$ is, but not the value for a specific $\tilde{p}^W - \tilde{p}^{so}$. Getting the value for the latter would require an estimation of the value of $\tilde{p}^W$ for the specific underlying security. If we knew the dealers' preferences for security, we could find the true value of a given security using expression (2.5.2). Under the assumptions from the environment considered in this paper, this would require the knowledge or an estimation of the preference parameters $\bar{v}$ and ρ . However, the estimation of such parameters may be complicated or even impossible in practice. For instance, one would need data on multiple points of the auctions' bid schedule (individual or aggregate) which might not be publicly available.²¹

Fortunately, we can back up p^W without the knowledge of preference parameters. It is clear from expressions (2.5.2), (2.6.4), (2.6.6) that the same set of parameters that determines

²¹Only three points are made public by the U.S. Treasury.

the Walrasian price also determines the price in the auction with market power, and the price of securities in the when-issued market. We can rearrange these expressions to get:

$$p^W = p^{so} + (1 - \gamma) (p^{wi} - p^{so}) + \frac{\gamma(2 - \gamma)}{1 - \gamma} E[p^{wi} - p^{so}].$$

Note that the variables that appear in the right hand side are p^{wi} , p^{so} , and γ . Therefore, the above expression implies that the true value for the security can be found if one knows the number of auction participants and has data on the price of the security in the auction and when-issued stages.²²

2.7.2. Volume. The environment considered in the previous sections illustrates how uniform price auctions give dealers incentives to trade when-issued securities. Nevertheless, it also offers predictions on the relationship between the variables in the model and total amount of securities being traded in the when-issued market.

Let $\Upsilon \equiv \frac{\sum_{i,t} (q_i^{wi,t})^2}{N}$ be the measure of interest for the volume *per dealer* in the when-issued market. The results from Corollary 6 imply that, in equilibrium:

$$(2.7.2) \quad \Upsilon = s_v^2 (1 - (1 - \gamma)^T)^2$$

where $s_v^2 \equiv \frac{1}{N} \sum_{i,t} \left(\frac{v_i}{\rho} - \frac{\bar{v}}{\rho} \right)^2$ is the degree of heterogeneity among dealers' true valuation for the securities.

Market power and the *per investor* volume in the when-issued market are negatively related. This can be readily seen from the positive relationship between Υ and γ . The intuition is the following. When dealers have a higher impact on the price of when-issued securities, they have incentives to behave less aggressively in the when-issued market. The larger the market power, the higher the shading at this market stage.

²²The parameter γ depends solely on the number of dealers participating in the auction. The price of the auction p^{so} is commonly made available by the Treasuries authorities throughout the world. Finally, data sets containing information relative to when-issued prices, p^{wi} , were used on many empirical studies, e.g., Bikhchandani et al. [2000], Barclay et al. [2006], Nyborg and Sundaresan [1996], Goldreich [2007]

The volume is also positively related with the length of the when-issued period. As seen in Subsection 2.6.2.2, dealers do not exploit all the trading gains on a single round in the when-issued market. They always end up with a fraction of the optimal portfolio. However, as T goes up, they have additional opportunities to meet up and exploit the remaining gains. Indeed, we saw that their positions converge to the optimal as T gets arbitrarily large.

2.8. Resale

Dealers are not restricted from trading the underlying securities after the auction takes place. In fact, the post-auction market for U.S. Treasury securities is one of the most liquid markets in the world. This raises the following question: would the conclusions from the previous sections still hold if dealers were allowed to trade after the auction? In this subsection, a post-auction market is added to the model considered above in order to check the robustness of the results presented earlier. As will be seen, the addition of a resale stage does not change the conclusions.

Suppose that, after the auction takes place, dealers can trade securities in the same fashion as they did in the when-issued market. Dealers submit demand (supply) schedules to a central broker and the pricing and allocation rule are the same as the ones described on Section 2.3 for the when-issued market. I will denote the variables for the resale market with superscripts R . We have the following result:

PROPOSITION 6. *Suppose there is a resale and a when-issued markets. For a given realization $\tilde{Q}$, the prices of the resale, auction, and when-issued markets in the linear SPE are given, respectively, by:*

$$\begin{aligned}
\tilde{p}^R &= \tilde{p}^W \\
\tilde{p}^{so,R} &= \tilde{p}^W - \frac{(1-\gamma)}{\gamma} \rho \frac{\tilde{Q}}{N} \\
p^{wi,R} &= (1-\gamma) E[p^W] + \gamma E[p^{so,R}].
\end{aligned}$$

The conclusions from the previous sections about the when-issued market are still valid. The equilibrium of the market is consistent with underpricing and dealers have incentives to trade when-issued securities prior to the auction. Moreover, the relationship between the magnitude of underpricing and the parameters of the model are unchanged with the additional market stage. The gap is increasing on the number of dealers and decreasing in the auction size and in the slope parameter of dealers' demand.

At first glance, one might suspect that adding a resale stage would have the same equilibrium properties as adding an additional round in the when-issued market. The set of dealers submitting bids, the pricing and allocation rules are exactly the same in both market stages. However, the timing at which dealers are trading the securities matters in equilibrium. Specifically, in the when-issued market, dealers anticipate that the Treasury will supply an inelastic amount of securities in the auction and that the resulting price will be artificially low. This fact drives down the equilibrium price in the when-issued stage as well. Dealers will not be willing to acquire securities for a price too far away from the auction price. In the resale market, however, the auction have already taken place. Dealers know that they will not have an opportunity to acquire securities for the low auction price anymore. As a result, the equilibrium price at this stage will be higher than in the when-issued market.

The addition of a resale market does not change the prices from the previous market stages. The equilibrium values for the price of securities in the when-issued and auction

stages take the exact same form as in the previous sections where resale was not allowed. Therefore, the analysis made before is robust to the inclusion of a resale stage after the auction takes place.²³

2.9. Conclusion

The model presented above shows how the structure of the Treasury market can give incentives to dealers to trade when-issued securities. It shows how dealers can use this specific forward contract in order to influence the behavior of their competitors, as well as his own, during the auction stage and how this can be beneficial for them. Moreover, the resulting equilibrium is consistent with two empirical facts from the inter-dealer market: underpricing and high volume of transactions.

It is worth emphasizing that the results presented in this paper rely solely on the way the market is structured. As pointed out above, the only reason dealers are trading when-issued securities is to influence the outcome of the auction stage. Therefore, this paper offers an alternative explanation for why dealers have incentives to enter into forward contracts instead of the more traditional speculation or/and hedging motives. Indeed, in the environment considered above, all market participants have symmetric and complete information.

A commonly used empirical strategy to measure the performance of an auction mechanism is the magnitude of the price gap from the when-issued and auction stages. The underlying assumption is that the price in the when-issued market is a good proxy for the true value of the security since dealers are trading in a free market environment at this stage. The price gap would then indicate the gap between how much dealers were willing to pay for the securities and how much they end up actually paying. However, for the specific case in which the auction follows a uniform pricing rule, I showed that the when-issued price is a biased proxy for the true value of a security. If one considers the security's "true value"

²³See [Coutinho \[2012a\]](#) for a more detailed analysis of equilibrium of divisible good auctions with a resale market.

as the one which would arrive in a perfect competitive market, the equilibrium price of a security when-issued price lies strictly below it since dealers anticipate they will be able to acquire it for a depressed price in the auction. A simple correction for the negative bias is proposed above.

The price gap between the two market stages is not used only to compare the performance of a specific auction, but also to compare the performance of different auction mechanisms. A recurrent question in the Treasury auction literature is whether uniform price auctions generate more revenue than discriminatory auctions. Although the analysis from this paper focused solely on the uniform price case, it did highlight how the dealers' strategies on the two market stages, and thus the resulting prices, are jointly determined in equilibrium. There is no straightforward reason to believe that the when-issued price would be invariant to the choice of which auction mechanisms the Treasury decides to use. Consequently, simply comparing the price gap between the two markets without taking into account that when-issued prices are endogenously determined may lead us to wrong conclusions.

Finally, even though the analysis focused on the market for Treasury securities, the conclusions drawn in this paper can be straightforwardly applied to any other market where agents trade forward contracts on homogeneous goods to be auctioned through a uniform price auction. Examples of such markets are the market for energy, IPOs and CDS auctions.

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CHAPTER 3

Divisible Good Auctions with Resale

3.1. Introduction

Many goods which, initially, are sold through auctions are also traded on secondary markets. In fact, there is no straightforward economic reason why this should not be the case besides possible contractual obligations. If the final allocation of an auction is not efficient, there will be room for gains of trade. Indeed, resale is common practice in many markets which has a primary market organized as an auction stage, e.g. goods sold through eBay, credit default swaps, IPOs, and etc. An extreme example of such market is the market for US Treasury securities, where the daily average transaction volume in the secondary market is over US\$194 billions.¹

The possibility that an individual can resell the good he acquires in the auction directly affects his bidding behavior. A bidder may participate in the auction even though he does not get any benefit from winning the good: if the expected payment from reselling the good is higher than the cost of acquiring it in the auction, he may enter for purely speculative reasons. On the other hand, some individuals may prefer to buy the good on the secondary market instead of competing for it in the auction: they anticipate that they have another chance to acquire the good in the secondary market, reducing their incentives to submit aggressive bids at the auction stage.

The total effect on revenue in the primary auction coming from adding a resale is therefore ambiguous. As argued above, a resale stage attracts speculators to the auction, which at first

¹As reported by Primary Dealers for the Federal Reserve of New York. Daily average transaction volume for the 3 years coupon US Government Coupon Securities in the period 1/5/2011-7/14/2011. [HTTP://WWW.newyorkfed.org/markets/statistics/msytd.pdf](http://WWW.newyorkfed.org/markets/statistics/msytd.pdf)

glance would be beneficial for the auctioneer. It increases competition in the auction, which is normally associated with a higher expected revenue for the auctioneer. However, since final buyers have less incentives to bid aggressively during the auction stage, the aggregate effect on the revenue is ambiguous.

The literature addressing the implications of resale on an auction's equilibrium outcome is not vast; it is worth noticing, however, that the topic has been subject of greater attention in recent years. For instance, [Hafalir and Krishna \[2008\]](#) find that the equivalence of expected revenue between first and second-price auctions for a single unit may not hold if resale is allowed. They consider a game where two asymmetric bidders compete in an auction and the winner makes a take-it-or-leave-it offer to the loser after the auction takes place. [Gupta and Lebrun \[1999\]](#) consider a similar model with the difference that the bidders valuations are disclosed between the auction and resale stages. [Cramton and Ausubel \[2001\]](#) and [Zheng \[2002\]](#) analyze optimal selling mechanisms when resale is allowed.

A common feature of auctions being considered on all papers cited above is the single unit restriction. An important extension of auction models, in general, is to consider situations where more than one unit of the good is being sold in the same auction. Besides the theoretical appeal, auctions of these type are widely used to sell multiple goods at the same time: bonds, electricity, initial shares, and telecommunication spectrum are a few practical examples.. The behavior of bidders in these type of auctions may differ considerably from the single unit case different since their strategy space is augmented by the quantity they acquire in addition to the price they pay. The same reason also complicates the theoretical analysis.

Nevertheless, the literature considering both, resale and multiple-unit auctions, is very incipient. This paper aims to shed light on this problem. I consider an environment where auction participants are allowed to trade the goods they have acquired in the auction on a secondary market. Under some assumptions (quadratic utility, uniform price auction,

competitive markets in the resale stage), I am able to find closed form equilibria when agents first compete to obtain some amount of a divisible good in the auction and then trade the units they won directly with each other.

I consider a model in the lines of [Wilson \[1979\]](#) auction of shares.² In this type of model, the auctioneer sells a positive amount of a perfectly divisible good and the bidders compete for shares of it. They submit a complete bid schedule determining how much of the good they are willing to acquire for all possible prices. The auctioneer aggregates the individual bids and determines the market-clearing price. I focus attention on uniform-price auctions, where participants pay the market-clearing price for all shares they acquire. [Back and Zender \[1993\]](#), [Ausubel and Cramton \[2002\]](#), and more recently, [Pycia et al. \[2010\]](#) and [Coutinho \[2012b\]](#) are examples of papers analyzing the properties of different auction frameworks using the divisible good assumption.

The novelty of this paper is the consideration of a resale stage after the share auction takes place. The possibility of resale opens room for an individual enter the auction solely to resell the good in the secondary market. To capture this type of behavior, I consider two distinct types of market participants, final investors and speculators. The good being auctioned brings direct benefits only to final investors. However, speculators can still benefit from participating in the auction if they sell their shares in the secondary market for a higher price than they initially paid.

I consider two different setups for the resale market. In the first, the market participants use Nash bargaining to split any possible gains from trade. Within this environment, I show that existence of a resale market has an ambiguous effect on the revenue generated by the auction. While a resale market may increase revenue due to more aggressive competition among bidders, it is possible to witness the exact opposite. The reason relies on the fact that,

² They are also called divisible good auctions.

instead of trying to out compete speculators during the auction stage, final investors have incentives to bid less aggressively and acquire the remaining shares in the secondary market. Depending on which of these two effects has a higher magnitude, a resale market can increase or decrease the auction revenue. I show how these effects depend on the bargaining power between the two types of participants and on the total supply of the good being auctioned.

In the second environment, I model the resale stage as a perfectly-competitive market. This allows an arbitrary number of speculators and final investors while maintaining a tractable analysis. The environment also provides additional comparative statics in a straightforward way. For instance, I show that adding a final investor can have an opposite effect on the equilibrium bid schedule as compared to the addition of an speculator. As showed by [Wilson \[1979\]](#), [Back and Zender \[1993\]](#), [Ausubel and Cramton \[2002\]](#) and others, uniform-price auction have equilibria in which bidders shade their bids. In order to keep the auction price at low levels, bidders have incentives to reduce the quantity they acquire at the auction. Since the generating force behind the existence of such equilibria is the market power of the bidders, the usual intuition is that an increase in the number of participants in the auction would reduce the magnitude of the shading of the bid schedules. However, I show that exactly the opposite can occur when the number of speculators increases. As a consequence, the equilibrium price and revenue depends not only on the total number of auction participants, but also on the specific composition of the bidders (whether they are speculators or final investors).

The distinction between investors and speculators provides further empirical implications on the shape of the bid schedules. Specifically, it provides novel implications on their skewness and kurtosis which are consistent with empirical observations. Specifically, [Keloharju et al. \[2005\]](#) document that, in the Finnish Treasury securities auctions, the skewness from the submitted bid schedules has a positive relationship with the total number of participants in the auction. The authors argue that their findings are not consistent with the market

power theory of uniform price auctions. The canonical models presented in [Kyle \[1989\]](#), [Back and Zender \[1993\]](#), [Wang and Zender \[2002\]](#) predict a negative (non-positive) relationship between skewness and number of bidders. Like these papers, I showed that an increase in the number of final investors is associated with a lower skewness for the equilibrium bid schedules. Nevertheless, I show that [Keloharju et al. \[2005\]](#) observation could be perfectly consistent with the theory if the additional participants are perceived as speculators instead of final investors.

The paper with greatest similarity to this work is [Pagnozzi \[2010\]](#). The author analyzes the equilibrium properties of a uniform-price auction when bidders are allowed to resell. I extend his work by considering a divisible-good framework and two different environments for the secondary market. The framework developed here provides implications on the shape of the bid schedule and additional comparative statics not captured by a discrete environment as the one considered in his paper.

The rest of the paper is organized as follows. In [Section 3.2](#), I present the main model: the auction and secondary market environments as well as the assumptions on agents' preferences. [Section 3.3](#) characterizes equilibrium under two different assumptions on the secondary market. [Section 3.4](#) extends the analysis from [Section 3.3](#) to the case where bidders submit non-linear bid schedules. Finally, [Section 3.5](#) concludes the paper.

3.2. Model

In this section I describe the basic model used in the rest of the paper. The environment resembles the auction of shares developed by [Wilson \[1979\]](#) and [Back and Zender \[1993\]](#). The main difference relies on the fact that I allow auction participants to resell the securities acquired in the auction.

There are two types of buyers in this economy with different preferences over two perfectly divisible goods: securities, q , and money m . The first type, final investors, directly benefits

from holding a positive quantity of securities. The preferences of an investor are represented by a quasi-linear utility function $U_I : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying $U_I(q, m) = u(q) + m$, where $u : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a concave, differentiable function of q . I also assume that investors have linear marginal utility. Specifically, $u(q)$ satisfies:

$$\frac{\partial u(q)}{\partial q} = v - \rho q.$$

The second type of buyers, the speculators, do not have any positive gains from holding the security on their own behalf. However, a speculator can benefit from holding a positive quantity of the securities since she can sell them to investors in the secondary market. Her utility function $U_S : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ is given by the net monetary value of realized transactions; $U_S(q, m) = m$.^{3, 4}

The model has two periods: the auction and the secondary market. In the auction stage, investors and speculators compete against each other for the securities offered by a third part (the auctioneer). After the auction takes place and the securities are distributed, market participants can trade them directly with each other in a secondary market.

I assume that investors have complete information. They all know the distribution of Q and their competitors utility functions $U_j(q, m)$, $j = I, S$. Moreover, the final outcome of the auction is disclosed to the public. This implies that the quantity each participant have before entering on the secondary market is common knowledge. Besides making the problem tractable, this last assumption is realistic in many setups. For instance, in the context of US Treasury security auctions, the Treasury reports information on the final allocation of the auction per investor class on the seventh business day of the month following a Treasury

³One interpretation for this environment is that there is subset of individuals who have a higher marginal valuation. For example, in the Treasury bills market primary dealers may have a higher valuation than regular investors since they are allowed to trade directly with the government. In Energy markets, some firms may have a lower cost than others due to technological differences.

⁴The restriction on the domain of U_i prevents agents from short-selling securities.

security auction. This report discloses how much was allocated to depository institutions, individuals, dealers and brokers, and a few other classifications of investors.⁵

3.2.1. Auction. There are many different auction mechanisms that could be used to sell a positive quantity of a homogeneous good. In this paper I will focus on uniform-price auctions.⁶ In this auction mechanism, bidders submit a complete bid schedule describing the quantity of securities they are willing to acquire for all possible prices. The auctioneer then aggregates the individual bid schedules and distributes the securities to the highest bids. In uniform price auctions, bidders pay the same price for all securities they end up acquiring.

Formally, an exogenous and stochastic quantity Q of a perfectly divisible security is auctioned by a third party (the auctioneer). Each agent submits a left continuous, weakly decreasing bid schedule $q_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, specifying the quantity demanded for each possible price. After collecting all individual bids, the auctioneer forms the aggregate bid, $\sum_{\mathcal{I}} q_i(p)$. The stop-out price p^{so} is defined as the value of $p \geq 0$ such that the aggregate bid is equal to the realized supply, i.e., $\sum_{\mathcal{I}} q_i(p^{so}) = Q$ ($p^{so} \equiv 0$ if $\sum_{\mathcal{I}} q_i(0) < Q$). Agent i gets $q_i(p^{so})$ units of the good, the value of his bid function evaluated at p^{so} . He pays the stop-out price for all units he acquires, $q_i(p^{so}) \times p^{so}$. The allocation and payment of this auction is illustrated in Figure 3.2.1.

Even though agents submit strictly decreasing bids in all equilibria described in this paper, a tie-breaking rule must be defined for the cases where aggregate demand is greater than supply at the stop-out price, i.e. when we have $\sum_{\mathcal{I}} q_i(p^{so}) > Q$. In this case, the segment of the aggregate demand curve above the stop-out price will be allocated in the same way

⁵For a detailed discussion of the information the Treasury discloses and how the securities are allocated in practice, see Fleming [2007].

⁶Uniform-price auctions are commonly used in practice. For instance, it has been chosen auction format the U.S. Treasury uses to sell its securities to the public since the mid 1990s. Vickrey and discriminatory are examples of other commonly analyzed divisible-good auctions.

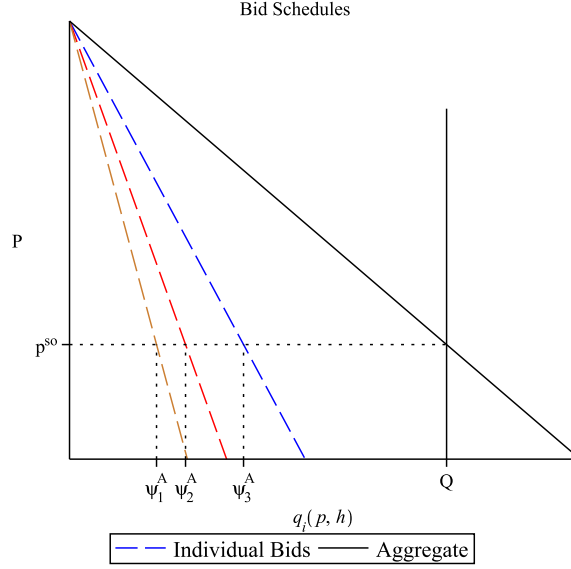


FIGURE 3.2.1. Payment on a uniform auction.

as before. The remaining units, given by $Q - \lim_{p \rightarrow p_+^{so}} \sum_I q_i(p)$ will be distributed to bidders on a pro-rata basis.⁷

The total supply being auctioned, Q , is stochastic and distributed according to a distribution function $F(Q)$ with full support on $[\underline{Q}, \overline{Q}]$. As showed by [Klemperer and Meyer \[1989\]](#) and [Back and Zender \[1993\]](#), stochastic supply considerably reduces the set of equilibrium strategies. In addition to this technical benefit, stochastic supply is realistic in many real-world markets. For instance, when auctioning Treasury notes, the U.S. government retains an unspecified share of the announced total quantity being auctioned. Moreover, the treasury allows some small bidders to submit *non-competitive* bids. This special type of bid specifies a quantity a bidder wins, independent of the stop-out price. It can be interpreted as an agent submitting a perfectly inelastic bid schedule. *Ex ante*, agents know neither how much the government will retain nor how much the non-competitive bidders will demand.

