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Perfectoid covers of abelian varieties and the weight-monodromy conjecture

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Perfectoid covers of abelian varieties and the weight-monodromy conjecture

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by

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The dissertation of Peter Wear is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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ABSTRACT OF THE DISSERTATION

Perfectoid covers of abelian varieties and the weight-monodromy conjecture

by

Peter Wear

Doctor of Philosophy in Mathematics

University of California San Diego, 2020

Professor Kiran S. Kedlaya, Chair

Deligne's weight-monodromy conjecture gives control over the poles of local factors of L-functions of varieties at places of bad reduction. His proof in characteristic p was a step in his proof of the generalized Weil conjectures. Scholze developed the theory of perfectoid spaces to transfer Deligne's proof to characteristic 0, proving the conjecture for complete intersections in toric varieties. Building on Scholze's techniques, we prove the weight-monodromy conjecture for complete intersections in abelian varieties.

Chapter 1

Introduction

1.1 Motivation: The Weil conjectures

The Weil conjectures [Wei49] motivated much of the development of arithmetic geometry. Let p be a prime, and let X be a variety over the finite field $\mathbb{F}_p$. Roughly speaking, X is the zero locus of some polynomials defined over $\mathbb{F}_p$. The variety X therefore has finitely many $\mathbb{F}_p$-points, and these aren't hard to compute by hand when the numbers are small enough: one can just plug in every possible value for every variable, and see when the polynomials all vanish. If we extend the base field to some larger (but still finite) field $\mathbb{F}_{p^r}$, computing the set $X(\mathbb{F}_{p^r})$ is still theoretically possible, but as r grows our naïve approach eventually becomes untenable. Still, one might wonder how these sets grow as r goes to ∞. Weil wrapped the numbers $\#X(\mathbb{F}_{p^r})$ up into a generating function, and made some remarkable conjectures (more generally, for any finite field $\mathbb{F}_q$).

The *zeta function of* X is defined via the formula

$$\zeta(X,s) = \exp\Big(\sum_{r=1}^{\infty} \frac{\#X(\mathbb{F}_{p^r})}{r} p^{-rs}\Big).$$

To simplify notation, we'll let $T = p^{-s}$, so that $\zeta(X,s)$ is a power series in $\mathbb{Q}[[T]]$. Weil conjectured that if X is smooth and projective with dimension d, this power series should represent a rational function. More precisely, he predicted a factorization

$$\zeta(X,T) = \frac{P_1(T)\cdots P_{2d-1}(T)}{P_0(T)\cdots P_{2d}(T)}$$

such that the (complex) roots of P_i are algebraic integers with absolute value $p^{i/2}$. He predicted a functional equation that ζ should satisfy. If X is the reduction of a smooth projective variety Y defined over the rational numbers $\mathbb{Q}$, he predicted that the degrees of the P_i should correspond to the Betti numbers of $Y(\mathbb{C})$.

These conjectures gave an analogy between these zeta functions and the Riemann zeta function. The statement about the roots of the P_i is the analogue of the Riemann hypothesis: it says that the zeros of the ith polynomial should all have real part $i/2$. The conjectures also suggested that cohomological techniques from topology should be developed for algebraic geometry, and that geometric ideas coming from the study of varieties over $\mathbb{C}$ should apply over more general fields. As a quick application, we note that one now only has to compute $\#X(\mathbb{F}_{p^r})$ for finitely many r to determine the polynomials $P_0,\ldots,P_{2d}$, and therefore the whole function $\zeta(X,s)$.

The theory of étale cohomology was developed for the express purpose of proving these conjectures. Grothendieck and his collaborators used étale cohomology to prove everything but the Riemann hypothesis in the 1960s. Deligne finished off the Riemann hypothesis in the 1970, again using étale cohomology, so the Weil conjectures are now a theorem.

It's natural to wonder what should happen when we relax the conditions on X. This is particularly useful if we want to study varieties over $\mathbb{Q}$ (or other number fields). The L-function of a variety Y is defined as a product of local zeta factors over each prime, and is a fundamental object of study in the Langlands program. There are many important conjectures about L-functions: they should come from automorphic representations, satisfy a functional equation, and contain lots of information about the variety Y. For example, the Birch and Swinnerton-Dyer conjecture says that the rank of the group of rational points on an elliptic curve E should be given by the order of the zero of the L-function $L(E,s)$ associated to E at $s=1$. A stronger version of the conjecture gives an interpretation of the leading coefficient of the Taylor series at 1 in terms of fundamental invariants of E.

When Y has a model over $\mathbb{Z}$ which reduces to a smooth projective variety over $\mathbb{F}_p$, the Weil conjectures give us the local factor at p, and tell us a lot about its behavior. The Riemann hypothesis implies that the poles of the ith local factors at such primes will have real part $i/2$. But there will generally be finitely many primes over which there is no such model, and we need to know information at every prime to get global information. Deligne's weight-monodromy conjecture concerns these primes: it implies that even at primes of bad reduction, all of the poles of the ith piece of the local zeta function will have real part at most $i/2$.

1.2 The weight-monodromy conjecture

We now give an overview of how these local zeta factors are constructed via étale cohomology, and give a precise statement of the weight-monodromy conjecture. For more details on the setup and proofs, see [Del80, Sections 1.6-1.8]; for the conjecture itself see [Del71]. Let X be a proper smooth d-dimensional variety over $\mathbb{Q}$ (for simplicity - everything works over a general number field), let p and ℓ be distinct primes, and let i be an integer between 0 and $2d$. Let $V := H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_\ell)$ be the ith ℓ-adic cohomology group: this is a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space. The absolute Galois group $G_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts on $X_{\overline{\mathbb{Q}}_p}$. This induces an action on V, so we obtain a Galois representation

$$\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}(V).$$

This is a fundamental object of study in arithmetic geometry. Roughly speaking, this representation is controlled by two parts: the action of Frobenius and the monodromy operator.

Frobenius comes from the residue field, and is used to define the weights of the representation, while the inertia in the Galois group leads to monodromy. As a first approximation, the weight-monodromy conjecture says that these two facets of the representation interact nicely.

Recall that the inertia subgroup $I_{\mathbb{Q}_p}$ of $G_{\mathbb{Q}_p}$ is defined via the short exact sequence

$$1 \to I_{\mathbb{Q}_p} \to G_{\mathbb{Q}_p} \to G_{\mathbb{F}_p} \to 1.$$

Fix an element $\Phi \in G_{\mathbb{Q}_p}$ which maps to geometric Frobenius: the map $\overline{\mathbb{F}}_p \to \overline{\mathbb{F}}_p$ sending $x \mapsto x^{1/p}$. The local zeta factor of X at p is defined as $\det(1 - p^{-s}\Phi | V^{I_{\mathbb{Q}_p}})$. This polynomial in p^{-s} doesn't depend on the choice of Φ because we are only acting on the part of V which is fixed by the action of inertia, and any two lifts of geometric Frobenius differ by inertia.

To motivate weight-monodromy and relate these concepts back to the Weil conjectures, we first consider the good reduction case and point out some nice features. That is, assume we have a proper smooth model $\mathfrak{X}$ over $\mathbb{Z}_p$, and let $\widetilde{\mathfrak{X}}/\mathbb{F}_p$ be the special fiber. Then by proper smooth base change, we have a Galois-equivariant isomorphism

$$H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_\ell) \cong H^i_{\text{ét}}(\widetilde{\mathfrak{X}}_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell),$$

where the right side has an action of $G_{\mathbb{F}_p}$. In this case, we see that the inertia group acts trivially on V. The local zeta factor is now $\det(1 - p^{-s}\Phi | V)$, which is precisely the polynomial $P_i(T)$ seen in the Weil conjectures. In this case, we have seen that the roots are algebraic integers with absolute value $p^{i/2}$ for any embedding into $\mathbb{C}$. We say that the roots are *Weil numbers of weight i*.

In the bad reduction case, the inertia group acts non-trivially on V. Grothendieck showed that there is a finite index subgroup of $I_{\mathbb{Q}_p}$ which acts on V via unipotent matrices. This is a general fact about Galois representations: nothing special is used about étale cohomology. The proof leads to a unique nilpotent operator $N : V \to V$ called the *monodromy operator*, coming from the action of the pro-ℓ part of the tame inertia. This operator gives rise to the *monodromy* *filtration* on V: a finite increasing filtration $\{V_j\}_{j\in\mathbb{Z}}$ such that $N(V_j) \subset V_{j-2}$, and such that N^j induces an isomorphism between the $(i-j)$th and $(i+j)$th graded pieces of the filtration.

The other half of the picture is the weight filtration. The image under ρ of our lift Φ of geometric Frobenius is a matrix $\rho(\Phi) \in \mathrm{GL}(V)$, which we can use to decompose V into eigenspaces. A few things are known about this decomposition, this time using very deep facts about étale cohomology. We can write the decomposition

$$V = \oplus_{k=0}^{2i} W'_k,$$

where the eigenvalues of Φ acting on W'_k are Weil numbers of weight k. If we chose a different lift of Frobenius, we get a different decomposition of V. However, the weights and dimensions remain the same, as well as the *weight filtration* defined by

$$W_j := \oplus_{k=0}^{j} W'_k.$$

For any choice of Φ, we have $N\Phi = p\Phi N$, so the monodromy operator interacts nicely with the weight filtration: $NW_j \subset W_{j-2}$.

Conjecture 1.2.1 (Deligne, [Del71])**.** *For a proper smooth variety over a local field, the weight filtration is the same as the monodromy filtration.*

Deligne proved his conjecture over characteristic p local fields (under a simplifying assumption which was later removed in [Ter98] and [Ito05]). In mixed characteristic, many special cases are known - see the introduction of [Sch12] for an overview - but the general conjecture is open. In his thesis [Sch12], Scholze introduced perfectoid spaces as a bridge between characteristics 0 and p. As the first of many applications, he proved the weight-monodromy conjecture for complete intersections in toric varieties by using perfectoid spaces to reduce to the characteristic p case that Deligne settled. The goal of this thesis is to extend Scholze's techniques from toric varieties to abelian varieties, resulting in the following theorem.

Theorem 1.2.2. *Let k be a local field of residue characteristic p. Let A be an abelian variety over k, let Y be a geometrically connected, proper smooth variety which is a set-theoretic complete intersection in A. Then the weight-monodromy conjecture holds for Y.*

When Y has dimension at least 3, the varieties that are provably handled by Theorem 1.2.2 are different from those provably handled by Scholze's theorem: see Remark 7.2.4 for details.

1.3 Scholze's approach via perfectoid spaces

Our proof of Theorem 1.2.2 follows the same path as Scholze's proof for toric varieties in [Sch12, Theorem 9.6]. We therefore sketch his proof here in the special case of a hypersurface Y in projective space $\mathbb{P}^n$ defined over $\mathbb{Q}_p$. We give more precise details and references on perfectoid spaces in Chapter 3. A more detailed version of this argument (in the context of abelian varieties) is given in Chapter 7.

Let $K = \widehat{\mathbb{Q}_p(p^{1/p^\infty})}$, the p-adic completion of the field obtained by adjoining all pth power roots of p to $\mathbb{Q}_p$. This is an example of a *perfectoid field*. Take the base change to Y_K and consider the corresponding G_K-representation on étale cohomology. This doesn't change the weights as any lift of geometric Frobenius in G_K gives one in $G_{\mathbb{Q}_p}$. It preserves the monodromy operator as it doesn't affect the pro-ℓ part of the Galois group. We can therefore work towards weight-monodromy after making this base change.

The *tilt* $K^\flat := \varprojlim_{x\mapsto x^p} K$ of K is a perfectoid field of characteristic p. In this case, $K^\flat \cong \widehat{\mathbb{F}_p((t^{1/p^\infty}))}$, where we are taking the t-adic completion. The starting point of the theory of perfectoid spaces is the Fontaine-Wintenberger isomorphism of Galois groups $G_K \cong G_{K^\flat}$. Deligne tells us that weight-monodromy holds for $G_{K^\flat}$-representations, and this isomorphism gives us a chance of turning these into the G_K-representations that we hope to study.

The theory of perfectoid spaces geometrizes this isomorphism. The spaces we need live in the category of adic spaces: a very general category of geometric spaces which are topologized using the metric topology coming from $\mathbb{Q}_p$. This is an analogue of considering a complex variety as a complex manifold, replacing the Zariski topology with the metric topology coming from $\mathbb{C}$. See Chapter 2 for more details and references on p-adic geometry.

To construct a concrete example of a perfectoid space, we consider the map of adic spaces $\varphi : \mathbb{P}_K^{n,\mathrm{ad}} \to \mathbb{P}_K^{n,\mathrm{ad}}$ defined on points by sending $(x_0 : \cdots : x_n) \mapsto (x_0^p : \cdots : x_n^p)$. Iterating this map to get an inverse system, we get a perfectoid space

$$\mathbb{P}_K^{n,\mathrm{perf}} \sim \varprojlim_{\varphi} \mathbb{P}_K^{n,\mathrm{ad}}.$$

As we discuss in Section 2.3, the category of adic spaces doesn't have inverse limits - this notation means that $\mathbb{P}_K^{n,\mathrm{perf}}$ is as close as we can get. This space is built out of adic spaces associated to the perfectoid K-algebra $K\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle$, where the angled brackets denote the p-adic completion.

While $\mathbb{P}_K^{n,\mathrm{perf}}$ isn't an inverse limit of adic spaces, we get inverse limits if we restrict to the underlying topological spaces or étale topoi. This allows topological and étale cohomological arguments to work in this setting. The tilt $(\mathbb{P}_K^{n,\mathrm{perf}})^\flat$ is given by the analogous construction $\mathbb{P}_{K^\flat}^{n,\mathrm{perf}}$. The story over $K^\flat$ is even simpler, as the map $\varphi_{K^\flat} : \mathbb{P}_{K^\flat}^{n,\mathrm{ad}} \to \mathbb{P}_{K^\flat}^{n,\mathrm{ad}}$ induces a homeomorphism of topological spaces and an isomorphism of étale topoi.

Putting all of this together, we get a projection map $\pi : \mathbb{P}_{K^\flat}^{n,\mathrm{ad}} \to \mathbb{P}_K^{n,\mathrm{ad}}$ on the level of either level of topological spaces and étale topoi, which is Galois-equivariant under the Fontaine-Wintenberger isomorphism. This suggests the following method for proving weight-monodromy: take a variety over K, embed it in projective space, pull back along π to get something over $K^\flat$, and apply Deligne's result. The issue is, the pullback won't be a variety: it'll be a transcendental fractally mess.

Scholze was able to fix this problem for hypersurfaces (and intersections of hypersurfaces). Huber showed that for our hypersurface $Y \subset \mathbb{P}_K^{n,\mathrm{ad}}$, there is a small open neighborhood $U \supset Y$ with the same ℓ-adic étale cohomology as Y. Scholze proved an approximation lemma, constructing a hypersurface Z inside $\pi^{-1}(U)$.

Working through the definitions, this gives a Galois-equivariant map

$$H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_\ell) \to H^i_{\text{ét}}(Z_{\overline{\mathbb{F}_p((t))}}, \overline{\mathbb{Q}}_\ell).$$

By Poincare duality and the compatibility of this map with cup products, the left side is a direct summand of the right side, which we know satisfies weight-monodromy.

1.4 Outline

With the preceding sketch in hand, we now have enough context give a full outline of this thesis. In Chapter 2, we give background on the various approaches to non-archimedean geometry that we will use throughout. We have already seen that adic spaces are needed to define perfectoid spaces, but much of our actual work will be done in the more classical categories of rigid analytic spaces and formal schemes. In particular, we recall Raynaud's uniformization of abelian varieties - the non-archimedean analogue of the description of complex abelian varieties as a quotient of $\mathbb{C}^d$ by a lattice - and Raynaud's description of rigid spaces as the generic fibers of formal schemes. We build off this in Chapter 3 to define perfectoid spaces, give some basic results, and explain how to construct perfectoid spaces as inverse limits of rigid spaces via what we call "F-towers": certain inverse systems of formal schemes.

In Chapter 4, we use F-towers to construct perfectoid covers of abelian varieties (more generally, of abeloids - the nonarchimedean analogue of complex tori) giving an alternate proof of the following theorem.

Theorem 1.4.1 (Blakestad, Gvirtz, Heuer, Shchedrina, Shimizu, W., Yao)**.** *Let A be an abelian variety (or abeloid) over a perfectoid field K of residue characteristic p with value group contained in $\mathbb{Q}$. There is a perfectoid group A_∞ such that $A_\infty \sim \varprojlim_{[p]} A$.*

We note that this theorem was already known when A has good reduction. The proof in [BGH+18] assumes that K is algebraically closed but doesn't have the restriction on the value group; we need the additional generality for our application to weight-monodromy. The cover A_∞ of A is our analogue of the perfectoid cover $\mathbb{P}_K^{n,\mathrm{perf}}$ of $\mathbb{P}_K^n$. In the case of projective space, the perfectoid cover was relatively easy to build out of explicit perfectoid K-algebras, obtained by iterating a lift φ of Frobenius. This isn't an option for abelian varieties: it's very hard to write down explicit formulas for them, and there isn't always a lift of Frobenius. The restriction on the value group allows us to construct formal models of A using methods from tropical geometry.

In Chapter 5, we begin our study of the tilt $A_\infty^\flat$. For $\mathbb{P}_K^{n,\mathrm{ad}}$, there was a natural interpretation of the tilt: as the perfectoid space sitting above $\mathbb{P}_{K^\flat}^{n,\mathrm{ad}}$. This again could be checked by explicitly working with the perfectoid K-algebras involved. There isn't such an obvious candidate for an abelian variety over $K^\flat$ that gives rise to $A_\infty^\flat$. Roughly speaking, a perfectoid space and its tilt will have isomorphic special fibers. To understand $A_\infty^\flat$, we take the special fiber of the F-tower used to define A_∞ and deform it to get the F-tower of an abeloid A' over $K^\flat$. This gives

Theorem 1.4.2 (Heuer, W.)**.** *Let A and A_∞ be as in Theorem 1.4.1. Then after a pro-p extension of K, there is a (non-unique) abeloid A' over $K^\flat$ such that $A_\infty^\flat \sim \varprojlim_{[p]} A'$.*

In Chapter 6, we explain how to transfer line bundles from A to (a suitable choice of) A' by constructing perfectoid covers of the associated $\mathbb{G}_m$-torsors. This allows us to algebraize the previous theorem: if we start with an abelian variety A, we can choose A' to be an abelian variety as well. Chapters 4, 5, and 6 are largely based on work done jointly with Ben Heuer. We plan to include many of these results in a future joint publication. We expect that the proofs and approach in this thesis will generally be more explicit and less conceptual than those in the published version.

In Chapter 7, we prove an analogue of Scholze's approximation lemma, which we use to prove Theorem 1.2.2. The cohomological aspects of the proof are similar to Scholze's argument, with one addition necessary. As the map $[p] : A' \to A'$ is not a homeomorphism and doesn't induce an isomorphism of étale sites, there is no straightforward analogue of Scholze's map $\pi : \mathbb{P}^n_{K^\flat} \to \mathbb{P}^n_K$ at finite level. Our computations therefore have to pass through the perfectoid cover $A^\flat_\infty$ more explicitly.

Chapter 2

Background on non-archimedean geometry

The motivation for non-archimedean geometry comes from the interplay between algebraic and analytic geometry over the complex numbers. Algebraic geometers often study complex algebraic varieties: the set of solutions of systems of polynomials with coefficients in $\mathbb{C}$. In nice cases, the complex points of such a variety can be given the structure of a complex manifold, called the analytification of the variety. These spaces have dramatically different topologies - the Zariski topology keeps track of where polynomials vanish, while the analytic topology comes from the metric topology on $\mathbb{C}$ - but still share lots of information. Serre's GAGA theorems (and generalizations) relate nice algebraic varieties, morphisms, and sheaves to complex manifolds, holomorphic maps, and analytic sheaves, in a way that respects cohomology.

A motivating example from complex geometry which will play an essential role in this thesis is the uniformization of elliptic curves. An elliptic curve over $\mathbb{C}$ is defined by an equation of the form $y^2 = x^3 + ax + b$ for some complex numbers a and b. The complex points of an elliptic curve can be given the structure of an abelian group. The analytification of an elliptic curve is a complex torus: it can be uniformized as an analytic quotient of the form $\mathbb{C}/\Lambda$ for a lattice Λ. This point of view is very useful: for example when studying moduli spaces of elliptic curves. It can be generalized to complex abelian varieties - algebraic varieties with commutative group laws - writing them all as higher dimensional complex tori.

The goal of non-archimedean geometry is to use these ideas to study varieties over a field which is complete with respect to a non-archimedean absolute value. Standard examples of non-archimedean fields include the p-adics $\mathbb{Q}_p$, their completed algebraic closure $\mathbb{C}_p$, and the characteristic p field $\mathbb{F}_p((t))$. The perfectoid fields $\mathbb{Q}_p\widehat{(p^{1/p^\infty})}$ and $\mathbb{F}_p\widehat{((t^{1/p^\infty}))}$ from the introduction are also examples. Such fields arise naturally in arithmetic geometry. As we saw in the construction of the local factors of L-functions in the introduction, they let us analyze varieties over global fields such as $\mathbb{Q}$ by focusing on one prime at a time.

When trying to transfer results and techniques from $\mathbb{C}$, technical issues arise from the non-archimedean valuation: all triangles are isosceles with respect to the metric topology, closed balls are open sets, and this causes far too many functions to be holomorphic if one tries to simple replace $\mathbb{C}$ by $\mathbb{C}_p$ in the definitions. Tate solved these issues in the 1960s, introducing rigid analytic spaces to study p-adic analogues of the uniformization of complex elliptic curves. He showed that elliptic curves over $\mathbb{C}_p$ can be written as quotients of the form $\mathbb{C}_p^\times/\langle q\rangle$ for $q \in \mathbb{C}_p$. Since then, many mathematicians have reformulated and generalized these ideas. Raynaud interpreted rigid spaces as generic fibers of formal schemes, and also generalized Tate's uniformization to abelian varieties. Berkovich's theory has nicer topological properties, while Huber's adic spaces are more general and flexible.

In this chapter, we give a brief introduction to the various theories of non-archimedean geometry that we will need throughout this thesis. We refer to [Bos14] for a more detailed introduction to the field, [Lüt16] for a complete overview of the applications to abelian varieties, and either [Sch12, Section 2] or [Wei19] for an introduction to adic spaces with an eye towards perfectoid spaces.

2.1 Rigid analytic spaces

See [Bos14] for a concise introduction to rigid analytic geometry, or [FvdP04] for an introduction with more examples and applications. Here we will just give some examples.

We recall the definition of a non-archimedean field and fix some notation.

Definition 2.1.1. *A non-archimedean absolute value on a field K is a map $|\cdot| : K \to \mathbb{R}_{\geq 0}$ satisfying the following properties for all a, b in K:*

1. $|a| = 0$ *if and only if* $a = 0$
2. $|ab| = |a||b|$
3. $|a+b| \leq \max\{|a|, |b|\}$.

The last condition is called the *non-archimedean triangle inequality*. Given a non-archimedean absolute value, we have the associated *valuation* $v : K \to \mathbb{R} \cup \{\infty\}$, setting $v(a) = -\log|a|$. We will be passing between these points of view regularly (and possibly implicitly) in the following.

Example 2.1.2. The field $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the absolute value associated to the *p-adic valuation*, which sends the (non-zero) rational number $p^n \frac{r}{s}$ to $-n$ (with $n \in \mathbb{Z}$ and r, s coprime to p), and sends 0 to ∞.

Notation 2.1.3. Throughout this thesis, K will always denote a non-archimedean field, complete with respect to the absolute value $|\cdot|_K$ (or $|\cdot|$ when no confusion seems likely) which corresponds to the valuation $v_K = v$. We let $\Gamma_K := v(K)$ denote the *value group* of K. We let K° denote the *power bounded* elements of K: the set $x \in K$ such that $\lim_{n\to\infty} |x^n| < \infty$. In this case, this is just the ring of integers, but we use similar notation later on for more general rings. We let $\widetilde{K}$ denote the residue field.

In algebraic geometry, varieties are built out of geometric spaces made from rings. A fundamental example is n-dimensional affine space $\mathbb{A}^n_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[T_1,\ldots,T_n])$. Here points correspond to prime ideals in the polynomial ring $\mathbb{C}[T_1,\ldots,T_n]$, and the closed points - tuples $(z_1,\ldots,z_n) \in \mathbb{C}^n$ - correspond to maximal ideals $(T_1 - z_1,\ldots,T_n - z_n)$. Quotienting out by ideals $\mathfrak{a}$ of the ring gives subspaces $V(\mathfrak{a}) = \mathrm{Spec}(\mathbb{C}[T_1,\ldots,T_n]/\mathfrak{a}) \subset \mathbb{A}^n_{\mathbb{C}}$. Gluing together spaces of this form leads to more general varieties. The ring $\mathbb{C}[T_1,\ldots,T_n]/\mathfrak{a}$ is the ring of functions on the affine variety $V(\mathfrak{a})$.