⁷[Kremer and Nyborg \[2004\]](#) compare equilibria of uniform-auctions with different tie-breaking rules.

Therefore, we can view total amount of securities that will be auctioned as stochastic. Alternatively, in many energy auctions, both supply and demand are determined through bid schedules. If the buyers (distributors) do not have full information about suppliers (generators) then they will be facing a stochastic supply. The same is true on the other side of the market: generators face stochastic demand if they do not have complete information about distributors.

3.2.2. Secondary Market Environments. In the secondary market, speculators and investors can trade with each other any securities they have acquired in the auction stage. If the final allocation of the auction stage is not efficient, it is possible to Pareto improve the allocation and share the remaining surplus among themselves.

I will consider two different environments for this market stage. In the first one, there is only one speculator participating in the auction and the gains from trade will be split through Nash bargaining. For simplicity, I will not allow investors to trade with each other at the resale stage. The speculator allocates the securities in the secondary market and retains a share θ of the surplus generated from all trades taking place in the secondary market. In the second environment, there is a finite number I of investors and N of speculators. In this environment, I assume that the price of the security is determined by perfect competition. Investors and speculators take the security's price as given, and simply choose how much to buy/sell.

3.3. Equilibrium

I use backward induction in order to characterize subgame perfect equilibria for the two-stage game.⁸ First, I find the best response in the resale stage for all agents, given any possible allocation. After that, taking the strategies in the resale stage as given, I describe the

⁸There might be more than one SPE equilibrium for this game. Here, I focus on a symmetric and linear equilibrium which might be unique on this class of equilibria.

equilibrium bid schedules of the agents in the auction stage. Throughout this section, I will focus on the special equilibrium for the auction where investors submit linear bid schedules. As pointed out by [Wang and Zender \[2002\]](#), there is a continuum of symmetric equilibria in uniform-price auctions such as the one considered in this paper. I focus on linear equilibria for expositionla reasons. However, the qualitative results still hold for non-linear equilibria. It is important to emphasize that linearity is not assumed *ex ante*, rather is a property of the equilibrium. In other words, given that his opponents are submitting linear bid schedules, the optimal response of a bidder is to submit a linear bid schedule himself. The linearity assumption is dropped in Section 3.4 in order to allow further empirical implications on the shape of the equilibrium bid schedule, such as its skewness.

Investors and speculators will be playing optimally in an *ex post* sense in all auction equilibria described below. In this type of equilibria, which were first described by [Klemperer and Meyer \[1989\]](#) and [Kyle \[1989\]](#), investor's bid schedules are best responses for any realization of Q . In other words, investors do not have an incentive to deviate from their strategies even if they know the exact amount of securities, Q , being supplied by the auctioneer prior to submitting their bids.

3.3.1. No Resale. In this subsection, I characterize the unique linear equilibrium of the auction when resale is not allowed. Even though there are positive gains of trade after the auction takes place, agents are restricted to end up with the securities they acquired in the auction on their portfolio. The equilibrium properties under this specific hypothesis will serve as a benchmark against which to compare the results founded in later sections, where we allow trade after the auction.

A weakly dominant strategy for speculators is to bid $q_S(P) = 0$. Their preference specification implies that they do not get any additional utility from holding a positive amount of securities on their portfolio. Moreover, by assumption the stop-out price cannot

be negative. Hence, winning a positive amount of securities makes them at most indifferent to not winning anything. If all speculators bid in this way, the auction's final outcome is not affected by their presence. Therefore, I will simply ignore speculators when characterizing the equilibrium of the game.

Investor i 's problem is to choose an entire bid schedule that solves:

$$\max_{q_i(\cdot)} E_{p^{so}} [u(q_i(p^{so})) - p^{so} q_i(p^{so})]$$

where the auction price satisfies the market clearing condition

$$(3.3.1) \quad \sum_{\mathcal{I}} q_i(\tilde{p}^{so}) + q_S(\tilde{p}^{so}) = \tilde{Q}.$$

We have the following result.

PROPOSITION 7. *In the unique linear equilibrium of the uniform price auction where resale is not allowed, investors submit the bid schedule*

$$(3.3.2) \quad q_i(P) = \frac{1}{\rho} \frac{I-2}{I-1} (v - P)$$

where I is the number of investors competing in the auction.

The equilibrium bid function is depicted in Figure 3.3.1. Since $\frac{I-1}{I-2} > 1$, the slope of the bid function is greater than the slope of the valuation function (in absolute terms). For any $q > 0$, the benefit that a marginal unit of the security brings to an investor is greater than the price he is willing to pay for it. This result comes from the fact that, in equilibrium, each bidder is a monopsonist over the residual supply left after we subtract his competitors' bid functions from quantity supplied Q . Acquiring an additional security in the auction will increase the stop-out price, which, in turn, increases the cost of all infra-marginal units this investors is already acquiring. Pointwise optimization implies that agents will equalize the

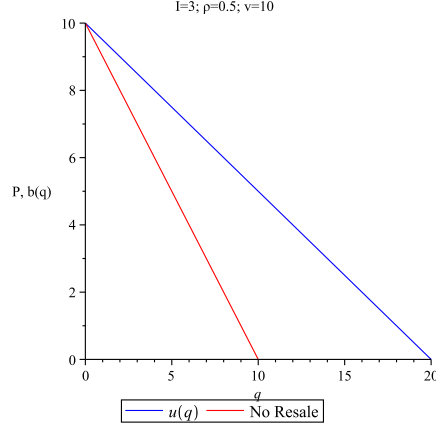


FIGURE 3.3.1. Equilibrium bid function when resale is forbidden

marginal valuation to the marginal cost for all levels of q . Since the marginal cost is greater than the price for any positive quantity of q , the optimal bid function will lie below the marginal valuation function.⁹

The gap between the true demand and the submitted bid by an investors is called *bid shading*. Positive bid shading implies that the auctioneer could potentially increase his revenue without changing the total quantity supplied. Indeed, the auctioneer could exogenously set any price between the equilibrium stop-out price and $u'(\frac{Q}{I})$, instead of running the auction, and investors would still be willing to acquire the entire supply Q . Therefore, bid shading can be used as a measure of how much revenue the auctioneer is losing due to bidders' market power in the auction stage.

3.3.2. Bargaining in the Secondary Market.

3.3.2.1. *Secondary Market Stage.* In the secondary market, an investor's demand for securities will depend on how much he acquires in the auction. For example, if he acquired q_i^A units of the good in the auction stage, his residual demand for securities is given by

⁹See Milgrom [2004] for a detailed explanation on this type of equilibria.

$P(q, q_i^A) = (v - \rho q_i^A) - \rho q$. The complete information assumption implies that the speculator knows how many securities each investor acquired in the auction stage, which implies that she also knows their residual demands in the secondary market.

The speculator then decides how to allocate the securities she acquired in the auction among the investors. She retains a fraction $\theta \in (0, 1)$ of the surplus generated by each bilateral trade with investors, and the remaining $(1 - \theta)$ goes to the investor. The investors' strategy space is restricted solely to the binary choice of accepting (or not) the speculator's offer. If the speculator sells q_i^R units to an investor who has arrived in the secondary market with q_i^A , the total surplus generated by this trade is:

$$\pi_i^D(q_i^A, \hat{q}_i^R) = \int_{q_i^A}^{q_i^A + \hat{q}_i^R} P(q, q_i^A) dq.$$

Under the specification of an investor's utility function described on the previous section we get:

$$(3.3.3) \quad \pi_i^D(q_i^A, \hat{q}_i^R) = \left(v - \rho \left(q_i^A + \frac{\hat{q}_i^R}{2} \right) \right) \hat{q}_i^R.$$

The speculator maximizes the sum of investors' payments $\sum_i \pi_i^D(q_i^A, \hat{q}_i^R)$ subject to the constraint that the total quantity she sells, $\sum_i \hat{q}_i^R$, is no greater than the amount she obtained in the auction stage, q_S^A . We have the following lemma:

LEMMA 1. Let $\{q_{(j)}^A\}_{j=1}^I$ be an weakly increasing ordering of $\{q_1^A, \dots, q_I^A\}$. Define ι as the highest $n \in \mathcal{I}$ such that $\frac{q_S^A + \sum_{j \leq n} q_{(j)}^A}{n} - q_{(n)}^A \geq 0$. Then, for all $j \leq \iota$ we have $q_{(j)}^R = \frac{q_S^A + \sum_{j \leq \iota} q_{(j)}^A}{\iota} - q_{(j)}^A$. If $j > \iota$, $q_{(j)}^R = 0$.

At the optimum, all investors buying from the speculator will leave the secondary market with the same quantity of the security. In other words, $q_i^R + q_i^A = q_j^R + q_j^A = \frac{q_S^A}{\iota}$ for all i, j

such that $q_i^R > 0$. To see this, note that they all must have the same marginal valuation for the security. If this was not the case, the speculator could increase her revenue by taking a small quantity from one investor whose marginal valuation is lower and transferring it to another investor with higher valuation. The difference in the payment received would be given by the difference between the two marginal valuations. The fact that investors have exactly the same utility function implies that they have the same downward sloping marginal valuation function as well. This implies that, in order for their marginal valuations be the same, they must hold the same final quantity of the security on their portfolio.

The way the speculator chooses to distribute her securities in the secondary market depends only on the auction allocation. Since she gets a constant share θ of the total gains generated by the trades at this stage, she chooses the allocation that maximizes the total surplus. However, the gains from trade depend solely on the specific way the securities were allocated in the auction stage. That is, given a specific set of auction allocations, $\{q_i^A\}_T$ the quantity the speculator will sell to each investor will not depend on the specific bargaining parameter θ .

3.3.2.2. Auction Stage. Now that we have already described how agents behave in the secondary market, we can turn our attention to the auction stage. As in the benchmark case where resale was not allowed, I will focus the analysis on equilibria where auction participants are submitting linear bid schedules.

An investor will have incentives to bid less aggressively in the auction when resale is permitted. This is due to the possibility of extracting a positive share of the surplus generated by the trade in the secondary market, which reduces the marginal gain of acquiring an additional unit in the auction stage. He anticipates that what he gets in the secondary market depends on the auction of the outcome in the specific way described on Lemma 1. This Lemma implies that the quantity he will end up trading in the secondary market is negatively related to the quantity he acquires in the auction. Therefore, he faces a trade off

between increasing the quantity he acquires in the auction and reducing the quantity he gets in the secondary market.

The marginal gains for an investor when he acquires an additional security at the auction stage depends both on the bargaining parameter and the amount he is already acquiring in the auction. If $\theta = 1$ – a perfectly discriminating environment – then all the surplus from trades in the resale stage goes to the speculator. In this case, the existence of a resale stage does not directly influence the investor's bid behavior in the auction stage.¹⁰ However, when $\theta < 1$ this is not the case. Acquiring an additional unit in the auction stage will change the amount received in the secondary market $\hat{q}_i^R$. Indeed, accordingly to Lemma 1, $\frac{d\hat{q}_i^R}{dq_i^A} = -1$ whenever $\hat{q}_i^R > 0$. A local increase in the quantity acquired in the auction is completely offset by a reduction in the quantity i holds in the secondary market. However, while the final amount of securities investor i would not change, his total utility depends on the specific amount he acquires in the auction stage. This is due to the fact that i gets the full surplus generated by the securities he acquires in the auction, whereas, he gets only a fraction $1 - \theta$ of the surplus generated in the secondary market. Investor i 's problem can be written as:

$$(3.3.4) \quad \max_{q_i(\cdot)} E_{p^{so}} \left[U(q_i(p^{so})) + (1 - \theta) \pi_i^D(q_i(p^{so}), \hat{q}_i^R) - p^{so} q_i(p^{so}) \right]$$

Let's turn our attention to the speculator's maximization problem. Let

$$\begin{aligned} \Pi(q^S, \{q_i^A\}_{i \in \mathcal{I}}) = & \max_{\{q_i^R\}} \sum_{i \in \mathcal{I}} \pi_i^{PD}(q_i^A, q_i^R) \\ & \text{s.t. } \sum_{i \in \mathcal{I}} q_i^R \leq q^S. \end{aligned}$$

¹⁰It does influence it indirectly, since a resale stage attracts the speculator to the auction, which increases the competition at this stage.

The function $\Pi(q^S, \{q_i^A\}_{\mathcal{I}})$ gives the maximum surplus that could be generated by trades in the secondary market if speculators and investors arrive in the secondary market with $(q^S, \{q_i^A\}_{\mathcal{I}})$ from the auction stage.

If investors are submitting the bid schedules $\{q_i^A(\cdot)\}_{i \in \mathcal{I}}$, the speculator problem can be written as:

$$(3.3.5) \quad \max_{q_S(\cdot)} E \left[\theta \times \Pi(q_S(p^{so}), \{q_i^A(p^{so})\}_{\mathcal{I}}) - p^{so} q_S(p^{so}) \right].$$

We have the following proposition.

PROPOSITION 8. *If $\theta > \underline{\Theta}$, there is a unique symmetric linear equilibrium in the auction stage for the above game where both investors and the speculator submit the following bid schedule:*

$$(3.3.6) \quad \theta q(P) = \frac{1}{\rho} \frac{I-1}{I} (\theta v - P).$$

A linear equilibrium will exist only if the the speculator's bargaining power θ is larger than a lower bound $\underline{\Theta}$. Intuitively, as the bargaining power of the speculator goes down, the shading of the equilibrium bid schedule goes up. In the limit, as θ approaches zero, expression (3.3.6) boils down to $P = 0$. However, it would be relatively cheap for a participant to deviate and acquire the total supply Q being offered in the auction. In this case, the above expression cannot be sustained in equilibrium. The exact expression determining $\underline{\Theta}$ is provided in an appendix.

I will restrict the analysis to the case where $\theta > \underline{\Theta}$. Proposition 8 shows that the shading in the bid schedule submitted by investors increases with the introduction of the speculator.

Moreover, it is straightforward to see that the shading is decreasing on θ . This result is intuitive. The lower θ is, the higher the surplus of trade in the secondary market goes to investors. Therefore, the net utility investors get from winning securities in the auction also decreases with θ , since they will be able to acquire them in the resale market from the speculator paying a lower price. This will give them incentives to bid less aggressively in the auction stage, which will tend to decrease the revenue for the auctioneer. However, the sole introduction of an additional participant in the auction increases the competition in the auction.

The corollary below gives the auction's revenue as a function of the parameters.

COROLLARY 8. *The revenue for the auctioneer for each realization of Q and with speculator's bargaining power given by $\theta > \underline{\theta}$ on the secondary is given by:*

$$R^{PD}(Q, \theta) = \theta \left(v - \rho \frac{I}{I-1} \frac{Q}{I+1} \right) \times Q.$$

Note that the revenue is strictly increasing on θ .

The existence of a resale market has an ambiguous effect on the the revenue generated by the auction. Indeed, the addition of a speculator to the market brings two effects. First, it increases the number of auction participants, increasing competition in the auction stage. This effect tends to increase the equilibrium price, and thus the revenue, from the auction. However, investors will have incentives to bid less aggressively in the auction since they are now able to acquire extra securities in the secondary market from the speculator. This has a negative impact on the auction revenue. Depending on the realization of total supply, one effect will dominate the other. Figure 3.3.2 illustrates a case where, for the same set of participants, the relative price will be higher or lower with the presence of the speculator depending on the realization of Q .

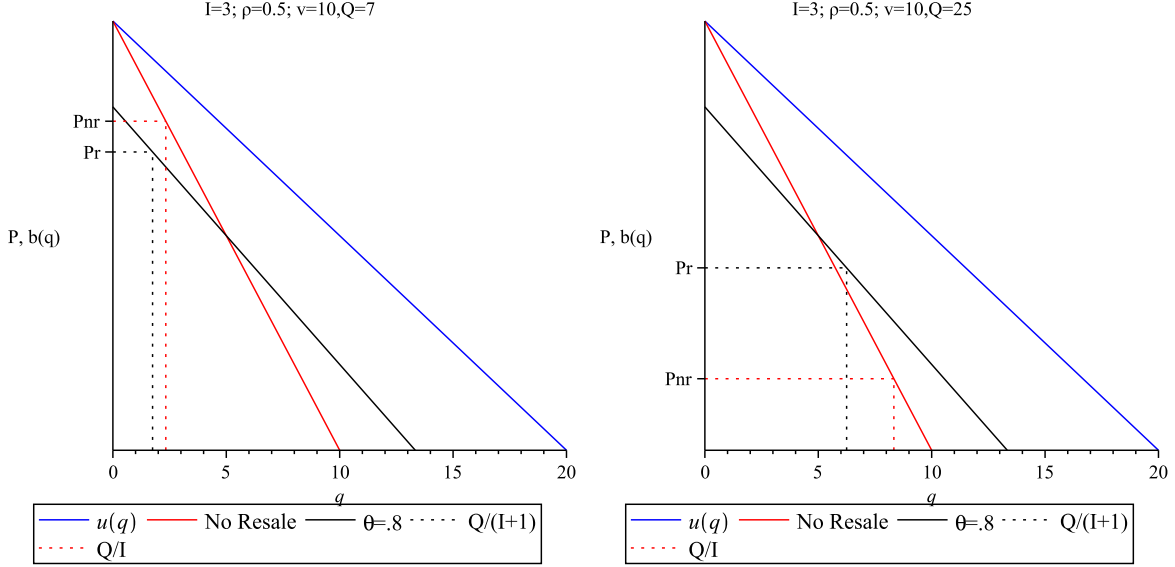


FIGURE 3.3.2. The existence of a resale market can decrease the revenue for the auctioneer. The equilibrium bid schedule for the benchmark no resale case is depicted in red, whereas the equilibrium bid schedule with resale is in black. The left hand panel shows that for a small realization of Q , the equilibrium price and thus the revenue is higher than with the possibility of resale.

The relationship between θ and Q determines both whether a linear equilibrium exists and, if it does, whether it generates a higher revenue than in the no resale benchmark equilibrium. As stated in Proposition 8, there will be a linear equilibrium only if $\theta > \underline{\theta}$. The lower bound $\underline{\theta}$ depends on Q ; their exact relationship is provided in an appendix. At the same time, it was argued above that relative revenue between the cases with and without resale depends on the realization of Q . Specifically, for low enough realizations, it is possible that the existence of a resale stage actually decreases the auction's revenue. Figure 3.3.3 illustrates the relationship between θ and Q . Holding θ fixed, the lower bound on the total supply, $\underline{Q}$, should be high enough that no participant has an incentive to deviate from the equilibrium bid schedule. However, for some intermediary values of Q , the auctioneer is actually harmed by the possibility of resale.

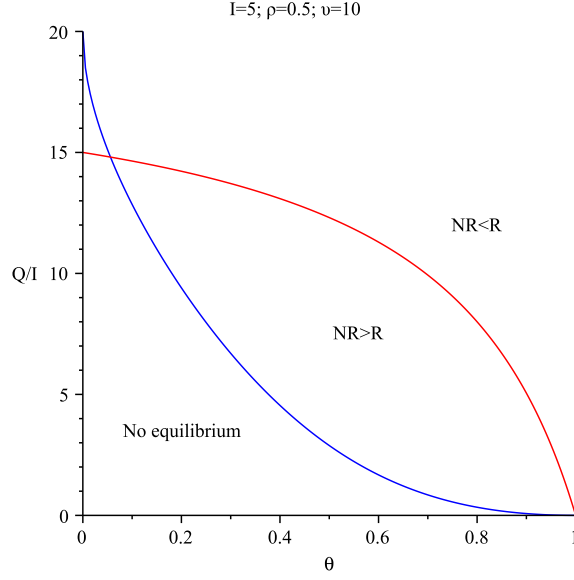


FIGURE 3.3.3. Regions where exists a linear equilibrium and where the revenue is higher (lower) with a resale market.

3.3.3. Competitive Resale Market. The assumption that there is only one speculator with a given amount of bargaining power is sufficient to find a linear equilibrium for the auction with resale. However, there is no special reason to believe that the market will have a unique participant acting as a Speculator. This section considers an alternative environment without this restriction: I analyze the auction problem with resale when there is a finite number of speculators. Moreover, I also relax the restriction that investors were not allowed to trade directly with each other on the secondary market.