Rigid spaces are built via a similar procedure, by gluing together affinoid spaces which are determined by their rings of functions. The fundamental example in this case doesn't just allow polynomial functions, it adds in power series that converge with respect to the non-archimedean absolute value on K.

Definition 2.1.4. *The n-dimensional* Tate algebra

$$K\langle T_1,\ldots,T_n\rangle := \Big\{ \sum_{I=(i_1,\ldots,i_n)\in\mathbb{Z}^n_{\geq 0}} a_I T_1^{i_1}\cdots T_n^{i_n} : \lim_{I\to\infty} |a_I| = 0 \Big\}$$

is the completion of $K[T_1,\ldots,T_n]$ *with respect to the* Gauss norm

$$|\cdot| : K[T_1,\ldots,T_n] \to \mathbb{R}_{\geq 0}; \quad \sum a_I T^I \mapsto \max\{|a_I|\}.$$

Points of the associated affinoid space $\mathrm{Sp}(K\langle T_1,\ldots,T_n\rangle)$ correspond to *maximal* ideals of the Tate algebra. Geometrically, they correspond to tuples $(z_1,\ldots,z_n) \in K^{\circ,n}$. So $|z_i| \leq 1$ for all i and our fundamental affinoid space corresponds to the closed unit disk. (Roughly speaking, if we try to evaluate our convergent power series at a point outside the disk, it might no longer converge.)

More generally, affinoid algebras are defined as quotients $K\langle T_1,\ldots,T_n\rangle/\mathfrak{a}$ for ideals $\mathfrak{a} \subset K\langle T_1,\ldots,T_n\rangle$, and rigid spaces are built by gluing affinoid spaces $\mathrm{Sp}(K\langle T_1,\ldots,T_n\rangle/\mathfrak{a})$. We give some one-dimensional examples, and hope that this is enough to give the flavor of the theory.

Example 2.1.5. Take an element $y \in K^\times$ with $|y| = c \leq 1$. The *closed annulus* $\mathrm{An}(c,1)$ - defined geometrically as the points $z \in K$ with $c \leq |z| \leq 1$ - is an affinoid space. Functions on $\mathrm{An}(c,1)$ are convergent Laurent series

$$K\left\langle T, \frac{c}{T} \right\rangle := \Big\{ \sum_{i \in \mathbb{Z}} a_i T^i : \lim_{i \to \infty} |a_i| = 0, \lim_{i \to -\infty} \frac{|a_i|}{c^i} = 0 \Big\}.$$

The notation and convergence conditions as $i \to \infty$ are saying that the term $\frac{c}{T}$ has "weight" 1, and ensure that the "negative tails" of these series will converge when we plug in points z with $|z| \geq c$. We can present this ring explicitly as a quotient of a Tate algebra

$$\mathrm{An}(c,1) := \mathrm{Sp}(K\langle T_1, T_2 \rangle / (T_1 T_2 - y)).$$

Whenever we make presentations of this type, the heuristic is that the variable T_1 will act as a dummy variable which fixes where the absolute value 1 is, and the variable T_2 (and any other variables) will correspond to the weighted monomials used in the Laurent series.

If we instead choose c not in the value group Γ of K, we can still define the ring of convergent Laurent series via the same formula, but this isn't an affinoid algebra over K. Taking an extension L of K such that $c \in \Gamma_L$, we get an affinoid algebra over L as before.

Example 2.1.6. Take an element $y \in K^\times$ with $|y| = c$. The *closed disk* $D(c,0)$ with radius c centered at 0 is an affinoid space. Functions on $D(c,0)$ are "convergent power series with radius c"

$$K\left\langle \frac{T}{c} \right\rangle := \Big\{ \sum_{i=0}^{\infty} a_i T^i : \lim_{i \to \infty} c^i |a_i| = 0 \Big\},$$

where the notation now gives $\frac{T}{c}$ weight 1. We can again present this ring explicitly as the quotient of a Tate algebra

$$D(c,0) := \mathrm{Sp}(K\langle T_1, T_2 \rangle / (T_1 - \frac{T_2}{y})).$$

If we take $y = 1$, we recover the expected fact that $D(1,0) \cong \mathrm{Sp}(K\langle T\rangle)$. For any y, we have an isomorphism of affinoid algebras

$$K\langle T_1, T_2\rangle/(yT_1 - T_2) \to K\langle T\rangle$$

$$T_1, T_2 \mapsto T, yT$$

which induces an isomorphism $D(1,0) \cong D(c,0)$ given on points by sending $z \mapsto yz$. On functions, it takes a power series $\sum_{i=0}^{\infty} a_i T^i$ of radius c to the power series $\sum_{i=0}^{\infty} a_i y^i T^i$ of radius 1.

Example 2.1.7. The affine line is *not* an affinoid space. One way to construct it as a rigid space is to take the union of an increasing sequence of closed disks. Explicitly, fix $y \in K^{\times}$ with $|y| = c > 1$, and glue the disks $D(c^n, 0)$ to each other via the maps

$$D(c^{n-1}, 0) \hookrightarrow D(c^n, 0)$$

$$K\langle T_1, T_2\rangle/(T_1 - \frac{T_2}{y^{n-1}}) \hookleftarrow K\langle S_1, S_2\rangle/(S_1 - \frac{S_2}{y^n})$$

$$T_1, T_2 \hookleftarrow yS_1, S_2.$$

The map on convergent power series sends $\sum_{i=0}^{\infty} a_i T^i$ to itself as we map S_2 to T_2, and these are the variables we use for our power series. Geometrically, any power series which converges on $D(c^n, 0)$ will still converge when restricted to the subspace $D(c^{n-1}, 0)$. We write the resulting space as $\mathbb{A}_K^{1,\mathrm{an}}$ - the analytification of the affine line. We'll see a different construction in Example 2.2.9.

This construction is the first step towards GAGA results for rigid spaces. More generally, there is a *rigid analytification* functor from the category of schemes of locally finite type over K to rigid spaces over K.

The general procedure for gluing affinoid spaces to get rigid spaces is more subtle than in algebraic geometry. Tate's point of view is that the non-archimedean topology has "too many" open sets, making it totally disconnected. He fixed this problem by defining a Grothendieck topology, allowing only *admissible* covers. We leave the details on this important subtlety to the references ([Bos14, Section 5.1, Definition 4], for example), and just give an example of a non-admissible cover.

Example 2.1.8. The open unit disc $D(1,0)^-$ - defined geometrically as the points $z \in K$ with $|z| < 1$ - is not an affinoid space. An admissible cover of $D(1,0)^-$ is given by the union of closed disks $D(c_n,0)$ for a sequence $\{c_n\}$ with $c_n < 1$ and $\lim_{n\to\infty} c_n = 1$.

We can therefore cover all the points of the closed unit disc $D(1,0)$ by taking the union of $D(1,0)^-$ and the unit circle $\mathrm{An}(1,1)$. This is not an admissible cover: in Example 2.3.5, we'll see that the associated adic space has extra points that aren't covered by any of these opens.

2.2 Formal geometry

Raynaud's perspective on non-archimedean geometry views rigid spaces over K as the generic fibers of formal schemes over K°: roughly speaking, completions of schemes over K° with respect to $|\cdot|$. We will reap many benefits from this perspective. We eventually want to construct perfectoid spaces at the top of inverse systems of rigid spaces. The category of rigid spaces doesn't admit inverse limits, but under reasonable circumstances the category of formal schemes does. We'll also use the formal picture over K° as an intermediate step when we tilt from K to $K^\flat$: a perfectoid space and its tilt have the same special fiber, which we will access through our formal models.

Much of the work in this paper therefore involves analyzing inverse systems of certain rigid spaces by choosing suitable formal models of each rigid space in the system, analyzing the corresponding inverse system of formal schemes, then passing to the generic fiber to make conclusions about the original inverse system. We now collect some basic tools from formal geometry we will need to do this. Good references for all of this material are [Bos14] and [Lüt16, Section 3]. As we will be dealing with relatively nice rigid spaces and formal schemes, we will not need to work in the most general setup possible, but some complications do arise as we can't assume the field we are working over to be discretely valued or algebraically closed.

Just as the absolute value on K extends to the Gauss norm on the Tate algebra in Definition 2.1.4, any reduced affinoid algebra has a supremum norm.

Definition 2.2.1. *Given an affinoid space $X = \mathrm{Sp}(A)$ for some $A = K\langle T_1, \ldots, T_n \rangle / \alpha$, the supremum norm $A \to \mathbb{R}$ is defined as*

$$|f|_{sup} := \max_{x \in X} \{|f(x)|_K\}.$$

That is, a function f on X is sent to the maximum absolute value it attains when evaluated at points of X.

Notation 2.2.2. We define the K°-algebra $A^\circ := \{f \in A : |f|_{\mathrm{sup}} \leq 1\}$, which comes with the maximal ideal $A^{\circ\circ} := \{f \in A : |f|_{\mathrm{sup}} < 1\}$. The quotient $\widetilde{A} := A^\circ / A^{\circ\circ}$ is a reduced finite type $\widetilde{K}$-algebra. We get a *reduction map* $\rho : \mathrm{Sp}(A) \to \mathrm{MaxSpec}(\widetilde{A})$ of topological spaces by sending $x \mapsto x \cap A^\circ \mod A^{\circ\circ}$. This map is surjective and the preimage of a Zariski open in $\mathrm{MaxSpec}(\widetilde{A})$ is an admissible open in $\mathrm{Sp}(A)$.

Example 2.2.3. If A is the Tate algebra $K\langle T_1, \ldots, T_n \rangle$, the supremum norm agrees with the Gauss norm and we have $A^\circ = K^\circ \langle T_1, \ldots, T_n \rangle \subset A$: the set of convergent power series with coefficients in K°. Similarly, $A^{\circ\circ}$ is the set of convergent power series with coefficients in $K^{\circ\circ}$, the maximal ideal of K°. Taking the quotient, we get convergent power series with coefficients in the residue field $\widetilde{K}$. Now all the elements have norm 1, so convergent power series are just polynomials, and we get

$$\widetilde{A} \cong \widetilde{K}[T_1, \ldots, T_n].$$

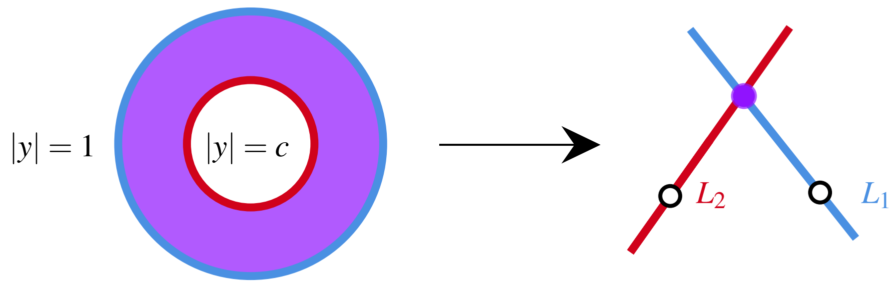

Figure 2.1: The reduction of an annulus

The points of $\mathrm{Sp}(K\langle T_1, \ldots, T_n\rangle)$ correspond to tuples in $K^{\circ,n}$, so the reduction map on points is just the quotient $K^{\circ,n} \to \widetilde{K}^n$.

Example 2.2.4. Continuing Example 2.1.5, we recall that

$$A = K\langle T_1, T_2\rangle/(T_1T_2 - y)$$

is the ring of functions on the annulus $\mathrm{An}(c, 1)$, where $|y| = c \leq 1$. In this case (but not in general) the sup norm agrees with the quotient norm induced by the Gauss norm. So

$$A^\circ = K^\circ\langle T_1, T_2\rangle/(T_1T_2 - y),$$

and the reduction $\widetilde{A}$ depends on whether $c = 1$ or $c < 1$. If $c = 1$, the reduction is isomorphic to $\widetilde{K}[T_1, T_2]/(T_1T_2 - 1)$: the affine line with the origin removed. The reduction map sends a point $z \in K^\circ$ with $|z| = 1$ to its reduction $\overline{z} \in \widetilde{K}^\times$.

If $c < 1$, the reduction is $\widetilde{K}[T_1, T_2]/(T_1T_2)$: two affine lines L_1 and L_2 intersecting at the origin in an ordinary double point. The reduction map when $c < 1$ sends the outer circle $\mathrm{An}(1, 1)$ to the smooth points on L_1 and the inner circle $\mathrm{An}(c, c)$ to the smooth points on L_2. The inside of the annulus is sent to the double point.

Formal models of rigid spaces are built by gluing the formal schemes associated to certain affinoid open covers for which the reduction map respects the Zariski topology.

Definition 2.2.5. *A K°-algebra A^+ is* admissible *if it has no K°-torsion and it is of* topologically finite type*: that is, it can be written as a quotient $K^\circ\langle T_1,\dots,T_n\rangle/\mathfrak{a}$. Fix any* pseudo-uniformizer $\varpi \in K^\circ$ *with $|0| < |\varpi| < 1$, then A° is ϖ-adically complete: we have*

$$A^+ = \varprojlim_{n\in\mathbb{N}} A^+/\varpi^{n+1}A^+.$$

The admissible formal scheme $\mathrm{Spf}(A^+)$ *has the same underlying topological space as the scheme* $\mathrm{Spec}(A^+/\varpi A^+)$ *with the Zariski topology (or* $\mathrm{Spec}(\widetilde{A})$ *if you prefer to work over a field) and structure sheaf*

$$\mathscr{O}_{\mathrm{Spf}(A^+)} := \varprojlim_{n\in\mathbb{N}} \mathscr{O}_{\mathrm{Spec}(A^+/\varpi^{n+1}A^+)}.$$

Having no K°-torsion is equivalent to being flat over K°. A theorem of Raynaud-Gruson (see [Bos14, Section 7.3, Theorem 4]) implies that admissible K-algebras are additionally of *topologically finite presentation*: the ideal α can be chosen to be finitely generated. (Note that when K is not discrete, K° is not noetherian.) For more details, see [Bos14, Sections 7,8]. For a more general construction of formal schemes, see [SW13, Section 2.2].

Definition 2.2.6. *A* formal analytic cover *of a rigid space X is an admissible cover $\{U_i\}$ of X by affinoid opens $U_i = \mathrm{Sp}(A_i)$ such that*

1. *For each algebra A_i, the K°-algebra A_i° is admissible as in Definition 2.2.5.*
2. *For each pair of opens U_i and U_j, the intersection $U_i \cap U_j \subset U_i$ is the preimage of a Zariski open subset of* $\mathrm{MaxSpec}(\widetilde{A}_i)$ *under the reduction map.*

We can now state precisely the construction that we'll need. This is essentially the content of [Lüt16, Remark 3.2.5].

Proposition 2.2.7. *The data of a formal analytic cover of X induces an admissible formal scheme $\mathfrak{X}$ which we call a* formal model *of X. Let $\{U_i\}$ be a formal analytic cover of X and $\{V_k\}$ be a* *formal analytic cover of Y, let $\mathfrak{X}$ and $\mathfrak{Y}$ be the corresponding formal models. Then a morphism of rigid spaces $\varphi : X \to Y$ extends to a morphism of formal models $\overline{\varphi} : \mathfrak{X} \to \mathfrak{Y}$ if for each U_i in the cover of X, there is some V_i in the cover of Y such that $\varphi(U_i) \subset V_i$ and $\varphi(U_i \cap U_j) \subset V_i \cap V_j$ for all i, j. Conversely, given any cover of $\mathfrak{X}$ by formal affine opens, the generic fiber is a formal analytic cover of X.*

We just sketch the construction and leave details to the references. The formal scheme $\mathfrak{X}$ is constructed by gluing together the affine formal schemes $\mathrm{Spf}(A_i^\circ)$ along the intersections $\mathrm{Spf}(A_{ij}^\circ)$. The morphism $\mathfrak{X} \to \mathfrak{Y}$ is defined by gluing the morphisms $\mathrm{Spf}(A_i^\circ) \to \mathrm{Spf}(B_i^\circ)$ induced by the morphisms $\mathrm{Sp}(A_i) \to \mathrm{Sp}(B_i)$. Raynaud showed the much more difficult and interesting statement that under reasonable conditions any rigid space has such a formal model, see [Bos14, Section 8.4]. Choosing a finer formal analytic cover of X corresponds to taking a blowup on the special fiber, see Figure 4.4 for an example.

Example 2.2.8. The analytification of $\mathbb{P}^1_K$ can be constructed from a formal analytic cover consisting of two unit disks. We let

$$D_0 = \{(z_0 : z_1) : |z_0| \leq |z_1|\}, \quad D_1 = \{(z_0 : z_1) : |z_0| \geq |z_1|\}$$

and glue along the boundary circle

$$D_{01} = \{(z_0 : z_1) : |z_0| = |z_1|\}.$$

The reduction of the corresponding model is $\mathbb{P}^1_{\widetilde{K}}$. However, there are many other possible formal models of $\mathbb{P}^{1,\mathrm{an}}_K$.

Example 2.2.9. The construction of $\mathbb{A}^{1,\mathrm{an}}_K$ given in Example 2.1.7 uses a cover which is admissible but not formal: the reduction map on $D(c^n, 0)$ sends $D(c^{n-1}, 0)$ to the origin in $\mathrm{Spec}(k[\frac{T}{c^n}])$, which is Zariski-closed.

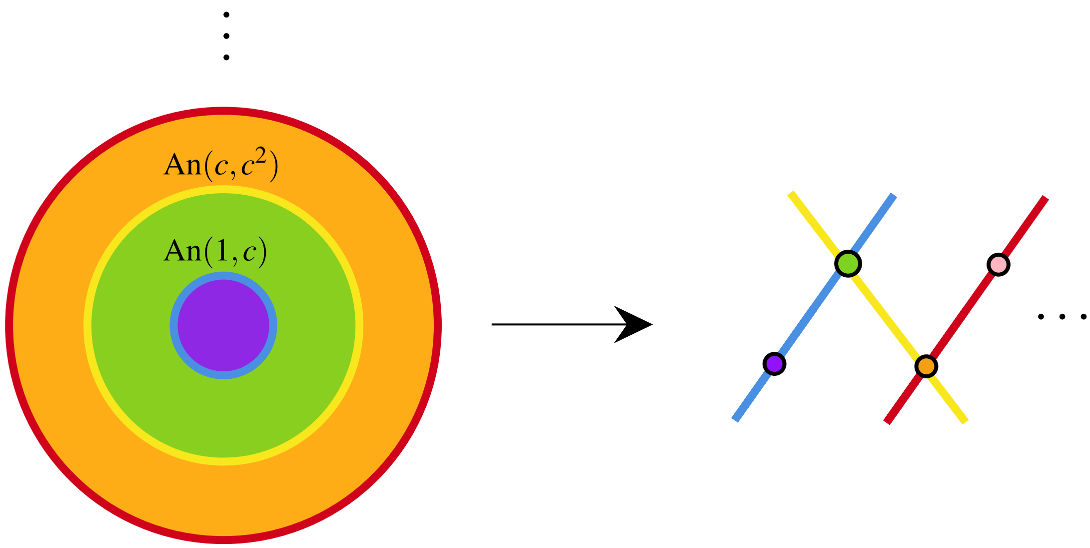

Figure 2.2: A reduction of $\mathbb{A}^1$

A formal analytic cover of $\mathbb{A}_K^{1,\mathrm{an}}$ is given by fixing some $y \in K$ with $|y| = c > 1$ and taking the unit disk $D(1,0)$ and the annuli $\mathrm{An}(c^n, c^{n+1})$ for all integers $n \geq 0$, glued along the border circles. The reduction of the disk is $\mathbb{A}^1_{\widetilde{K}}$, and the reduction of each annulus is two copies of $\mathbb{A}^1_{\widetilde{K}}$ glued at the origin. These reductions are glued to form a infinite chain of $\mathbb{P}^1_{\widetilde{K}}$s, indexed by $\mathbb{N}$.

Example 2.2.10. A key example for us will be the rigid analytification of the affine torus $\mathbb{G}_{m,K}$. This is just $\mathbb{A}_K$ with the origin removed, so a formal analytic cover is given by fixing some $y \in K$ with $|y| = c > 1$ and taking the annuli $\mathrm{An}(c^n, c^{n+1})$ for all integers n, again glued along border circles. The reduction is now a two-sided infinite chain of $\mathbb{P}^1_k$s, indexed by $\mathbb{Z}$.

As promised, this category admits inverse limits. (Note that we are not assuming that our formal schemes are admissible here.)

Lemma 2.2.11. *Let $\mathfrak{X}_i$ be an inverse system of formal schemes over K° with affine transition maps. Then the inverse limit $\mathfrak{X} = \varprojlim \mathfrak{X}_i$ exists in the category of formal schemes over K°. If all the $\mathfrak{X}_i$ are flat, so is $\mathfrak{X}$.*

Proof. As the transition maps are affine, we can reduce to the case where all the $\mathfrak{X}_i$ are affine. Say $\mathfrak{X}_i = \mathrm{Spf}(A_i)$, then we have $\mathfrak{X} = \mathrm{Spf}(\widehat{\varinjlim A_i})$, where we are taking the ϖ-adic completion of the direct limit. Flatness is equivalent to being torsion-free over K°, so it is preserved by this process. $\square$

2.3 Adic spaces

When we construct perfectoid spaces, we will need to work with more general rings than the affinoid algebras used to build rigid spaces. Quotients of Tate algebras are enough to work with finite-type varieties and their analytifications, but we will soon see that perfectoid K-algebras are infinitely ramified and non-noetherian. Huber's theory of adic spaces ([Hub94], [Hub96]) geometrizes much more general rings, and is therefore well-suited to the perfectoid theory.

Adic spaces are built from affinoid spaces, which are again geometric spaces determined by rings, now with some additional data.

Definition 2.3.1. *A* Huber ring *is a topological ring A containing some open subring A_0 and finitely generated ideal $I \subset A_0$ such that A_0 has the I-adic topology. As usual, we let A° denote the ring of power bounded elements of A (now bounded with respect to the topology on A). A* Huber pair (A,A^+) *is a Huber ring A along with an open, integrally closed subring $A^+ \subset A^\circ$. If we can choose $I = (\varpi)$ for a topologically nilpotent unit ϖ, we say that A is* Tate *and (A,A^+) is a* Tate-Huber *pair. When we are working over a (non-trivial) non-archimedean field K, all rings will be Tate as we can take a pseudo-uniformizer from K: this simplifies the theory considerably.*

Example 2.3.2.

1. Any ring A can be given the discrete topology, in which case (A,A) is a Huber pair topologized by the zero ideal. This will let us construct adic spaces from schemes.

2. Given a non-archimedean field K, we get the Tate-Huber pair (K,K°) as K° is topologized by the ideal (ϖ) for any $\varpi \in K^\times$ with $|\varpi| < 1$.

3. The Tate algebra $K\langle T_1, \ldots, T_n\rangle$ is a Tate-Huber ring as the subring $K^\circ\langle T_1, \ldots, T_n\rangle$ is topologized by the ideal (ϖ), and is both open and integrally closed. This gives the Tate-Huber pair $(K\langle T_1, \ldots, T_n\rangle, K^\circ\langle T_1, \ldots, T_n\rangle)$. More generally, the affinoid algebras A from Section 2.1 give Tate-Huber pairs (A, A°) over (K, K°).

4. Similarly, $(K^\circ\langle T_1, \ldots, T_n\rangle, K^\circ\langle T_1, \ldots, T_n\rangle)$ is a Huber pair (but it is not Tate as ϖ is no longer a unit), and the formal schemes from Section 2.2 give Huber pairs (A°, A°) over (K°, K°).

Definition 2.3.3. *Given a Huber pair (A, A^+), the affinoid adic space $\mathrm{Spa}(A, A^+)$ is defined as the set of equivalence classes of continuous valuations $|\cdot| : A \to \Lambda \cup \{0\}$ such that $|f| \leq 1$ for any $f \in A^+$, where Λ is a totally ordered abelian group. Given a point $x \in \mathrm{Spa}(A, A^+)$ we write $f \mapsto |f(x)|$ for the corresponding valuation. The topology of $\mathrm{Spa}(A, A^+)$ is generated by subsets of the form*

$$U\left(\frac{f}{g}\right) := \{x : |f(x)| \leq |g(x)| \neq 0\}$$

for $f, g \in A$.