The problem that arises from this framework is how to determine the final allocation on the secondary market. Depending on the strategy space, bargaining power, and other variables, the final outcome could take many forms. In order to keep the analysis simple, I consider the case where agents act competitively, i.e., take prices as given (demand-side and supply-side). As in the case with only one perfect discriminating speculator, I will be able to characterize a symmetric linear equilibrium for the auction.

The structure of the market is similar to the one described on previous sections. Agents will first participate in a uniform-price auction and then they will be able to exchange shares on a secondary market. The difference now is that this secondary market will not be composed by a single monopolist, but by a finite number of competitive individuals.

For tractability, I will retain the assumption that the final outcome of the auction is disclosed to the public. Therefore, prior to entering the resale stage, all agents know how many units each of their competitors have on their hands. One can interpret this stage as an exchange economy with each individual being endowed with what they won on the auction.

3.3.3.1. *Resale.* The investors' utility specification implies that they each have a downward-sloping demand function for the security. Moreover, the possibility that an agent consumes a negative quantity of the security (short-selling) is also ruled out by the restriction on the domain of the utility functions.

Suppose a speculator arrives with quantity q_S^A on the secondary market. Since she doesn't incur any utility from holding a positive amount of securities on her portfolio she would like to sell them all. Therefore, her net utility is $p^R q_S^A - p^{so} q_S^A$, where the first term is how much she makes on the secondary market by selling q_S^A and the second term is the amount she had to pay to acquire quantity q_S^A in the first stage auction.

The optimality conditions derived from the investors' maximization problem on the secondary market imply that the marginal utility of acquiring additional securities should equal to p^R . Since all investors have the same utility function, they all obtain $q_i^R = \frac{v-p^R}{\rho}$ regardless of how many units they acquired in the auction.

The above argument implies that the final outcome of the secondary market depends only on the realization of the total supply of securities, Q . Each investor will end up with a quantity $q_i^R = \frac{Q}{I}$ implying that the market clearing price is $p^R = v - \rho \frac{Q}{I}$. No matter the outcome of the auction, both the equilibrium price and allocation will remain constant. This

point is crucial to understand the bid behavior in the auction stage, which we investigate in the next section.

3.3.3.2. *"First-stage" Auction.* We now turn attention to the auction stage. I will characterize a symmetric linear sub game perfect equilibrium for this auction under the assumption that the secondary market is competitive. As in the case of a single speculator, I proceed by backward induction. We have the following proposition:

PROPOSITION 9. *Suppose we have I investors and N speculators, $I + N \geq 3$, and the secondary market is perfectly competitive. Then there exists a symmetric linear equilibrium on the auction stage where both investors and speculators submits the following demand schedule:*

$$(3.3.7) \quad q(P) = (v - P) \frac{1}{\rho} \frac{I}{I + N} \frac{I + N - 2}{I + N - 1}.$$

As in the case of a single perfectly discriminating speculator, investors and speculators submit the same bid schedule in equilibrium. With the specification considered here, both types of agents face exactly the same marginal valuation and marginal costs of acquiring an additional security in the auction stage. Since these are the dimensions they consider in order to determine their bid schedules, symmetry arrives naturally.

The competitive market assumption implies that any individual can buy or sell a security at the resale price p^R . Those who want to sell securities in this stage take as given an infinitely-elastic demand curve with p^R as its intercept. The same is true for the agents buying the good, but instead of inelastic demand they face a infinitely inelastic supply curve.

The gains from arriving at the secondary market with a quantity q' are given by $p^R q'$ for both types of agents. Consider first the case of a speculator. As argued in the previous section, if $p^R > 0$ she will sell all her securities in the secondary market. She receives $p^R q'$ for

the sale. The investor's case is also straightforward. Relying on the argument of the previous section, each investor obtains a final quantity $\frac{Q}{I}$ of securities, regardless the outcome of the auction. The quantity he wins in the auction will change only the amount of securities he should buy (or sell) in the secondary market. For instance, if he wins q' in the auction, his utility is $u\left(\frac{Q}{I}\right) - p^R\left(\frac{Q}{I} - q'\right)$. In the case where he obtains no shares in the auction, his utility is $u\left(\frac{Q}{I}\right) - p^R\frac{Q}{I}$. The difference between these two values is exactly $p^R q'$. Therefore, the value *per share* a security brings to an individual is the resale price p^R , no matter the functional form of his utility function.

The cost of acquiring a particular quantity q' in the auction is also going to be the same for both speculators and investors in equilibrium. The stop-out price depends on q' and the aggregate bid from his competitors. However, all bidders face the same aggregate bid function in a symmetric equilibrium. Therefore, the value of $b_{-i}(Q - q')q'$ will be the same for all agents, and all q' .

3.3.3.3. Additional participants: The addition of an investor affects the revenue from the auction in two ways. The first effect is simply due to an increase in competition for the Q securities: the new investor reduces the market power of existing participants in auction, which is the source of underpricing in this stage. This fact implies that the stop-out price will get closer to the true value of the security, which is given by the price on the secondary market, p^R in the competitive secondary market environment. Secondly, the price in the secondary market itself increases when we add a new investor. Indeed, from the previous section we know that the price on the resale stage is given by the marginal valuation of the investors evaluated at $\frac{Q}{I}$, i.e., $p^R = v - \rho\frac{Q}{I}$. It is apparent that p^R is strictly increasing with respect to I .¹¹

¹¹If the total number of investors participating in the secondary market is fixed, an additional investor participating in the auction will have the same impact on revenue as an additional speculator. The price in the secondary market depends only on the total number of investors who will trade in this second stage: the fraction of those who participates in the auction is irrelevant for the determination of p^R .

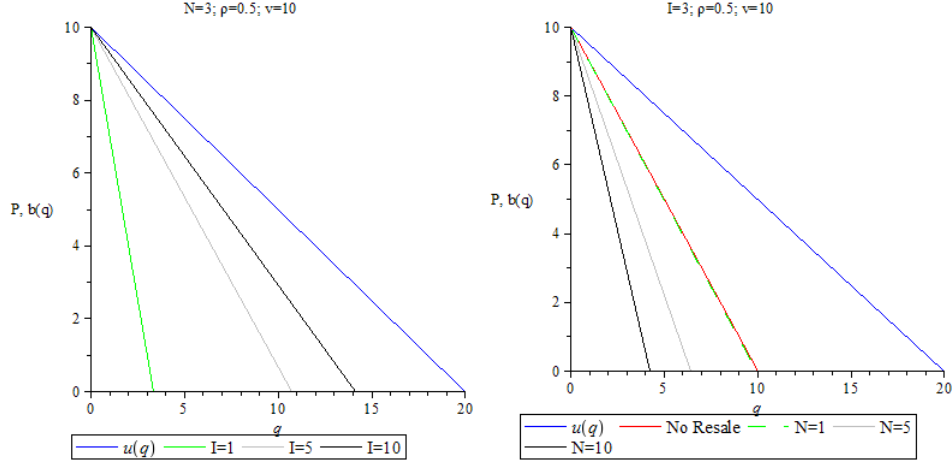


FIGURE 3.3.4. Bid schedules with a competitive secondary market environment

An additional speculator brings only the first effect. The price on the secondary market p^R is independent on the number of speculators participating in this market. However it affects the equilibrium's stop-out price, and consequently the revenue, by increasing the competition for securities in the auction.

Figure 3.3.4 illustrates how equilibrium bid schedules change with the number of investors and speculators. The specific type of an additional participant have opposite effects on the equilibrium bid schedules. Whereas an additional investor makes the participants bid more aggressively, an additional speculator increases the bid shading from the equilibrium bid schedules.

Revenue considerations: The gap between the price observed in the auction and the price in the secondary market is commonly known as *underpricing*. It gives a *per share* measure of how much the auctioneer is losing due to strategic behavior of bidders in the auction. For the market setup considered above, the underpricing is illustrated in the following corollary below:

COROLLARY 9. *The difference between the secondary price and the stop-out price in the unique linear equilibrium of the uniform-price auction followed by a competitive resale stage*

is given by:

$$U_{I,N}(Q) = \rho \frac{1}{I+N-2} \frac{Q}{I}.$$

The following corollary illustrates the revenue generated by the auction:

COROLLARY 10. *Suppose there are I investors and N speculators competing in an uniform price auction. If the secondary market is perfectly competitive, the revenue generated by auctioning a random quantity Q of the security is given by:*

$$R_{I,N}(Q) = (p^R - U_{I,N}(Q)) \times Q.$$

It is straightforward to see that the revenue generated by the auction is strictly increasing in both the number of speculators and the number of investors in the auction. However, the specific type of participant determines the extent of increase of the revenue. Note that the payoff a single unit of the security acquired in the auction brings to a participant is exactly the price by which he/she can buy/sell it on the secondary market, p^R . As argued before, an additional investor affects not only the degree of competition in the auction, but also the competition in the secondary market. The latter effect increases the equilibrium value of p^R , increasing the valuation of securities acquired in the auction for all participants. An additional speculator, in turn, affects only the competition for securities in the auction stage. The secondary price is completely independent of the number speculators in the economy. Therefore, the auctioneer's revenue is much more sensitive to the number of investors than to the number of speculators participating in the auction.

The above argument can be readily seen in the limiting cases. For instance, if we let $N \rightarrow \infty$, the stop-out price converges to the resale price in the secondary market, $v - \rho \frac{Q}{I}$. Meanwhile, if we let $I \rightarrow \infty$, the stop-out price (and the secondary market price) converges to v .

The fact that the equilibrium outcome is differently affected by the presence of investors and speculators has further implications. First, there is no straightforward relationship between the number of auction participants and the revenue generated in the auction: we have seen that the composition of the auction participants turns out to have an important role as well. For instance, consider the case where there are three participants in the auction, but all of them are investors ($I = 3, N = 0$). For a given realization of the total supply, the revenue generated in equilibrium is:

$$R_{3,0} = \left(v - \frac{2}{3}\rho Q \right) \times Q.$$

Now, consider the case where we have four participants, but two of them are investors and two are speculators ($I = N = 2$). For the same realization of total supply, this auction would generate:

$$R_{2,2} = \left(v - \frac{3}{4}\rho Q \right) \times Q < R_{3,0}$$

Even though the number of auction participants is 33% higher in the second scheme, the revenue generated will be lower since there are fewer investors.

If the secondary market is competitive, the addition of a resale stage increases the revenue for the auctioneer. Keeping the number of investors, I , constant, the presence of a secondary market increases the auctioneer's revenue by:

$$R_{I,N}(Q) - R_{I,0}(Q) = \rho \frac{Q^2}{I} \times \frac{N}{(I-2)(I+N-2)}.$$

This example illustrates that auctioneers should be specifically interested on attracting final investors. Although additional speculators increases the revenue of the auction, the magnitude is lower than if the same number of additional investors were attracted. If for some reason the presence of speculators decreases the number of investors willing to participate in the market, the auctioneer can be better off if resale is not allowed.

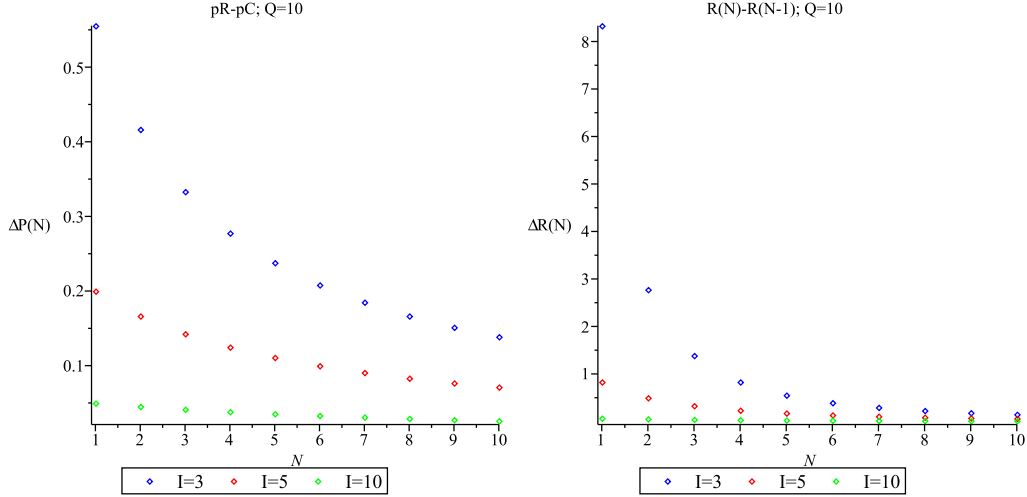


FIGURE 3.3.5. Equilibrium stop-out prices and revenues of a uniform-price auction with a competitive secondary market.

Arbitrage gains: The arbitrage gains a speculator gets in equilibrium are given by $(p^R - p^{so}) \frac{Q}{I+N}$. As the number of speculators increases, the arbitrage opportunities disappear. Her arbitrage gains can be rewritten as:

$$\rho \frac{Q^2}{I(I+N)} \frac{1}{I+N-2}.$$

Hence, they are strictly decreasing in I . The equilibrium difference between p^R and p^{so} shrinks as $I \rightarrow \infty$, and the quantity she wins in the auction, $\frac{Q}{I+N}$, also vanishes in this limit.

3.4. Non-linear equilibrium

Up to this point, we have focused attention on equilibria where bidders submit linear bid schedules in the first-stage auction stage. However, this is only one of many possible bid schedules constituting an equilibrium for the auction. In this section, I extend the analysis above and consider non-linear equilibria for the case where the resale market is competitive. The purpose of this section is twofold. First, it checks the robustness of the qualitative

results derived in the previous section. Second, it provides additional intuitions about the shape of the equilibrium bid schedules, which is consistent with empirical observations.

The setup of the auction and resale market is exactly the same in Section 3.3.3 above. We have the following proposition.

PROPOSITION 10. *Suppose there are I investors and N speculators, such that $I + N \geq 3$, and the secondary market is perfectly competitive. Then there exists a symmetric equilibrium in the auction stage where all investors and speculators submit the following inverse bid schedule:*

$$(3.4.1) \quad P(q) = v \left[1 - \left(\frac{q}{q_0} \right)^{I+N-1} \right] - q \left[1 - \left(\frac{q}{q_0} \right)^{I+N-2} \right] \frac{I+N}{I} \frac{I+N-1}{I+N-2} \rho.$$

where $q_0 \in \left[0, \frac{v}{\rho} \frac{I}{I+N} \right]$ is the quantity demanded at price $p^{so} = 0$.

A recurrent question, both in the theoretical and empirical literature, is how the outcome of an auction depends on the number of participants. The environment considered in this paper provides implications not only on how the outcome changes with the absolute number, but also with the composition of the participants. Indeed, as in the linear case discussed above, the sensitivity of the bid schedule with respect to an additional participant will depend on whether he is a speculator or a final investor.

The distinction between investors and speculators has empirical implications. For instance, [Keloharju et al. \[2005\]](#) document that, in the Finnish Treasury securities auctions, the skewness of the submitted bid schedules has a positive relationship with the total number of participants in the auction. Without a resale market, this result is hard to reconcile with the market power theory of bidding. The canonical models presented in [Kyle \[1989\]](#), [Back and Zender \[1993\]](#), and [Wang and Zender \[2002\]](#) predict a negative (non positive) relationship between skewness and number of bidders. Nevertheless, the equilibrium bid schedule described in Proposition 10 below can reconcile the market power theory with the empirical

observations from [Keloharju et al. \[2005\]](#). It will be true that as the number of investors increase, the skewness of the equilibrium bid schedules becomes more negative. However, this may not be the case as we increase the number of speculators.

Figure 3.4.1 illustrates the equilibrium bid schedule for different combinations of investors/speculators acquiring securities in the auction. Keeping q_0 constant, the left and right panels show what happens to the equilibrium bid schedule when the number of speculators and investors, respectively, participating in the market increases. One can readily see that, depending on the specific type of participant, we can get opposite implications in the skewness of the equilibrium bid schedules. As in the canonical [Wang and Zender \[2002\]](#) model without a resale market, the right panel illustrates that the skewness of equilibrium bid schedule gets more negative as the number of final investors increases.¹² However, the left panel illustrates that the opposite effect can arise if the additional participants are speculators instead. This implies that the observed pattern in the Finnish Treasury auctions documented by [Keloharju et al. \[2005\]](#) is consistent with the market power theory of divisible good auctions so long as the extra participants are perceived as speculators instead of final investors.

Another implication of Proposition 10 is that the set of possible bid schedules constituting a symmetric equilibrium depends on the number of speculators. The restriction that $P(\cdot)$ is weakly decreasing generates an upper bound on the value that the initial quantity $q_0 \equiv P(0)$ can take. Indeed, it is easy to show that q_0 must be no greater than $\frac{v}{\rho} \frac{I}{I+N}$ in order for the equilibrium bid schedule to be downward-sloping.¹³ As the number of speculators increases, this upper bound shrinks; the left panel of Figure 3.4.1 illustrate this concept. If the number of speculators gets sufficiently high, the expression for the equilibrium bid schedule given in the proposition above can be strictly increasing on some intervals, which is disallowed in

¹²When there are no speculators in the auction, i.e. when $N = 0$, the above expression for the equilibrium bid schedules is identical to the equilibrium described in Proposition 3.4 of [Wang and Zender \[2002\]](#).

¹³The linear equilibrium is a special case in which the initial condition is $q_0 = \frac{v}{\rho} \frac{I}{I+N} \frac{I+N-2}{I+N-1}$.

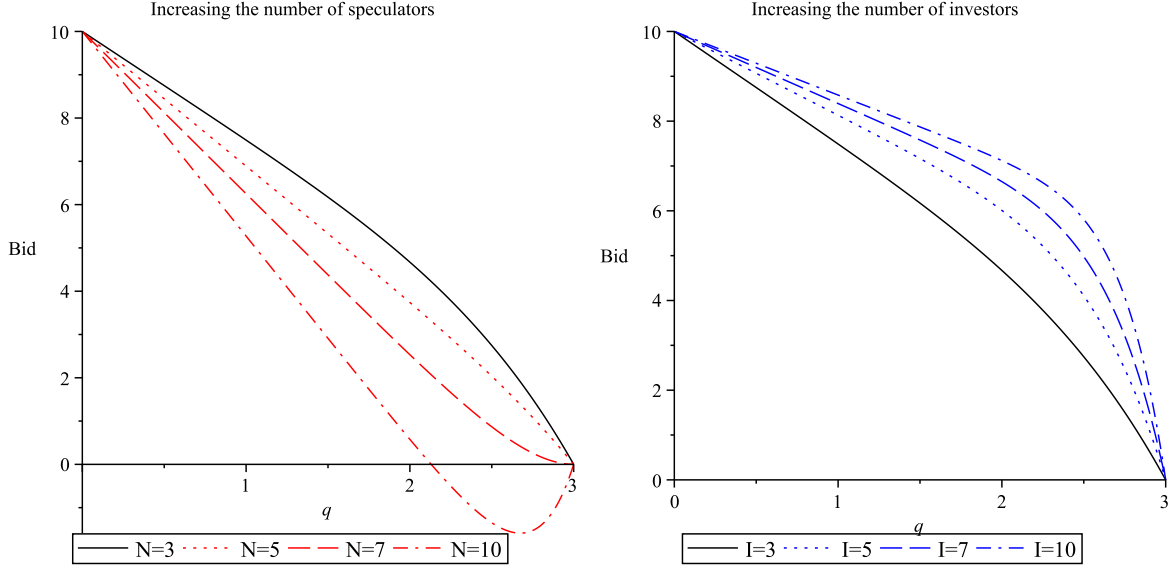


FIGURE 3.4.1. The left panel shows how the equilibrium bid schedules change when the number of speculators increase, while maintaining the number of investors fixed. Conversely, the right panel shows the bid schedules when the number of investors varies, while the number of speculators is maintained constant. The equilibrium bid schedule is differently affected depending whether the extra participants are either speculators or final investors. In both panels, the parameters are such that $v = 10$, $\rho = 1$, $q_0 = 3$.

equilibrium. Therefore, the addition of speculators to the auction reduces the set of initial conditions for which bid schedules can be sustained in equilibrium.

3.5. Conclusion

The existence of a secondary market has a direct impact on the behavior of participants in an auction. Indeed, the possibility of future resale gives speculators incentives to acquire goods in the auction just for the sake of reselling them in the secondary market. Even though this increases competition in the auction stage, the effects on equilibrium price and revenue are ambiguous. Indeed, adding speculators to the auction gives the original participants incentives to bid less aggressively since they will have another chance to buy any remaining demand at the secondary market.

I considered two environments for the secondary market. In the first one, I made the assumption that participants use Nash Bargaining to split all possible gains from trading the goods at this stage. I showed how the existence of a linear equilibrium in the auction depends on the specific bargaining power of the speculator. Moreover, I showed that depending on the realization of the total supply being auctioned, the existence of a secondary market could actually decrease the revenue generated in the auction. In the second environment, the resale market is assumed to be perfectly competitive. This assumption enabled me to develop comparative statics in a tractable framework for an arbitrary number of participants.