Example 2.3.4. Let A be a ring with the discrete topology, let (A, A) be the corresponding Huber pair. Then any prime ideal $\mathfrak{p} \in A$ gives the continuous valuation

$$|\cdot|_{\mathfrak{p}} : A \to \{0, 1\}$$

defined by sending f to 0 when $f \in \mathfrak{p}$. This induces a homeomorphism of topological spaces $\mathrm{Spec}(A) \to \mathrm{Spa}(A, A)$, as the distinguished open $D(g) \subset \mathrm{Spec}(A)$ corresponds to the open $U(\frac{0}{g}) \subset \mathrm{Spa}(A, A)$.

Example 2.3.5. We give some examples of points of $\mathrm{Spa}(K\langle T\rangle, K^\circ\langle T\rangle)$, the adic unit disk. A more complete description of this space (along with a picture) can be found in [Sch12, Example 2.20].

- The valuation sending

$$\sum a_i T^i \mapsto |a_0|_K$$

corresponds to the origin: the maximal ideal $(T) \in \mathrm{Sp}(K\langle T\rangle)$;

- The valuation sending

$$\sum a_i T^i \mapsto \max_i\{|a_i|_K\}$$

is called the Gauss point. It corresponds to the Gauss norm from Definition 2.1.4 (the notation may be a bit confusing as the words norm and valuation are somewhat conflated here). There is a similar valuation corresponding to the supremum norm on any closed or open disk contained in the unit disk. The previous example corresponds to a closed disk of radius 0.

  Such points are already present in Berkovich's theory, and explain why the cover of $D(1,0)$ given in Example 2.1.8 is not admissible: the Gauss point isn't in $D(1,0)^-$ or $\mathrm{An}(1,1)$.

- A more exotic point which only appears for adic spaces comes from when the valuation ring has higher rank. Letting $\Lambda = \mathbb{R}_{\geq 0} \times \lambda^{\mathbb{Z}}$, ordered so that $1-\varepsilon < |\lambda| < 1$ for all ε, we have a valuation sending

$$\sum a_i T^i \mapsto \max_i\{|a_i|_K \lambda^i\}.$$

  This defines a point infinitesimally below the Gauss point.

More general adic spaces are constructed by gluing together opens of the form $\mathrm{Spa}(A,A^+)$. We note, however, that this process is more subtle than in the case of schemes: the structure presheaf is harder to define and might not be a sheaf in general. As this will not be a problem for us, we ignore this completely, but we sketch the construction over (K,K°) here. See the references for details and proofs.

Given an affinoid adic space $\mathrm{Spa}(A,A^+)$ over (K,K°), let $f_1,\dots,f_n \in A$ be elements generating the unit ideal, and let $g \in A$. We call the subspace

$$U\left(\frac{f_1,\ldots,f_n}{g}\right) = \{x : |f_i(x)| \leq |g(x)| \neq 0 \text{ for all } i\}$$

a rational subset. It is the intersection of the subsets $U(f_i/g)$ generating the topology of $\mathrm{Spa}(A,A^+)$ from Definition 2.3.3.

The structure presheaf is first defined on rational subsets. For any rational subset $U \subset X = \mathrm{Spa}(A,A^+)$, there is a complete Huber pair $(A,A^+) \to (\mathscr{O}_X(U), \mathscr{O}_X^+(U))$ such that the affinoid adic space $\mathrm{Spa}(\mathscr{O}_X(U), \mathscr{O}_X^+(U))$ is homeomorphic to U, and which is universal for all maps $\mathrm{Spa}(B,B^+) \to \mathrm{Spa}(A,A^+)$ factoring over U. Explicitly, $\mathscr{O}_X(U)$ is the completion of $A[f_1/g,\ldots,f_n/g] \subset A[g^{-1}]$, and $\mathscr{O}_X^+(U)$ is the completion of the integral closure of $A^+[f_1/g,\ldots,f_n/g] \subset A[g^{-1}]$.

This defines the presheaves $\mathscr{O}_X$ and $\mathscr{O}_X^+$ on X on rational subsets. In general, we define

$$\mathscr{O}_X(V) = \varprojlim_{U \subset W \text{ rational}} \mathscr{O}_X(U)$$

and define $\mathscr{O}_X^+$ similarly. It is not clear if these presheaves are actually sheaves, but this is the case when the rings involved are noetherian or perfectoid.

Example 2.3.6. Gluing the types of Huber pairs in Example 2.3.2(3) lets us geometrize to a fully faithful functor from rigid spaces over K to adic spaces over (K,K°). Gluing the types of Huber pairs in Example 2.3.2(4) lets us geometrize to a fully faithful functor from locally noetherian formal schemes to adic spaces. Doing this, Raynaud's interpretation of rigid spaces as the generic fibers of formal schemes as described in Proposition 2.2.7 is really writing the rigid space X/K as the adic generic fiber of a formal scheme $\mathfrak{X}/K^\circ$. That is, $X^{\mathrm{ad}} = \mathfrak{X}^{\mathrm{ad}}_\eta := \mathfrak{X}^{\mathrm{ad}} \times_{\mathrm{Spa}(K^\circ,K^\circ)} \mathrm{Spa}(K,K^\circ)$.

Inverse limits do not generally exist in the categories of rigid spaces or adic spaces. As a replacement, we recall the definition of tilde-limits from [SW13, Section 2.4] along with some useful properties.

Definition 2.3.7 ([SW13, Definition 2.4.1]). *Let X_i be a filtered inverse system of adic spaces with quasicompact and quasiseparated transition maps. Let X be an adic space with a compatible family of morphisms $\phi_i : X \to X_i$. Then we write $X \sim \varprojlim X_i$ and call X the* tilde-limit *of the inverse system if the induced map of topological spaces $|X| \to \varprojlim |X_i|$ is a homeomorphism, and there is an open cover of X by affinoids* $\mathrm{Spa}(A, A^+)$ *such that the maps*

$$\varinjlim_{\mathrm{Spa}(A_i, A_i^+) \subset X_i} A_i \to A$$

have dense image, where the direct limit runs over the affinoid opens of X_i such that the restriction of ϕ_i to $\mathrm{Spa}(A, A^+)$ *factors through* $\mathrm{Spa}(A_i, A_i^+) \subset X_i$.

We caution that it's not generally clear if any affinoid subspace of X should satisfy this density condition.

Proposition 2.3.8 ([SW13, Proposition 2.4.2]). *Let (A_i, A_i^+) be a direct system of complete Tate-Huber pairs over (K, K°) with compatible rings of definition $A_{i,0}$, all with ideal of definition (ϖ). Let $(A, A^+) = (\varinjlim A_i, \varinjlim A_i^+)$ be the Tate-Huber pair with ring of definition $A_0 = \varinjlim A_{i,0}$ and ideal of definition (ϖ). Then*

$$\mathrm{Spa}(A, A^+) \sim \varprojlim \mathrm{Spa}(A_i, A_i^+).$$

Proposition 2.3.9 ([SW13, Proposition 2.4.3]). *Given $X \sim \varprojlim X_i$ as in Definition 2.3.7, let $Y_i \hookrightarrow X_i$ be an open immersion for some fixed X_i in the inverse system. Let $Y_j := Y_i \times_{X_i} X_j$ for $j \geq i$ and $Y := Y_i \times_{X_i} X$. Then $Y \sim \varprojlim_{j \geq i} Y_j$.*

To construct tilde-limits of inverse systems of rigid spaces, we construct a corresponding inverse system of well-behaved formal models of these spaces, use the fact that inverse limits do often exist in the category of formal schemes, then take the adic generic fiber of the limit.

Lemma 2.3.10. *Let $\mathfrak{X}_i$ be a filtered inverse system of formal schemes over K° with affine transition maps, let $\mathfrak{X}$ be the inverse limit constructed in Lemma 2.2.11. Let $X_i = (\mathfrak{X}_i)_\eta$ and $X = \mathfrak{X}_\eta$ be the adic generic fibers as in Example 2.3.6. Then $X \sim \varprojlim X_i$. We have*

$$\Gamma(X) \cong (\varinjlim \Gamma(\mathfrak{X}_i))^\wedge \otimes_{K^\circ} K. \tag{2.1}$$

Proof. As the transition maps are affine, they are qcqs. Fix some finite level $\mathfrak{X}_0$ and cover it by formal affines $\mathrm{Spf}(A_0)$. As the transition maps are affine, these pull back to an affine cover $\mathrm{Spf}(A)$ of $\mathfrak{X}$, which gives an affinoid cover $\mathrm{Spa}(A[\frac{1}{\varpi}], A)$ of X. By definition, we can check the tilde-limit property on these pieces. This now follows directly from Proposition 2.3.8 once we note that the construction remains a tilde-limit after taking the ϖ-adic completion. To get the isomorphism of global sections, we note that $\Gamma(\mathfrak{X}_\infty) \cong (\varinjlim \Gamma(\mathfrak{X}_i))^\wedge$ [Sta20, Tag 01Z0]. Taking the adic generic fiber gives the desired isomorphism. $\square$

For our purposes, a key result is that tilde-limits behave like inverse limits on étale cohomology. This will let us transfer cohomological results between characteristics by passing through perfectoid tilde-limits.

Proposition 2.3.11 ([Sch12, Theorem 7.17])**.** *In the situation of Definition 2.3.7, the étale topos $X^\sim_{ét}$ of X is the projective limit of the étale topoi $(X_i)^\sim_{ét}$.*

Unpacking this statement a bit more concretely, we have

Corollary 2.3.12 ([Sch12, Corollary 7.18])**.** *In the situation of Definition 2.3.7, if F_i is a sheaf of abelian groups on X_i, F_j the preimage on X_j for $j \geq i$, and F the preimage on X, then we have bijections*

$$\varinjlim_j H^n(X_{j,ét}, F_j) \to H^n(X_{ét}, F)$$

for all $n \geq 0$.

2.4 Raynaud extensions

Let A be an abelian variety over K, implicitly considered as a rigid analytic space (or adic space, if you like). In this section, we recall Raynaud's uniformization of A as a quotient of a semi-abelian variety E with good reduction by a lattice. We refer to [Lüt16, Section 6] for a nice exposition of the full construction; the original paper is [BL91]. We will be working closely with this construction throughout, and will go into more detail later on.

The example that motivated Tate to develop his theory is the uniformization of elliptic curves over K with bad reduction. We go through this example before moving on to the general case. We will carry the Tate curve example throughout the entire thesis. We feel that it often gives good intuition for the general case of abelian varieties, and we will be able to make everything much more explicit in this case.

Over $\mathbb{C}$, all elliptic curves can be uniformized as quotients $E_\omega \cong \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\omega)$ for some $\omega \in \mathbb{C}-\mathbb{R}$. This can't work over K, as there are no non-trivial discrete additive subgroups: for any non-zero element $y \in K$, the sequence $\{p^n y\}$ converges to 0. To fix this, we exponentiate: the isomorphism $\mathbb{C}/\mathbb{Z} \to \mathbb{C}^\times$ gives an isomorphism $E_\omega \cong \mathbb{C}^\times/(q^{\mathbb{Z}})$ for $q = e^{2\pi i\omega}$. As the multiplicative group $K^\times$ has discrete subgroups, this perspective transfers over more naturally.

Example 2.4.1. Fix an element $q \in K^\times$ with $|q| = c < 1$. We want to construct a rigid space with points $K^\times/q^{\mathbb{Z}}$. Extending the base field if needed, we assume that q has a square root in K. We can then glue the annuli

$$\mathrm{An}(c^{1/2},1) = \mathrm{Sp}(K\langle T, c^{1/2}/T\rangle) = \mathrm{Sp}(K\langle T_1,T_2\rangle/(T_1T_2 - q^{1/2}))$$

and

$$\mathrm{An}(c,c^{1/2}) = \mathrm{Sp}(K\langle S/c^{1/2}, c/S\rangle) = \mathrm{Sp}(K\langle S_1,S_2,S_3\rangle/(S_1 - q^{1/2}S_2, S_1S_3 - q^{1/2}))$$

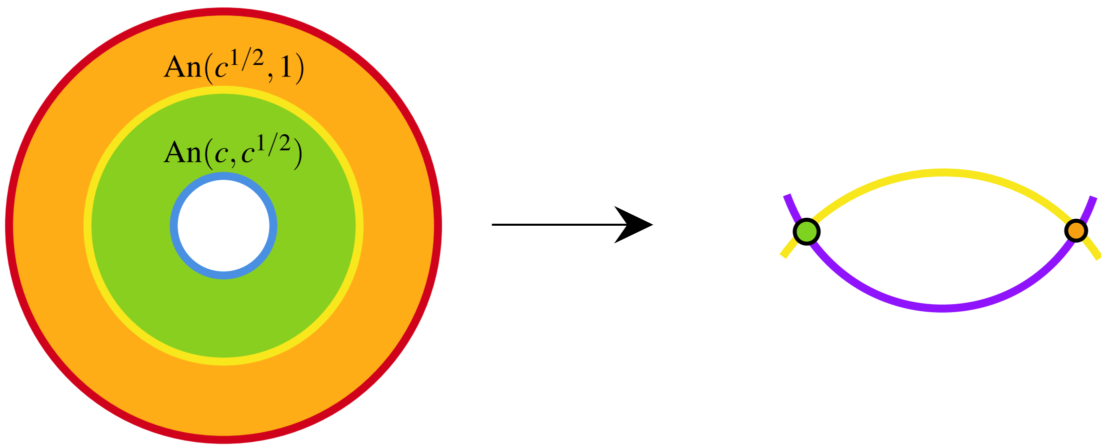

Figure 2.3: Gluing the red and blue circles gives a formal analytic cover of the Tate curve. The special fiber of the corresponding formal model is two $\mathbb{P}^1$s intersecting twice.

along their boundary circles. The inner circles $\mathrm{An}(c/2,c/2)$ are glued via the identity, while the outer circles are glued along the map $\mathrm{An}(1,1)\to\mathrm{An}(c,c)$ given on points by multiplication by q. On rings this is

$$K\langle S/c,c/S\rangle = K\langle S_1,S_2,S_3\rangle/(S_1-qS_2,S_1S_3-q)\xrightarrow{\cong} K\langle T_1,T_2\rangle/(T_1T_2-1)=K\langle T,1/T\rangle$$

$$S_1,S_2,S_3\mapsto T_1,T_1/q,q/T_2.$$

This is a formal analytic cover, so it gives rise to a formal model of the Tate curve. More generally, we will see in Section 4.1 that we get a formal model from any decomposition of the interval $[c,1]\subset\mathbb{R}$ into closed intervals with endpoints in Γ.

Not every elliptic curve can be uniformized in this fashion. The classical q-expansion formula for the j-invariant still holds in this case, so

$$j(q)=q^{-1}+744+196884q+\cdots$$

is given by a Laurent series which converges when $|q|<1$. By the non-archimedean triangle inequality, we have $|j(q)|=|q^{-1}|>1$, so only elliptic curves with j-invariant satisfying this

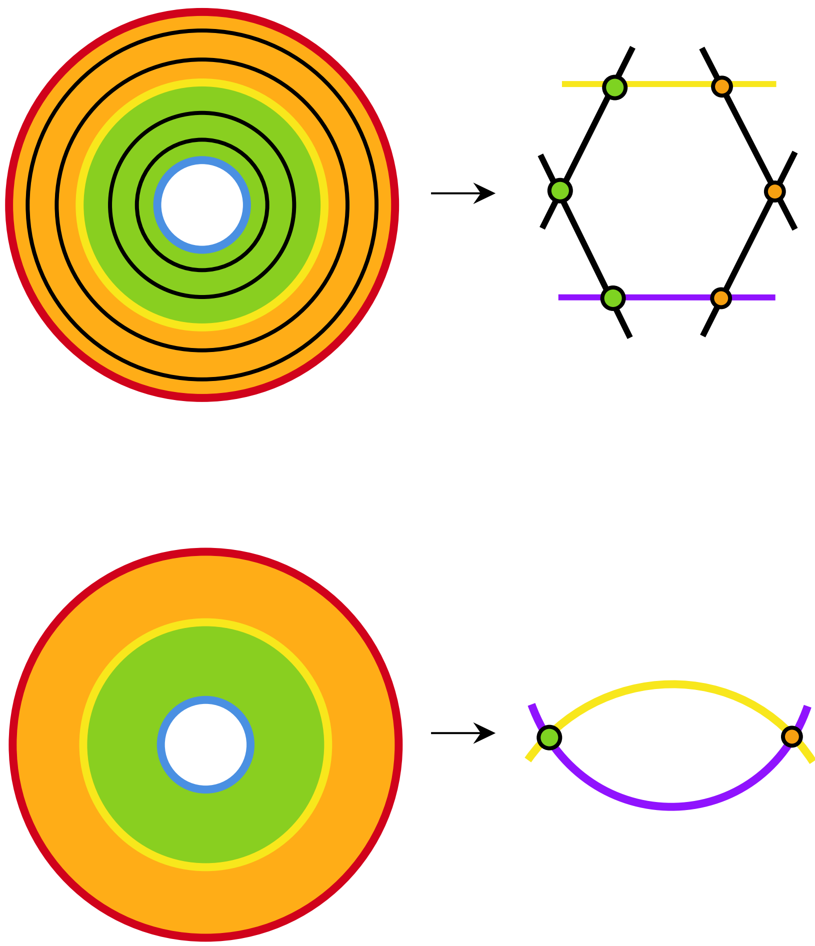

Figure 2.4: Refining the formal analytic cover corresponds to blowing up the special fiber.

inequality occur. These are precisely the elliptic curves with *bad reduction*: that can't arise as the generic fiber of an elliptic curve over K°.

The story for abelian varieties combines both possibilities: the uniformization will have an abelian part with good reduction, and a torus part with bad reduction. We first need a bit more background on rigid and formal tori. We take this opportunity to point out our general naming scheme.

Notation 2.4.2. The r-dimensional split rigid torus is $(\mathbb{G}_m^{an})^r$ and will generally be denoted T. Taking products of Example 2.2.10, we get an example of a formal model $\mathfrak{T}$ of T: a formal scheme over K° with $\mathfrak{T}_\eta \cong T$. This construction will be considered in more depth and generalized in Chapter 4. In general, we will use normal letters for spaces over K, and fractur letters for formal models of spaces over K. We emphasize that $\mathfrak{T}$ is not a formal torus. The one dimensional formal torus is $\overline{\mathbb{G}}_{m,K^\circ} = \mathrm{Spf}(K^\circ\langle T, T^{-1}\rangle)$. Its generic fiber is the unit circle $\mathrm{An}(1,1)$, so in general we have an embedding $(\overline{T})_\eta \hookrightarrow T$. We denote formal tori by $\overline{T}$. In general, spaces with straight lines over them will be the ones that live "naturally" over K°. As we have seen, the special fiber $\widetilde{\mathfrak{T}}$ is very different from the special fiber $\widetilde{T}$. In the one-dimensional case, $\widetilde{\mathfrak{T}}$ is an infinite chain of $\mathbb{P}^1$s, while $\widetilde{T} = \mathrm{Spec}(\widetilde{K}[T, T^{-1}]) = \mathbb{G}_{m,\widetilde{K}}$. Special fibers will always be denoted with a tilde on top.

Let $\overline{B}$ be a formal abelian scheme over K°, so

$$\overline{B} := \varinjlim_{n \in \mathbb{N}} \widetilde{B}_n$$

where $\widetilde{B}_n = \overline{B} \times_{K^\circ} K^\circ/\varpi^{n+1}$ is an abelian scheme over K°/ϖ^{n+1}. Fix a strict exact sequence of formal groups over K°

$$1 \to \overline{T} \to \overline{E} \xrightarrow{\pi} \overline{B} \to 1 \tag{2.2}$$

(strict means that the induced sequence of tangent bundles is also exact, but we will not be using this notion explicitly).

Let E be the pushforward of $\overline{E}_\eta$ by the embedding $\overline{T}_\eta \hookrightarrow T$. Then we get a commutative diagram of short exact sequences of rigid groups.

$$\begin{tikzcd}
1 \arrow[r] & \overline{T}_\eta \arrow[r] \arrow[d, hook] & \overline{E}_\eta \arrow[r, "\overline{\pi}_\eta"] \arrow[d, hook] & \overline{B}_\eta \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & T \arrow[r] & E \arrow[r, "\pi"] & B \arrow[r] & 1
\end{tikzcd} \tag{2.3}$$

Definition 2.4.3. *We call the short exact sequence at the bottom of diagram (2.3) a* Raynaud extension*. Here T is a split rigid torus of rank r, B is an abelian variety with good reduction, and E is a semi-abelian variety.*

Just as complex abelian varieties are uniformized by taking the quotient of $\mathbb{C}^r$ by a lattice, rigid abelian varieties are uniformized by taking the quotient of some Raynaud extension E by a lattice. To make this more precise, we must define what it means to be a lattice in E. This will require a bit more information about the geometry of the above short exact sequences.

The map $\overline{\pi}$ in (2.2) admits local sections $\overline{s} : \overline{V_i} \to \overline{E}$ for some open cover $\{\overline{V}_i\}$ of $\overline{B}$, so we can cover $\overline{E}$ by formal open subschemes of the form $\overline{T} \times \overline{V}_i$. These formal opens are glued on the overlaps by maps

$$\overline{\phi}_{ij} : \overline{T} \times (\overline{V}_i \cap \overline{V}_j) \to \overline{T} \times (\overline{V}_i \cap \overline{V}_j)$$

such that $\overline{\phi}_{ij}(t,v) = (\overline{\psi}_{ij}(v)t, v)$ for some maps $\overline{\psi}_{ij} : (\overline{V}_i \cap \overline{V}_j) \to \overline{T}$.

The local sections of $\overline{\pi}$ extend to local sections of π, so E can be covered by opens isomorphic to $T \times V_i$ where $V_i := (\overline{V}_i)_\eta$. The gluing data is equivalent to the maps

$$\psi_{ij} : V_i \cap V_j = (\overline{V}_i \cap \overline{V}_j)_\eta \to \overline{T}_\eta.$$

For a description of all this in terms of Ext groups of schemes, see [Ser88, VII §3].

The valuation on K gives a group homomorphism $\ell : T(K) \cong (K^{\times})^r \to \mathbb{R}^r$ sending the point $(x_1, \dots, x_r)$ to $(v(x_1), \dots, v(x_r)) = (-\log(|x_1|), \dots, -\log(|x_r|))$. We extend the map $\ell : T(K) \to \mathbb{R}^r$ to a map $\ell : E(K) \to \mathbb{R}^r$ as follows. Locally on K-points, this is the map $T \times V_i \xrightarrow{pr_1} T \xrightarrow{\ell} \mathbb{R}^r$. On intersections the gluing maps come from the $\overline{T}_\eta$-action on T. The map ℓ is invariant for this action as ℓ sends $\overline{T}_\eta$ to the origin in $\mathbb{R}$. We can therefore can glue the local maps together to get a group homomorphism $\ell : E(K) \to \mathbb{R}^r$ as desired.

Definition 2.4.4. *A* lattice *in E is a subgroup $M \subset E(K)$ such that ℓ restricts to a bijection from M to a lattice Λ in $\mathbb{R}^r$. The* rank *of M is the rank of Λ, M has* full rank *if it has rank r. Note that these definitions all specialize to the case $E = T$.*

Theorem 2.4.5 ([Lüt16, Theorem 6.4.8])**.** *Let A be an abelian variety over K. After a suitable finite separable extension L/K there is a unique Raynaud extension E over L and full rank lattice M in E such that $A_L \cong E/M$.*

In the category of rigid spaces, E is simply connected (See [FvdP04, Example 7.4.9(1)] or [BL91, Theorem 1.2]), so this result is as strong as possible.

More generally, define an abeloid variety to be a proper, smooth, connected, rigid group variety. This is the non-archimedean analogue of a complex torus.

Theorem 2.4.6 ([Lüt16, Theorem 7.6.4])**.** *Any abeloid variety over K admits a Raynaud uniformization after a suitable finite separable extension of K.*

Chapter 3

Background on perfectoid spaces

The original reference for this material is [Sch12]. Other good introductions include [BCC$^+$19] and [Bha17]. Fix a prime p. The theory of perfectoid spaces was inspired by the following (slightly modified) theorem

Theorem 3.0.1 (Fontaine-Wintenberger)**.** *The absolute Galois groups of the fields* $\mathbb{Q}_p(p^{1/p^\infty}) := \cup_{n>0}\mathbb{Q}_p(p^{1/p^n})$ *and* $\mathbb{F}_p((t^{1/p^\infty}))$ *are isomorphic.*

This isomorphism gives a link between arithmetic information in characteristic 0 and p after an infinitely ramified extension. Perfectoid fields and spaces generalize this idea, allowing for more geometric objects and concepts to transfer back and forth. In this chapter, we give a brief overview of perfectoid fields, then geometrize to perfectoid spaces. We then explain how we will be building perfectoid covers in this thesis, and give an exposition of a Galois descent result in this context.

3.1 Perfectoid fields

Definition 3.1.1. *Let K be a non-archimedean field with residue characteristic p, complete with respect to the norm $|\cdot|$. Fix a* pseudo-uniformizer $\varpi \in K$ *with $|p| < |\varpi| < |1|$. We say that K is* perfectoid *if* $|\cdot|$ *is not discrete and the Frobenius map on* K°/ϖ *is surjective.*

Example 3.1.2. The field $K = \widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ is perfectoid, where the completion is taken with respect to the p-adic valuation. The valuation is not discrete as the limit of $|p^{1/p^n}| = (1/p)^{1/p^n}$ as $n \to \infty$ is 1. The power bounded elements K° are $\widehat{\mathbb{Z}_p(p^{\frac{1}{p^\infty}})}$. Taking $\varpi = p$, we get $K^\circ/\varpi \cong \mathbb{F}_p[t^{\frac{1}{p^\infty}}]/t$ by sending p to t. Frobenius is surjective on this ring as it is generated as an $\mathbb{F}_p$-algebra by the elements t^{1/p^n}, which all have pth roots.

Other examples of perfectoid fields include $\widehat{\mathbb{Q}_p(\mu_{p^\infty})}, \widehat{\mathbb{F}_p((t^{1/p^\infty}))}$ (complete with respect to the t-adic valuation), and any algebraically closed non-archimedean field. In characteristic p, perfectoid fields are exactly the non-discrete, perfect fields.

Given a perfectoid field, the *tilt* is defined as $K^\flat := \varprojlim_{x \mapsto x^p} K$, so elements of $K^\flat$ are sequences $(x_0, x_1, \dots)$ of elements of K with $x_n^p = x_{n-1}$. This is a characteristic p perfectoid field, where the multiplication law is given by component-wise multiplication, and the addition law is given by $(x_n) + (y_n) = (z_n)$, where

$$z_n = \lim_{m \to \infty} (x_{m+n} + y_{m+n})^{p^m}.$$

As perfectoid fields in characteristic p are perfect, we have $K^\flat = K$ for such fields.

There is a projection map $\sharp : K^\flat \to K$ sending $(x_n) \mapsto x_0$: this is multiplicative but not additive. Its image is the elements of K which have p^nth roots for all n. The absolute value on $K^\flat$ is given by $|x|_{K^\flat} = |x^\sharp|_K$. The fields K and $K^\flat$ have the same value group. We can therefore choose our pseudo-uniformizer ϖ of K to be in the image of $\sharp$ and choose our pseudo-uniformizer of $K^\flat$ to be an element $\varpi^\flat$ of $K^\flat$ with $(\varpi^\flat)^\sharp = \varpi$. A key relation between K and $K^\flat$ is the isomorphism

$$K^\circ/\varpi \cong K^{\flat\circ}/\varpi^\flat \tag{3.1}$$

Example 3.1.3. The tilt of $K = \widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ is isomorphic to $\widehat{\mathbb{F}_p((t^{1/p^\infty}))}$, where the element

$(p, p^{1/p}, \dots) \in K^\flat$ is identified with the pseudo-uniformizer t. Then $t^\sharp = p$, and we have

$$K^{\flat\circ}/\varpi^\flat \cong \mathbb{F}_p\widehat{[[t^{1/p^\infty}]]}/t \cong \mathbb{F}_p[t^{\frac{1}{p^\infty}}]/t \cong K^\circ/\varpi$$

as expected.

The tilting equivalence for perfectoid fields generalizes Theorem 3.0.1, and is a model for the more general tilting equivalences that we'll see next.

Theorem 3.1.4 ([Sch12, Theorem 3.7]). *Every finite extension L of a perfectoid field K is again perfectoid, with tilt $L^\flat/K^\flat$ such that $[L:K] = [L^\flat : K^\flat]$. This induces an equivalence of categories between the finite extensions of K and the finite extensions of $K^\flat$, and therefore an isomorphism* $\mathrm{Gal}(\overline{K}/K) \cong \mathrm{Gal}(\overline{K}^\flat/K^\flat)$.

Notation 3.1.5. For the remainder of this thesis, K will always denote a perfectoid field, and ϖ will always denote a pseudo-uniformizer in the image of the sharp map, with $|p| \leq |\varpi| < 1$.

3.2 Perfectoid spaces

Perfectoid spaces were developed to geometrize the tilting equivalence for perfectoid fields. As the non-archimedean norm plays a key role in tilting, perfectoid spaces must live within some framework of non-archimedean geometry. The rings involve will be highly ramified and non-noetherian, so the category of rigid analytic spaces is not general enough for our purposes. For this reason, we instead work with adic spaces.

For the purposes of this thesis, we will not need the full generality of perfectoid rings and adic spaces. Our goal is to work with explicit limits of well-behaved spaces coming from rigid geometry. We can therefore fix a perfectoid field K and restrict our attention to the following nice situation.

Definition 3.2.1. *A* perfectoid K-algebra *is a complete K-algebra R such that R° is open and bounded, and the Frobenius map on R°/ϖ is surjective. An* affinoid perfectoid *space is an adic space* $\mathrm{Spa}(R,R^+)$ *with R perfectoid. A* perfectoid space *over K is an adic space which is locally affinoid perfectoid.*

Example 3.2.2. The *perfectoid Tate algebra* $K\langle T_1^{1/p^\infty},\dots,T_n^{1/p^\infty}\rangle$ is a perfectoid K-algebra with power bounded elements $K^\circ\langle T_1^{1/p^\infty},\dots,T_n^{1/p^\infty}\rangle$ and special fiber $K^\circ/\varpi[T_1^{1/p^\infty},\dots,T_n^{1/p^\infty}]$. The affinoid perfectoid space $\mathrm{Spa}(K\langle T_1^{1/p^\infty},\dots,T_n^{1/p^\infty}\rangle, K^\circ\langle T_1^{1/p^\infty},\dots,T_n^{1/p^\infty}\rangle)$ is called the n-dimensional perfectoid closed disk.

Perfectoid K-algebras and affinoid perfectoid spaces can again be tilted, with

$$(R,R^+)\mapsto (R^\flat,R^{\flat+}) := (\varprojlim_{x\mapsto x^p} R, \varprojlim_{x\mapsto x^p} R^+).$$

The map $\sharp : R^\flat \to R$ and topology on $R^\flat$ are defined analogously to the case of fields, and there is again a crucial isomorphism $R^+/\varpi \cong R^{\flat+}/\varpi^\flat$. This tilting process glues, so perfectoid spaces in general tilt.

Theorem 3.2.3 ([Sch12]). *Tilting induces an equivalence between the following pairs of categories:*

1. *Perfectoid K-algebras and perfectoid $K^\flat$-algebras;*
2. *Affinoid perfectoid spaces over K and $K^\flat$;*
3. *Finite étale algebras over a perfectoid K-algebra R and finite étale algebras over $R^\flat$;*
4. *Perfectoid spaces over* $\mathrm{Spa}(K,K^\circ)$ *and* $\mathrm{Spa}(K^\flat,K^{\flat\circ})$*;*
5. *(Finite) étale covers of a perfectoid space X and of its tilt $X^\flat$.*

Example 3.2.4. The tilt of the perfectoid Tate algebra over K is the perfectoid Tate algebra over $K^\flat$.

3.3 Constructing perfectoid covers via F-towers

Our overall goal is to transfer information about the étale cohomology of varieties from characteristic p to characteristic 0. Theorem 3.2.3 lets us make this transfer for perfectoid spaces, so we need to relate the cohomology of varieties to that of perfectoid spaces. Proposition 2.3.11 says that tilde-limits act like inverse limits on étale topoi, and therefore étale cohomology. Following Scholze, our approach is to construct inverse systems of varieties with a perfectoid tilde-limit. In [Sch12], Scholze does this for inverse systems built by iterating Frobenius lifts on toric varieties. As we don't generally have Frobenius lifts for abelian varieties, we need a more general setup. Roughly speaking, we need to make sure that the rings at the top of the tower will have all their pth power roots mod ϖ. At each level of the tower, we add more pth roots by requiring that our maps factor through relative Frobenius.

Definition 3.3.1. *Let S be a scheme in characteristic p, let $\mathrm{Frob}_S : S \to S$ be the (absolute) Frobenius map: the pth power map on rings. Let $f : X \to S$ be a map, then Frob_X is not a map over S: it fits into the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{Frob}_X} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ S & \xrightarrow{\mathrm{Frob}_S} & S. \end{array}$$

The relative Frobenius *map $F_{X/S}$ is the map over S which is "as close as possible" to Frobenius. We just need to add one piece to the above diagram: the fiber product $X^{(p)} := X \times_{S,\mathrm{Frob}_S} S$.*

$$\begin{tikzcd} X \arrow[dr, "F_{X/S}"] \arrow[drr, bend left, "\mathrm{Frob}_X"] \arrow[ddr, bend right, "f"'] & & \\ & X^{(p)} \arrow[r, "\mathrm{Frob}_S"] \arrow[d, "f"] & X \arrow[d, "f"] \\ & S \arrow[r, "\mathrm{Frob}_S"] & S \end{tikzcd} \tag{3.2}$$

The following formalism is modeled off of Scholze's construction of perfectoid covers of Shimura varieties in [Sch15, III.2].

Definition 3.3.2. *Let X be a rigid analytic space over K. An F-tower over X consists of an inverse system of formal schemes $\{\mathfrak{X}_n\}_{n\geq 1}$ satisfying the following conditions:*

1. *The generic fiber of $\mathfrak{X}_1$ is X,*
2. *The formal schemes $\mathfrak{X}_n$ are all flat over K°,*
3. *The transition morphisms $\mathfrak{X}_n \to \mathfrak{X}_{n-1}$ are affine and on the mod ϖ special fiber factor through the relative Frobenius map*

$$F_{\widetilde{\mathfrak{X}}_n/(K^\circ/\varpi)} : \widetilde{\mathfrak{X}}_n \to \widetilde{\mathfrak{X}}_n^{(p)}.$$

To ease notation, we write $\widetilde{F}_n$ for this relative Frobenius map from now on.

We write $\{X_n\}$ for the adic generic fiber of the F-tower. If the formal schemes are all admissible, this is an inverse system of rigid spaces over X.

Proposition 3.3.3. *Given an F-tower $\{\mathfrak{X}_n\}$ of a rigid space X, there is a unique perfectoid space $X_\infty \sim \varprojlim X_n$. We have $X_\infty = (\varprojlim \mathfrak{X}_n)_\eta$.*

Proof. The inverse limit $\mathfrak{X}_\infty := \varprojlim \mathfrak{X}_n$ exists in the category of formal schemes by Lemma 2.2.11, and the generic fiber X_∞ is a tilde-limit for $\{X_n\}$ by Lemma 2.3.10.

We check perfectoidness of X_∞ locally: by [Sch12, Lemma 5.6], it is enough to cover $\mathfrak{X}_\infty$ by affine formal schemes $\mathrm{Spf}(R_\infty^+)$ for flat K° algebras R_∞^+ such that absolute Frobenius induces an isomorphism $R_\infty^+/\varpi^{1/p} \to R_\infty^+/\varpi$. Take a formal affine open $\mathrm{Spf}(R_0^+) \subset \mathfrak{X}_0$. As the transition maps of our F-tower are affine and all the formal schemes are flat, the preimage in $\mathfrak{X}_n$ (respectively, $\mathfrak{X}_\infty$) is a flat formal affine open $\mathrm{Spf}(R_n^+)$ (respectively, $\mathrm{Spf}(R_\infty^+)$).

We first claim that relative Frobenius is an isomorphism on R_∞^+/ϖ. We can construct the inverse on the direct system of rings, as condition (3) gives the following diagram

$$\begin{tikzcd}[column sep=small]
 & & (R_2^+/\varpi)^{(p)} \arrow[dl, "\widetilde{F}_2"'] & & & (R_1^+/\varpi)^{(p)} \arrow[dl, "\widetilde{F}_1"'] & \\
\cdots & R_2^+/\varpi \arrow[l] & & R_1^+/\varpi \arrow[ll] \arrow[ul, "\widetilde{V}_2"'] & & & R_0^+/\varpi \arrow[lll] \arrow[ul, "\widetilde{V}_1"']
\end{tikzcd}$$

which induces the desired inverse map $V : R_\infty^+/\varpi \to (R_\infty^+/\varpi)^{(p)}$ to relative Frobenius on the limit.

As K is perfectoid, absolute Frobenius induces an isomorphism $K^\circ/\varpi^{1/p} \to K^\circ/\varpi$, and therefore an isomorphism $R_\infty^+/\varpi^{1/p} \to (R_\infty^+/\varpi)^{(p)}$. We conclude from the diagram (3.2) that absolute Frobenius induces the desired isomorphism.

By the next Lemma, any inverse system has at most one perfectoid tilde-limit (even though it could have multiple tilde-limits), so X_∞ is the unique perfectoid tilde-limit of the F-tower $\{\mathfrak{X}_n\}$. $\square$

Lemma 3.3.4 ([SW13, Proposition 2.4.5])**.** *If an inverse system of adic spaces over (K, K°) admits a perfectoid tilde limit, it is unique.*

Example 3.3.5. Let $\mathbb{P}^1_K$ be the analytification of the projective line. In Example 2.2.8, we constructed a formal model by gluing Spf($K^\circ\langle T\rangle$) to Spf($K^\circ\langle T^{-1}\rangle$) along the boundary circles. The map of formal models sending $T \mapsto T^p$ and $T^{-1} \mapsto T^{-p}$ reduces to relative Frobenius mod ϖ and satisfies all the conditions of Definition 3.3.2. We can therefore iterate this map to get an F-tower of $\mathbb{P}^1_K$. This corresponds to gluing two perfectoid closed unit disks Spa($K\langle T^{1/p^\infty}\rangle, K^\circ\langle T^{1/p^\infty}\rangle$) and Spa($K\langle T^{-1/p^\infty}\rangle, K^\circ\langle T^{-1/p^\infty}\rangle$) (as in Example 3.2.2) along the perfectoid unit circle. On points, this is iterating the map $(z_0 : z_1) \mapsto (z_0^p : z_1^p)$. This construction works more generally for toric varieties, as described in [Sch12, Section 8].

Notation 3.3.6. In this thesis, we will be applying this construction to construct F-towers of various rigid analytic groups G. In this case, on the generic fiber our inverse system will always come from iterating the multiplication map $[p] : G \to G$. We therefore denote the nth transition map of the tower as $[\mathfrak{p}]_n : \mathfrak{G}_{n+1} \to \mathfrak{G}_n$. Note however that the formal models will generally not be group schemes.

We can apply this formalism when G is an abelian variety with good reduction.

Example 3.3.7 ([PS16, Lemme A.16]). Let B be (the analytification of) an abelian variety with good reduction over K, so there is a formal abelian scheme $\overline{B}/K^\circ$ with rigid generic fiber B. Iterating the multiplication map $[\overline{p}] : \overline{B} \to \overline{B}$ gives an F-tower by standard facts about abelian varieties. In particular, the mod ϖ special fiber is an abelian scheme in characteristic p, so the map $[\widetilde{p}]$ factors as Frobenius and Verschiebung. We therefore get a perfectoid cover $B_\infty \sim \varprojlim_{[p]} B$.

We note that this construction is functorial, and therefore that G_∞ is a perfectoid group.

Proposition 3.3.8 ([BGH$^+$18, Lemma 2.10]). *Let G be a rigid group with a perfectoid cover $G_\infty \sim \varprojlim_{[p]} G$. Then there is a unique way to endow G_∞ with the structure of a group object in the category of perfectoid spaces such that all projections $G_\infty \to G$ are group homomorphisms. Given another rigid group H with perfectoid tilde-limit $H_\infty \sim \varprojlim_{[p]} H$ and a group homomorphism $H \to G$, there is a unique group homomorphism $H_\infty \to G_\infty$ commuting with all projection maps.*

Proof. This follows directly from functoriality of the construction and uniqueness of perfectoid tilde-limits. □

To transfer information at finite level between characteristics 0 and p, we need to understand how F-towers interact with tilting. By the tilting equivalence 3.2.3 (more precisely, [Sch12, Theorem 5.2]), we have the following:

Lemma 3.3.9. *Let $\{\mathfrak{X}_n\}$ be an F-tower for a rigid space X, let X_∞ be the perfectoid cover of X given by Proposition 3.3.3. Then X_∞ can be reconstructed from the mod ϖ special fiber $\widetilde{\mathfrak{X}}_\infty$ of the F-tower.*

Proof. Part of the tilting equivalence gives an equivalence of categories between perfectoid K-algebras and perfectoid $K^{\circ a}/\varpi$-algebras, where the a means that we are working in the world of "almost mathematics" as in [GR03]. We have been trying to avoid talking about this technicality as much as possible, so we won't go into detail here: see [Sch12, Section 4] for an overview. The actual K°/ϖ-algebras used to build $\widetilde{\mathfrak{X}}_\infty$ give $K^{\circ a}/\varpi$-algebras, which lift to K-algebras under the equivalence of categories. This process glues by [Sch12, Proposition 6.17], so we get the desired reconstruction. $\square$

This implies that different abelian varieties will have the same perfectoid cover, a fact which we will exploit in Chapter 6 when we tilt line bundles.

Example 3.3.10. Let B and B' be abelian varieties with good reduction over K, let $\overline{B}$ and $\overline{B}'$ be the corresponding formal abelian schemes over K°, let B_∞ and B'_∞ be the perfectoid covers constructed in Example 3.3.7. If $\overline{B} \times_{K^\circ} K^\circ/\varpi \simeq \overline{B}' \times_{K^\circ} K^\circ/\varpi$, then we have $B_\infty \simeq B'_\infty$.

This allows us to understand the tilt of a perfectoid cover of X by constructing a suitable F-tower $K^\flat$.

Lemma 3.3.11. *Let $\{\mathfrak{X}_n\}$ be an F-tower for a rigid space X, let $\{\widetilde{\mathfrak{X}}_n\}$ denote the mod ϖ special fiber of the tower, let X_∞ be the perfectoid cover of X given by Proposition 3.3.3. Let $\{\mathfrak{X}'_n\}$ be an F-tower for a rigid space X' over $K^\flat$, let $\{\widetilde{\mathfrak{X}}'_n\}$ denote the mod $\varpi^\flat$ special fiber of the tower, let X'_∞ be the perfectoid cover of X' given by Proposition 3.3.3.If the isomorphism $K^\circ/\varpi \cong K^{\flat\circ}/\varpi^\flat$ induces an isomorphism of the special fibers of the two F-towers, then $(X_\infty)^\flat \cong (X'_\infty)$.*

Proof. This follows directly from Theorem 3.2.3 and Lemma 3.3.9. See [Sch15, Section III.2.3] for an example of this method in the case of Shimura varieties. $\square$

Remark 3.3.12. It is not clear how common it is for rigid spaces X to admit F-towers. It is also not clear, given an F-tower over K, if it should be possible in general to construct an F-tower over $K^\flat$ with isomorphic special fiber. In this thesis, we answer these questions for abelian varieties (and some other group schemes along the way). Answering these questions for more general varieties should make it possible to prove more cases of weight-monodromy.

We'll need one more lemma later on:

Lemma 3.3.13 ([BGH+18, Lemma 2.8]). *Let* (A_i, A_i^+) *and* (B_i, B_i^+) *be direct systems as in Proposition 2.3.8. If there are perfectoid tilde-limits* $\mathrm{Spa}(A, A^+) \sim \varprojlim \mathrm{Spa}(A_i, A_i^+)$ *and* $\mathrm{Spa}(B, B^+) \sim \varprojlim \mathrm{Spa}(B_i, B_i^+)$*, then*

$$\mathrm{Spa}(A, A^+) \times_{\mathrm{Spa}(K, K^\circ)} \mathrm{Spa}(B, B^+) \sim \varprojlim (\mathrm{Spa}(A_i, A_i^+) \times_{\mathrm{Spa}(K, K^\circ)} \mathrm{Spa}(B_i, B_i^+))$$

is also a perfectoid tilde-limit.

3.4 Galois descent for perfectoid tilde-limits

In this section, we give an exposition of the following proposition:

Proposition 3.4.1 ([She17, Corollary 2.3.5]). *Let* L/K *be a finite Galois extension of perfectoid fields with Galois group* G*. Let* X_i *be a filtered inverse system of quasi-compact adic spaces over* $\mathrm{Spa}(K, K^\circ)$ *with finite transition maps, and let* $X_{L,i}$ *be the base change of* X_i *to* $\mathrm{Spa}(L, L^\circ)$*. Assume that there is a perfectoid space* $X_{L,\infty}$ *over* $\mathrm{Spa}(L, L^\circ)$ *such that* $X_{L,\infty} \sim \varprojlim X_{L,i}$*. Then there is a perfectoid space* X_∞ *over* $\mathrm{Spa}(K, K^\circ)$ *such that* $X_\infty \sim \varprojlim X_i$*.*

When applied to the inverse system $\varprojlim_{[p]} A$, we see that we can construct our tilde-limit after a finite Galois extension of the base field and then descend to the desired tilde-limit. We can therefore assume that A admits a Raynaud uniformization when constructing its perfectoid cover A_∞.

Before we prove the proposition, we give a useful lemma that will allow us to reduce to the affinoid case.

Lemma 3.4.2 ([She17, Lemma 2.2.4]). *Let* X_i *be a filtered inverse system of quasi-compact adic spaces with finite transition maps* $\phi_{i,j} : X_i \to X_j$*, let* X_∞ *be a perfectoid space with* $X_\infty \sim \varprojlim X_i$*. Then there exists some adic space* X_n *in the inverse system and affinoid cover* $\{U_{\alpha,n}\}$ *of* X_n *such* *that the pullback of the cover along the map $\phi_n : X_\infty \to X_n$ gives a perfectoid affinoid cover $\{U_{\alpha,\infty}\}$ of X_∞.*

Proof. We note that X_∞ is quasi-compact as it is an inverse limit of spectral spaces with spectral maps by [Sta20, Tag 08YF], so we can cover X_∞ by finitely many perfectoid affinoids and restrict our attention to a single perfectoid affinoid $W_\infty \subset X_\infty$.

Topologically, we have $|X_\infty| \cong \varprojlim |X_i|$ by definition of $\sim$. We can therefore find some X_m in the inverse system and open set $W_m \subset X_m$ such that $|W_\infty|$ is the pullback of $|W_m|$ along the map $\phi_m : X_\infty \to X_m$. Cover W_m by affinoids $V_{j,m}$. As the transition maps are finite, affinoids pull back to affinoids and so each $V_{j,m}$ pulls back to an inverse system of affinoids defining an open $V_{j,\infty}$ of W_∞ by Proposition 2.3.9. It is not obvious (though certainly plausible) that $V_{j,\infty}$ will be affinoid. Even if this were the case, it is not known that any affinoid subspace of a perfectoid space can be written as $\mathrm{Spa}(A, A^+)$ for some perfectoid ring A.

We do know that rational localizations of perfectoid rings remain perfectoid and form a basis for the topology, so we can cover W_∞ by finitely many rational localizations $U_{\alpha,\infty} := (W_\infty)_{f_1,\ldots,f_n,g}$ that are each contained in some $V_{j,\infty}$. Refining our cover further if necessary, we can assume that these affinoids satisfy the density property in the definition of tilde-limits. (We believe this is necessary as the definition doesn't promise that every affinoid satisfies the property, just that some cover does. The property is preserved under rational localization, so we can always make this refinement.)

By [KL15, Remark 2.4.7] there is some $\varepsilon > 0$ such that we can change the f_i, g to f_i', g' with $|f_i - f_i'| < \varepsilon, |g - g'| < \varepsilon$ without changing the rational localization. By the density property, we can now choose X_n in the inverse system such that each $V_{j,m}$ pulls back along the map $\phi_{n,m} : X_n \to X_m$ to an affinoid $V_{j,n} = \mathrm{Spa}(B_{j,n}, B_{j,n}^+)$ with the $f_i, g \in B_{j,n}$. Taking the corresponding rational localizations gives the desired affinoid cover $\{U_{\alpha,n}\}$ pulling back to the affinoid perfectoid cover $\{U_{\alpha,\infty}\}$. $\square$

Proof of Proposition 3.4.1. The following result of Hansen gives the definition of the space X_∞

and shows that it is perfectoid.

Proposition 3.4.3 ([Han16, Theorem 1.4.ii]). *Let X be a perfectoid space with the action of a finite group G. Suppose that X has a covering by G-stable open affinoid perfectoid subspaces* $\mathrm{Spa}(R,R^+)$*, that X lives over* $\mathrm{Spa}(K,K^\circ)$ *for some perfectoid field K, and the G-action is K-linear. Then the quotient X/G in the category of v-ringed spaces is a perfectoid space. The perfectoid space X/G is covered by the perfectoid affinoids* $\mathrm{Spa}(R^G,R^{+G})$.