I showed how the equilibrium bid schedules are sensitive not only to the number of participants, but also to the composition of participants. Additional speculators have a much lower impact on the auction's price than additional investors. As a consequence, the auction's revenue is not necessarily monotone on the number of participants, as suggested by previous theoretical literature. Moreover, I show that the shape of the equilibrium bid schedules is also sensitive to the composition of the participants. For instance, while the addition of final investors to the auction decreases the skewness of the bid schedules, the opposite can occur when additional speculators enter the market. These results can explain the initially puzzling empirical observations from the Finnish Treasury auctions documented by [Keloharju et al. \[2005\]](#).

An implicit assumption in the model I have analyzed above is that the auctioneer behaves passively. He sells a quantity Q drawn from an exogenous distribution even though it may be possible to increase the revenue generated in the auction if he chooses a different value for Q . For instance, the government may be acting strategically when deciding how many treasury securities will be retained on their primary auction. One possible extension of my analysis is to consider the case where the auctioneer acts in a strategic way. After investors

and speculators submit their bid schedule, the auctioneer would choose the total amount of securities which maximizes his revenue.¹⁴

¹⁴[Back and Zender \[2001\]](#) consider a strategic auctioneer in a model where agents have constant marginal valuation for the good and resale does not take place.

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CHAPTER 4

Illiquidity Discounts with Endogenous Debt Financing (with Antonio Bernardo)

4.1. Introduction

Complex assets often trade in markets consisting of a small number of highly specialized investors. These investors often make use of substantial debt financing to take advantage of tax benefits (e.g., private equity) or to increase the return on limited pools of equity financing (e.g., hedge funds). However, debt financing increases the likelihood that the asset will have to be liquidated prematurely because of the inability to meet margin calls or restrictive debt covenants. Early liquidation of the asset may be costly, especially if the value of the asset to other potential owners is relatively low. Consequently, optimal debt financing and illiquidity discounts (the difference between the market price of the asset and its fundamental value) are jointly determined in equilibrium.

We develop a model in which a single asset is initially sold in a second-price auction among N potential bidders. Each of the bidders knows his own private valuation for the asset if held until maturity and the distribution of valuations for the other bidders. For example, in a private equity setting, valuations may differ among potential management groups bidding to acquire a firm. We first consider the possibility the winning bidder faces an exogenous shock to his income that may force him to liquidate the asset prior to maturity. If this occurs he sells the asset in a second-price auction, but this time among the $N - 1$ remaining bidders. We show that the optimal bidding strategy in the initial auction shades down the bid relative to the value of the asset if held until maturity by (i) the expected

loss due to early liquidation and (ii) the value of the outside option to participate in the second auction if the winning bidder is forced to liquidate early. Consequently, the illiquidity discount is large when there are fewer bidders with disperse valuations for the asset.

We then consider a setting in which the probability of early liquidation is endogenous, depending on the proportion of debt used to finance the purchase of the asset. We assume debt financing provides a tax benefit (tax deductibility of interest payments) relative to equity, but increases the likelihood of early liquidation. In this setting, the choice of debt financing impacts the asset valuation and may reverse the asset pricing implications of the model with exogenous early liquidation. For example, when the number of bidders is large the costs of early liquidation are small as in the model with exogenous liquidation; however, this encourages bidders to increase the proportion of debt financing which increases the likelihood of early liquidation. We show that the latter effect dominates the former so that, contrary to the model with exogenous liquidation, the illiquidity discount increases when there are more potential bidders for the asset.

We also find that the optimal proportion of debt financing is lower when the winning bidder's valuation is higher because a higher valuation results in a greater expected loss in the event of liquidation (and hence the cost of debt is greater). This finding is contrary to traditional signaling models predicting a positive relation between debt and valuations. We also show that optimal bids (and hence equilibrium prices) include a "bankrupt your opponent" strategy: by bidding more aggressively a losing bidder increases the cost to the winning bidder and improves the chances that there will be a second opportunity to purchase the asset because of early liquidation.

This paper is related to the work of ? and ? examining how illiquidity impacts optimal debt choice. ? argue that asset values will decline significantly if an industry is hit with an adverse shock because the potential buyers of the asset are also adversely impacted. They argue that this cost of financial distress should limit the optimal debt choice. We extend

this analysis to consider how the optimal debt choice impacts asset values. ? argue that an investor’s debt choice imposes externalities on other investors through their impact on equilibrium asset illiquidity. They focus on the fragility of equilibrium with such externalities and show that small changes in the environment can have a large impact on asset liquidity [?].

We also show here that early liquidation costs are an important determinant of optimal debt choice, but we go further to analyze how this impacts equilibrium expected prices. Thus, our work complements existing models of illiquidity discounts with exogenous debt (e.g., ?; ?) and constraints on raising equity.

The framework developed in this paper is also related to the literature considering non-cash auctions, where auction participants can use different instruments to finance their bids (e.g., ?; ?). Our main contribution is to disentangle how the trade-off between tax benefits and early liquidation affects the participant’s bidding behavior in second price auctions. Moreover, the dynamic aspect of our model provides new insights on the bidding behavior of participants when the occurrence of a second auction depends on the outcome from a first auction. Therefore, our work is also related to the sequential auctions literature [?].

4.2. Model

We consider a model with a single asset for sale and N risk-neutral investors who have the necessary expertise to consider purchasing it. For example, the asset could be a firm and the N investors represent other firms in the industry with the managerial and organizational capabilities to run it. The investors have (private) valuations for the asset, denoted by $\mathbf{M} = (M_1, \dots, M_N)$, where the valuations are independent and identically distributed according to an increasing distribution function $F : [0, \omega] \rightarrow [0, 1]$. It is assumed that F is continuously differentiable, has full support, and $E[M_i] < \infty$. Investor $i \in \mathcal{N} = \{1, \dots, N\}$ knows his own valuation $M_i = \mu_i$ and only the distribution of the other investors’ valuations.

The model consists of three periods. In period 0, the initial owner of the asset uses a second price auction to sell the asset to the N investors. The investor who submits the highest bid acquires the asset and pays the second highest bid for it. In period 1, the new asset owner may suffer an income shock that is sufficiently large to force him to liquidate the asset. We denote the probability of early liquidation by $(1 - q) < 1$. If forced to liquidate, the asset is once again auctioned but this time to the remaining $N - 1$ investors who did not win the auction in period 0. In period 2, the asset pays off μ_i to the asset holder, where i denotes the identity of the asset holder at the end of period 1.

We consider two specifications for the probability of early liquidation, $1 - q$. In our first specification, the probability of early liquidation is exogenous. In our second specification, the probability of early liquidation increases in the amount of debt raised to finance the purchase of the asset in period 0. This captures the fact that debt financing increases the likelihood that assets will have to be liquidated in the future because, for example, the asset owner is unable to satisfy margin calls or debt covenants. In the endogenous liquidation setting, investors trade-off the benefits and costs of debt financing. On one hand, debt provides a tax benefit due to the tax deductibility of interest payments, a feature of the tax code that encourages debt financing. Specifically, we assume that debt has a per-dollar effective tax benefit (relative to equity) denoted by τ . On the other hand, debt increases the likelihood of early liquidation at a price below the asset's fundamental value. Specifically, we assume that a higher debt/value ratio decreases the probability of keeping the asset until its maturity, denoted by $q(\cdot)$. For technical reasons, we assume that $q' < 0$, $q'' < 0$, $q(0) = 1$ and $q(1) = 0$.

We assume the lending market is competitive and that the market expected interest rate is given exogenously by $\bar{r}$. The promised interest rate may exceed the expected interest rate because the lender may not receive the promised payments in the event the asset is

liquidated early. The promised interest rate, denoted r^* , will therefore depend on the level of debt financing because of its impact on the probability of early liquidation.¹

Therefore, equilibrium in the lending market implies:

$$q \cdot (1 + r^*) D + (1 - q) \cdot E [\min \{P_1, (1 + r^*) D\}] = D (1 + \bar{r}).$$

The first term on the LHS represents the full promised payment to the lender if the asset is not liquidated early. The second term on the LHS represents the expected partial payment to the lender if the asset is liquidated early. The expected payment to the lender must yield the expected return $\bar{r}$.

We will focus on a pure strategies symmetric equilibrium. An equilibrium bidding strategy consists of a vector of strategies (β_0, β_1) for periods 0 and 1, respectively. We conjecture that both strategy functions are (strictly) increasing and differentiable. The period 0 bidding strategy, $\beta_0 : [0, \omega] \rightarrow R^+$, depends only on an investor's own valuation for the asset. The strategy function in period 1 may depend on the price of the asset in period 0. The conjecture that β_0 is invertible (increasing) implies that the value of the winning bid, b_0^w , in the first stage is publicly known and is given by $\beta_0^{-1}(b_0^w)$. Therefore, we can write the bidding function at period 1 as $\beta_1 : [0, \omega]^2 \rightarrow R^+$.

To recap, the timing of our model is as follows:

- Period 0:
 - The asset is sold to investors through a second price auction.
 - The sale price is given by $P_0 \equiv$ second highest bid.
 - In the model with endogenous early liquidation, the winner chooses the level of debt, $D \in [0, \mu_i]$.

¹We ignore the possibility of adverse selection in the lending market. One justification for this assumption is that the lender does due diligence to learn the value of the asset to the borrower prior to lending. Although the lender is assumed to know the value of the asset to the borrower, the lender does not have the expertise to generate value from the asset so the lender will choose to liquidate the asset in the event the borrower defaults.

- Period 1:
 - With prob. $(1 - q)$, the asset holder is forced to liquidate.
 - * The asset is sold to the remaining $N - 1$ bidders through a second price auction.
 - * The sale price is given by $P_1 \equiv$ second highest bid.
 - With prob. q the initial buyer keeps the asset until period 2.
- Period 2:
 - The asset yields the payoff μ_i to the asset holder $i \in \mathcal{N}$.

4.3. Exogenous probability of liquidation

We begin by considering the case where the probability of early liquidation is exogenously given, i.e., independent of the debt choice by the winning investor. Moreover, without loss of generality, we assume there is no per-dollar tax benefits. With these assumptions, investors will be indifferent between debt and equity financing.

Period 1: We begin by considering the equilibrium price for the asset at period 1. The price is determined by an auction in which the participants are the investors who did not acquire the asset in the previous period. Suppose, without loss of generality, that investor 1 acquired the asset at period 0. The participants of the auction at period 1 consists of all investors in $\mathcal{N}_{-1}$.

Since there is no forced liquidation after period 1, the investor who ends up holding the asset at the end of the period will hold it until it pays off in period 2. This implies that, from investor $i \in \mathcal{N}_{-1}$ perspective, the asset is worth exactly μ_i at this time. Therefore, one can use a standard argument to show that it is a weakly dominant strategy for an investor to submit his own valuation for the asset. Let $Y_k^{(N-1)}$ denote the k -th highest valuation among the $N - 1$ investors other than investor 1. We have the following lemma.

LEMMA 2. *The price of the asset at period 1 is given by:*

$$P_1 = Y_2^{(N-1)}.$$

Since all investors submit their own private valuations as a bid in equilibrium, the expected price at period 1 is simply the second highest valuation among the auction participants. Therefore, the expected payoff of investor i , conditional on winning the auction at this period, is given by:

$$E \left[\mu_i - Y_2^{(N-1)} \mid Y_2^{(N-1)} \leq \mu_i \right].$$

In equilibrium, investor i ends up acquiring the security if and only if he has the highest valuation among the $(N - 1)$ remaining investors, i.e. $\mu_i \geq Y_2^{(N-1)}$ and pays $P_1 = Y_2^{(N-1)}$ for it.

There are two possible cases in which investor 1 will deliver the asset to the period 1 auction winner. With probability $(1 - q)$, investor 1 suffers an income shock and does not have a choice but to liquidate his position. This will happen independently of the realized P_1 . However, he would be willing to do so even if he does not incur the income shock, as long as $P_1 > \mu_i$. Although this never happens on the equilibrium path of the dynamic game, we have to consider this possibility to fully characterize the equilibrium.

Period 0: An investor's optimal strategy at period 0 should take into account the possibility of early liquidation. Indeed, early liquidation affects how much the asset is worth to investors at this stage in two ways. First, the investor who acquires the asset in period 0 may end up being forced to sell it for a lower price in period 1. Second, early liquidation implies that investors might have a second chance to acquire the asset in period 1. Both effects reduce the willingness of an investor to pay a high price for the asset at period 0.

If investor 1 acquires the asset at period 0 and the highest valuation among the remaining investors is $Y_1^{(N-1)} = y$, his expected continuation value can be written as:

$$W(\mu_i, y) \equiv q \cdot E \left[\max \{ \mu_i, P_1 \} | Y_1^{(N-1)} = y \right] + (1 - q) \cdot E \left[P_1 | Y_1^{(N-1)} = y \right].$$

The first term on the RHS of the above expression is the payoff when he does not suffer the income shock, and thus is not forced to liquidate his position. However, he will prefer to do so whenever P_1 is greater than his own valuation for the asset, i.e., whenever $P_1 > \mu_i$. The second term on the RHS of the above expression is the payoff when he suffers the income shock, which occurs with probability $(1 - q)$ in which case he sells the asset at the price P_1 .

Now consider the continuation value for a losing bidder i after period 0. If the winning bidder has the highest valuation, $Y_1^{(N-1)} = y$, he will not sell the asset if he does not suffer the income shock. However, with probability $(1 - q)$, the winning investor will be forced to liquidate his position. In this case, we know by the analysis of Period 1, that an investor i will get the asset if and only if he has the highest valuation among the remaining investors. In other words, if $\mu_i \geq Y_2^{(N-1)}$. Therefore, investor i 's expected payoff before entering period 1 can be written as:

$$L(\mu_i, y) \equiv (1 - q) \cdot E \left[\mu_i - Y_2^{(N-1)} | Y_2^{(N-1)} \leq \min \{ \mu_i, y \} \right] \cdot Pr \left(Y_2^{(N-1)} \leq \mu_i | Y_2^{(N-1)} \leq y \right).$$

We have the following proposition:

PROPOSITION 11. *Given that investors are submitting their true valuation in period 1 and q be sufficiently high, the unique symmetric pure strategy equilibrium in period 0 is given by:*

$$\beta_0^{II} = v(\mu_i, \mu_i)$$

where $v(\mu_i, y) \equiv W(\mu_i, y) - L(\mu_i, y)$ is the net gain from acquiring a security in period 0 given i 's valuation and that $Y_1^{(N-1)} = y$.

This is a standard result in auction theory [?]. The intuition for why the bid in the above constitutes an equilibrium is the following. Since v is increasing in its first argument,

we have $v(\mu_i, y) - v(y, y) > 0$ for all $y < \mu_i$, and $v(\mu_i, y) - v(y, y) < 0$ for all $y > \mu_i$. Therefore, investor i maximizes his expected payoff by submitting $\beta_1^{II}(\mu_i)$. The lower bound assumption on the probability of keeping the asset until period 2 is made so as to guarantee that $v(\mu, \mu)$ is strictly increasing in μ , consistent with our initial conjecture.

Looking closer at the equilibrium bid we see that the investor shades his bid in two ways relative to his valuation:

$$\beta_0(\mu_i) = \mu_i - \underbrace{(1-q) E \left[\mu_i - Y_2^{(N-1)} \middle| Y_2^{(N-1)} \leq \min \{ \mu_i, y \} \right]}_{\text{Loss due to early liquidation}} - \underbrace{(1-q) E \left[\mu_i - Y_2^{(N-1)} \middle| Y_2^{(N-1)} \leq \min \{ \mu_i, y \} \right]}_{\text{Outside option}}.$$

In standard second price auctions, bidders submit their own valuation for the asset. This is the case here except the investor's valuation at period 0 must account for (i) the possibility he will be forced to liquidate the asset at a price below his own valuation in period 1, and (ii) the lost option value from not waiting to have a chance to buy the asset at a lower price in period 1.

Equilibrium Prices: The possibility of early liquidation has a negative impact on the

COROLLARY 11. *The price of the asset in each period is given by:*

$$\begin{aligned} E[P_1] &= E[Z_2^{(N)}] - E[\pi(Z_2^{(N)})] \\ E[P_2] &= E[Z_3^{(N)}] \end{aligned}$$

where $Z_k^{(N)}$ is the k -th highest valuation among all N investors and $\pi(z) \equiv 2 \cdot (1-q) \cdot \int_0^z \frac{F(y)^{N-2} dy}{F(z)^{N-2}}$ is the illiquidity discount.

For what follows, we define the illiquidity discount as the difference between the price in period 0 with and without the possibility of early liquidation. It is clear that the discount on

the price is strictly increasing in $1 - q$. As argued above, the possibility of early liquidation negatively impacts the investor's valuation of the asset. In the extreme case when there is no possibility of early liquidation ($q = 1$), the illiquidity discount is zero. The discount also depends on the magnitude of the loss an investor suffers when he is forced to liquidate. For example, this will be large when there are fewer potential bidders (N is small) because the expected value of the asset to the remaining bidders is lower. Therefore, if one interprets the number of bidders as a proxy for market depth then equilibrium asset prices are lower when market depth is lower. Furthermore, the option to wait to acquire the asset in period 1 is also greater when the asset price is expected to decline by a greater amount, causing investors to shade their bids down even more in period 0.

The comparative statics are clearly demonstrated in the case where valuations are uniformly distributed, i.e., $\mu_i \sim U[0, \omega]$. In this case, the expected illiquidity discount takes the form:

$$E[\pi(z)] = 2(1 - q) \frac{\omega}{N + 1}$$

The illiquidity discount depends of three factors. First, the discount increases in the probability of early liquidation. Second, it increases in the dispersion of investors' private valuation, which is proportional to the range of the support of the distribution of private valuations, ω . Third, it decreases in the number of potential investors in the market, given by N . Indeed, the net loss for the asset holder when he is forced to liquidate is given by the difference between his own valuation for the asset and the price he gets from selling it, $P_1 = Z_3^{(N)}$. In equilibrium, the asset holder has the highest valuation for the asset among his peers. Therefore, the net loss takes the form of $Z_1^{(N)} - Z_3^{(N)}$. It is easy to see that this difference is positively related to ω and negative related to N .

The equilibrium yields several empirical implications. For example, the range of valuations ω is likely to be large for assets that are hard to value (e.g., young technology firms) and

for assets that require complementary assets (e.g., business software whose value depends on complementary client relationships). In such cases, we should expect large illiquidity discounts. Conversely, if the asset is highly tangible and can be financed by secured debt (e.g., hotels) we should not expect a high degree of dispersion in valuations and, therefore, the asset prices should not contain a large illiquidity discount. Furthermore, the illiquidity discount is smaller when the number of potential buyers (market depth) is larger. Thus, for example, assets in less concentrated industries should have lower illiquidity discounts.

Our model also yields implications for the change in the asset price between the two periods. For the uniform distribution case, the difference in the price of the asset in periods 0 and 1 is given by:

$$E[P_0] - E[P_1] = \underbrace{\frac{\omega}{N+1}}_{\text{Difference in Valuations}} - \underbrace{2(1-q)\frac{\omega}{N+1}}_{\text{Illiquidity discount}}.$$

Clearly, if there is no possibility of early liquidation, then the highest valued investor owns the asset in both periods and the price difference between the two periods is zero. However, in the event of early liquidation there are two forces moving the price between periods 0 and 1. On one hand, the price declines because the asset is exchanged from a higher-to-lower valued owner. On the other hand, the initial price is lower in expectation of the possibility of early liquidation. If the ex ante probability of early liquidation is large the price declines between the two periods, on average; however, if this probability is small the price increases between the two periods, on average.²

4.4. Optimal capital structure and endogenous early liquidation

In this section, we allow investors to optimally choose the capital structure (the proportion of debt and equity) to finance the purchase of the asset. We assume debt has a per-dollar

²This is an artifact of our modeling assumption that the second buyer does not face the probability of early liquidation; otherwise, the price would always decline on average in the event of early liquidation.

effective tax benefit over equity, denoted by $\tau\bar{r}$. However, a higher debt/value ratio decreases the probability of holding the asset until its maturity, denoted by $q(\cdot)$. Moreover, we assume that $q' < 0$, $q'' < 0$, $q(0) = 1$ and $q(1) = 0$.

For simplicity, we only consider the optimal capital structure decision for the winning bidder in period 0.³ With this assumption, the bidding strategies in period 1 are exactly the same as in the exogenous liquidation model, therefore, 2 can be readily applied here.

As in the previous section, we will focus on a strictly increasing equilibrium $\beta_0 : [0, \omega] \rightarrow \mathbb{R}_+$. Suppose that all investors $i \in \mathcal{N}_{-1}$ are following such a strategy. If investor 1 acquires the asset at period 0, and the highest valuation among the remaining investors is $Y_1^{(N-1)} = y$, his expected continuation value can be written as:

$$W(\mu_1, y) \equiv q\left(\frac{D(\mu_1, y)}{\mu_1}\right) \cdot E\left[\max\{\mu_1, P_1\} | Y_1^{(N-1)} = y\right] + \left(1 - q\left(\frac{D(\mu_1, y)}{\mu_1}\right)\right) \cdot E\left[P_1 | Y_1^{(N-1)} = y\right] + \tau\bar{r}D(\mu_1, y).$$

The continuation value is similar to what we derived in the previous section except now investor i chooses a debt level that (i) can affect the probability of early liquidation $1 - q$ and (ii) brings extra value through tax benefits. Let $D(\mu_i, y)$ denote the optimal debt level an investor uses to finance the asset purchase in the case his type is μ_i and the price paid in the period 0 auction is $P_0 = \beta_0(y)$.⁴ Thus, $D(\mu_i, y)$ satisfies:

$$D(\mu_i, y) = \arg \max_{D'} q\left(\frac{D'}{\mu_i}\right) \cdot E\left[\max\{\mu_i, P_1\} | Y_1^{(N-1)} = y\right] + \left(1 - q\left(\frac{D'}{\mu_i}\right)\right) \cdot E\left[P_1 | Y_1^{(N-1)} = y\right] + \tau\bar{r} \cdot D'.$$

We will focus on the interesting case where the optimal debt value is interior. For these cases, we have that:

$$(4.4.1) \quad -q'\left(\frac{D(\mu_i, y)}{\mu_i}\right) = \frac{\tau\bar{r}\mu_i}{E\left[\max\{\mu_i, P_1\} - P_1 | Y_1^{(N-1)} = y\right]}.$$

³We ignore the capital structure decision in period 1 because there is no longer the potential for early liquidation and, therefore, there is no interesting tradeoff between debt and equity. This greatly simplifies our analysis without compromising the main qualitative results.

⁴Recall, the assumption of strictly increasing equilibrium strategies implies that investor i can infer $Y_1^{(N-1)} = y$ from observing $\beta_0(y)$.

The optimal level of debt depends on the magnitude of $P_0 = \beta(y)$, and the per dollar tax benefit. The above FOC implies that:

LEMMA 3. *If $1 > \mu_i \geq y > 0$, and $\tau\bar{r}$ is low enough then,*

$$\begin{aligned}\frac{d}{d\mu} \left[\frac{D(\mu, y)}{\mu} \right] &< 0 \\ \frac{d}{dy} \left[\frac{D(\mu, y)}{\mu} \right] &> 0 \\ \frac{d}{dN} \left[\frac{D(\mu, y)}{\mu} \right], \frac{d}{d\tau\bar{r}} \left[\frac{D(\mu, y)}{\mu} \right] &> 0\end{aligned}$$

As is clear from 4.4.1, the optimal debt is increasing in the per dollar tax benefits, $\tau\bar{r}$, and decreasing in the expected early liquidation loss $E[\max\{\mu_i, P_1\} - P_1 | Y_1^{(N-1)} = y]$. This is intuitive: an investor would tend to increase debt financing if the tax benefits are higher, and reduce it if the losses associated with early liquidation are higher. Interestingly, optimal debt financing decreases when the winning bidder's valuation is higher. The reason is that the expected loss due to early liquidation is higher when the winning bidder's valuation is higher. This result counters the prediction that debt financing signals high valuations as is standard in the corporate finance literature. The expected early liquidation loss also depends on the valuation of the highest of the remaining bidders' valuations. Given the equilibrium in period 1, we know that the price this investor will get for the asset in case of a liquidation is bounded above by y . Therefore, the investor expects lower prices for lower values of y . The number of investors in the market N also has a positive effect on the optimal debt level. As discussed in the previous section, more investors means more competition for the asset in period 1, which increases the expected price P_1 .

Now, consider the case where investor 1 submits $b < \beta(y)$ at period 0. In this case, he does not win the auction at period 0, but still has a chance of acquiring the asset at period 1. This will happen if (i) the asset holder is forced to liquidate his position; and (ii) investor 1 has the highest valuation among the remaining investors. His expected continuation value

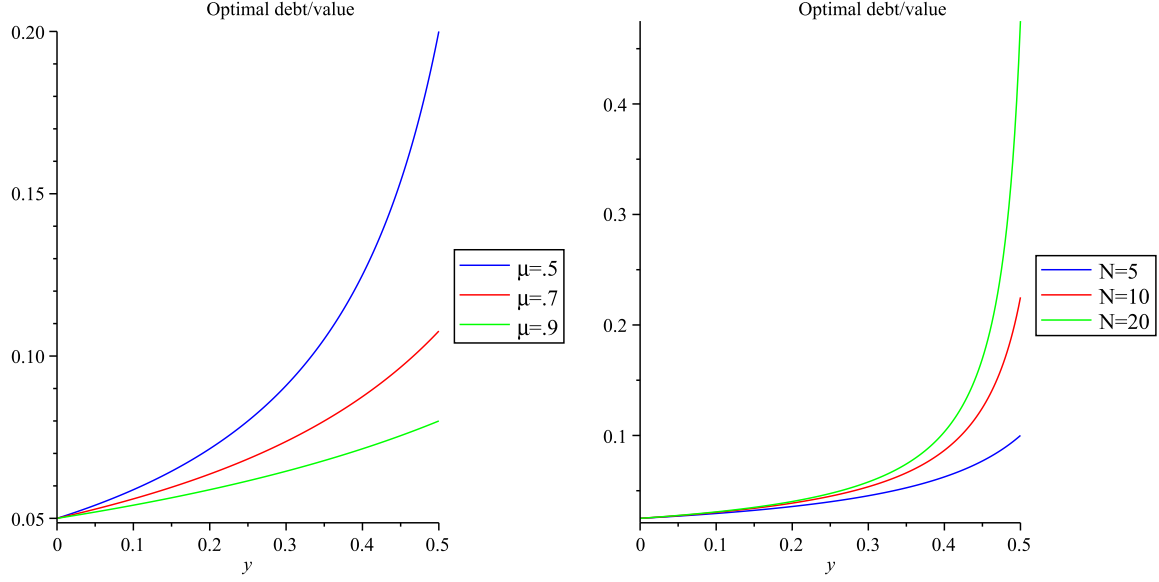


FIGURE 4.4.1. Optimal $\frac{D}{\mu_i}$ as a function of $Y_1^{(N-1)} = y$. The left panel shows how the optimal debt/value changes with investor own valuation μ_i . The right panel shows how the optimal debt/value changes with the number of investors N .

given that the asset holder valuation is $Y_1^{(N-1)} = y$ can be written as:

$$(4.4.2) \quad L(\mu_i, y, b) \equiv E \left[\left(1 - q \left(\frac{D(y, \psi)}{y} \right) \right) (\mu_i - P_1) \middle| P_1 \leq \min \{ \mu_i, y \} \right] \cdot Pr \left(P_1 \leq \mu_i \middle| Y_1^{(N-1)} = y \right).$$

where $\psi \equiv \max \left\{ \beta^{-1}(b), Y_2^{(N-1)} \right\}$.

Conditional on losing in period 0, investors would like to submit the highest bid possible because, even though the investor does not acquire the asset, he would like the winning bidder to pay a higher price, take on more debt, and increase the probability of early liquidation. By increasing the bid, an investor increases the likelihood of another auction at period 1. To see this, note that the continuation value is (weakly) increasing in the bid submitted by investor i . Indeed, Lemma 3 implies that the debt level the asset holder takes, $\frac{D(y, \psi)}{y}$, is increasing in ψ , implying that the probability of early liquidation will be increasing in ψ as

well. Therefore, whenever b is high enough so as $\psi = b$, the continuation value for investor i is strictly increasing in b .

The economic environment at period 0 resembles one of a costly signaling game. The strictly increasing strategy conjecture implies that when i submits $\beta(\mu_i)$, he will be revealing μ_i to the winner of the auction at period 0. In other words, his bid would serve as a signal of i 's own type to the asset holder. By the discussion in the above paragraph, one might wonder why i does not submit a bid as high as possible in this case. What prevents him from doing so is that a higher bid increases the probability he actually ends up acquiring the asset for a price above his own valuation.

Suppose all investors in $j \in \mathcal{N}_{-i}$ are following a strictly increasing strategy $\beta(\cdot)$. At period zero, investor i faces the following maximization problem:

$$(4.4.3) \quad \max_{\mu'} \int_0^{\mu'} (W(\mu_i, y) - \beta(y)) dF_1(y) + \int_{\mu'}^1 L(\mu_i, y, \beta(\mu')) dF_1(y).$$

The FOC from the above problem, evaluated at the equilibrium bid is given by the following:

$$\beta(\mu_i) = [W(\mu_i, \mu_i) - L(\mu_i, \mu_i, \beta(\mu_i))] + \int_{\mu_i}^1 \frac{d}{d\mu} L(\mu_i, y, \beta(\mu_i)) \frac{dF_1(y)}{dF_1(\mu_i)}.$$

The first represents the net value for i from acquiring the asset at period 0 when $Y_1^{(N-1)} = \mu_i$. The second term represents how investor i 's bid will influence his payoff in case he loses the auction. As discussed earlier, even in the case he loses the auction at period 0, investor i can affect his payoff by increasing his bid because it impacts the price paid and debt choice by the winning bidder which, in turn, impacts the likelihood he will have a second chance to purchase the asset.

Let the second term in the FOC be denoted as "Bankrupt your opponent" component. We have the following lemma:

LEMMA 4. *Suppose that the optimal debt is always interior. A symmetric equilibrium at $T = 1$ satisfies:*

$$\beta_1^{II}(\mu_i) = v(\mu_i, \mu_i) + \mathcal{B}(\mu_i)$$

where

- $v(\mu_i, \mu_i) \equiv W(\mu_i, \mu_i) - L(\mu_i, \mu_i, \beta(\mu_i))$ - net gain from acquiring the security.
- $\mathcal{B}(0) = \mathcal{B}(\omega) = 0, \mathcal{B}(\cdot) \geq 0$ - "Bankrupt your opponent" component.

PROOF. In the appendix. The proof is incomplete. For the complete proof for a special case, see Proposition 12. □

The "bankrupt your opponent" effect present in the equilibrium above is similar in spirit to ? who considers an environment where cash constrained bidders compete for a good with uncertain value in a second price auction.⁵ After the true value of the good is observed, the auction winner can decide whether to keep the good or to default. He will keep it as long as the realized value is larger than the price he would have to pay (the second highest bid). In case he defaults, the good returns to the auctioneer who then uses another second price auction to sell it to the remaining participants.

As in our framework, losing bidders internalize that the chances there will be a second opportunity to purchase the asset improves as they submit a higher bid in the first auction. However, the mechanism in which this happens is different from ours. In ?, a larger bid submitted by a losing bidder (weakly) increases the price the auction winner will have to pay if he chooses not to default. Therefore, the larger bid reduces the set of realizations of the good's value in which the winner keeps the asset for himself, increasing the probability of the occurrence of a second auction. In our framework, however, this relationship occurs through the signaling. By submitting a larger bid, a losing bidder sends a signal to the

⁵He also analyzes first price auctions in this paper.

winner that he has a high valuation. The winning bidder interprets this signal to mean he will suffer a smaller loss in case he is forced to liquidate his position, inducing him to take on more debt, and increasing the probability a second auction will occur.

Uniform distribution: Equilibrium prices can be determined in the model with endogenous debt for the special case where valuations are uniformly distributed.

PROPOSITION 12. *In addition to the assumptions of Lemma 4, suppose that investors' valuation are i.i.d. according to the uniform distribution $\mu_i \sim U[0, \omega]$ for all $i \in \mathcal{N}$. Moreover, suppose that the early liquidation probability satisfies $q(x) = 1 - x^2$, $x \in [0, 1]$. We have that:*

$$\begin{aligned}
E[P_0] - E[P_1] = & \underbrace{E[Z_2^{(N)}] - E[Z_3^{(N)}]}_{\text{Difference in valuations}} - \underbrace{E[Z_2^{(N)}] \left(\frac{\tau\bar{r}}{2}\right)^2 (N-1)}_{\text{Loss due to early liquidation}} - \underbrace{E[Z_2^{(N)}] \left(\frac{\tau\bar{r}}{2}\right)^2 (N-1)}_{\text{Outside option}} \\
& + \underbrace{\tau\bar{r} \left(\frac{(N-1)Z_2^{(N)}}{2}\right)}_{\text{Debt is cheaper}} + \underbrace{E[\mathcal{B}(Z_2^{(N)})]}_{\text{Bankrupt your opponent}}. \\
E[P_1] = & E[Z_3^{(N)}]
\end{aligned}$$

The first term is simply the expected difference between the second highest and the third highest valuations, i.e., $E[Z_2^{(N)}] - E[Z_3^{(N)}]$. This would be the equilibrium difference in price without the early liquidation and tax benefits of debt. The second term is the illiquidity discount. As in the previous section, the discount will consist of two parts: the first comes from the fact that early liquidation gives investors the incentive to be less aggressive at period 0 and the second comes from the fact that losing the auction at period 0 is not that bad since you may end up acquiring the asset at period 1. The third term is the tax benefit of debt evaluated at the expected amount of debt taken on by the winning bidder. Finally, the last term represents the increase in the bid to increase the probability the winner at period 0 will be forced to liquidate his position.

As in the previous section, we define the illiquidity discount as the sum of loss due to early liquidation discount and the outside option components. In the specific environment considered in Proposition 12, the discount takes the following form (in expectation).

$$E[\pi(z)] = 2 \left(\frac{\tau \bar{r}}{2} \right)^2 \frac{(N-1)^2}{N+1}$$

The illiquidity discount is increasing in the tax benefit. An increase in the per dollar tax benefits will induce investors to increase the share of the cost being financed through debt, as is clear from Lemma 3 above. *Ceteris paribus*, any increase in the proportion of debt financing increases the probability of early liquidation q . And, we have seen in the previous section, the illiquidity discount increases with the probability of early liquidation, q .

Interestingly, the illiquidity discount and the number of participants are positive related in equilibrium. In a first moment, the fact that a larger number of participants is associated with a lower loss for the asset holder conditionally on him being forced to liquidate his position prematurely gives the impression that N and the early liquidation component should be positively related. This was indeed the case in Section 4.3, where the probability of early liquidation was treated as exogenously given. However, as pointed out by Lemma 3, the optimal debt level from the winner of the auction in period 0 is strictly increasing in the number of participants. A higher debt level, by its turn, increases the probability he will be forced to liquidate his position in period 1. This effect will be dominant in the environment considered in Proposition 12, implying on a positive relationship between early liquidation component and the number of participants in the market. Conversely, the outside option component from the illiquidity discount is also increasing in the number of participants. The idea is similar to the above: as N increases, the optimal debt level from the winner increases, improving the chance that a second auction will take place.

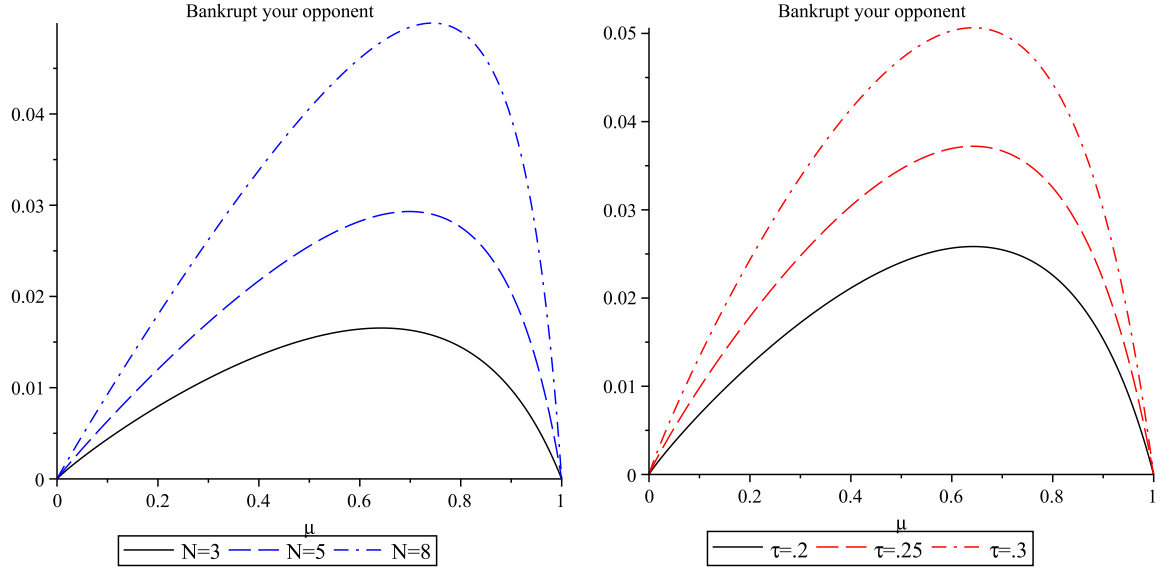


FIGURE 4.4.2. Both panels illustrate the "bankrupt your opponent" component for the case where the private valuations are uniformly distributed. The left panel illustrate how it changes as the number of investors in the market increases. The right hand panel illustrate its relationship with the per-dollar tax benefits, $\tau\bar{r}$.

Figure 4.4.2 illustrates the "bankrupt your opponent" component for the uniform distribution case. First, notice that the incentive to increase the bid to bankrupt your opponent increases then decreases in the investor's valuation. The reason is that the cost of raising the bid is relatively small at low valuations (winning the auction and overpaying for the asset). Furthermore, the incentive to increase the bid is increasing in the number of potential bidders, N , and the tax benefits of debt, $\tau\bar{r}$. The reason is that both of these factors increase the winning bidder's incentive to take on debt, implying that the marginal benefit of increasing your bid to bankrupt your opponent increases without a commensurate increase in the marginal cost.

4.5. Conclusion

We show that the optimal financing of an asset and its illiquidity discount are determined jointly and depend on the tax benefits of debt, the value of the asset to the winning bidder, the number of potential bidders, and the ex ante dispersion of valuations among the bidders. Our analysis is conducted in the context of an auction for a single asset; however, illiquidity discounts often depend on how many similar assets are being sold at the same time and the financial constraints of potential buyers. Therefore, a fruitful area for future research would consider multiple asset auctions and the impact of interdependent financing decisions on equilibrium asset prices.

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APPENDIX A

Chapter 2

A.1. Proofs

Proposition 1. Given history h satisfying assumption 2, dealer i maximization problem at the auction stage can be written as:

$$\max_{q_i(\cdot)} E \left[u_i \left(q_i(p) + \theta_i^{wi} \right) - p q_i(p) \right].$$

Point wise maximization leads to the result of the proposition. To see this, assume that the realization $\tilde{p}^W$ is known prior to i choosing the quantity to acquire. In this case, his maximization problem is:

$$\max_{q_i} u_i \left(q_i + \theta_i^{wi} \right) - \tilde{p}^W q_i.$$

The solution of this problem is given by $q_i^A(\tilde{p}^W)$, described on 2.5.1. Therefore, dealer i can acquire the ex post optimal quantity for all realizations by submitting the bid schedule $q_i^A(\cdot)$.

A.1.1. Proposition 2.

PROOF. Let's consider the problem for dealer i . Since we are searching for a linear equilibrium, it is natural to start assuming that the other dealers are submitting linear bid schedules:

$$q_j(p) = \alpha_j - \beta_j p.$$

for all $j \neq i$.