We check that we can let $X = X_{L,\infty}$ and $G = \mathrm{Gal}(L/K)$ in the above theorem. Using Lemma 3.4.2, we obtain an open affinoid cover of some finite level $X_{L,m}$ which pulls back to an open affinoid perfectoid cover of $X_{L,\infty}$ satisfying the density property for tilde-limits. We refine this cover to get a G-stable one by choosing a rational subcover of $X_{L,m}$ which is the pullback of an open affinoid cover $\{\mathrm{Spa}(R_m,R_m^+)\}$ of X_m. By definition, $X_{L,\infty}$ lives over $\mathrm{Spa}(K,K^\circ)$ and the action of $\mathrm{Gal}(L/K)$ is certainly K-linear, so we obtain the desired quotient X_∞.

It remains to show that $X_\infty \sim \varprojlim X_n$. The open affinoids $\mathrm{Spa}(R_m,R_m^+) \subset X_m$ pull back to inverse systems of open affinoids $\mathrm{Spa}(R_{L,i},R_{L,i}^+) \subset X_{L,i}$ with perfectoid tilde-limit $\mathrm{Spa}(R_{L,\infty},R_{L,\infty}^+) \subset X_{L,\infty}$. Let $(R_i,R_i^+) := (R_{L,i}^G,R_{L,i}^{G+}) \subset X_i$ for all i, let $(R_\infty,R_\infty^+) := (R_{L,\infty}^G,R_{L,\infty}^{G+}) \subset X_\infty$. Then by the following lemma, $\mathrm{Spa}(R_\infty,R_\infty^+)$ is a perfectoid affinoid tilde-limit for $\varprojlim \mathrm{Spa}(R_i,R_i^+)$ and therefore X_∞ is a perfectoid tilde-limit for $\varprojlim X_i$.

$\square$

Lemma 3.4.4. *Let K be a perfectoid field, let (R_i,R_i^+) be a filtered direct system of complete Tate-Huber pairs over (K,K°). Assume that there is a perfectoid affinoid Tate-Huber pair (R,R^+) over (K,K°) with* $\mathrm{Spa}(R,R^+) \sim \varprojlim \mathrm{Spa}(R_i,R_i^+)$ *such that $\varinjlim R_i$ is dense in R. Suppose that there is a finite group G acting K-linearly on all of the R_i and R, all compatibly with the transition morphisms and the morphisms of the tilde-limit. Then we have* $\mathrm{Spa}(R^G,R^{G+}) \sim \varprojlim \mathrm{Spa}(R_i^G,R_i^{G+})$.

Proof. We must show that $\varinjlim R_i^G$ is dense in R^G. Following the structure of [KL16, Theorem

3.3.26], we start with two easier cases, then combine them to get the general result.

We first assume that the characteristic of K doesn't divide the order of G. Given $r \in R^G$, there is a sequence $\{r_n\}$ converging to $\frac{r}{|G|}$ with each r_n coming from some R_{i_n}. For all g in G, $\{gr_n\}$ converges to $g(\frac{r}{|G|}) = \frac{r}{|G|}$, so $\{\sum_{g\in G} gr_n\}$ is a sequence of elements of $\varinjlim R_i^G$ converging to a.

We next assume that K is characteristic p and $|G| = p^k$. By [KL16, Theorem 3.3.26], R^G is perfectoid, so it is perfect. We therefore can take arbitrary pth roots, so for any $r \in R^G$ we can fix a p^kth root of r and a sequence $\{r_n\}$ converging to r^{1/p^k} with each r_n coming from some R_{i_n}. Then for all g in G, $\{gr_n\}$ converges to $g(r^{1/p^k}) = r^{1/p^k}$, so $\{\prod_{g\in G} gr_n\}$ is a sequence of elements of $\varinjlim R_i^G$ converging to r.

Finally, we assume that K is characteristic p and $|G| = p^k m$ where p does not divide m. Let H be a p-Sylow subgroup of G. Given $r \in R^G$, we can apply the previous paragraph to write $\frac{r}{m}$ as the limit of H-stable elements r_n each coming from some R_{i_n}. Choose left coset representatives $g_1, \dots, g_m$ for H in G. As the r_n are H-stable, the element $\sum_{j=1}^m g_j r_n$ is independent of the choice of representatives and is therefore G-stable. We conclude that $\{\sum_{j=1}^m g_j r_n\}$ is a sequence of elements of $\varinjlim R_i^G$ converging to r. $\square$

Chapter 4

Perfectoid covers of abeloids

Let K be a perfectoid field with value group $\Gamma = |K^{\times}|$ such that $\Gamma \subset \mathbb{Q}$. For example, this holds whenever K is a perfectoid subfield of $\mathbb{C}_p$. In this section, we reprove the main theorem of [BGH+18], constructing for any abeloid A over K a perfectoid space $A_\infty \sim \varprojlim_{[p]} A$. We give a sketch of the proof here.

After a finite extension of K, we can uniformize A as a quotient E/M for E a Raynaud extension by Theorem 2.4.5. By Proposition 3.4.1, we can construct our perfectoid cover after this extension, then descend to get the desired cover. We construct an F-tower as in Definition 3.3.2 for the torus part T of E, then use this to construct an F-tower for E, giving a perfectoid space $E_\infty \sim \varprojlim_{[p]} E$. We choose formal models that all have compatible actions of M, which allows us to take the quotient by this action to get formal models of A, which we show form an F-tower for A.

To construct our models for a torus T of rank r, we use techniques from tropical geometry: the theory of polytopal domains in rigid tori as described in [Gub07]. As we described in Section 2.4, we have a log map $\ell : T(K) \to \Gamma^r \subset \mathbb{R}^r$. The preimage of any Γ-rational polytope in $\mathbb{R}^r$ gives the K-points of an affinoid subspace of T. We can therefore construct formal analytic covers of T as in Definition 2.2.6 out of polytope decompositions of $\mathbb{R}^r$. Given a lattice Λ in

$\mathbb{R}^r$, if we take Λ-invariant decompositions, we get formal models with a lattice action. Such models were first considered by Mumford in [Mum72] and have applications in tropical geometry, see [EKL06], [Gub07], [Rab12] for details.

To move from models of T to models of E, we take the pushout as in diagram 2.3. When we do this, the lattice action is retained, allowing us to quotient out by the action of M to get models of A. We remark that the assumption $|K^\times| \subset \mathbb{Q}$ is required to ensure that Γ-rational Λ-invariant decompositions of $\mathbb{R}^r$ exist. One can combine the theory of polytope domains with the proof of [BGH$^+$18] to prove the theorem over any perfectoid field K. This will not result in formal models of the A, which will be necessary for Section 5, so we do not go into more detail here.

4.1 Formal models of tori

Let T be a split rigid analytic torus of rank r over K, so $T \cong (\mathbb{G}_{m,an})^r$. Let M be a lattice in T with rank r as in Definition 2.4.4: that is, the log map $\ell : T(K) \to \mathbb{R}^r$ sends M bijectively to a lattice Λ. Fix coordinates $x_1, \ldots, x_r$ on $T = \mathbb{G}_{m,an}^r$ and $u_1, \ldots, u_r$ on $\mathbb{R}^r$ so that the log map sends x_i to u_i.

In this section, we construct an F-tower $\{\mathfrak{T}_n\}$ of T such that the translation action of M on T extends to an action on each $\mathfrak{T}_n$, and such that these actions are compatible with the morphisms $[\mathfrak{p}]_n$. This will allow us to construct an F-tower for T/M, though we will wait until next section to do this in the more general setting of Raynaud extensions.

We briefly recall some definitions from convex geometry. (We note, however, that one can get by just thinking about covering $\mathbb{R}^r$ by sufficiently small hypercubes as in Remark 4.1.3.)

Definition 4.1.1. *Let Γ be a subgroup of $\mathbb{R}$. Let Λ be a lattice in $\mathbb{R}^r$.*

- *A* polytope *Δ is a bounded subset of $\mathbb{R}^r$ which is the intersection of finitely many closed half-spaces.*

- *We say that* Δ *is* Γ-rational *if each half-space can be defined by an equation* $\sum_1^r e_i u_i \geq c$ *with all the* $e_i \in \mathbb{Z}$ *and* $c \in \Gamma$.

- *A* closed face *of* Δ *is either all of* Δ *or the intersection of* Δ *with the boundary of a closed half-space containing* Δ.

- *A* vertex *of* Δ *is a zero-dimensional closed face of* Δ.

- *An* open face *of* Δ *is a closed face without its properly contained closed faces.*

- *A* polytopal complex $\mathscr{C}$ *in* $\mathbb{R}^r$ *is a locally finite set of polytopes such that*

  1. *For any* Δ *in* $\mathscr{C}$*, every closed face of* Δ *is also in* $\mathscr{C}$*;*

  2. *For any* Δ_1, Δ_2 *in* $\mathscr{C}$*, either* $\Delta_1 \cap \Delta_2$ *is empty or is a shared closed face.*

- *A polytopal complex is* Γ-rational *if every polytope in the complex is* Γ*-rational.*

- *A* polytopal decomposition *of* $S \subset \mathbb{R}^r$ *is a polytopal complex* $\mathscr{C}$ *such that* $\bigcup_{\Delta \in \mathscr{C}} \Delta = S$.

- *A* Λ-periodic, Γ-rational decomposition of $\mathbb{R}^r$ *is a* Γ*-rational polytopal decomposition* $\mathscr{C}$ *of* $\mathbb{R}^r$ *such that the translation action of* Λ *on* $\mathbb{R}^r$ *permutes the polytopes in* $\mathscr{C}$.

We will be using the following decomposition in our constructions:

Definition 4.1.2. *Given a rational number* α*, let* $\Delta_\alpha := [0, \alpha]^r \subset \mathbb{R}$ *be an r-dimensional hypercube. Given an element* $\mathbf{e} = (e_1, \ldots, e_r) \in \mathbb{Z}^r$*, let* $\mathbf{e}\Delta_\alpha$ *be the translation of* Δ_α *by* $(e_1\alpha, \ldots, e_r\alpha)$*. As we are assuming that* $\Gamma \subset \mathbb{Q}$*, the set* $\{\mathbf{e}\Delta_\alpha\}_{\mathbf{e}} \in \mathbb{Z}^r$ *is a* Γ*-rational polytope decomposition of* $\mathbb{R}^r$*. If we choose* α *so that every element* $\lambda \in \Lambda$ *is an integer multiple of* α*, this decomposition is also* Λ*-invariant. (For example, picking a basis* $\lambda_1, \ldots, \lambda_r$ *of* Λ*, we can let d be the least common multiple of the denominators of the components of the* λ_i *and take* $\alpha = \frac{1}{d}$*.) In this case, we say that* α *divides the lattice* Λ*.*

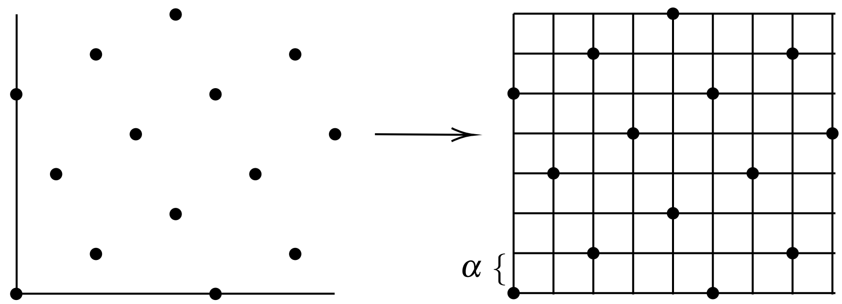

Figure 4.1: Constructing a hypercube decomposition in $\mathbb{R}^2$.

In [Mum72], Mumford showed that formal models for T can be built out of Γ-rational polytopal decompositions of $\mathbb{R}^r$. He showed that the models coming from Λ-periodic, Γ-rational decompositions of $\mathbb{R}^r$ lead to models of T/M. We now summarize the key steps of his construction. As he did not use the language of rigid analytic geometry to do this, our treatment largely follows that of [Gub07], which uses Berkovich spaces. We do not assume K to be algebraically closed, which complicates things somewhat.

Remark 4.1.3. The reader who just wants to get to the F-towers and perfectoid spaces can safely skip ahead to Proposition 4.1.9, as our models will be based off of the decomposition in Example 4.1.2. (Everything in between was written when I was using Example 4.1.15 for the models, which made things harder...)

Notation 4.1.4. Throughout the section, we write r-tuples of objects in bold. For example, $\mathbf{e} \in \mathbb{Z}^r$ is shorthand for the tuple $(e_1, \ldots, e_r)$, and $\mathbf{x}^{\mathbf{e}}$ is shorthand for $x_1^{e_1} \cdots x_r^{e_r}$.

We check that polytopes give rise to the admissible formal schemes from Definition 2.2.5. This will allow us to build models of T by gluing them as in 2.2.7.

Proposition 4.1.5. *Let Δ be a Γ-rational polytope. Then*

1. *The preimage $\ell^{-1}(\Delta) \subset T(K)$ is the K-points of an affinoid open U_Δ in T.*

2. *We have $U_\Delta = \mathrm{Sp}(A_\Delta)$, where*

$$A_\Delta := \Big\{ \sum_{\mathbf{e}\in\mathbb{Z}^r} a_{\mathbf{e}} \mathbf{x}^{\mathbf{e}} : a_{\mathbf{e}} \in K,\ \lim_{|\mathbf{e}|\to\infty} |a_{\mathbf{e}}| e^{-\mathbf{u}\cdot\mathbf{e}} = 0\ \forall\ \mathbf{u} \in \blacksquare \Big\}.$$

3. *If $\Delta = \{\mathbf{u}\}$ is a single point, the supremum norm on A_Δ can be computed via the formula*

$$|\sum_{\mathbf{e}\in\mathbb{Z}^r} a_{\mathbf{e}} \mathbf{x}^{\mathbf{e}}|_{\mathbf{u}} = \sup_{\mathbf{e}\in\mathbb{Z}^r} |a_{\mathbf{e}}| e^{-\mathbf{u}\cdot\mathbf{e}}.$$

   In this case, the supremum norm is multiplicative.

4. *In general, let V_Δ be the (finite) set of vertices of Δ. The supremum norm on U_Δ can be computed via the formula*

$$|f|_{sup} = \max_{\mathbf{v}\in V_\Delta} |f|_{\mathbf{v}}.$$

5. *The K°-algebra A_Δ° is admissible.*

Proof. Parts (1) and (2) follow from a combination of [EKL06, Proposition 3.1.5] and [EKL06, Proposition 3.1.8c]. An alternate proof of the parts (1)-(4) is [Gub07, Proposition 4.1]. When K is algebraically closed (more generally, stable) or discretely valued, part (5) is [BGR84, §6.4.1]. As these conditions do not hold for the perfectoid fields we are interested in, we make a more direct argument.

To show (5), we construct an epimorphism $\alpha : K\langle y_1, \ldots, y_s\rangle \to A_\Delta$ such that the residue norm $|\cdot|_\alpha$ agrees with the supremum norm $|\cdot|_{\mathrm{sup}}$. In this case, $A_\Delta^\circ = \mathrm{Im}(R\langle y_1, \ldots, y_s\rangle)$, so it is topologically finite type.

Lemma 4.1.6. *For any vertex v of Δ, let $C_v \subset \mathbb{R}^r$ be the cone spanned by the outward pointing normal vectors of the $(r-1)$-dimensional faces of Δ meeting at v. Then the set of monomials $\mathbf{x}^{\mathbf{e}}$ with supremum norm attained at v corresponds to the set of points $\mathbf{e}$ in $C_v \cap \mathbb{Z}^r$.*

Proof. This follows directly from the formula in (3). See [Rab12] for a more detailed exposition of these types of results, including some helpful pictures. $\square$

For each of the finitely many vertices $v_1, \ldots, v_k$ of Δ, choose a finite set of tuples $\mathbf{e}_{i,1}, \ldots, \mathbf{e}_{i,\ell_i}$ in $C_{v_i} \cap \mathbb{Z}^r$ such that the semigroup generated by the $\mathbf{e}_{i,j}$ is all of $C_{v_i} \cap \mathbb{Z}^r$. For each $\mathbf{e}_{i,j}$, choose an element $b_{i,j} \in K$ with $|b_{i,j}| = e^{v \cdot \mathbf{e}_{i,j}} = |\mathbf{x}^{\mathbf{e}_{i,j}}|_{\text{sup}}$ (the second equality follows from (4)). Let S be the set of all the $\mathbf{e}_{i,j}$ for all vertices of Δ. For each $e_{i,j} \in S$, let $y_{i,j}$ be a formal variable. We claim that the map

$$\alpha : K\langle y_{1,1}, \ldots, y_{k,\ell_k} \rangle \to A_\Delta, \quad y_{i,j} \mapsto \frac{\mathbf{x}^{\mathbf{e}_{i,j}}}{b_{i,j}}$$

is the desired epimorphism. This is a well defined morphism of affinoid algebras as each $y_{i,j}$ is mapped to an element of supremum norm 1.

Lemma 4.1.7. *For each monomial $\mathbf{x}^{\mathbf{e}}$, there is a monomial $b_{\mathbf{e}}\mathbf{y}^{\mathbf{q}}$ with $\alpha(b_{\mathbf{e}}\mathbf{y}^{\mathbf{q}}) = \mathbf{x}^{\mathbf{e}}$ and $|b_{\mathbf{e}}| = |\mathbf{x}^{\mathbf{e}}|_{sup}$.*

Proof. Let v_i be a vertex of Δ such that the supremum norm of $\mathbf{x}^{\mathbf{e}}$ is attained at v_i. By assumption, we can write $\mathbf{e} = \sum_{j=1}^{\ell_i} a_j \mathbf{e}_{i,j}$ for some non-negative integers a_j, so we can write

$$\mathbf{x}^{\mathbf{e}} = \prod_{j=1}^{\ell_i} \mathbf{x}^{a_j \mathbf{e_{i,j}}} = \alpha(\prod_{j=1}^{\ell_i} (b_{i,j} y_{i,j})^{a_j}).$$

As $|\cdot|_{v_i}$ is multiplicative, the norms $|\cdot|_{v_i}$ and $|\cdot|_{\text{sup}}$ agree for monomials with exponents $\mathbf{e}$ and $\mathbf{e}_{i,j}$, and the Gauss norm on $K\langle y_{1,1}, \ldots, y_{k,\ell_k} \rangle$ is multiplicative, so we are done. $\square$

The lemma implies that $|\cdot|_{\text{sup}}$ agrees with $|\cdot|_\alpha$ on all monomials in A_Δ. This gives us surjectivity of α as for any $\sum a_{\mathbf{e}} \mathbf{x}^{\mathbf{e}} \in A_\Delta$, the corresponding power series $\sum a_{\mathbf{e}} b_{\mathbf{e}} y^{\mathbf{q}}$ converges with respect to the Gauss norm on $K\langle y_{1,1}, \ldots, y_{k,\ell_k} \rangle$. By Part (4), we have $|\sum a_{\mathbf{e}} \mathbf{x}^{\mathbf{e}}|_{\text{sup}} = |\sum a_{\mathbf{e}} b_{\mathbf{e}} y^{\mathbf{q}}|_{\text{sup}}$, which implies that $|\cdot|_{\sup} \geq |\cdot|_\alpha$. As it is always the case that $|\cdot|_{\sup} \leq |\cdot|_\alpha$, the two norms must agree.

$\square$

Write $\mathfrak{U}_\Delta := \operatorname{Spf}(A_\Delta^\circ)$ for the affine formal scheme corresponding to U_Δ and $\widetilde{U}_\Delta$ for the mod ϖ reduction of $\mathfrak{U}_\Delta$.

Proposition 4.1.8. *Given an inclusion of Γ-rational polytopes $\Delta' \subset \Delta$, the corresponding morphism $U_{\Delta'} \to U_\Delta$ induces an open immersion $\widetilde{U}_{\Delta'} \to \widetilde{U}_\Delta$ of reductions if and only if Δ' is a closed face of Δ.*

Proof. [Gub07, Proposition 4.4(e)] $\square$

Combining these results, we see that a Γ-rational polytopal decomposition $\mathscr{C}$ of $\mathbb{R}^r$ gives rise to a formal analytic cover in the sense of Definition 2.2.6, and therefore a formal model $\mathfrak{T}_\mathscr{C}$ of T. We want to extend a morphism $f : T \to T$ of rigid spaces to a morphism of formal models $\mathfrak{f} : \mathfrak{T}_\mathscr{C} \to \mathfrak{T}_{\mathscr{C}'}$. By Proposition 2.2.7, it is enough to ensure that for each $\Delta \in \mathscr{C}$, there is some $\Delta' \in \mathscr{C}'$ such that $f(U_\Delta) \subset U_{\Delta'}$.

In our case, we want to extend the translation morphisms $\tau_m : T \to T$ for all lattice elements $m \in M$. On $\mathbb{R}^r$, these maps induce the translation maps $+\lambda : \mathbb{R}^r \to \mathbb{R}^r$ for $\lambda \in \Lambda$. We conclude that:

Proposition 4.1.9. *There is a formal model for T extending the action of M corresponding to any Λ-periodic, Γ-rational decomposition of $\mathbb{R}^r$.*

Any such model leads to a model of T/M by identifying any open U_Δ in our cover with the opens $U_{\lambda+\Delta}$ for all $\lambda \in \Lambda$.

Example 4.1.10. The construction of the Tate curve $\mathbb{G}_m/q^\mathbb{Z}$ with $|q| = c$ in Example 2.4.1 corresponds to the decomposition of $\mathbb{R}$ into intervals of length $c/2$. More generally, any sequence $c = c_0 < c_1 < \cdots < c_n = 1$ of elements of Γ gives rise to a Λ-periodic, Γ-rational decomposition

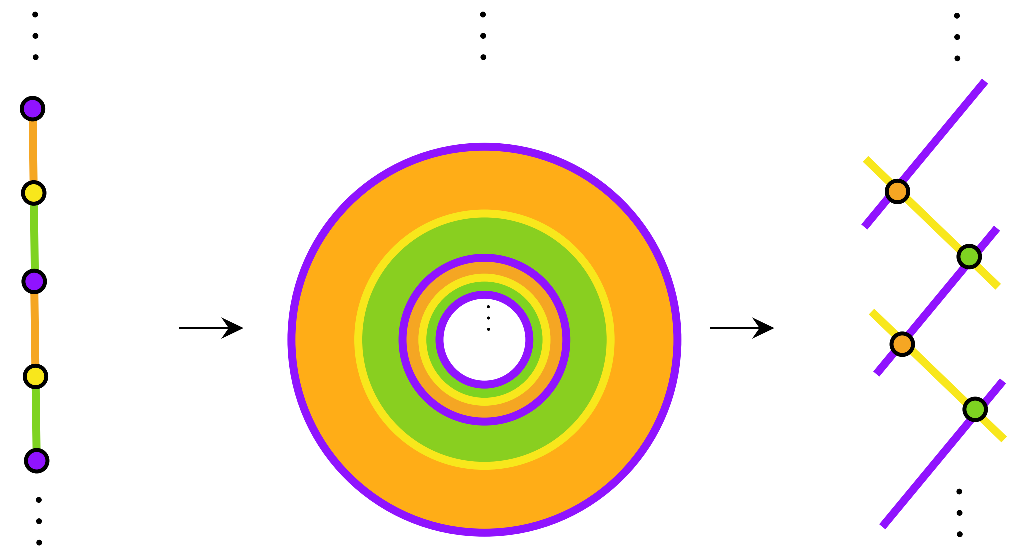

Figure 4.2: Left: the decomposition of $\mathbb{R}$ corresponding to a Tate curve. Center: the corresponding formal affinoid cover of $\mathbb{G}_m$. Right: the special fiber of the corresponding formal model.

of $\mathbb{R}$, and therefore a formal model of $\mathbb{G}_m/q^{\mathbb{Z}}$. This is given by translating the intervals $[c_{i-1}, c_i]$ by $e\lambda$ for all $1 \leq i < n$ and $e \in \mathbb{Z}$.

The models for T that we will be using in our construction come from the decompositions in Example 4.1.2.

Definition 4.1.11. *Let α be a rational number dividing the lattice Λ in the sense of Definition 4.1.2, and let $\{\mathbf{e}\Delta_\alpha\}_{\mathbf{e}} \in \mathbb{Z}^r$ be the corresponding Λ-periodic, Γ-rational polytope decomposition of $\mathbb{R}^r$. By Proposition 4.1.9, this gives a formal model of T extending the action of M. We denote this model by $\mathfrak{T}_\alpha$.*

These models behave well with respect to the multiplication by p map.