Let $b_{-i}(q)$ be the aggregate inverse bid schedule without considering dealer i , i.e., $\sum_{j \neq i} q_j(b_{-i}(q)) \equiv q$. If $j \neq i$ submit bid schedules as above, $b_{-i}(q)$ is uniquely determined for all $q \in \mathfrak{R}$ and has the following form:

$$(A.1.1) \quad b_{-i}(q) = \frac{1}{\sum_{j \neq i} \beta_j} \left[\sum_{j \neq i} \alpha_j - q \right].$$

Market clearing at the auction stage implies that $\sum_{j \neq i} q_j(\tilde{p}^{so}) = Q - q_i(\tilde{p}^{so})$, where $\tilde{p}^{so}$ is the realized stop-out price. Thus, we can write the maximization problem of dealer 1 as:

$$\max_{q_i^A(\cdot)} E \left[u_i(q_i(p, h) + \theta_i^{wi}) - b_{-i}(Q - q_i(p, h)) q_i^A(p, h) \right] - \sum_{t=1}^{T-1} p^t (\theta_i^t - \theta_i^{t+1}).$$

Since θ_i^t and p^t are already given by the time the auction takes place for $\tau = 1, \dots, T$, we can ignore the last summation term of the objective function. As in the no market power benchmark case, I will use point wise maximization to characterize the solution from the above problem. For any realization $\tilde{Q}$, dealer i chooses q_i^A that satisfies:

$$v_i - \rho(q_i^A + \theta_i^{wi}) + b'_{-i}(\tilde{Q} - q_i^A) q_i^A - b_{-i}(\tilde{Q} - q_i^A) = 0.$$

Solving for q_i^A and using the fact that $\tilde{p}^{so} = b_{-i}(\tilde{Q} - q_i^A)$ and that $b'_{-i}(\tilde{Q} - q_i^A) = -\frac{1}{\sum_{j \neq i} \beta_j}$ we arrive on:

$$q_i^A = \frac{1}{\rho + \frac{1}{\sum_{j \neq i} \beta_j}} (v_i - \rho \theta_i^{wi} - \tilde{p}^{so}).$$

Note that q_i^A is the optimal quantity for a *given* realization of $\tilde{p}^{so}$. This price is not known before the auction takes place which, at first glance, would prevent agents to condition their

bids directly on $\tilde{p}^{so}$. However, they can do it indirectly by submitting the bid schedule:

$$(A.1.2) \quad q_i^A(P, h) = \frac{1}{\rho + \frac{1}{\sum_{j \neq i} \beta_j}} (v_i - \rho \theta_i^{wi} - P).$$

In a linear equilibrium we must have all dealers submitting bids in the above form. This implies that the β s should satisfy

$$\beta_i = \frac{1}{\rho + \frac{1}{\sum_{j \neq i} \beta_j}}.$$

for $i = 1, \dots, N$. The unique solution of the above set of equations is symmetric and gives us $\beta_i = \frac{\gamma}{\rho}$ for all i . Substituting the equilibrium value for β_i on A.1.2 implies that the intercepts of the bid schedules in the linear equilibrium are given by:

$$\alpha_i = \frac{\gamma}{\rho} (v_i - \rho \theta_i^{wi}).$$

Let $\tilde{\psi}_i^A(\theta_i^{wi})$ be the demand from dealer i given a realization of $\tilde{p}^{so}$ as a function of his when-issued position θ_i^{wi} . Condition 3 implies that $\tilde{\psi}_i^A(\theta_i^{wi}) \geq 0$ for all i . $\square$

To find the expression for the stop-out price, sum up equation (2.6.3) across all dealers. The aggregate bid schedule is given by:

$$\sum_{\mathcal{I}} q_i^A(\tilde{p}^{so}, h) = \frac{\gamma}{\rho} (N\bar{v} - N\tilde{p}) - \gamma \sum_{\mathcal{I}} \theta_i^{wi}.$$

Since the when-issued market is a zero supply market in all periods, we must have that $\sum_{\mathcal{I}} \theta_i^{wi} = 0$. Thus, the last term of the right hand side of the above equation can be ignored. Using the market clearing condition of the auction stage, $\sum_{\mathcal{I}} q_i^A(\tilde{p}^{so}, h) = \tilde{Q}$, solving it for $\tilde{p}^{so}$ and using the definition of $\tilde{p}^W$, gives us 2.6.4.

A.1.2. Proposition 3.

PROOF. The first order condition from the maximization problem given by 2.6.5 implies that:

$$(A.1.3) \quad -p^{wi} + E \left[u'_i \left(\psi_i^F \left(\theta_i^{wi} \right) \right) \left(\frac{d\psi_i^F \left(\theta_i^{wi} \right)}{d\theta_i^{wi}} \right) - p^{so} \frac{d\psi_i^A \left(\theta_i^{wi} \right)}{d\theta_i^{wi}} \right] = 0.$$

Using Corollary (2), summing up (A.1.3) across $i = 1, \dots, I$ and dividing the result by I , gives us:

$$p^{wi} = (1 - \gamma) E \left[\bar{v} - \rho \frac{\sum_{\mathcal{I}} \psi_j^F \left(\theta_j^{wi} \right)}{N} \right] + \gamma E [p^{so}].$$

Market clearing, in the auction and when-issued stages, implies on $\sum_{j=1}^I \tilde{\psi}_j^F \left(\theta_j^{wi} \right) = \tilde{Q}$ for all realizations of Q . Equation (2.6.6) follows directly from the definition of p^W , given by (2.5.2). If we substitute p^{wi} of equilibrium in the first order condition, we get:

$$(1 - \gamma) E \left[u'_i \left(\psi_i^F \left(\theta_i^{wi} \right) \right) - p^W \right] = 0.$$

for $i = 1, \dots, N$. Substituting (2.6.3) and (2.5.2) on the above expression and solving it for θ_i^{wi} , gives us the equilibrium positions described by θ_i^c on equation (2.6.7). $\square$

A.1.3. Proposition 5.

PROOF. I will use an inductive argument in order to prove the Proposition. First, I will find the linear equilibrium for $t = 1$. Then, assuming that Corollaries 6 and 5 hold for all rounds $t < \tau$, I show that there is a linear equilibrium at τ where dealers submit bid schedules as 2.6.10.

$t = 1$:

Suppose dealers arrive at $t = 1$ holding positions $\{\theta_i^2\}_{i=1}^N$ from the previous rounds of the when-issued market. As in the auction stage, I will focus on linear equilibrium. Therefore, I start assuming that all dealers $j \neq i$ are submitting bid schedules at this round in the

following form:

$$q_j^{wi,1}(p, h^1) = \alpha_j^1 - \beta_j^1 p.$$

Let $b_{-i}^1(\cdot)$ be the aggregate inverse demand without dealer i at this round. Market clearing conditions imply that, at any round t of the when-issued market, $\sum_{j \neq i} q_j^t(p^t, h^t) = -q_i^t(p^t, h^t)$. If dealers $j \neq i$ are submitting the bid schedules above, the price at round 1 will be uniquely determined by the quantity dealer i decides to acquire, i.e., $p^1 = b_{-i}^1(-q_i^1)$.

Therefore, we can write dealer i 's problem as:

$$\max_{\theta_i^1(\cdot)} E \left[u_i \left(\psi_i^F(\theta_i^1) \right) - p^{so} \psi_i^A(\theta_i^1) \right] - b_{-i}(\theta_i^2 - \theta_i^1) (\theta_i^1 - \theta_i^2).$$

Note that, for all realizations of $\tilde{Q}$, $\tilde{p}^{so}$ does not depend on θ_i^1 at all, as seen by (2.6.4).

Therefore, the FOC from the above maximization problem implies that

$$E \left[\left(v_i - \rho \left(\psi_i^F(\theta_i^1) \right) \right) \left(\frac{d\psi_i^F(\theta_i^1)}{d\theta_i^1} \right) - p^{so} \frac{d\psi_i^A(\theta_i^1)}{d\theta_i^1} \right] + b'_{-i}(\theta_i^2 - \theta_i^1) (\theta_i^1 - \theta_i^2) - b_{-i}(\theta_i^1 - \theta_i^2) = 0.$$

is equal to zero. The results from Corollary (2) and the fact that $p^1 = b_{-i}(\theta_i^2 - \theta_i^1)$ for all i allow us to rewrite the above FOC as:

$$(1 - \gamma) E \left[v_i - \rho \left(\psi_i^F(\theta_i^1) \right) \right] + \gamma E[p^{so}] + b'_{-i}(\theta_i^2 - \theta_i^1) (\theta_i^1 - \theta_i^2) - p^1 = 0.$$

From Proposition (2), it is clear $\psi_i^F(\theta_i^1) = \psi_i^F(0) + (1 - \gamma) \theta_i^1$. Substituting this relationship into the above equation and solving it for θ_i^1 gives us:

$$\theta_i^1 = \frac{1}{(1 - \gamma)^2 \rho - b'_{-i}(-\theta_i^1)} \left((1 - \gamma) E \left[v_i - \rho \psi_i^F(0) \right] + \gamma E[p^{so}] - b'_{-i}(\theta_i^1 - \theta_i^2) \theta_i^2 - p^1 \right).$$

Equation (A.1.1) implies that $b'_{-1}(\theta_i^1 - \theta_i^2) = -\frac{1}{\sum_{j \neq i} \beta_j}$. Thus,

$$\theta_i^1 = \frac{1}{(1-\gamma)^2 \rho + \frac{1}{\sum_{j \neq i} \beta_j^1}} \left((1-\gamma) E[v_i - \rho \psi_i^F(0)] + \gamma E[p^{so}] + \frac{1}{\sum_{j \neq i} \beta_j} \theta_i^2 - p^1 \right).$$

In a linear equilibrium, the above equation must hold for all $i = 1, \dots, N$. Therefore, we have I equations of the form:

$$\beta_i^1 = \frac{1}{(1-\gamma)^2 \rho + \frac{1}{\sum_{j \neq i} \beta_j^1}}, \quad i \in \mathcal{I}.$$

The unique solution for the β^1 s gives us $\beta_i^1 = \frac{\gamma}{(1-\gamma)^2 \rho}$ for all i . Substituting the equilibrium β on the above equations and using the fact that $q_i^{wi,1} \equiv \theta_i^1 - \theta_i^2$, the intercepts of the bid function will be given by:

$$\alpha_i = \frac{\gamma}{(1-\gamma)^2 \rho} \left((1-\gamma) E[v_i - \rho \psi_i^F(0)] + \gamma E[p^{so}] \right) - \gamma \theta_i^2.$$

From the equilibrium described on Proposition (2) we have that:

$$\begin{aligned} v_i - \rho \tilde{\psi}_i^F(0) &= v_i - \rho \frac{\gamma}{\rho} (v_i - \tilde{p}^{so}) \\ &= (1-\gamma) \left(v_i - \rho \frac{\tilde{Q}}{N} \right) + \gamma \tilde{p}^W \\ (A.1.4) \quad &= (1-\gamma) \rho \theta_i^c + \tilde{p}^W. \end{aligned}$$

where I used (2.6.4) from the first to the second line. From the second to the third line, I added and subtracted $(1-\gamma) \frac{\sum_{\mathcal{I}} v_i}{N}$ and used the definition of $\tilde{p}^W$. Substituting the above expression into α_i and using the definition of $p^{wi,c}$, we find that the bid schedule at $t = 1$ satisfies (2.6.10).

$t = \tau ::$

Suppose that for all $t < \tau$, 6 and 5 holds. The maximization problem of dealer i at τ can be written as:

$$(A.1.5) \quad \max_{\theta_i^\tau(\cdot)} E \left[u_i \left(\psi_i^F \left(\theta_i^1 \right) \right) - p^{so} \psi_i^A \left(\theta_i^1 \right) \right] - \sum_{t=1}^{\tau} p^t \left(\theta_i^t - \theta_i^{t+1} \right).$$

The prices and positions for all $t > \tau$ are already given when i chooses θ_i^t . Moreover, the assumption that corollary 5 implies that $p^t = p^{wi,c}$ for all $t < \tau$, thus, we have that:

$$\sum_{t=1}^{\tau} p^t \left(\theta_i^t - \theta_i^{t+1} \right) = p^{wi,c} \left(\theta_i^1 - \theta_i^\tau \right) + p^\tau \left(\theta_i^\tau - \theta_i^{\tau+1} \right).$$

Let $b_{-i}^\tau(q)$ be the inverse linear bid schedule without dealer i 's bid built in the same way as in the previous step of this proof. The FOC from (A.1.5) will be given by:

$$\left\{ E \left[u'_i \left(\psi_i^F \left(\theta_i^1 \right) \right) \left(\frac{d\psi_i^F \left(\theta_i^1 \right)}{d\theta_i^1} \right) - p^{so} \frac{d\psi_i^A \left(\theta_i^1 \right)}{d\theta_i^1} \right] - p^{wi,c} \right\} \frac{d\theta_i^1}{d\theta_i^\tau} + p^{wi,c} + \frac{db_{-i}^\tau}{d\theta_i^\tau} \times \left(\theta_i^\tau - \theta_i^{\tau+1} \right) - p^\tau = 0.$$

The term on chains can be rewritten as:

$$\begin{aligned} E \left[u'_i \left(\psi_i^F \left(\theta_i^1 \right) \right) \left(1 - \gamma \right) + p^{so} \gamma \right] - p^{wi,c} &= E \left[\left(v_i - \rho \psi_i^F(0) \right) \left(1 - \gamma \right) - \rho \left(1 - \gamma \right)^2 \theta_i^1 + \gamma p^{so} \right] - p^{wi,c} \\ &= E \left[\left(1 - \gamma \right) p^W + \left(1 - \gamma \right)^2 \rho \left(\theta_i^c - \theta_i^1 \right) + \gamma p^{so} \right] - p^{wi,c} \\ &= \left(1 - \gamma \right)^2 \rho \left(\theta_i^c - \theta_i^1 \right) \\ &= \left(1 - \gamma \right)^{\tau+1} \rho \left(\theta_i^c - \theta_i^\tau \right). \end{aligned}$$

From the first to the second, I used equation (A.1.4). From the third to the fourth I used Corollary 6 which implies that:

$$\theta_i^1 = \gamma \theta_i^c \sum_{t=0}^{\tau-2} \left(1 - \gamma \right)^t + \left(1 - \gamma \right)^{\tau-1} \theta_i^\tau.$$

This same Corollary implies that $\frac{d\theta_i^1}{d\theta_i^\tau} = (1 - \gamma)^{\tau-1}$. Substituting these relationships on the FOC, we can use the same steps used to find a linear equilibrium at $t = 1$ to find that dealers submit bid schedules satisfying (2.6.10) for $t = \tau$. $\square$

A.1.4. Corollary 5.

PROOF. Remember that the when-issued market is a zero-supply market in each round, so $\sum_{\mathcal{I}} q_i^{wi,t}(p^{wi,t}) = 0$ and $\sum_{\mathcal{I}} \theta_i^{t+1} = 0$. Summing up (2.6.10) across all dealers and using the market clearing condition gives us:

$$(A.1.6) \quad 0 = \frac{\gamma}{(1 - \gamma)^{2t} \rho} \left\{ I \times (p^{wi,c} - p^\tau) + (1 - \gamma)^{2t} \rho \sum_{\mathcal{I}} \theta_i^c \right\}.$$

But $\sum_{\mathcal{I}} \theta_i^c = 0$, which implies that $p^\tau = p^{wi,c}$. $\square$

A.1.5. Corollary 6.

PROOF. I will do the proof for dealer i . Substituting the equilibrium stop-out price on his bid schedule gives us

$$\begin{aligned} q_i^{wi,t}(p^{wi,c}) &= \frac{\gamma}{(1 - \gamma)^{2t} \rho} (A_i^t - p^{wi,c}) - \gamma \theta_i^{t+1} \\ &= \gamma \theta_i^c - \gamma \theta_i^{t+1}. \end{aligned}$$

But $\theta_i^t = q_i^{wi,t}(p^{wi,c}) + \theta_i^{t+1}$, which gives the result of the corollary. $\square$

A.1.6. Corollary 7.

PROOF. From Corollary 6, we have that, in the unique linear SPE, the position that dealer i arrives at the auction from the WI market is given by

$$\theta_i^1 = \theta_i^c [1 - (1 - \gamma)^T].$$

Therefore, $\lim_{T \rightarrow \infty} \theta_i^1 = \theta_i^c$, i.e., as T goes to infinity, θ_i^1 converges to the position i arrives in the auction on the perfect competitive case. Nevertheless, we know from Corollary 3 that this specific position leads to an efficient allocation after the auction takes place and the WI contracts are settled. $\square$

A.1.7. Proposition 6.

PROOF. The characterization of the equilibrium with the additional resale market will follow the same steps as the ones used in proposition 5. The problem is very similar to the previous without resale market. With the exception of changes in the market clearing conditions, it is exactly the same as if we were adding a period in the when-issued market.

Resale market::

Let $\{\theta_i^A\}_{\mathcal{I}}$ be the positions that dealers arrive at the resale stage. Dealer i 's maximization problem at this stage is exactly the same as the auction's in the case without the resale stage:

$$\max_{q_i^R(\cdot)} u_i \left(q_i^R(p) + \theta_i^A \right) - p q_i^R(p).$$

Therefore, the unique linear equilibrium has dealers submitting bid schedules as:

$$(A.1.7) \quad q_i^R(P, \theta_i^A) = \frac{\gamma}{\rho} (v_i - P) - \gamma \theta_i^A.$$

Using the market clearing conditions $\sum_{\mathcal{I}} q_i^R = 0$ and $\sum_{\mathcal{I}} \theta_i^A = Q$, we get that equilibrium price in the resale stage is p^W .

Auction::

Dealer i maximizes:

$$\max_{q_i^A(\cdot, h)} E \left[u_i \left(\psi_i^R + \theta_i^A \right) - p^R \psi_i^R - p^{so} q_i^A(p, h) \right].$$

where θ_i^{wi} is the position he arrives from the when-issued market and $\theta_i^A \equiv q_i^A(p, h) + \theta_i^{wi}$. Once more, I use point wise optimization in order to characterize the equilibrium in the auction. For a given realization of $\tilde{Q}$, the first order condition of the above problem satisfies:

$$(A.1.8) \quad \left(v_i - \rho \left(\psi_i^R + \theta_i^A \right) \right) (1 - \gamma) + \gamma p^R - \frac{dp^{so}}{dq_i^A} q_i^A = p^{so}.$$

Suppose that dealer $j \neq i$ submits a linear demand schedule given by:

$$q_j(p, h) = \alpha_j - \beta_j p$$

Summing across $j \neq i$ and using the market clearing condition, we have that the stop-out price in the auction will satisfy $p^{so} = \frac{1}{\sum_{j \neq i} \beta_j} \left(\sum_{j \neq i} \alpha_j - (Q - q_i^A) \right)$. This implies that:

$$(A.1.9) \quad Q = q_i^A + \sum_{j \neq i} \alpha_j - p^{so} \sum_{j \neq i} \beta_j$$

$$(A.1.10) \quad \frac{dp^{so}}{dq_i^A} = \frac{1}{\sum_{j \neq i} \beta_j}.$$

Substituting (A.1.10) and (A.1.12) into the FOC and rearranging the terms, we get:

$$(A.1.11) \quad \left(v_i - \rho (1 - \gamma) \theta_i^{wi} \right) (1 - \gamma) - \rho (1 - \gamma) \psi_i^R(0) + \gamma p^R - \left(\rho (1 - \gamma)^2 + \frac{1}{\sum_{j \neq i} \beta_j} \right) q_i^A - p^{so} = 0.$$

where $\psi_i^R(0)$ is the amount of securities i would acquire in the resale stage if he arrives with $\theta_i^A = 0$. Note, by (A.1.7), that

$$(A.1.12) \quad \psi_i^R(\theta_i^A) = \psi_i^R(0) + (1 - \gamma) \theta_i^A.$$

In the equilibrium of the resale stage, we have that

$$\begin{aligned}
\psi_i^R(0) &= \frac{\gamma}{\rho} [v_i - p^W] \\
&= \frac{\gamma}{\rho} \left[v_i - \left(\bar{v} - \rho \frac{Q}{N} \right) \right] \\
&= \frac{\gamma}{\rho} \left[v_i - \left(\bar{v} - \frac{\rho}{N} \left(q_i^A + \sum_{j \neq i} \alpha_j - p^{so} \sum_{j \neq i} \beta_j \right) \right) \right].
\end{aligned}$$

where I used (A.1.9) in the last line. Substituting the above relationship on (A.1.11), and solving it to q_i^A gives us:

$$q_i^A = \alpha_i - \beta_i p^A.$$

where

$$\begin{aligned}
\alpha_i &= \frac{1}{(1-\gamma)^2 \rho + \sum_{j \neq i} \frac{1}{\beta_j} + \frac{\rho}{N} (1 - (1-\gamma)^2)} \left[(v_i - \rho \theta_i^{wi}) (1-\gamma)^2 + (1 - (1-\gamma)^2) \left(\bar{v} - \frac{\rho}{N} \sum_{j \neq i} \alpha_j \right) \right] \\
\beta_i &= \frac{\left[1 - \frac{\rho}{N} ((1 - (1-\gamma)^2)) \sum_{j \neq i} \beta_j \right]}{(1-\gamma)^2 \rho + \sum_{j \neq i} \frac{1}{\beta_j} + \frac{\rho}{N} (1 - (1-\gamma)^2)}.
\end{aligned}$$

The symmetric solution from the above system of equations gives:

$$\begin{aligned}
\alpha_i &= \frac{\gamma}{\rho} \left[(v_i - \rho \theta_i^{wi}) \omega + (1 - \omega) \bar{v} \right] \\
\beta_i &= \frac{\gamma}{\rho}.
\end{aligned}$$

where $\omega \equiv \frac{1-\gamma}{(1-\gamma)+\gamma+\gamma(1-\gamma)}$.

Finally, substituting the equilibrium values for β on A.1.10, and noticing that $\sum_{\mathcal{I}} (\psi_i^R + \theta_i^A) = Q$, averaging up the FOC described on A.1.8 gives us the equilibrium price for the auction as described in the proposition.

When-issued market::

The maximization problem in the WI market is written as:

$$\max_{\theta_i^{wi}(\cdot)} E \left[u_i \left(\psi_i^R + \theta_i^A \right) - p^R q_i^R - p^{so} \left(\theta_i^A - \theta_i^{wi} \right) \right] - b_{-i} \left(-\theta_i^{wi} \right) \times \theta_i^{wi}.$$

In the when-issued market, the realization of Q is not know yet. Therefore, the FOC from the above problem can be written as:

$$(A.1.13) \quad E \left[u'_i \left(\psi_i^R + \theta_i^A \right) (1 - \gamma) + \gamma p^R \right] (1 - \gamma) + \gamma E [p^{so}] - b'_{-i} \left(-\theta_i^{wi} \right) \times \theta_i^{wi} - p^{wi} = 0.$$

Note that

$$\begin{aligned} \sum_{\mathcal{I}} u'_i \left(\psi_i^R \right) &= \sum_{\mathcal{I}} \left(v_i - \rho \psi_i^R \right) \\ &= \sum_{\mathcal{I}} v_j - \rho Q \\ &= N \times p^W. \end{aligned}$$

where I used the market clearing condition in the resale stage from the first to the second line. Using the above relationship and the market clearing condition in the WI market ($\sum_{\mathcal{I}} \theta_i^{wi} = 0$) we get the equilibrium price of the WI market by summing up (A.1.13) across dealers and solving for p^{wi} . $\square$

A.2. Example continued

How is p^{wi} determined in equilibrium? We've seen that both agents are better off by trading the when-issued security if $p^{wi} \in (v_2, v_1)$. But how would investors get into an agreement about p^{wi} ? I will consider a very simple environment for the when-issued market that leads to this result. Suppose that dealers compete for an unit of a when-issued security through a double auction game. Each player i submits a bid b_i in this stage. The dealer who submitted the lowest bid will supply a when-issued security to the other dealer, who submitted the higher bid in exchange for a payment of $p^{wi} = (b_i + b_j)/2$. In case of a tie, the

when-issued security goes to 1. Moreover, I will assume that after bids are computed, one or both dealers can decide to withdraw from the when-issued market without any cost.

Before characterizing the equilibrium for the full game, I need to determine what would happen in the auction for all possible positions dealers can arrive from the when-issued market. The case where 1 acquired one unit of the when-issued security was illustrated in the previous section. In that case, the payoff of the two dealers were given by:

$$\begin{aligned} U_1^{wi} &= 2v_1 - p^{wi} \\ U_2^{wi} &= p^{wi}. \end{aligned}$$

What would happen if 2 was the winner of the when-issued security instead? As in the main text, I will assume that dealers will play the Pareto dominant equilibrium of the game:

CLAIM 3. *There is an equilibrium of the auction where dealers submit $\beta_1^{wi} = (\pi, 0)$ and $\beta_2^{wi} = (v_2, 0)$. The equilibrium stop-out price is $p^{so} = 0$ and payoffs are given by:*

$$\begin{aligned} U_1' &= p^{wi} \\ U_2' &= 2v_2 - p^{wi}. \end{aligned}$$

If one of the dealers decides to withdraw from the market, then there is no when-issued trade. The equilibrium of the auction in this case was described in Claim 1 and dealers get the payoff:

$$U_i'' = v_i.$$

It is straightforward to see that there is a continuum of equilibria with $b_1 = b_2 \in (v_1, v_2)$. Dealer $i \in \mathcal{I}$ would not have an incentive to pay more than v_i for the when-issued security. Moreover, i would benefit from withdrawing the when-issued market whenever $p^{wi} \leq v_i$,

where I am assuming he prefers to withdraw when he is indifferent from trading the when-issued security. Any price between these two values is an equilibrium for the game described above. Dealer 1 would get the when-issued security and neither of them would benefit from withdrawing from when-issued market.

A.3. Underpricing

Discriminatory		Uniform		Benchmark	Authors
Underp.	# Obs.	Underp.	# Obs.		
1.3	364	-		Quotes	Spindt and Stolz (1992)
0.37	66	-		Quotes	Simon (1994)
0.27	76	0.09	15	Transactions	Nyborg and Sundaresan (1996)
0.55	66	0.21	44	Quotes	Malvey and Archibald (1998)
0.8	130	-		Quotes	Bikhchandani et al. (2000)
0.61	105	0.40	178	Transactions	Goldreich (2007)

TABLE 1. Underpricing = Avg Auction Yield - When-Issued Yield. All values are in units of basis points.

APPENDIX B

Chapter 3

B.1. Proofs

B.1.1. Proof of Proposition 7. Let $b_{-i}(q)$ be the aggregate inverse bid schedule without investor i , giving the price at which the sum of the demand from all other competitors is q . We are searching for an equilibrium where investors submit linear bid schedules. Assume $b_{-i}(q)$ also is a linear function. Explicitly, suppose that:

$$b_{-i}(q) = v - \mu_i q.$$

For an arbitrary $\hat{Q}$, investor i chooses a quantity $\hat{q}_i^A$ to solve the problem:

$$\max_{\hat{q}_i^A} \left(v - \frac{\rho}{2} \hat{q}_i^A \right) \hat{q}_i^A - b_{-i}(\hat{Q} - \hat{q}_i^A) \hat{q}_i^A.$$

The solution of the above problem is:

$$(B.1.1) \quad q_i^A = (\rho + \mu_i)^{-1} \left(v - b_{-i}(\hat{Q} - q_i^A) \right).$$

Note that q_i^A is the optimal quantity for a *given* realization of the *stochastic* quantity being auctioned, Q . This quantity is not known before the auction takes place which prevents agents to condition their bids directly on Q . However, they can do this indirectly by submitting the inverse bid schedule

$$(B.1.2) \quad b_i(q) = v - (\rho + \mu_i) q.$$

Indeed, take an arbitrary value $\hat{Q} \in [0, \bar{Q}]$ and suppose that investor i submits the above bid. By definition, $p^{so} = b_i(q'_i) = b_{-i}(\hat{Q} - q'_i)$. Therefore, we must have:

$$v - (\rho + \mu_i) q = v - \mu_i (\hat{Q} - q'_i)$$

The above equation implies that q'_i satisfies [B.1.1](#) when $Q = \hat{Q}$. Since an investor can always attain his *ex post* best response, the above bid schedule will be a solution for his *ex ante* expected utility maximization problem.

If all agents follow the above strategy, we will have I equations of the form:

$$\mu_i = \left(\sum_{j \neq i} (\rho + \mu_j)^{-1} \right)^{-1}$$

Finding the unique solution to this set of equations and substituting it into [\(B.1.1\)](#) gives us the bid functions in the proposition.

B.1.2. Proof of Lemma 1. The speculator faces the following problem:

$$s.t \max_{\sum_i \hat{q}_i^R \leq q^S; \hat{q}_i^R \geq 0} \theta \cdot \left[\sum_{\mathcal{I}} U(q_i^A + \hat{q}_i^R) - U(q_i^A) \right].$$

Let $\theta\lambda_i$ and $\theta\phi$ be the Lagrange multipliers with respect to the constraints $\hat{q}_i^R \geq 0$ and $\sum_{\mathcal{I}} \hat{q}_i^R \leq q^S$ respectively. The Kuhn-Tucker conditions are sufficient for optimality of the above problem and can be described as:

$$\begin{aligned} \forall i, v - \rho(q_i^A + \hat{q}_i^R) &= \phi - \lambda_i \\ \forall i, \lambda_i \hat{q}_i^R &= 0 \\ \mu(q^S - \sum_i \hat{q}_i^R) &= 0. \end{aligned}$$

This set of linear equations gives us the Lemma's result.

B.1.3. Proof of Proposition 8.

Speculator: Start with the speculator's problem. Once again, I will focus on an *ex post* (symmetric) optimal linear equilibrium. Let $b_{-s}(q)$ be the aggregate inverse bid schedule without the speculator, of the form:

$$b_{-s}(Q - q'_s) = \theta [v - \mu_s (Q - q'_s)].$$

For a given realization of Q , the speculator problem can be rewritten as:

$$\max_{\hat{q}_s} \theta \times \Pi \left(\hat{q}_s, \left\{ q_i^A(\hat{q}_s) \right\}_{\mathcal{I}} \right) - b_{-s}(Q - \hat{q}_s) \hat{q}_s.$$

The symmetry assumption implies that all investors will end up acquiring the same amount of securities, which implies that $q_i^A = \frac{Q - \hat{q}_s}{I}$ for all $i \in \mathcal{I}$. The FOC of the speculator's problem gives us:

$$\theta \cdot \frac{d\Pi \left(\hat{q}_s, \frac{Q - \hat{q}_s}{I} \right)}{d\hat{q}_s} - \theta \mu \hat{q}_s - p^{so} = 0.$$

Note that,

$$\begin{aligned} \frac{d\Pi \left(\hat{q}_s, \frac{Q - \hat{q}_s}{I} \right)}{d\hat{q}_s} &= \frac{d}{d\hat{q}_s} \left[I \times \sum_{i \in \mathcal{I}} U \left(q_i^A + \hat{q}_i^R \right) - U \left(q_i^A \right) \right] \\ &= \frac{d}{d\hat{q}_s} \left[I \times \sum_{i \in \mathcal{I}} U \left(\frac{Q}{I} \right) - U \left(\frac{Q - q'_s}{I} \right) \right] \\ &= U' \left(\frac{Q - \hat{q}_s}{I} \right). \end{aligned}$$

Substituting this result into the the FOC we have:

$$\theta \times \left(v - \rho \left(\frac{Q - \hat{q}_s}{I} \right) \right) - \theta \mu_s - p^{so} = 0.$$

The second order condition of the maximization problem is satisfied whenever:

$$(B.1.3) \quad \frac{\rho}{I} - 2\mu_s < 0.$$

The assumption on the format of the residual supply function implies that $(Q - \hat{q}_s) = \frac{1}{\theta\mu_s} (\theta v - p^{so})$. Therefore, we can rewrite the FOC as:

$$(B.1.4) \quad q_s = \frac{\theta\mu_s - \frac{\rho}{I}}{(\theta\mu_s)^2} (\theta v - p^{so}).$$

Investor: Let's turn to the investor problem. Suppose that investor i faces the residual supply

$$b_{-i}(Q - q_i) = \theta [v - \mu_i(Q - q_i)],$$

and that all investors $j \in \mathcal{I}_{-i}$ are submitting the same bid schedule. We can rewrite i 's maximization problem, described in 3.3.4, as:

$$\max_{\hat{q}_i} U(\hat{q}_i) + (1 - \theta) [U(\hat{q}_i + q_i^R) - U(q_i')] - b_{-i}(Q - \hat{q}_i) \hat{q}_i$$

where q_i^R is determined according to Lemma 1. Investors are restricted to acquire a quantity lower than the total supply. Therefore, the FOC from the above problem is written as

$$(B.1.5) \quad \theta U' (q_i^A) + (1 - \theta) U' (q_i^A + q_i^R) \frac{d(q_i^A + q_i^R)}{dq_i^A} - \theta \mu_i q_i^A + b_{-i}(Q - q_i^A) - \sigma q_i^A = 0,$$

where σ is the Lagrange multiplier associated with the constraint $q_i^A \leq Q$. Note that the symmetry assumption implies that the speculator distributes securities even across all investors $j \in \mathcal{I}_{-i}$ in the secondary market. Therefore, Lemma 1 implies that:

$$\begin{aligned} \frac{d(q_i^A + q_i^R)}{dq_i^A} &= \frac{d \left[\frac{q^S(p^{so}) + \sum q_{(j)}^A(p^{so})}{I} \right]}{dq_i^A} I_{q_i^R > 0} + I_{q_i^R = 0}. \\ &= I_{q_i^R = 0}. \end{aligned}$$

The first term on the right hand side represents the case in which q_i^A is not high enough so as to make $q_i^R = 0$. It follows from the market clearing condition, $q^S(p^{so}) + \sum q_{(j)}^A(p^{so}) = Q$.

In this case, any marginal increase in the quantity i acquires in the auction will be offset by a decrease in the quantity the speculator sells to him in the secondary market. The second term represents the case in which investor i is acquiring a large quantity of securities in the auction, and the speculator does not allocate any security to him in the secondary market. Since i is not getting anything in the secondary market, any increase in the quantity he acquires in the auction will be translated to an increase in the final quantity of securities he holds in his final portfolio.

The FOC [B.1.5](#) can be rewritten as:

$$\theta (v - \rho q_i^A) + (1 - \theta) (v - \rho q_i^A) I_{q_i^R=0} - \theta \mu_i q_i^A + p^{so} - \sigma q_i^A = 0.$$

The equation above can have two solutions, depending whether q_i^R is larger or equal than zero:

$$\begin{aligned} \text{(B.1.6)} \quad q_i^{A,1} &= \frac{1}{\sigma + \theta (\rho + \mu_i)} (\theta v - p^{so}) \\ q_i^{A,2} &= \frac{1}{\sigma + \rho + \theta \mu_i} (v - p^{so}). \end{aligned}$$

Note that we cannot have a symmetric equilibrium if i is acquiring $q_i^{A,2}$. He acquires $q_i^{A,2}$ only if he is not getting any security in the secondary market, which occurs only if $q_i^{A,2}$ is larger than the quantity the other investors are acquiring in the auction. To see why this is the case, [Figure B.1.1](#) illustrates the marginal valuation and the residual supply functions for an investor when all his competitors submit the equilibrium bid schedule [3.3.6](#). Note that the marginal valuation function has a point of discontinuity. This is essentially the point at which the speculator stop trading with this investor in the secondary market. Therefore, the residual supply could cross the marginal valuation curve in two different points. Each one of them represents a local maxima for agent i . However, for the equilibrium described on [proposition 8](#) to hold, the global maximum must be the first point at which the residual

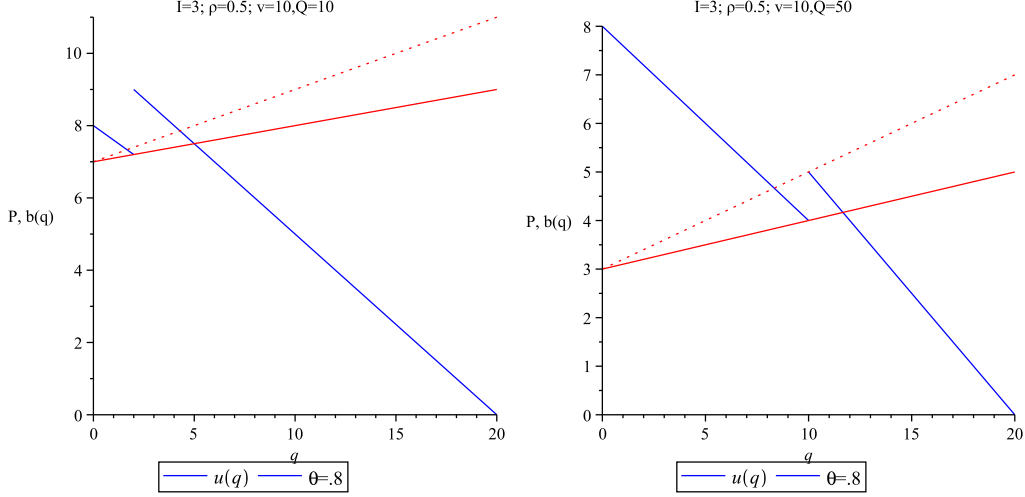


FIGURE B.1.1. Marginal valuation and residual supply function

supply function crosses the marginal valuation. This will be the case only when $\theta > \underline{\Theta}$. It is straightforward to see that $\underline{\Theta}$ is strictly increasing in $\underline{Q}$; if $\underline{Q}$ is big enough, then the equilibrium holds. If $\hat{\theta} < \underline{\Theta}$, investors will be better off by deviating to the other intersection point for some realizations of $\underline{Q}$.

Since we are focusing on symmetric equilibrium, I will restrict the analysis to the case where $q_i^{A,1}$ is the solution to the i 's maximization problem and the supply constraint is not binding.

If all investors submit bid schedules according to (B.1.6) and the speculator submits a bid schedule as in (B.1.4), the equilibrium conditions for μ_S and μ_i are:

$$\begin{aligned}\mu_i &= \left(\frac{\mu_S - \frac{\rho}{I}}{\mu_S^2} + \sum_{j \neq i, s} (\rho + \mu_j)^{-1} \right)^{-1} \\ \mu_s &= \left(\sum_{j \neq s} (\rho + \mu_j)^{-1} \right)^{-1}.\end{aligned}$$

The only solution of the above equation which is symmetric within each type has

$$\mu_s = \mu_i = \frac{\rho}{I-1}.$$

Substituting the solution into the equilibrium bid schedules gives us [3.3.6](#).

Equilibrium conditions: We still have to check two necessary conditions for [\(3.3.6\)](#) to be an equilibrium. First, we need to check if the SOC for the speculator's maximization problem is satisfied. Second, we need to check whether [\(B.1.6\)](#) is indeed the solution to the investor's maximization problem.

It is straightforward to see that the SOC for the speculator's problem is satisfied simply by plugging the equilibrium value of μ_s into [\(B.1.3\)](#).

For the investor's problem, we need to check whether acquiring $q_i^{A,1}$ in the auction gives him a higher payoff than acquiring $q_i^{A,2}$. This happens, in equilibrium, whenever:

$$\begin{aligned} p' \left(q_i^{A,2} - \frac{Q}{I+1} \right) + (p' - p^{so}) \frac{Q}{I+1} &\geq U(q_i^{A,2}) - U\left(\frac{Q}{I+1}\right) \\ &\quad - (1-\theta) \left(U\left(\frac{Q}{I}\right) - U\left(\frac{Q}{I+1}\right) \right). \end{aligned}$$

The parameter $\underline{\Theta}$ is defined as the bargaining power such that the restriction above holds with equality at $\underline{Q}$.

B.1.4. Proof of Proposition [9](#).

PROOF. As argued above, the equilibrium price in the resale stage for any realization of Q is given by $p^R = v - \rho \frac{Q}{I}$. Moreover, the final allocation of securities will also be independent of the result of the auction stage and will be such that all investors hold $\frac{Q}{I}$ units. Thus the allocation of the auction stage will determine only how many units each agent will sell/buy in the resale stage.

The utility of an investor who acquires $\hat{q}_i^A$ units in the auction and faces a linear inverse bid function $b_{-i}(q)$ is given by:

$$\left(v - \frac{\rho Q}{2I}\right) \frac{Q}{I} - b_{-i}(Q - \hat{q}_i^A) \hat{q}_i^A + p^R \left(\hat{q}_i^A - \frac{Q}{I}\right)$$

Suppose that $b_{-i}(q) = v - \mu_i q$. Substituting p^R for $v - \rho \frac{Q}{I}$ and noticing that the stop-out price is given by $b_{-i}(Q - \hat{q}_i^A)$, the first order condition related to the maximization of the above utility with respect to $\hat{q}_i^A$ gives

$$v - \rho \frac{Q}{I} - P = \mu_i \hat{q}_i^A.$$

We know that $P = b_{-i}(Q - \hat{q}_i^A) = v - \mu_i (Q - \hat{q}_i^A)$. Substituting this relationship on the above equation, we get

$$P = v - \mu_i \frac{\mu_i + \frac{\rho}{I}}{\mu_i - \frac{\rho}{I}} \hat{q}_i^A.$$

A speculator facing $b_{-s}(q)$ chooses $\hat{q}_s^A$ in order to maximize

$$p^R \hat{q}_s^A - b_{-s}(Q - \hat{q}_s^A) \hat{q}_s^A.$$

If $b_{-s}(q) = v - \mu_s q$, the first order condition for the speculator is exactly equal to the one we get when maximizing the investor's problem. Thus:

$$P = v - \mu_s \frac{\mu_s + \frac{\rho}{I}}{\mu_s - \frac{\rho}{I}} \hat{q}_s^A.$$

If all agents bid according to the above bid schedule, there will be $I + N$ equations as:

$$\mu_j = \left(\sum_{k \neq j} \left(\mu_k \frac{\mu_k + \frac{\rho}{I}}{\mu_k - \frac{\rho}{I}} \right)^{-1} \right)^{-1}, \quad j = 1, \dots, I, I+1, \dots, I+N.$$

The symmetric solution for this set of equations gives us [3.3.7](#). □

B.1.5. Proof of Proposition 10. As in the linear case, the equilibrium price and final allocations of the resale market are independent of the outcome from the auction stage. Moreover, I will keep the focus on equilibria where the bid schedules are ex post optimal for all realizations of Q , downward-sloping and differentiable.

In the auction stage, an investor i , facing inverse residual supply $b_{-i}(q)$, chooses a quantity $\hat{q}_i^A$ that solves:

$$\max_{\hat{q}_i^A} u\left(\frac{Q}{I}\right) - b_{-i}\left(Q - \hat{q}_i^A\right) \hat{q}_i^A + p^R\left(\hat{q}_i^A - \frac{Q}{I}\right).$$

Conversely, a speculator s , facing residual supply $b_{-s}(q)$, solves the following problem:

$$\max_{\hat{q}_s^A} \left(p^R - b_{-s}(q)\right) \times \hat{q}_s^A.$$

In a symmetric equilibrium, all agents submit bid schedules $q(\cdot)$. Let $b(\cdot)$ denote the residual supply that an agent k (either an investor or a speculator) faces. Since the outcome of the resale market does not depend on the outcome of the auction, the FOC from the above maximization problems are the same, and are given by

$$(B.1.7) \quad p^R - b(q_k^A) = -b'(q_k^A) q_k^A.$$

The market-clearing condition implies that for any q_k^A this agent acquires in the auction stage, we have

$$(I + N - 1) q(b(q_k^A)) = Q - q_k^A.$$

Taking the derivative of both sides with respect to q_k^A and rearranging terms, we arrive at

$$b'(q_k^A) = -\frac{I + N - 1}{q'(P)},$$

where $P = b(q_k^A)$. Substituting this expression into the FOC given in (B.1.7) and using the equilibrium condition $q_k^A = q(P)$, we have

$$(B.1.8) \quad p^R = P + (I + N - 1) \frac{q(P)}{q'(P)}.$$

Now, the market clearing condition in the *resale* market implies that $Q = \frac{I}{\rho} (v - p^R)$. Substituting this relationship into the market clearing condition of the *auction* stage, we have that:

$$p^R = v - \frac{(I + N)}{I} \times q(P).$$

Together with (B.1.8), the above expression implies that the optimal bid schedule satisfies the following ODE:

$$(I + N - 1) q'(P) = \frac{q(P)}{v - \rho \frac{I+N}{I} q(p^{so}) - p^{so}}.$$

Together with the initial condition $q(0) = q_0$, the equilibrium bid schedule given in the proposition is a solution for the above ODE.