Proposition 4.1.12. *The morphism $[p] : T \to T$ extends to a morphism $[\mathfrak{p}]_n : \mathfrak{T}_{\alpha/p} \to \mathfrak{T}_\alpha$ of formal models. This morphism reduces mod ϖ to relative Frobenius.*

Proof. This follows from [Gub07, Proposition 6.4a], which shows that $[p]$ extends to $[\mathfrak{p}]_n$ and describes the morphism locally on polytopes as the morphism $U_{\Delta_{\alpha/p}} \to U_{\Delta_\alpha}$ sending $x_i \mapsto x_i^p$. $\square$

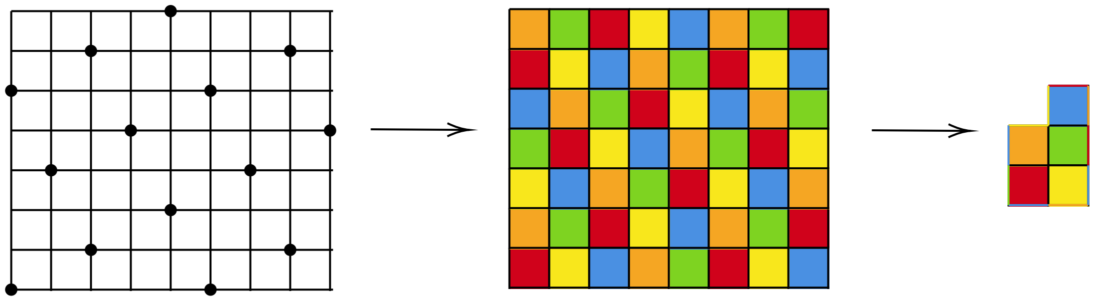

Figure 4.3: Quotienting the decomposition in Figure 4.1 by the lattice identifies squares of the same color. The polytope data gives a formal analytic cover, and therefore a model of T/M as in Proposition 4.1.9.

Corollary 4.1.13. *The data $\{\mathfrak{T}_{\alpha/p^n}\}$ is an F-tower for T, so it gives rise to a perfectoid space $T_\infty \sim \varprojlim_{[p]} T$.*

We note that all of the models in this tower are already defined over K° (that is, with no field extension needed) as the perfectoidness of K implies that $|K^\times|$ is p-divisible. The generic fiber of the formal torus $\overline{T}_\eta$ described in Notation 2.4.2 corresponds to the polytope domain $U_{\{0\}} \subset T$. The translation action of $\overline{T}$ on T therefore induces translation by 0 on $\mathbb{R}^r$, so it fixes any polytope decomposition. This gives:

Proposition 4.1.14. *For any $n \geq 1$, the action $\overline{T}_\eta \times T \to T$ extends to a morphism of formal models $\overline{T} \times \mathfrak{T}_{\alpha/p^n} \to \mathfrak{T}_{\alpha/p^n}$. This action is compatible with the models for $[p]$ from Proposition 4.1.12 in that the following diagram commutes:*

$$\begin{tikzcd}
\overline{T} \times \mathfrak{T}_{\alpha/p^{n+1}} \arrow[r] \arrow[d, "{[p]\times[\mathfrak{p}]_n}"'] & \mathfrak{T}_{\alpha/p^{n+1}} \arrow[d, "{[\mathfrak{p}]_n}"] \\
\overline{T} \times \mathfrak{T}_{\alpha/p^n} \arrow[r] & \mathfrak{T}_{\alpha/p^n}.
\end{tikzcd}$$

Proof. On the generic fiber, all of the maps are well defined and the diagram commutes by definition. On the special fiber, the action of $\overline{T}$ is trivial, so the diagram commutes by functoriality. □

Example 4.1.15. Here is another class of examples. Fix a fundamental domain $D \subset \mathbb{R}^r$ for $\mathbb{R}^r/\Lambda$ with vertices in Γ and an integer $n \geq 1$. We get a Λ-periodic, Γ-rational decomposition

$\{\frac{1}{p^n}\lambda D\}_{\lambda\in\Lambda}$. Here λ acts on D by translation and $\frac{1}{p^n}$ acts by scaling. Note that this decomposition is Λ-periodic by definition, and that it is Γ-rational as we are assuming that $\Gamma \subset \mathbb{Q}$.

The corresponding models again behave well with respect to multiplication by p, and all of the above constructions carry over. This gives a different formal model of T_∞, and if we carried this model through this thesis, would give different formal models of the various perfectoid covers we end up constructing. While different decompositions will lead to different F-towers, the resulting perfectoid space T_∞ is unique by [SW13, Proposition 2.4.5].

4.2 Formal models of abeloids

Let A be an abeloid variety over K. By Theorem 2.4.6, after a suitable finite separable extension of K, there is a Raynaud extension

$$1 \to T \to E \xrightarrow{\pi} B \to 1 \tag{4.1}$$

and a full rank lattice $M \subset E(K)$ such that $A \cong E/M$. By Proposition 3.4.1, we can construct a perfectoid tilde-limit for A after a finite Galois extension of K, then descend to get the desired perfectoid tilde-limit over K. We can therefore assume that our Raynaud extension and lattice are already defined over K.

By Definition 2.4.4, the image of M under the log map is a lattice Λ in $\mathbb{R}^r$. In this section, we use the formal models of T constructed in Section 4.1 to construct formal models of E that retain an action of M. Quotienting out by this action, we obtain formal models of A, which we use to construct an F-tower for A.

Recall from Section 2.4 that E was constructed as a pushout $T \times^{\overline{T}_\eta} \overline{E}_\eta$. We construct models for E as pushouts of our models for T.

Proposition 4.2.1. *Let α be rational number dividing Λ as in Definition 4.1.2. For any integer*

$n > 1$, let $\mathfrak{T}_{\alpha/p^n}$ be the corresponding model for T defined in Definition 4.1.11.

1. *The formal scheme $\mathfrak{E}_{\alpha/p^n} := \mathfrak{T}_{\alpha/p^n} \times^{\overline{T}} \overline{E}$ is a flat formal model of the rigid space E.*

2. *There is a morphism $\mathfrak{E}_{\alpha/p^n} \to \overline{B}$ which is a fiber bundle and a formal model of $E \to B$.*

3. *For any integer $n > 1$, there is an affine morphism $[\mathfrak{p}]_n : \mathfrak{E}_{\alpha/p^{n+1}} \to \mathfrak{E}_{\alpha/p^n}$ which is a formal model of $[p] : E \to E$ and with reduction mod ϖ factoring through relative Frobenius.*

We therefore obtain an F-tower $\{\mathfrak{E}_{\alpha/p^n}\}$ for E, giving us a perfectoid space $E_\infty \sim \varprojlim_{[p]} E$.

Proof. This follows from standard results on fiber bundles, transferred over to to the category of adic spaces. We don't go into detail here as everything is about to be proven more explicitly in Proposition 4.2.2, but we want to point out that this result can be obtained by more abstract means.

□

The models $\mathfrak{E}_{\alpha/p^n}$ come with an action of M, which gives us models of A leading to an F-tower for A. To see this, we explicitly construct formal analytic covers for $\mathfrak{E}_{\alpha/p^n}$ and apply Proposition 2.2.7 to show that these covers lead to the desired models. The precise requirements for our formal analytic covers are summarized in the following:

Proposition 4.2.2. *For every integer $n \geq 1$, these exists a formal analytic cover (in the sense of Definition 2.2.6) $\mathscr{U}_n = \{U_{i,n}\}_{i \in I_n}$ of E such that:*

1. *The formal model of E corresponding to $\mathscr{U}_n$ is $\mathfrak{E}_{\alpha/p^n}$.*

2. *For all $m \in M$ and $U_{i,n}$ in $\mathscr{U}_n$, there is some $U_{j,n}$ in $\mathscr{U}_n$ such that the translation map $\tau_m : E \to E$ restricts to an isomorphism $U_{i,n} \to U_{j,n}$. If m is not the identity, then $U_{i,n} \cap U_{j,n} = \emptyset$.*

3. *For all $n \geq 1$ and $U_{i,n+1}$ in $\mathscr{U}_{n+1}$, there is some $U_{j,n}$ in $\mathscr{U}_n$ such that the map $[p] : E \to E$ restricts to a morphism $U_{i,n+1} \to U_{j,n}$. Furthermore, the mod ϖ reduction of this morphism factors through relative Frobenius.*

Proof. We first construct the cover $\mathscr{U}_1$. The map $\pi : E \to B$ sends each $m \in M$ to a point $b_m \in B(K)$ which extends to a point $\overline{b}_m \in \overline{B}(K^\circ)$. As $\overline{B}$ is a formal abelian scheme, we get an isomorphism $\times \overline{b}_m : \overline{B} \to \overline{B}$. Choose a formal open cover $\{\overline{V}_{j,1}\}$ of $\overline{B}$ such that for any $m \in M$, the morphism $\times \overline{b}_m$ permutes the $\overline{V}_{j,1}$. Note that it is enough to choose generators $m_1, \dots, m_r$ of M and ensure that the corresponding morphisms permute the $\overline{V}_{j,1}$. Taking the generic fiber, we get an affinoid cover of B which is permuted by multiplication by b_m for any $m \in M$. Refining the cover if needed, we may assume that π splits over this cover, so E is locally of the form $T \times V_{j,1}$. We can now define the cover $\mathscr{U}_1$ of E to contain all opens of the form $U_{\mathbf{e}\Delta_\alpha} \times V_{j,1}$ for $U_{\mathbf{e}\Delta_\alpha}$ as in Definition 4.1.11 and $V_{j,1}$ in our cover of B.

Now to construct $\mathscr{U}_n$, we first cover B by the preimages $V_{j,n}$ of the affinoid opens $V_{j,1}$ under the morphism $[p^n] : B \to B$. This is again a cover of B by affinoid opens permuted by morphisms τ_{b_m}. By iterating the following lemma, we see that π again splits over each $V_{j,n}$.

Lemma 4.2.3. *Let $\{V_j\}$ be a formal analytic cover of the abelian variety B over which the map $\pi : E \to B$ splits. Then π also splits over the cover $\{[p]^{-1}(V_j)\}$.*

Proof. This can be checked over $\overline{B}$. As in [Lüt16, Proposition A.2.5], we can construct the splitting on the special fiber $\widetilde{B}$ and then use smoothness to lift to $\overline{B}$. On the special fiber, multiplication by p factors through relative Frobenius, so we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{T} \arrow[r] \arrow[d, "\cong{,}V"] & \widetilde{E} \arrow[r] \arrow[d, "V"] & \widetilde{B} \arrow[r] \arrow[d, "V"] & 0 \\
0 \arrow[r] & \widetilde{T}^{(p^{-1})} \arrow[r] \arrow[d, "F"] & \widetilde{E}^{(p^{-1})} \arrow[r] \arrow[d, "F"] & \widetilde{B}^{(p^{-1})} \arrow[r] \arrow[d, "F"] & 0 \\
0 \arrow[r] & \widetilde{T} \arrow[r] & \widetilde{E} \arrow[r] & \widetilde{B} \arrow[r] & 0
\end{tikzcd}$$

where the composition of the vertical maps is $[p]$. Relative Frobenius is functorial, so the local sections lift to the middle row. The square on the top right is a fiber product, so the local sections pull back to the top row, where they are defined on the cover $\{[p]^{-1}(\widetilde{V}_j)\}$. □

We now define the cover $\mathscr{U}_n$ of E to contain all opens of the form $U_{\mathbf{e}\Delta_\alpha/p^n} \times V_{j,n}$. Note that the $\mathscr{U}_n$ are all formal analytic covers as each open is the product of opens in formal analytic covers for T and B. Part (1) follows as the cover $\mathscr{U}_n$ was constructed so that the corresponding affine formal schemes are subschemes of $\mathfrak{E}_{\alpha/p^n}$.

For part (2), we see that $\tau_m : E \to E$ restricts to an isomorphism

$$\tau_m : U_{\mathbf{e}\Delta_\alpha/p^n} \times V_{j,n} \xrightarrow{\simeq} U_{\ell(m)+\mathbf{e}\Delta_\alpha/p^n} \tau_{b_m} V_{j,n},$$

and the target is again an open in $\mathscr{U}_n$ as $p^n\ell(m) + \lambda \in \Lambda$ and $b_m V_{j,n}$ is again in our cover of B. If m is not the identity, the polytopes $\mathbf{e}\Delta_\alpha/p^n$ and $\ell(m) + \mathbf{e}\Delta_\alpha/p^n$ are disjoint, so the corresponding open subsets of T and therefore E are disjoint.

For part (3), we see that $[p] : E \to E$ restricts to a morphism

$$[p] : U_{\mathbf{e}\Delta_\alpha/p^n} \times V_{j,n} \to U_{\mathbf{e}\Delta_\alpha/p^{n-1}} \times V_{j,n-1}$$

as multiplication by p acts by scaling the polytope $\mathbf{e}\Delta_\alpha/p^n$ and we chose our covers of B so that $[p](V_{j,n}) = V_{j,n-1}$. The target is therefore an open in $\mathscr{U}_{n-1}$ as desired. To see that the reduction factors through relative Frobenius, we use Proposition 4.1.12 for the torus part and Example 3.3.7 for the abelian part. □

The following theorem now follows directly.

Theorem 4.2.4. *For every integer $n \geq 1$, let $\mathfrak{E}_{\alpha/p^n}$ be the model of E constructed in Proposition 4.2.1. Then*

1. *For every $m \in M$ and $n \geq 1$, the morphism $\tau_m : E \to E$ extends to a morphism $\tau_{\mathfrak{m}} : \mathfrak{E}_{\alpha/p^n} \to \mathfrak{E}_{\alpha/p^n}$.*

2. *Taking the quotient of $\mathfrak{E}_{\alpha/p^n}$ by the action of M defined in (1), we obtain models $\mathfrak{A}_{\alpha/p^n}$ of A such that the analytic quotient map $E \to A$ extends to a map of models $\mathfrak{E}_{\alpha/p^n} \to \mathfrak{A}_{\alpha/p^n}$.*

3. *The models for τ_m in (1) commute with the models for $[p]$ in Proposition 4.2.1 in that the commutative diagram*

$$\begin{array}{ccc} E & \xrightarrow{\tau_m} & E \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ E & \xrightarrow{\tau_m^p} & E \end{array}$$

extends to a commutative diagram

$$\begin{array}{ccc} \mathfrak{E}_{\alpha/p^{n+1}} & \xrightarrow{\tau_{\mathfrak{m}}} & \mathfrak{E}_{\alpha/p^{n+1}} \\ {\scriptstyle [\mathfrak{p}]_n}\big\downarrow & & \big\downarrow{\scriptstyle [\mathfrak{p}]_n} \\ \mathfrak{E}_{\alpha/p^n} & \xrightarrow{\tau_{\mathfrak{m}^{\mathfrak{p}}}} & \mathfrak{E}_{\alpha/p^n}. \end{array}$$

4. *The models for $[p] : E \to E$ induce models of $[p] : A \to A$ in that the commutative diagram*

$$\begin{array}{ccc} E & \longrightarrow & A \\ \big\downarrow{\scriptstyle [p]} & & \big\downarrow{\scriptstyle [p]} \\ E & \longrightarrow & A \end{array}$$

extends to a commutative diagram of formal models

$$\begin{array}{ccc} \mathfrak{E}_{\alpha/p^{n+1}} & \longrightarrow & \mathfrak{A}_{\alpha/p^{n+1}} \\ \big\downarrow{\scriptstyle [\mathfrak{p}]_n} & & \big\downarrow{\scriptstyle [\mathfrak{p}]_n} \\ \mathfrak{E}_{\alpha/p^n} & \longrightarrow & \mathfrak{A}_{\alpha/p^n}. \end{array}$$

5. *The mod ϖ reduction of $[\mathfrak{p}]_n : \mathfrak{A}_{\alpha/p^{n+1}} \to \mathfrak{A}_{\alpha/p^n}$ factors through relative Frobenius.*

Proof. Parts (1), (3), and (4) are all formal consequences of Proposition 4.2.2 and Proposition 2.2.7, obtained by checking that all the maps in question preserve the data of the formal analytic covers. The model $\mathfrak{A}_{\alpha/p^n}$ in Part (2) is constructed from the formal analytic cover of A obtained by using the translation maps $\tau_m : E \to E$ to identify the opens $U_{i,n}$ in $\mathscr{U}_n$ that are sent to the same place in A. For Part (5), we note that $[\mathfrak{p}]_n : \mathfrak{A}_{\alpha/p^{n+1}} \to \mathfrak{A}_{\alpha/p^n}$ is locally on the source the same as $[\mathfrak{p}]_n : \mathfrak{E}_{\alpha/p^{n+1}} \to \mathfrak{E}_{\alpha/p^n}$, and the latter map factors through relative Frobenius. $\square$

Restating the above Theorem, we recover the main theorem of [BGH$^+$18].

Theorem 4.2.5. *We have an F-tower $\{\mathfrak{A}_{\alpha/p^n}\}$ for A, giving us a perfectoid space $A_\infty \sim \varprojlim_{[p]} A$.*

This chapter contains material from joint work with Ben Heuer which is currently being prepared for submission for publication.

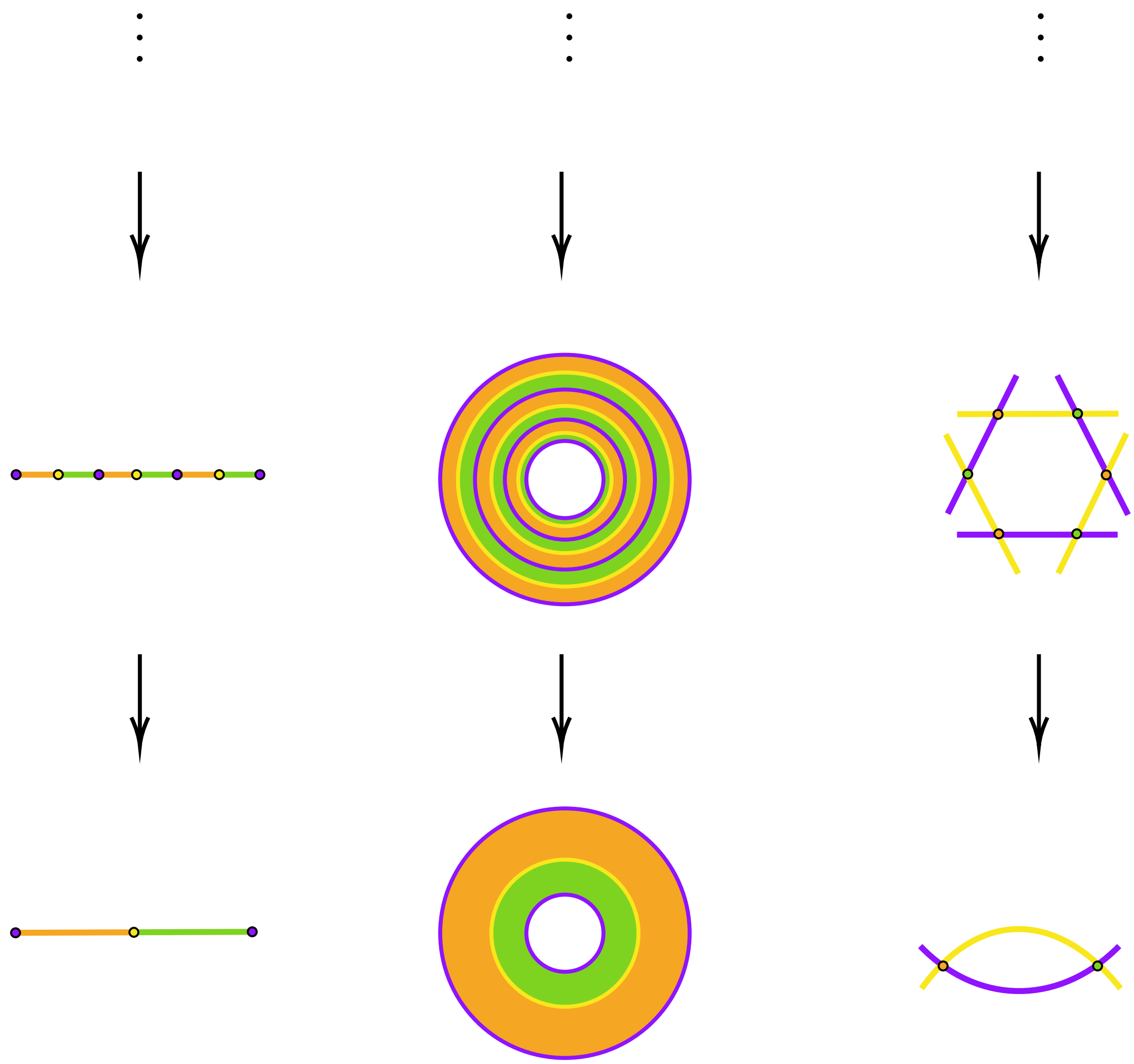

Figure 4.4: The first two levels of the F-tower for a Tate curve, extending Figure 2.4. Left: a tower of polytope decompositions of $[1,c]$, coming from the quotient of the decompositions in Corollary 4.1.13 by $c^{\mathbb{Z}}$. Center: The corresponding tower of formal affinoid covers of $\mathbb{G}_m/q^{\mathbb{Z}}$. Right: The corresponding tower of special fibers. The maps respect the coloring, so the preimage of the orange annulus is the three disjoint orange annuli, and the preimage of the purple $\mathbb{P}^1$ on the special fiber is three disjoint purple $\mathbb{P}^1$s above.

Chapter 5

Tilting perfectoid covers of abeloids

We continue to assume that K is a perfectoid field with value group $\Gamma \subset \mathbb{Q}$. Given an abeloid A over K, in Theorem 4.2.5 we constructed an F-tower for A, giving us a perfectoid cover A_∞ of A. In this chapter, after taking a pro-p extension of K, we construct an abeloid A' over $K^\flat$ such that the corresponding perfectoid cover $(A')_\infty$ is the tilt $A_\infty^\flat$ of A_∞. By Lemma 3.3.11, if a perfectoid space is built using an F-tower over K°, its tilt can be built using an F-tower over $K^{\flat\circ}$ such that the isomorphism $K^\circ/\varpi \cong K^{\flat\circ}/\varpi^\flat$ induces an isomorphism of the special fibers of the two F-towers.

To construct A', we first use the deformation theory of abelian varieties to construct a Raynaud extension E' over $K^\flat$ such that the special fibers of $\overline{E}$ and $\overline{E}'$ agree. This implies that $E_\infty^\flat = E'_\infty$. Choosing a compatible system of pth roots of M, we get a subgroup $M_\infty \subset E_\infty(K)$. This is where we need the pro-p extension of K. This subgroup tilts to a subgroup $(M_\infty)^\flat \subset E_\infty^\flat(K^\flat) = E'_\infty(K^\flat)$, which we show projects down to a lattice $M' \subset E'(K^\flat)$. Taking the quotient, we obtain the desired abeloid $A' := E'/M'$.

5.1 Deformation theory of Raynaud extensions

Let E be a Raynaud extension over K, recall from Section 2.4 that this is equivalent to a strict exact sequence of formal groups over K°

$$1 \to \overline{T} \to \overline{E} \xrightarrow{\overline{\pi}} \overline{B} \to 1. \tag{5.1}$$

In this section, we combine deformation theoretic results of Illusie with a theorem of Grothendieck to construct a corresponding Raynaud extension E' over $K^\flat$ such that the mod ϖ special fiber of the sequence (5.1) is isomorphic to the mod $\varpi^\flat$ special fiber of the corresponding sequence for E'.

A common setup in deformation theory is as follows: let R be a ring, let I be an ideal such that $I^2 = 0$. Given some object - an algebra, scheme, morphism, etc - living over R/I, we want to classify the corresponding objects living over R that agree with the original object when reduced mod I. In nice cases, such questions are controlled by the cotangent bundle. We need to work with non-noetherian rings, so we instead use the cotangent complex, which is more than general enough. We will be working with $R = K^\circ/\varpi^{i+1}$ and $I = (\varpi^i)$, successively lifting (semi)abelian schemes from K°/ϖ^i to K°/ϖ^{i+1}.

For any map $S \to R$ of rings, the cotangent complex $L_{S/R}$ is a complex of S-modules living in the derived category $D(S)$. The cohomology in degree zero is $\Omega^1_{S/R}$, and if $R \to S$ is smooth, the complex is concentrated in degree zero, so we recover the classical deformation theory results in this case. For details on the construction, see [Ill71] or [Sta20, Tag 08P5].

To deform Raynaud extensions, we use the following theorems of Illusie on deforming algebras over $R_0 := R/I$.

Theorem 5.1.1 ([Ill71, III.2.1.2.3])**.** *Let S_0 be a flat R_0-algebra. There is an obstruction class $o(S_0)$ in $\mathrm{Ext}^2(L_{S_0/R_0}, S_0 \otimes_{R_0} I)$ which vanishes exactly when S_0 can be lifted to a flat R-algebra S such that $S \otimes_R R_0 = S_0$. If a lift exists, the set of isomorphism classes of such lifts forms a torsor* *under* $\mathrm{Ext}^1(L_{S_0/R_0}, S_0 \otimes_{R_0} I)$*. Each lift has automorphism group* $\mathrm{Hom}(L_{S_0/R_0}, S_0 \otimes_{R_0} I)$*.*

Theorem 5.1.2 ([Ill71, III.2.1.2.2])**.** *Let S, S' be flat R algebras, let S_0, S'_0 be their reductions to R_0. Given a morphism $f_0 : S_0 \to S'_0$, there is an obstruction class $o(f_0)$ in* $\mathrm{Ext}^1(L_{S_0/R_0}, S'_0 \otimes_{R_0} I)$ *which vanishes exactly when f_0 can be lifted to a morphism $f : S \to S'$. If a lift exists, then the set of all lifts forms a torsor under* $\mathrm{Hom}(L_{S_0/R_0}, S'_0 \otimes_{R_0} I)$*.*

Let X_0 be a smooth scheme over $\mathrm{Spec}(R_0)$. We want to understand when we can find a smooth scheme X over $\mathrm{Spec}(R)$ such that $X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R_0) \cong X_0$; we call such a scheme a lift of X_0 from R_0 to R. Let $T_{X_0} = \mathscr{H}om(\Omega^1_{X_0}, \mathscr{O}_{X_0}) = \mathrm{Der}_{K^\circ}(\mathscr{O}_{X_0}, \mathscr{O}_{X_0})$.

Proposition 5.1.3. *There is an obstruction class $o(X_0)$ in $H^2(X_0, T_{X_0}) \otimes_{R_0} I$ which vanishes exactly when X_0 can be lifted to R. If a lift exists, the set of isomorphism classes of such lifts forms a torsor under $H^1(X_0, T_{X_0}) \otimes_{R_0} I$. The obstruction is functorial in the following sense: given an isomorphism $\psi_0 : Y_0 \to X_0$ of smooth* $\mathrm{Spec}(R_0)$*-schemes, one has a diagram*

$$H^2(Y_0, T_{Y_0}) \otimes_{R_0} I \to d\psi_0 H^2(Y_0, \psi_0^* T_{X_0}) \otimes_{R_0} I \leftarrow \psi_0^* H^2(X_0, T_{X_0}) \otimes_{R_0} I$$

and $d\psi_0(o(Y')) = \psi_0^*(o(X'))$.

Sketch. The proof is the same as [Oor71, Proposition 2.2.5], but the added generality of above theorems allows us to remove the noetherian hypotheses. Because X_0 is smooth, we can cover it with affine opens with unique lifts (up to isomorphism) from R_0 to R as the corresponding Ext groups from Theorem 5.1.1 vanish. To lift X_0, it remains to lift the gluing data, which Theorem 5.1.2 shows is controlled by T_{X_0}. □

With this slightly more general statement, the arguments in [Oor71] now allow us to deform abelian schemes.

Proposition 5.1.4. *Let $\widetilde{B}$ be an abelian scheme over* $\mathrm{Spec}(K^\circ/\varpi)$*. Then there is an abelian formal scheme $\overline{B}$ over* $\mathrm{Spf}(K^\circ)$ *with mod ϖ special fiber $\widetilde{B}$.*

Note that I am not yet making any claims about algebraization, and that $\overline{B}$ is certainly not unique.

Sketch. See [Oor71, Theorem 2.2.1] for a full proof. We first note that by [MFK94, Proposition 6.15], any deformation of an abelian scheme is again an abelian scheme: that is, the group structure lifts. It is therefore enough to lift $\widetilde{B}$ to a formal scheme. We can do this inductively, lifting from K°/ϖ to K°/ϖ^2, then to K°/ϖ^3, and so on, via Proposition 5.1.3 if we can show that the corresponding obstruction classes vanish. This vanishing follows from functoriality and the group structure: we sketch the idea when the characteristic of K is not 2. One checks that the inverse map $\widetilde{B} \to \widetilde{B}$ sends $o(\widetilde{B}) \in H^2(\widetilde{B}, T_{\widetilde{B}}) \otimes_{K^\circ/\varpi} (\varpi)$ to $-o(\widetilde{B})$, so the obstruction must be 0 in this case. The characteristic 2 case is a bit more involved, but follows from a similar trick. $\square$

This result allows us to deform semi-abelian schemes without much extra work.

Proposition 5.1.5. *Let $\widetilde{E}$ be a semi-abelian scheme over* $\mathrm{Spec}(K^\circ/\varpi)$. *Then there is a semi-abelian formal scheme $\overline{E}$ over* $\mathrm{Spf}(K^\circ)$ *with mod ϖ special fiber $\widetilde{E}$.*

Proof. Given an exact sequence of group schemes over $\mathrm{Spec}(K^\circ/\varpi)$

$$1 \to \widetilde{T} \to \widetilde{E} \to \widetilde{B} \to 1,$$

Proposition 5.1.4 lets us lift the abelian scheme $\widetilde{B}$ to an abelian formal scheme $\overline{B}/\mathrm{Spec}(K^\circ)$. The split torus $\widetilde{T}$ lifts to the formal torus $\overline{T} \cong \mathbb{G}^r_{m,K^\circ}$. It remains to lift $\widetilde{E}$ to an element $\overline{E}$ of $\mathrm{Ext}(\overline{B}, \overline{T})$. By [Lüt16, Theorem A.2.8], $\mathrm{Ext}(\overline{B}, \overline{T})$ is canonically isomorphic to the set of K°-valued points of $(\overline{B}^\vee)^r$, where $\overline{B}^\vee$ is the dual abelian scheme of $\overline{B}$. Similarly, $\mathrm{Ext}(\widetilde{B}, \widetilde{T})$ is canonically isomorphic to the set of K°/ϖ-valued points of $(\widetilde{B}^\vee)^r$. Under these isomorphisms, the reduction map $\mathrm{Ext}(\overline{B}, \overline{T}) \to \mathrm{Ext}(\widetilde{B}, \widetilde{T})$ is the reduction map

$$\overline{B}^\vee(K^\circ)^r \to \overline{B}^\vee(K^\circ/\varpi)^r \cong \widetilde{B}^\vee(K^\circ/\varpi)^r.$$

As this map is surjective, every semi-abelian K°/ϖ-variety $\widetilde{E}$ lifts to a formal semi-abelian K°-variety $\overline{E}$. $\square$

In particular, if we start with a Raynaud extension E over K, we can consider the mod ϖ reduction $\widetilde{E}$ of $\overline{E}$ as a semi-abelian scheme over $K^{\flat\circ}/\varpi^\flat$. Applying Proposition 5.1.5, we deform $\widetilde{E}$ to a formal scheme $\overline{E}'$ corresponding to a Raynaud extension E' over $K^\flat$.

Theorem 5.1.6. *With E and E' defined as in the previous paragraph, we have $E_\infty^\flat \cong (E')_\infty$.*

Proof. As $\overline{E}$ and $\overline{E'}$ both have the same special fiber, the torus parts T and T' of E and E' must have the same rank r. The log maps ℓ and ℓ' for E and E' therefore both map to $\mathbb{R}^r$. The images of ℓ and ℓ' in $\mathbb{R}^r$ are the same as the tilting equivalence identifies the value groups $|K^\times|$ and $|(K^\flat)^\times|$. We can therefore define an F-tower for E' using models $\mathfrak{E}'_{\alpha/p^n}$ constructed from the same lattice Λ and α dividing Λ from Definition 4.1.11 as we used for E.

This F-tower defines the perfectoid cover $(E')_\infty$ of E'. By Lemma 3.3.11, it suffices to show that the mod $\varpi^\flat$ special fiber of this F-tower is isomorphic to the mod ϖ special fiber of the F-tower constructed in Proposition 4.2.1. Recall that the models $\mathfrak{E}_{\alpha/p^n}$ are defined as pushouts $\mathfrak{T}_{\alpha/p^n} \times^{\overline{T}} \overline{E}$, and that the morphisms $[\mathfrak{p}]_n : \mathfrak{E}_{\alpha/p^{n+1}} \to \mathfrak{E}_{\alpha/p^n}$ are determined by the morphisms $[\mathfrak{p}]_n : \mathfrak{T}_{\alpha/p^{n+1}} \to \mathfrak{T}_{\alpha/p^n}$, $[p] : \overline{T} \to \overline{T}$, and $[p] : \overline{E} \to \overline{E}$. Moreover, this construction commutes with reducing to the mod ϖ special fiber, so it is enough to check that the special fibers of these morphisms agree with the corresponding morphisms on the $K^\flat$ side.

The mod ϖ special fiber of $\overline{T}$ is the split torus of rank r over K°/ϖ, and the mod ϖ special fiber of $\overline{T'}$ is the split torus of rank r over $K^{\flat\circ}/\varpi^\flat$. As $K^\circ/\varpi \cong K^{\flat\circ}/\varpi^\flat$, these tori are isomorphic, and the multiplication by p maps on $\overline{T}$ and $\overline{T'}$ both reduce to the multiplication by p map on the special fibers, so they are the same as well. By Proposition 5.1.5, the same is true for the special fibers of $\overline{E}$ and $\overline{E'}$.

We next check that mod ϖ special fiber $\widetilde{\mathfrak{T}}_{\alpha/p^n}$ of a model of the rank r split torus T over K agrees with the mod $\varpi^\flat$ special fiber $\widetilde{\mathfrak{T}}'_{\alpha/p^n}$ of the corresponding model of the rank r split torus

T' over $K^\flat$. By the construction of Section 4.1, this claim follows from the following lemma:

Lemma 5.1.7. *Let Δ be a Γ-rational polytope, let $\mathrm{Sp}(A_\Delta) = \ell^{-1}(\Delta)$, let $\mathrm{Sp}(A'_\Delta) = (\ell'^{-1})(\Delta)$ as in Proposition 4.1.5. Then the mod ϖ special fiber $\widetilde{A^\circ_\Delta}$ is isomorphic to the mod $\varpi^\flat$ special fiber $\widetilde{A'^\circ_\Delta}$.*

Proof. Recall from Proposition 4.1.5 that the elements of A_Δ and A'_Δ can be represented as convergent power series in r variables with coefficients in K and $K^\flat$ respectively, and that the supremum norm on these algebras can be computed via a formula that depends only on the norms of these coefficients. We can therefore define a map $\sharp : A'_\Delta \to A_\Delta$ sending $\sum_{\mathbf{e}\in\mathbb{Z}^r} a_{\mathbf{e}}\mathbf{x}^{\mathbf{e}}$ to $\sum_{\mathbf{e}\in\mathbb{Z}^r} a^\sharp_{\mathbf{e}}\mathbf{x}^{\mathbf{e}}$, where $\sharp : K^\flat \to K$ is defined as in Section 3.1. This does not preserve the ring structure, but it does preserve the supremum norm. We can therefore restrict to a map $\sharp : A'^\circ_\Delta \to A^\circ_\Delta$, and as the sharp map sends $\varpi^\flat$ to ϖ, a map $\sharp : \widetilde{A'^\circ_\Delta} \to \widetilde{A^\circ_\Delta}$. This map is an isomorphism as $\sharp : K^\flat \to K$ induces the isomorphism $K^{\flat\circ}/\varpi^\flat \to K^\circ/\varpi$. □

By Proposition 4.1.12, the reduction of the morphisms $[\mathfrak{p}]_n$ on each model is relative Frobenius. Now that we know the reductions of the models over K and $K^\flat$ agree, we conclude that the reductions of the morphisms also agree. □

5.2 Tilting the F-tower for A

Let us summarize our current progress. Given an abeloid A over K, uniformized as a quotient E/M, we constructed an F-tower for A by constructing an F-tower for E for which the action of the lattice M on E extends to an action on the formal tower. We then constructed a Raynaud extension E' over $K^\flat$ and an F-tower for E' such that the mod ϖ and mod $\varpi^\flat$ special fibers of the towers for E and E' agree, implying that $E^\flat_\infty \cong (E')_\infty$. In this section, we construct a lattice $M' \subset E'$ such that the action of M' on E' extends to an action on the formal tower for E', and such that the mod $\varpi^\flat$ special fiber of this action agrees with the mod ϖ special fiber of the action of M on E. Letting $A' := E'/M'$ be the corresponding abeloid over $K^\flat$, we conclude that $A_\infty^\flat \cong (A')_\infty$.

To construct M', we fix lattices $M^{1/p^n} \subset E$ such that for any $n \geq 1$, the map $[p] : E \to E$ restricts to an isomorphism $M^{1/p^n} \to M^{1/p^{n-1}}$. This is equivalent to fixing a subgroup $M_\infty \subset E_\infty$ such that the projection map $E_\infty \to E$ to the bottom of the inverse system restricts to an isomorphism $M_\infty \to M$. This requires us to make a pro-p extension of K to ensure that these lattices exist. We noted in Proposition 3.3.8 that E_∞ is a perfectoid group, so for point $m_\infty \in M_\infty$, translation by m_∞ induces an isomorphism $\tau_{m_\infty} : E_\infty \to E_\infty$. This map fits into the commutative diagram

$$\begin{tikzcd}
E_\infty \arrow[r, "\tau_{m_\infty}"] \arrow[dd, bend right=40, "q_2"'] \arrow[ddd, bend right=60, "q_1"'] & E_\infty \arrow[dd, bend left=40, "q_2"] \arrow[ddd, bend left=60, "q_1"] \\
\vdots \arrow[d, "{[p]}"] & \vdots \arrow[d, "{[p]}"'] \\
E \arrow[r, "\tau_m^{1/p}"] \arrow[d, "{[p]}"] & E \arrow[d, "{[p]}"'] \\
E \arrow[r, "\tau_m"] & E.
\end{tikzcd} \tag{5.2}$$

Here all the rectangles are fiber diagrams, including the large rectangles with E_∞ in the top row. We can therefore think of the choice of pth power roots of m as being equivalent to fixing a choice of isomorphism $E_\infty \simeq E \times_{\tau_m, E} E_\infty$, where the fiber product is taken with respect to the maps $\tau_m : E \to E$ and $q_1 : E_\infty \to E$.

By the tilting equivalence, all of the isomorphisms $E_\infty \xrightarrow{\tau_{m_\infty}} E_\infty$ correspond to isomorphisms $E_\infty^\flat \xrightarrow{\tau^\flat_{m_\infty}} E_\infty^\flat$. Here $m_\infty^\flat$ is the image of m_∞ under the homeomorphism $E_\infty(K) \cong E_\infty^\flat(K^\flat)$. In

Theorem 5.1.6, we constructed a Raynaud extension E' over $K^\flat$ such that there is an isomorphism

$$\iota : E^\flat_\infty \to (E')_\infty \sim \varprojlim_{[p]} E'.$$

Let $q' : E'_\infty \to E'$ be the induced map to the bottom of the inverse system. We can now define $M' \subset E'$ by tracing M through these maps.

Lemma 5.2.1. *The lattice $M \subset E$ is mapped bijectively to a lattice $M' := q'_1 \circ \iota \circ \flat \circ q_1^{-1}(M_\infty)$ in E' with $\ell'(M') = \ell(M) \subset \mathbb{R}^r$.*

Proof. We claim that the following diagram commutes:

$$\begin{tikzcd}
E_\infty(K) \arrow[rr, "\iota\circ\flat"] \arrow[d, "q_1"] & & E'_\infty(K^\flat) \arrow[d, "q'_1"] \\
E(K) \arrow[dr, "\ell"] & & E'(K^\flat) \arrow[dl, "\ell'"'] \\
& \mathbb{R}^r. &
\end{tikzcd} \tag{5.3}$$

Assuming this, the lemma is clear as $\ell \circ q_1$ maps M_∞ bijectively to $\Lambda \subset \mathbb{R}^r$, so the same is true when we traverse the diagram in the other direction. To see commutativity of the diagram, it is enough to fix a point $\mathbf{c}$ in $\mathbb{R}^r$ and show that the preimage of $\mathbf{c}$ in $E_\infty(K)$ is mapped to the preimage of $\mathbf{c}$ in $E'_\infty(K^\flat)$. By our assumption that the value group of K is contained in $\mathbb{Q}$, we can assume that $\mathbf{c}$ is in $\mathbb{Q}^r$.

We can refine the polytopal decompositions of $\mathbb{R}^r$ used to construct E_∞ without changing the diagram (5.3): this corresponds to choosing finer formal analytic coverings of E and E', which correspond to blowups on the special fiber that aren't noticed on the generic fiber. So we can go back to Definition 4.1.11 and replace our choice of rational number α dividing Λ with some α' with larger denominator such that c is a vertex of the corresponding decomposition of $\mathbb{R}^r$. The claim now follows as in Theorem 5.1.6: the tilt of the perfectoid affinoid lying over any U_Δ in the formal cover of E is the perfectoid affinoid lying over the corresponding U'_Δ in the formal cover of E', and we can now take $\Delta = \{\mathbf{c}\}$. □

Note that in the above Lemma, it was essential that we fixed the maps q_1 and q'_1 going all the way to the bottom of the inverse systems, and that we defined E_∞ and E'_∞ using the same decomposition of $\mathbb{R}$. If we hadn't done this, our lattices M and M' would be off by a factor of p^n.

We are now ready to prove the main theorem of the chapter.

Theorem 5.2.2. *Let $A = E/M$ be an abeloid over a perfectoid field K with value group contained in $\mathbb{Q}$. Let A' be the abeloid E'/M' over $K^\flat$ for E' as in Theorem 5.1.6 and M' as in Lemma 5.2.1, defined after a suitable pro-p extension of K. Defining A_∞ and $(A')_\infty$ as in Theorem 4.2.5, we have $A_\infty^\flat \cong (A')_\infty$.*

Proof. Let m be an element of M, let m' be the corresponding element of M' from Lemma 5.2.1. By part (3) of Theorem 4.2.4, the diagram (5.2) and the analogous diagram for m' extend to diagrams of formal models

$$
\begin{tikzcd}
\mathfrak{E}_\infty \arrow[r, "\tau_{\mathfrak{m}_\infty}"] \arrow[dd, bend right, "q_2"'] \arrow[ddd, bend right=60, "q_1"'] & \mathfrak{E}_\infty \arrow[dd, bend left, "q_2"] \arrow[ddd, bend left=60, "q_1"] \\
\vdots \arrow[d, "{[\mathfrak{p}]_2}"] & \vdots \arrow[d, "{[\mathfrak{p}]_2}"'] \\
\mathfrak{E}_{\frac{1}{p^2}D} \arrow[r, "\tau_{\mathfrak{m}^{1/p}}"] \arrow[d, "{[\mathfrak{p}]_1}"] & \mathfrak{E}_{\frac{1}{p^2}D} \arrow[d, "{[\mathfrak{p}]_1}"'] \\
\mathfrak{E}_{\frac{1}{p}D} \arrow[r, "\tau_{\mathfrak{m}}"] & \mathfrak{E}_{\frac{1}{p}D}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{E}'_\infty \arrow[r, "\tau_{\mathfrak{m}^\flat_\infty}"] \arrow[dd, bend right, "q'_2"'] \arrow[ddd, bend right=60, "q'_1"'] & \mathfrak{E}'_\infty \arrow[dd, bend left, "q'_2"] \arrow[ddd, bend left=60, "q'_1"] \\
\vdots \arrow[d, "{[\mathfrak{p}']_2}"] & \vdots \arrow[d, "{[\mathfrak{p}']_2}"'] \\
\mathfrak{E}'_{\frac{1}{p^2}D} \arrow[r, "\tau_{(\mathfrak{m}')^{1/p}}"] \arrow[d, "{[\mathfrak{p}']_1}"] & \mathfrak{E}'_{\frac{1}{p^2}D} \arrow[d, "{[\mathfrak{p}']_1}"'] \\
\mathfrak{E}'_{\frac{1}{p}D} \arrow[r, "\tau_{\mathfrak{m}'}"] & \mathfrak{E}'_{\frac{1}{p}D}.
\end{tikzcd}
\tag{5.4}
$$

We claim that the mod ϖ special fiber of the first diagram is isomorphic to the mod $\varpi^\flat$ special fiber of the second. By the tilting equivalence, this is true for the maps $\times\mathfrak{m}_\infty$ and $\times\mathfrak{m}^\flat_\infty$. By construction, this is true for all the vertical maps. As all squares are fiber diagrams, we conclude that this must be true for all the other horizontal maps. In particular, as $M' \subset M'^{1/p^n}$ for all n, we have actions of M' on every $\mathfrak{E}'_{\alpha/p^n}$ agreeing with the M action on $\mathfrak{E}_{\alpha/p^n}$ on the special fiber. Applying parts (2) and (4) of Theorem 4.2.4, we obtain an F-tower for A' satisfying the condition in Lemma 3.3.11: for all $n \geq 1$, the commutative diagrams

$$\begin{tikzcd} \mathfrak{E}_{\alpha/p^{n+1}} \arrow[r] \arrow[d, "{[\mathfrak{p}]_n}"] & \mathfrak{A}_{\alpha/p^{n+1}} \arrow[d, "{[\mathfrak{p}]_n}"] \\ \mathfrak{E}_{\alpha/p^n} \arrow[r] & \mathfrak{A}_{\alpha/p^n} \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{E}'_{\alpha/p^{n+1}} \arrow[r] \arrow[d, "{[\mathfrak{p}']_n}"] & \mathfrak{A}'_{\alpha/p^{n+1}} \arrow[d, "{[\mathfrak{p}']_n}"] \\ \mathfrak{E}'_{\alpha/p^n} \arrow[r] & \mathfrak{A}'_{\alpha/p^n} \end{tikzcd}$$

have isomorphic special fibers. We conclude that $A^{\flat}_{\infty} \cong A'_{\infty}$.

$\square$

This chapter contains material from joint work with Ben Heuer which is currently being prepared for submission for publication.

Chapter 6

Line bundles

In this chapter, we show how to transfer a line bundle L on an abeloid A/K to a line bundle L' on a suitable abeloid $A'/K^\flat$ as constructed in Theorem 5.2.2. This process has two applications. If A is an abelian variety, it has an ample line bundle L. We will see that the corresponding L' on A' must also be ample, giving us a refinement of Theorem 5.2.2: if A is an abelian variety, we can choose A' to be an abelian variety as well. In Chapter 7, we'll see that moving line bundles from A to A' is the key input in our analogue of Scholze's approximation lemma, which is itself a key step in the proof of weight-monodromy.

We first give an overview of the classification of line bundles on Raynaud extensions in Section 6.1. We use this in Section 6.2 to show that there is some model $\mathfrak{A}_\alpha$ in the F-tower for A constructed in Theorem 4.2.5 such that L extends to $\mathfrak{A}_\alpha$. This is based off of an argument of Gubler in the totally degenerate case. Having this model gives us a well-defined metric on L, which will let us approximate global sections L.

In Section 6.3, we transfer line bundles to the tilt by developing the concept of $\mathbb{G}_{m,\infty}$-torsors on perfectoid spaces. These give rise to to systems of pth power roots of line bundles. Any line bundle over a perfectoid space of characteristic p extends to a $\mathbb{G}_{m,\infty}$-torsor, which we can then untilt to get a $\mathbb{G}_{m,\infty}$-torsor in characteristic 0. We show that every line bundle L on A pulls back to a line bundle on A_∞ which arises in this way. It is not clear in general if all line bundles on perfectoid spaces in characteristic 0 arise in this way. This is a question of Dorfsman-Hopkins: $\mathbb{G}_{m,\infty}$-torsors are a geometric interpretation of ideas we learned from his thesis, in particular the cohomological untilting map of line bundles from [DH19, Section 5.1].

An important subtlety arises when we try to transfer the data of a line bundle L from A to A'. When L is ample, it defines a polarization: an isogeny $\varphi_L : A \to A^\vee$. Tilting L will therefore involve constructing an isogeny $\varphi_{L'} : A' \to (A')^\vee$ (or at least data equivalent to such a map). There are many choices involved in the construction of A' in Theorem 5.2.2. If these choices are made at random for A and then again for $A^\vee$, there is no reason to expect the resulting abelian varieties A' and $(A^\vee)'$ over $K^\flat$ to be duals. To get dual varieties over $K^\flat$, we need to make compatible choices. We use $\mathbb{G}_{m,\infty}$-torsors to do this, constructing and tilting a perfectoid cover of the Poincaré bundle over $B \times B^\vee$, and showing that the choices made in this process induce compatible choices for A and $A^\vee$.