B.2. Endogenous Entry

If for some reason the presence of speculators decreases the number of investors willing to participate in the market, the auctioneer can be better off if resale is not allowed. Indeed, we have seen that the presence of speculators has a negative impact on the equilibrium payoff of investors participating in the market, so it is reasonable to consider that this could derail the entrance of new investors. For instance, suppose there is a fixed cost c in order to become an investor, while it costs nothing to a speculator to enter the market. This implies that speculators would enter until the price of the security in the auction be the same as the price

in the resale market. The payoff of an investor would be given by:

$$\begin{aligned}\pi_{I,N}(Q) &= u\left(\frac{Q}{I}\right) - p^R \frac{Q}{I} + (p^R - p^{so}) q_i^A - c \\ &= \frac{\rho}{2} \left(\frac{Q}{I}\right)^2 - c\end{aligned}$$

In equilibrium, the number of investors would $I^R = \left\lfloor \sqrt{\frac{\rho}{2c} E[Q^2]} \right\rfloor$, i.e. the largest integer not greater than $\sqrt{\frac{\rho}{2c} E[Q^2]}$. With free entry of speculators, the equilibrium price in the auction would be the same as in the resale market and given by:

$$p^{so} = p^R = v - \rho \frac{Q}{I^R}$$

Now, suppose that resale is forbidden. In this case, only final investors would participate in the auction. The equilibrium payoff for an investor is given by:

$$\begin{aligned}\pi_{I,0}(Q, q_i^A) &= u\left(\frac{Q}{I}\right) - p^R \frac{Q}{I} + (p^R - p^{so}) q_i^A - c \\ &= \rho \frac{I}{2(I-2)} \left(\frac{Q}{I}\right)^2 - c.\end{aligned}$$

With no resale, the number of investors in the market is given by $I^{NR} = \left\lfloor 1 + \sqrt{1 + \frac{\rho}{2c} E[Q^2]} \right\rfloor$, and the auction price is

$$p^{so} = v - \rho \frac{I^{NR} - 1}{I^{NR} - 2} \frac{Q}{I^{NR}}.$$

Therefore, the revenue when resale is permitted is larger than when resale is not permitted if and only if

$$I^R \geq I^{NR} \frac{I^{NR} - 2}{I^{NR} - 1}.$$

It is easy to show that the above always hold, implying that resale increases revenue in this specif setup.

APPENDIX C

Chapter 4

C.1. Proofs

C.1.1. Proof of Proposition 11.

PROOF. First, note that there is no reason why an investor would submit a bid $b > \beta_0^{II}(1)$, since he would acquire the asset anyway. Now, suppose that all investors $j \neq i$ are following β_1^{II} . The payoff for i when his valuation is μ_i and he submit bid b is:

$$\begin{aligned} \Pi(b, \mu_i) &= \int_0^{\beta_1^{-1}(b)} \left(W(\mu_i, y) - \beta_0^{II}(y) \right) g(y) dy + \int_{\beta_1^{-1}(b)}^1 L(\mu_i, y) g(y) dy \\ &= \int_0^{\beta_1^{-1}(b)} (v(\mu_i, y) - v(y, y)) g(y) dy + \int_0^1 L(\mu_i, y) g(y) dy \end{aligned}$$

where $g(y)$ is the probability density of $Y_1^{(N-1)}$. The FOC implies:

$$\left(v(\mu_i, \beta_1^{-1}(b^*)) - v(\beta_1^{-1}(b^*), \beta_1^{-1}(b^*)) \right) g(\beta_1^{-1}(b^*)) = 0$$

which implies that $b^* = v(\mu_i, \mu_i)$.

We still need to check whether $v(\mu_i, \mu_i)$ is strictly increasing, consistent with our initial conjecture. The assumptions on F implies that $\beta_0(\cdot)$ is differentiable and is given by:

$$\beta'(\mu_i) = 1 - (2 - q) \left(1 - \frac{d}{dx} E[Y_1^{(N-1)} | Y_1^{(N-1)} = x]_{x=\mu_i} \right).$$

Let $\kappa \equiv \inf_{\mu_i \in [0, \omega]} \frac{d}{dx} E \left[Y_1^{(N-1)} \middle| Y_1^{(N-1)} = x \right]_{x=\mu_i}$. The above expression is strictly positive as long as

$$q > 1 - \frac{1}{2(1 - \kappa)}.$$

□

C.1.2. Proof of Corollary 11.

PROOF. Proposition 11 implies that for a given realization $\{\mu_i\}_{\mathcal{I}}$, the equilibrium price in period 0 is $v(z_2^{(N)}, z_2^{(N)})$. By definition of v , we have:

$$\begin{aligned} v(z, z) &= q \cdot z + (1 - q) \cdot E \left[Y_2^{(N-1)} \middle| Y_2^{(N-1)} < z \right] - (1 - q) \cdot E \left[z - Y_2^{(N-1)} \middle| Y_1^{(N-1)} = z \right] \\ &= z - 2 \cdot (1 - q) \cdot E \left[z - Y_2^{(N-1)} \middle| Y_1^{(N-1)} = z \right] \\ &= z - 2 \cdot (1 - q) \cdot E \left[z - Y_1^{(N-2)} \middle| Y_1^{(N-2)} \leq z \right] \\ &= z - 2 \cdot (1 - q) \cdot \left[z - \int_0^z \frac{y(n-2)f(y)F(y)^{N-3}dy}{F(z)^{N-2}} \right] \\ &= z - 2 \cdot (1 - q) \cdot \left[z - \left(z - \int_0^z \frac{F(y)^{N-2}dy}{F(z)^{N-2}} \right) \right] \\ &= z - 2 \cdot (1 - q) \cdot \left[\int_0^z \frac{F(y)^{N-2}dy}{F(z)^{N-2}} \right]. \end{aligned}$$

□

C.1.3. Proof of Lemma 3.

PROOF. Let $F_1^{(N-2)}(x|x \leq y)$ be the distribution of the highest valuation among the remaining $N - 2$ investors, conditionally on them being less than y . We have that:

$$\begin{aligned}
E \left[\max \{ \mu_i, P_1 \} - P_1 | Y_1^{(N-1)} = y \right] &= \int_0^{\min\{\mu_i, y\}} \mu_i dF_1^{(N-2)}(x|x \leq y) + \int_{\min\{\mu_i, y\}}^y x dF_1^{(N-2)}(x|x \leq y) - \int_0^y x dF_1^{(N-2)}(x|x \leq y) \\
&= \mu_i \cdot F_1^{(N-2)}(\mu_i|x \leq y) - \int_0^{\min\{\mu_i, y\}} x dF_1^{(N-2)}(x|x \leq y) \\
&= \begin{cases} \mu_i - \int_0^y x dF_1^{(N-2)}(x|x \leq y) & \text{if } \mu_i \geq y \\ \int_0^{\mu_i} x dF_1^{(N-2)}(x|x \leq y) & \text{otherwise.} \end{cases}
\end{aligned}$$

If we have an interior solution for the optimal debt level, we have that:

$$-q' \left(\frac{D}{\mu_i} \right) = \frac{\tau \bar{r} \mu_i}{E \left[\max \{ \mu_i, P_1 \} - P_1 | Y_1^{(N-1)} = y \right]}.$$

The right hand side of the above expression is decreasing in μ_i and increasing in $N, \tau \bar{r}$ and y for any $1 > \mu_i, y > 0$. Therefore, the assumptions on the probability q imply on the relationships stated in the Lemma. $\square$

C.1.4. Proof of Lemma 4. (Incomplete).

PROOF. Note that, evaluated at equilibrium, the effect on the continuation value of losing due to an increase in the signal μ' is given by:

$$\frac{d}{d\mu} L(\mu_i, y, \beta(\mu)) \Big|_{\mu=\mu_i} = E \left[-q' \left(\frac{D(y, \mu_i)}{y} \right) \cdot \frac{d}{d\mu} \left[\frac{D(y, \mu)}{y} \right] \Big| \left(\mu_i - Y_2^{(N-1)} \right) \Big| Y_2^{(N-1)} \leq y \right] \cdot F_2^{(N-1)}(\mu_i|y).$$

Moreover, the assumption of an interior solution for the optimal debt implies that

$$-q' \left(\frac{D(y, \mu_i)}{y} \right) = \frac{\tau \bar{r} y}{E \left[y - Y_2^{(N-2)} \Big| Y_2^{(N-2)} \leq \mu_i \right]}.$$

Substituting the above expressions into the FOC of the problem, we have that the optimal bid should satisfy:

$$\begin{aligned}
\beta(\mu_i) &= W(\mu_i, \mu_i) - L(\mu_i, \mu_i, \beta(\mu_i)) + \\
\text{(C.1.1)} \quad &\tau \bar{r} \cdot \int_{\mu_i}^{\omega} \left(\frac{E[\mu_i - Y_2^{(N-1)} | Y_2^{(N-1)} \leq \mu_i]}{E[y - Y_2^{(N-1)} | Y_2^{(N-1)} \leq \mu_i]} \frac{d}{d\mu} \left[\frac{D(y, \mu_i)}{y} \right] F_2^{(N-1)}(\mu_i | y) \right) \frac{dF_1(y)}{dF_1(\mu_i)}.
\end{aligned}$$

Equation C.1.1 gives the FOC that the equilibrium bid should satisfy. The "bankrupt your opponent" component is given by:

$$\mathcal{B}(\mu_i) = \tau \bar{r} \cdot \int_{\mu_i}^{\omega} \frac{d}{d\mu} L(\mu_i, y, \beta(\mu_i)) \frac{dF_1(y)}{dF_1(\mu_i)}.$$

which is always positive since, by Lemma 3, $\frac{d}{d\mu'} \left[\frac{D(y, \mu')}{y} \right] \geq 0$, which increases the continuation value of losing, for any μ_i and y . The expression above boils down to zero whenever $\mu_i = 1$, implying that $\mathcal{B}(1) = 0$. Moreover, $L(0, y, \beta(z)) = 0$ for any $z \in [0, \omega]$, which implies that $\mathcal{B}(0) = 0$.

It remains to prove that μ_i is a global maximum for (4.4.3) and $\beta^{II}(\cdot)$ is strictly increasing. Let's start with the former. Given that $j \in \mathcal{N}_{-i}$, the derivative of the objective function evaluated at $\mu' \in [0, \omega]$ is given by:

$$\left[W(\mu_i, \mu') - L(\mu_i, \mu', \beta(\mu')) - \beta^{II}(\mu') \right] dF_1(\mu') + \tau \bar{r} \cdot \int_{\mu_i}^1 \frac{d}{d\mu} L(\mu_i, y, \beta(\mu')) dF_1(y).$$

Substituting the equilibrium bid, described in the Lemma, we can rewrite the above as:

$$\begin{aligned}
&[W(\mu_i, \mu') - L(\mu_i, \mu', \beta(\mu')) - (W(\mu', \mu') - L(\mu', \mu', \beta(\mu')))] dF_1(\mu') \\
&+ \tau \bar{r} \cdot \int_{\mu'}^1 \left[\frac{d}{d\mu} L(\mu_i, y, \beta(\mu')) - \frac{d}{d\mu} L(\mu', y, \beta(\mu')) \right] dF_1(y).
\end{aligned}$$

It suffices to prove that the above is positive for $\mu' < \mu_i$ and negative when $\mu' > \mu_i$.

$$q\left(\frac{D(\mu_i, \mu')}{\mu_i}\right) \cdot E[\mu_i - P_1 | Y_1^{(N-1)} = \mu'] + E[P_1 | Y_1^{(N-1)} = \mu'] + \tau \bar{r} D(\mu_i, y)$$

For $\mu_i > \mu'$, we can write

$$\begin{aligned} W(\mu_i, \mu') - L(\mu_i, \mu', \beta(\mu')) &= q(\mu_i, \mu') \mu_i + (1 - q(\mu_i, \mu')) \cdot E[P_1 | Y_1^{(N-1)} = \mu'] + \tau \bar{r} D(\mu_i, \mu') \\ &\quad - E[(1 - q(\mu', \mu'))(\mu_i - P_1) | P_1 \leq \mu'] \\ &= \mu_i + \tau \bar{r} D(\mu_i, \mu') - (2 - q(\mu_i, \mu') - q(\mu', \mu')) \cdot E[(\mu_i - P_1) | P_1 \leq \mu'] \end{aligned}$$

Taking the derivative of the above expression in respect to μ_i we have:.

$$\begin{aligned} &1 - \tau \bar{r} D_1(\mu_i, \mu') - \left(-q'(\mu_i, \mu') \frac{D_1 \mu_i - D}{\mu_i^2} - q(\mu', \mu')\right) \cdot E[(\mu_i - P_1) | P_1 \leq \mu'] - (2 - q(\mu_i, \mu') - q(\mu', \mu')) \\ &= 1 - \tau \bar{r} D_1(\mu_i, \mu') - \left(-q'(\mu_i, \mu') \frac{D_1 \mu_i - D}{\mu_i^2} - q(\mu', \mu')\right) \cdot E[(\mu_i - P_1) | P_1 \leq \mu'] - (2 - q(\mu_i, \mu') - q(\mu', \mu')) \end{aligned}$$

But, from equation (4.4.2),

□

C.1.5. Proof of Proposition 12.

PROOF. The expression 4.4.1 can be rewritten as:

$$D(y, \mu_i) = \frac{1}{2} \frac{\tau \bar{r} y^2}{y - \frac{N-2}{N-1} \cdot \mu_i}.$$

We can write the value of losing as:

$$L(\mu_i, y, \beta(\mu_i)) = E \left[\left[\frac{1}{2} \frac{\tau \bar{r} y}{y - \frac{N-2}{N-1} \cdot \mu_i} \right]^2 (\mu_i - Y_2^{(N-1)}) \middle| Y_2^{(N-1)} \leq \min\{\mu_i, y\} \right] \cdot \left(1 + \left(\left(\frac{\mu_i}{y} \right)^{N-2} - 1 \right) I_{\frac{\mu_i}{y} < 1} \right).$$

which implies that:

$$\begin{aligned} L(\mu_i, \mu_i, \beta(\mu_i)) &= \left[\frac{\tau \bar{r} (N-1)}{2} \right]^2 \frac{\mu_i}{N-1} \cdot \left(\frac{\mu_i}{\mu_i} \right)^{N-2} \\ &= \mu_i \left(\frac{\tau \bar{r}}{2} \right)^2 (N-1). \end{aligned}$$

and the value of winning is written as:

$$W(\mu_i, y) = \left[1 - \left(\frac{1}{2} \frac{\tau \bar{r} \mu_i}{\mu_i - \frac{N-2}{N-1} \cdot y} \right)^2 \right] \cdot E \left[\max \left\{ \mu_i, Y_2^{(N-2)} \right\} \middle| Y_1^{(N-1)} = y \right] + \left(\frac{1}{2} \frac{\tau \bar{r} \mu_i}{\mu_i - \frac{N-2}{N-1} \cdot y} \right)^2 \cdot \frac{N-2}{N-1} y + \tau \bar{r} \cdot \left(\frac{1}{2} \frac{\tau \bar{r} \mu_i^2}{\mu_i - \frac{N-2}{N-1} \cdot y} \right).$$

Evaluated at equilibrium:

$$\begin{aligned} W(\mu_i, \mu_i) &= \left[1 - \left(\frac{(N-1) \tau \bar{r}}{2} \right)^2 \right] \mu_i + \left(\frac{(N-1) \tau \bar{r}}{2} \right)^2 \frac{N-2}{N-1} \mu_i + \tau \bar{r} \left(\frac{(N-1) \tau \bar{r} \mu_i}{2} \right) \\ &= \mu_i - \mu_i \left(\frac{\tau \bar{r}}{2} \right)^2 (N-1) + \tau \bar{r} \left(\frac{\tau \bar{r} (N-1) \mu_i}{2} \right). \end{aligned}$$

Therefore, the net value of acquiring the security in the first stage paying $\beta(\mu_i)$ for it is given by:

$$W(\mu_i, \mu_i) - L(\mu_i, \mu_i, \beta(\mu_i)) = \mu_i - \mu_i \left(\frac{\tau \bar{r}}{2} \right)^2 (N-1) + \tau \bar{r} \left(\frac{\tau \bar{r} (N-1) \mu_i}{2} \right) - \mu_i \left(\frac{\tau \bar{r}}{2} \right)^2 (N-1).$$

The first parameter is simply the expected value the security will bring for you. Since there is a probability of earlier liquidation, the value is strictly lower than μ_i . The second parameter is the per dollar value of tax benefits i is gets evaluated at his optimal debt level when he pays $\beta(\mu_i)$ for the security. Moreover, the derivative of the loss function evaluated at μ_i is written as:

$$\begin{aligned} \left. \frac{d}{d\mu} L(\mu_i, y, \beta(\mu)) \right|_{\mu=\mu_i} &= E \left[-q' (D(y, \mu_i)) \cdot D_2(y, \mu_i) (\mu_i - Y_2^{(N-1)}) \middle| Y_2^{(N-1)} \leq \mu_i \right] \cdot \left(\frac{\mu_i}{y} \right)^{N-2} \\ &= E \left[\left[\frac{(\tau \bar{r} y)}{\left(y - \frac{N-2}{N-1} \cdot \mu_i \right)} \cdot \frac{1}{2} \frac{N-2}{N-1} \frac{(\tau \bar{r} y)}{\left(y - \frac{N-2}{N-1} \cdot \mu_i \right)^2} \right] \cdot (\mu_i - Y_2^{(N-1)}) \middle| Y_2^{(N-1)} \leq \mu_i \right] \cdot \left(\frac{\mu_i}{y} \right)^{N-2} \\ &= \frac{1}{2} \frac{(\tau \bar{r} y)^2}{\left(y - \frac{N-2}{N-1} \cdot \mu_i \right)^3} \cdot \frac{N-2}{N-1} \cdot \frac{\mu_i}{N-1} \left(\frac{\mu_i}{y} \right)^{N-2}. \end{aligned}$$

implying on

$$\begin{aligned}
\beta(\mu_i) &= W(\mu_i, \mu_i) - L(\mu_i, \mu_i, \beta(\mu_i)) + \int_{\mu_i}^1 \left(\frac{1}{2} \frac{(\tau \bar{r} y)^2}{\left(y - \frac{N-2}{N-1} \cdot \mu_i\right)^3} \frac{N-2}{N-1} \frac{\mu_i}{N-1} \left(\frac{\mu_i}{y}\right)^{N-2} \right) \cdot \left(\frac{y}{\mu_i}\right)^{N-2} dy \\
&= W(\mu_i, \mu_i) - L(\mu_i, \mu_i, \beta(\mu_i)) + \int_{\mu_i}^1 \frac{1}{2} \frac{(\tau \bar{r} y)^2}{\left(y - \frac{N-2}{N-1} \cdot \mu_i\right)^3} \cdot \frac{N-2}{N-1} \frac{\mu_i}{N-1} dy \\
&= \mu_i - 2\mu_i \left(\frac{\tau \bar{r}}{2}\right)^2 (N-1) + \tau \bar{r} \left(\frac{\tau \bar{r} (N-1) \mu_i}{2}\right) + \\
&\quad + \mu_i \frac{\tau \bar{r}}{2} \frac{N-2}{(N-1)^2} \cdot \left[\frac{(N-1)^2}{2} \left[1 - \frac{1}{N-1 - (N-2) \mu_i} \right] + \ln \left(\frac{N-1 - (N-2) \mu_i}{\mu_i} \right) + \frac{(N-2)^2 (1 - \mu_i)}{N-1 - (N-2) \mu_i} \right]
\end{aligned}$$

The expression in the last line is the "bankrupt your opponent" component. It is easy to see that it will always be positive and equal to zero whenever $\mu_i \in \{0, 1\}$.

Finally, we need to check whether the bid function described above is strictly increasing μ_i , therefore consistent with the initial conjecture. It turns out that, as long as $\tau \bar{r}$ is low enough so the debt solution is interior, the expression above is strictly increasing. $\square$