6.1 Line bundles on Raynaud extensions

We first summarize a bit more background on Raynaud extensions. Let A be an abelian variety over an algebraically closed, non-archimedean field K. As described in Section 2.4, there is a short exact sequence of rigid groups

$$1 \to T \to E \xrightarrow{\pi} B \to 1 \tag{6.1}$$

for a rigid torus T of rank r and an abelian variety B with good reduction, and a lattice $h : M \hookrightarrow E$ such that $A \cong E/M$. Let $M^\vee := \mathrm{Hom}(T, \mathbb{G}_m)$ denote the character group of the torus, so $M^\vee$ is a free abelian group of rank r. Let $B^\vee$ denote the dual abelian variety of B, let $P_{B\times B^\vee}$ denote the Poincarè bundle on $B \times B^\vee$. We denote the associated $\mathbb{G}_m$-torsor by $P^\times_{B\times B^\vee}$.

By [Lüt16, Theorem A.2.8] and the discussion before [Lüt16, Definition 6.1.2], the data of the Raynaud extension (6.1) is equivalent to a group homomorphism $\phi^\vee : M^\vee \to B^\vee$. To see this equivalence explicitly, note that for any character $m^\vee \in M^\vee$, taking the pushout of (6.1) by $m^\vee$ gives a commutative diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d, "m^\vee"] & E \arrow[r, "\pi"] \arrow[d, "\langle \cdot, m^\vee \rangle"] & B \arrow[d, equal] \\ \mathbb{G}_m \arrow[r] & P^\times_{B\times\phi^\vee(m^\vee)} \arrow[r] & B. \end{tikzcd} \tag{6.2}$$

This diagram *defines* the image $\phi^\vee(m^\vee) \in B^\vee$ and the map $\langle \cdot, m^\vee \rangle$, as there is an isomorphism $\mathrm{Ext}(B, \mathbb{G}_m) \cong B^\vee$. The line bundle $\phi^\vee(m^\vee)$ must be translation-invariant because of the group law on E: any group homomorphism $M^\vee \to \mathrm{Pic}_B$ will define an extension of B by T, but the extension will only admit a group law if the image of lands in $\mathrm{Pic}^0(B) = B^\vee$. See [Ser88, VII.3, Theorem 6] and the surrounding material for more details.

To go in the other direction, given a group homomorphism $\phi^\vee : M^\vee \to B^\vee$ and choosing a basis $m_1^\vee, \ldots, m_r^\vee$ of $M^\vee$, we can write

$$E \cong P^\times_{B\times\phi^\vee(m_1^\vee)} \times_B \cdots \times_B P^\times_{B\times\phi^\vee(m_r^\vee)}. \tag{6.3}$$

Once a decomposition for E has been fixed as above, we can describe $\langle \cdot, m^\vee \rangle$ more explicitly. Expand $m^\vee = e_1 m_1^\vee + \cdots + e_r m_r^\vee$ in terms of our basis. Given $z \in E$, choose an open of B containing $\pi(z)$ over which the T-torsor E trivializes, so that we can represent z as an element $(t_1, \ldots, t_r)$ in the copy of T over $\pi(z)$. Then

$$\langle z, m^\vee \rangle = t_1^{e_1} \cdots t_r^{e_r}$$

in the copy of $\mathbb{G}_m$ over $\pi(z)$ in the trivialization of $P^\times_{B\times\phi^\vee(m^\vee)}$. As in the discussion before Definition 2.4.4 changing the trivialization will not change the absolute value of the t_i, so we have a well-defined valuation map

$$\ell : E(K) \to \mathbb{R}^r; (t_1, \ldots, t_r) \mapsto (-\log|t_1|, \ldots, -\log|t_r|).$$

The above description shows that $\langle \cdot, m^\vee \rangle$ “factors over $\mathbb{R}$", giving an affine function $F_{m^\vee} : \mathbb{R}^r \to \mathbb{R}$ so that the following diagram commutes:

$$\begin{tikzcd} E(K) \arrow[r, "\ell"] \arrow[d, "\langle \cdot, m^\vee \rangle"] & \mathbb{R}^r \arrow[d, "F_{m^\vee}"] \\ P^\times_{B \times \phi^\vee(m^\vee)} \arrow[r, "\ell"] & \mathbb{R}. \end{tikzcd} \tag{6.4}$$

We call $F_{m^\vee}$ the *tropicalization* of $\langle \cdot, m^\vee \rangle$.

Composing the inclusion $h : M \to E$ with $\pi : E \to B$ gives a group homomorphism $\phi : M \to B$. As $(B^\vee)^\vee \cong B$, this defines a Raynaud extension

$$1 \to T^\vee \to E^\vee \xrightarrow{\pi^\vee} B^\vee \to 1 \tag{6.5}$$

where the dual torus $T^\vee$ is again a rigid torus of rank r. By [Lüt16, Proposition 6.1.8], the map h is equivalent to a non-degenerate bilinear form $\langle \cdot, \cdot \rangle : M \times M^\vee \to P^\times_{B \times B^\vee}$ living over $\phi \times \phi^\vee$. That is, a commutative diagram

$$\begin{tikzcd} & P^\times_{B \times B^\vee} \arrow[d] \\ M \times M^\vee \arrow[ur, "\langle \cdot, \cdot \rangle"] \arrow[r, "\phi \times \phi^\vee"] & B \times B^\vee \end{tikzcd} \tag{6.6}$$

such that for any fixed $m^\vee \in M^\vee$, the restriction $\langle \cdot, m^\vee \rangle : M \to P^\times_{B \times \phi^\vee(m^\vee)}$ is the restriction of the middle vertical arrow in (6.2), and the symmetric condition is true for $m \in M$. By non-degenerate bilinear form, we mean that the tropicalization $F : M \times M^\vee \to P^\times_{B \times B^\vee} \to \mathbb{R}^r$ which restricts to $F_{m^\vee}$ for any fixed $m^\vee \in M^\vee$ is a non-degenerate bilinear form.

By [Lüt16, Theorem 6.3.3] the dual lattice uniformizes the dual abelian variety $A^\vee =$

$E^\vee/M^\vee$. We remark that the notation $E^\vee$ may be a little misleading: this is not some canonical dual associated to the semi-abelian variety E, it also depends on the lattice $M \subset E$.

We are now ready to sketch the construction of line bundles on A, following [Lüt16, Example 6.2.1] (but note that we are not bothering to rigidify our line bundles). All (cubical) line bundles on E arise as pullbacks $\pi^*(N)$ of line bundles on B. Any line bundle L on A is given by the M-linearization of a (cubical) line bundle $\pi^*(N)$ on E: a compatible family of isomorphisms

$$\{c_m : \pi^*N \to \tau_m^*\pi^*N : m \in M\}.$$

Here τ_m is the multiplication by m map, and the compatibility condition is

$$\tau_{m_1}^*(c_{m_2}) \cdot c_{m_2} = c_{m_1+m_2} \tag{6.7}$$

for all $m_1, m_2 \in M$. For many choices of line bundle N and lattice M, there will not be any such isomorphisms - we need to add some conditions to see when families of c_m exist.

Recall that the line bundle N on B induces a homomorphism $\varphi_N : B \to B^\vee$, sending $b \mapsto \tau_b^* N \otimes N^{-1}$. Each isomorphism c_m on E corresponds to an isomorphism between the trivial line bundle on E and the line bundle

$$(\pi^*N)^{-1} \otimes \tau_m^*\pi^*N = \pi^*(N^{-1} \otimes \tau_{\phi(m)}^*N) = \pi^*(\varphi_N(\phi(m))).$$

We therefore need to determine which translation-invariant line bundles on B pull back to the trivial bundle on E, and require the composition $\phi \circ \varphi_N : M \to B^\vee$ to map to points representing these bundles. Given a point $b^\vee \in B^\vee$ corresponding to a line bundle $N_{b^\vee}$ on B and a $\mathbb{G}_m$-torsor $P^\times_{B\times b^\vee} \to B$, the pullback of $N_{b^\vee}$ to $P^\times_{B\times b^\vee}$ is trivial. This follows from a more general fact: given a $\mathbb{G}_m$-torsor $\pi_L : L^\times \to X$ on a space X, the pullback $\pi_L^*(L^\times) \cong L^\times \times_X L^\times$ is the trivial $\mathbb{G}_m$-torsor on $L^\times$, as a section is given by the diagonal map $L^\times \to L^\times \times_X L^\times$.

We conclude that for any $m^\vee \in M^\vee$, the line bundle represented by $\phi^\vee(m^\vee) \in B^\vee$ pulls back to the trivial bundle on $P^\times_{B\times\phi^\vee(m^\vee)}$. Pulling further back along the map

$$\langle \cdot, m^\vee \rangle : E \to P^\times_{B\times\phi^\vee(m^\vee)}$$

of (6.2), we get the trivial bundle on E. This turns out to be the only way to construct such bundles, so our first compatibility condition on M and N is that there must be a map $\sigma : M \to M^\vee$ making the following diagram commute:

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, "\sigma"] & B \arrow[d, "\varphi_N"] \\ M^\vee \arrow[r, "\phi^\vee"] & B^\vee. \end{tikzcd} \tag{6.8}$$

The compatibility relation (6.7) forces σ to be a group homomorphism.

Given an isomorphism c_m of line bundles, we can get other isomorphisms by rescaling the lines by a fixed unit $r(m) \in K^\times$. Writing this more canonically, the second condition is a trivialization $r : M \to \phi^*N$ of the $\mathbb{G}_m$-torsor ϕ^*N on M satisfying

$$r(m_1+m_2) \otimes r(m_1)^{-1} \otimes r(m_2)^{-1} = \langle m_1, \sigma(m_2)\rangle \tag{6.9}$$

for all $m_1, m_2 \in M$. The factor $\langle m_1, \sigma(m_2)\rangle$ is needed to make translations match up for non-translation-invariant bundles. It defines a symmetric, non-degenerate bilinear form on M. Taking valuations, we get a bilinear form

$$\langle \cdot, \cdot \rangle_\sigma : M \times M \to \mathbb{R} \tag{6.10}$$

which extends to a bilinear form $F_\sigma : \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{R}$.

We now make c_m more explicit. Choose a basis for $M^\vee$, giving coordinates for E as in

(6.3), and expand $\sigma(m) = e_1 m_1^\vee + \cdots e_r m_r^\vee$. Over a trivialization of $\pi^*(N)$, c_m takes the fiber over a point $z \in E$ corresponding to $(t_1, \ldots, t_r)$ in the copy of T over $\pi(z)$, translates it to the fiber over $\tau_m(z)$, and scales the copy of $\mathbb{G}_m$ by the unit $r(m) t_1^{e_1} \cdots t_r^{e_r}$ in $K^\times$. Again taking valuations, we get an affine function $z_m : \mathbb{R}^r \to \mathbb{R}$ satisfying

$$z_m(\ell(z)) := \ell(c_m(z)) = \ell(r(m)) + F(\ell(z), \ell(\sigma(m))). \tag{6.11}$$

This is the tropicalization of c_m.

This is enough data to classify all line bundles on A (with a lot more work to make this rigorous, see [Lüt16, Sections 6.2,6.3]). Restating this for further use, we have

Proposition 6.1.1. *A line bundle L on an abelian variety A is determined by the data of a line bundle N on B, group homomorphism $\sigma : M \to M^\vee$, and trivialization $r : M \to \phi^* N$ such that the diagram (6.8) commutes and (6.9) holds. Furthermore, L is ample precisely when N is ample and the bilinear form (6.10) is positive definite.*

Working through the above construction, we note how this description interacts with multiplication by p in the Picard group.

Corollary 6.1.2. *Given a line bundle L on A corresponding to the data (N, σ, r) as in Proposition 6.1.1, the line bundle $L^{\otimes p}$ corresponds to the data $(N^{\otimes p}, p\sigma, r^p)$.*

Example 6.1.3. Translation-invariant line bundles on A correspond to the case where N is a translation-invariant line bundle on B and σ is the trivial morphism. In this case, ϕ_N is trivial, so (6.8) commutes. The right hand side of (6.9) is now 0, so r is a group homomorphism $M \to \phi^* N$. Choosing a basis $m_1, \ldots, m_r$ of M, we get coordinates of $E^\vee \cong P_{\phi(m_1) \times B^\vee} \times_{B^\vee} \cdots \times_{B^\vee} P_{\phi(m_r) \times B^\vee}$ as in (6.3), so that r determines a point $(r(m_1), \ldots, r(m_r)) \in E^\vee$. So as expected, points of $E^\vee$ give rise to translation-invariant bundles on A via the surjection $E^\vee \to A^\vee$.

Example 6.1.4. We specialize Proposition 6.1.1 to the Tate curve $\mathbb{G}_m/q^{\mathbb{Z}}$. The abelian part of the Raynaud extension $\mathbb{G}_m$ is trivial, so there is no line bundle N to worry about (equivalently, tori have trivial Picard group). The lattice is one-dimensional, so the group homomorphism σ and trivialization r are completely determined by the image of the generator q. In other words, once we choose an automorphism c_q of the trivial line bundle on $\mathbb{G}_m$, everything else will fall into place.

As the lattice is one-dimensional, $M^\vee = \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$, where the identity map $m^\vee$ is sent to 1. The homomorphism σ sends q to $dm^\vee$ for some integer d: this is the degree of the corresponding line bundle L. The map r can send q to any element $r(q)$ in $K^\times$. When $d = 0$ and we are dealing with Pic^0, the line bundle is therefore determined by an element of $K^\times$. We get the trivial line bundle precisely when $r(q) \in q^{\mathbb{Z}}$, giving the expected self-duality $\mathrm{Pic}^0(\mathbb{G}_m/q^{\mathbb{Z}}) \cong \mathbb{G}_m/q^{\mathbb{Z}}$.

The takeaway from these examples is that r controls Pic^0, σ controls the part of the Néron-Severi group coming from the lattice M, and N controls the part of Néron-Severi coming from B.

6.2 Formal models of line bundles

As a first step to moving line bundles from A to A', we need to extend a line bundle L on A to a formal line bundle $\mathfrak{L}_n$ on $\mathfrak{A}_{\alpha/p^n}$ for some n. Any line bundle on B (and therefore E) extends to a formal bundle on the corresponding formal (semi-)abelian scheme by [Lüt16, Lemma 6.2.2]. We still need to extend the M-linearization to the formal model. Gubler does this very explicitly in the totally degenerate case in [Gub07, Proposition 6.6]. In this section, we summarize his construction and see that it extends to the general case.

Fix a line bundle L on an abelian variety $A \cong E/M$, with associated data (N, σ, r) as in Proposition 6.1.1. For each $m \in M$, we have the associated affine function z_m from (6.11): the tropicalization of the isomorphism $c_m : \pi^*N \to \tau_m^*\pi^*N$. Fix a formal analytic cover $\mathscr{U}_n$ of E as in Proposition 4.2.2 such that π^*N is trivial over this cover. Explicitly, we have a $\pi(M)$-invariant cover $\overline{V}_{j,n}$ of $\overline{B}$ over which N is trivial and a rational number α dividing the lattice, and write the typical open of $\mathscr{U}_n$ as a product $U_{\mathbf{e}\Delta_{\alpha/p^n}} \times V_{j,n}$ for $\mathbf{e} \in \mathbb{Z}^r$. In Theorem 4.2.4, we saw that the action of M extends to the corresponding formal model $\mathfrak{E}_{\alpha/p^n}$ of E, so quotienting gives a formal model $\mathfrak{A}_{\alpha/p^n}$ of A.

Proposition 6.2.1. *With notation as in the preceding paragraph, formal models $\mathfrak{L}_{\alpha/p^n}$ of L on $\mathfrak{A}_{\alpha/p^n}$ can be constructed from pairs $(\mathfrak{N}, f)$ where $\mathfrak{N}$ is a formal model of N on the formal abelian scheme $\overline{B}$, and $f : \mathbb{R}^r \to \mathbb{R}$ is a continuous function such that:*

1. *When restricted to a polytope $\Delta = \mathbf{e}\Delta_{\alpha/p^n}$ in our cover of $\mathbb{R}^r$, we have $f(u) = m_\Delta \cdot u + c_\Delta$ for some $m_\Delta \in \mathbb{Z}^r$, $c_\Delta \in \Gamma$, where $\cdot$ denotes dot product;*

2. *For $\lambda \in \Lambda = \ell(M)$, we have $f(u+\lambda) = f(u) + z_\lambda(u)$.*

Proof. When A is totally degenerate, this is a weaker version of [Gub07, Proposition 6.6]. Gubler shows in this case that formal models of L are in bijection with these functions f (the line bundle N being trivial in this case). We expect a statement along these lines to hold in general, but have not checked this as we will not need it in this thesis. Note that for a given model $\mathfrak{A}_{\alpha/p^n}$ and line bundle L, there might not be a function f satisfying the necessary requirements. We will see that for n large enough, f must exist.

To extend L to $\mathfrak{L}_{\alpha/p^n}$, we use the data of $(\mathfrak{N}, f)$ to construct a suitable frame of L: units $g_{i,n}$ on each $U_{i,n}$ in the cover such that the transition functions $\frac{g_{i_1,n}}{g_{i_2,n}}$ on the intersection $U_{i_1,n} \cap U_{i_2,n}$ have supremum norm 1, and therefore give transition functions of a formal model.

We first choose a frame $(h_{j,n})$ of $\mathfrak{N}$ on the cover $\overline{V}_{j,n}$ of $\overline{B}$. As $\mathfrak{N}$ is a formal line bundle, the transition functions are formal units and therefore have supremum norm 1. For each open $U_{\Delta_i} \times V_{j_i,n}$ in our cover, we choose any $a_\Delta \in K^\times$ with $\ell(a_\Delta) = c_\Delta$. Then $a_\Delta \mathbf{z}^{\mathbf{e}}$ is a unit in A_{Δ_i}, the ring of holomorphic functions on Δ_i from Proposition 4.1.5.

We claim that setting $g_{i,n} = a_\Delta \mathbf{z}^{\mathbf{e}} \otimes h_{j,n}$ gives an M-invariant frame of $\mathfrak{N}$. As $h_{j,n}$ does not affect the supremum norm, Gubler's argument carries over to this case. By construction, for any point $z \in U_{\Delta_i} \times V_{j_i,n}$, we have $v(g_{i,n}(z)) = f(\ell(u))$. As f is continuous, the absolute values of the frames agree on intersections, so the transition functions have absolute value 1 and give a formal model as desired. By condition (2), pulling back along τ_m sends the frame $g_{i,n}$ on $U_{\Delta_i} \times V_{j_i,n}$ to a frame on $\tau_m^*(U_{\Delta_i} \times V_{j_i,n})$ with the correct absolute value, so our frame descends to the quotient.

$\square$

Example 6.2.2. For this example, let $K = \mathbb{C}_p$. We give some examples of formal models of line bundles on the Tate curve $C = \mathbb{G}_m/\langle p^2 \rangle$ constructed in Example 2.4.1. We hope this helps illuminate the role of the function f. As we saw in Example 6.1.4, a line bundle L on $\mathbb{G}_m/\langle p^2 \rangle$ is determined by its degree d and an element $r(p^2)$ of $K^\times$.

1. Let L be the trivial bundle, corresponding to $d = 0, r(p^2) = 1$. A formal analytic cover of C over which L is trivial is given by the polytope cover

$$U_{[0,1]} = \mathrm{Sp}(K\langle T, \frac{p}{T}\rangle), U_{[1,2]} = \mathrm{Sp}(K\langle \frac{S}{p}, \frac{p^2}{S}\rangle),$$

which are glued along the inner circles $U_{\{1\}}$ by the map sending $T \mapsto S$, and along the outer circles by the map

$$\tau_{p^2} : U_{\{0\}} \to U_{\{2\}}$$

$$K\langle \frac{S}{p^2}, \frac{p^2}{S}\rangle \to K\langle T, \frac{1}{T}\rangle$$

$$S \mapsto p^2 T.$$

The tropicalization of the (identity) isomorphism c_{p^2} is the zero map $z_{p^2} : \mathbb{R} \to 0$, so the functions $f : \mathbb{R} \to \mathbb{R}$ that could lead to formal models of L are periodic and determined their behavior on the closed interval $[0,2]$. They must be affine on $[0,1]$ and $[1,2]$ with integer slopes, and must satisfy $f(0) = f(2) = c_0$ for some $c_0 \in \mathbb{Q} \cong \Gamma$.

Letting $f(u) = u$ for $u \in [0,1]$ and $f(u) = 2 - u$ for $u \in [1,2]$, we get a function that will lead to a non-trivial model of the trivial line bundle. A frame corresponding to this f is given by $g_{01} = T$ on $U_{[0,1]}$ and $g_{12} = \frac{p^2}{S}$ on $U_{[1,2]}$. As L is trivial, the restriction maps on the intersections are the same as the gluing maps. Writing everything in terms of T, our chosen frame gives the transition map $\frac{T^2}{p^2}$ on $U_{\{1\}}$, which always has absolute value 1 when $|T| = |p|$ - the defining property of $U_{\{1\}}$, so we get a formal unit. Writing the other transition on $U_{\{0\}}$, we first send $g_{12} = \frac{p^2}{S}$ to $\frac{1}{T}$, then take the quotient to get the transition map T^2, which again has absolute value 1 on $U_{\{0\}}$ and is therefore a formal unit. As our transition maps are now made from formal units, we get a formal model of our line bundle. However, this is not the trivial line bundle - the unit T on $U_{[0,1]}$ is not a formal unit as it has absolute value less than 1 on $U_{(0,1]}$

2. Let L be the line bundle corresponding to $d = 0, r(p^2) = 1 + p$. We can use the same formal analytic cover of C, function f, and frame as in the previous example. Restriction maps of this L on the cover are the normal gluing map on the inner circles $U_{\{1\}}$, but are given by sending $S \mapsto p^2(1+p)T$ on the outer circles. The transition functions corresponding to the previous frame are now $\frac{T^2}{p^2}$ on $U_{\{1\}}$ and $\frac{T^2}{1+p}$ on $U_{\{0\}}$.

3. Let L be the line bundle corresponding to $d = 0, r(p^2) = p^{1/p}$. The formal analytic cover from the previous examples still trivializes L, but now there is no function f satisfying the necessary and therefore no formal line bundle extending L to the corresponding model. In this case, the tropicalization z_{p^2} of c_{p^2} is the constant function sending u to $1/p$, so we must have $f(2) = f(0) + 1/p$ by condition (2) of Proposition 6.2.1. But by condition (1), f must be continuous and linear on $[0,1]$ and $[1,2]$ with integer slopes, so this is impossible.

   To fix this, we refine the polytope decomposition used to create the formal analytic cover, instead using the affinoids corresponding to the intervals $[\frac{\lambda}{p}, \frac{\lambda+1}{p}]$ for $\lambda \in \{0, \ldots, 2p-1\}$.

We can now choose f to have slope 1 on the interval $[0, 1/p]$ and slope 0 on $[1/p, 2]$.

As the tropicalization maps z_m all take values in $v(K) = \Gamma \subset \mathbb{R}$, if we assume that $\Gamma \subset \mathbb{Q}$, Example 6.1.4(3) is the only reason the function f might not exist. Refining the formal analytic cover will always give a model over which f exists. As any line bundle on the abelian variety B extends to a formal line bundle on the formal abelian scheme $\overline{B}$, we can extend line bundles to the formal models in our towers, completing our main goal of the section.

Proposition 6.2.3. *A line bundle L on A extends to the formal model $\mathfrak{A}_\alpha$ if for every $m \in M$, $v(r(m))$ is an integer multiple of α.*

Combining this with Corollary 6.1.2, we get:

Corollary 6.2.4. *If the line bundle $L^{\otimes p}$ on A extends to the formal model $\mathfrak{A}_\alpha$, then L extends to the model $\mathfrak{A}_{\alpha/p}$.*

6.3 Perfectoid line bundles

In this section, we explain how to take a line bundle L on A and construct a line bundle L' on a suitable A' with the same special fiber. One might expect this to follow directly from applying Proposition 6.2.1 to A', but transferring the homomorphism $\sigma : M \to M^\vee$ to the tilt side such that diagram (6.8) still commutes seems quite subtle. When we tilted lattices in Section 5.2, we made an arbitrary choice of pth roots M^{1/p^n} of M. If the choices we make when tilting the dual lattice $M^\vee$ aren't compatible with the choices we make when tilting M, we won't get nice duality properties on the tilt side. To solve this issue, we need to tilt both M and $M^\vee$ simultaneously. We saw in diagram (6.5) that the lattice $M \times M^\vee$ injects into the Poincare torsor $P^\times_{B \times B^\vee}$. Our plan is to construct and tilt a perfectoid cover of this torsor, with $M \times M^\vee$ inside it, and construct both A' and $A'^\vee$ from this.

This will require use to work with perfectoid covers of line bundles, extending $\mathbb{G}_m$-torsors on A to $\mathbb{G}_{m,\infty}$-torsors on A_∞. This type of construction is very closely related to the arguments [DH19]. Perfectoid line bundles are already used implicitly in Scholze's approximation lemma [Sch12, Proposition 8.7], so we will also need them when we generalize it from toric varieties to abelian varieties in Proposition 7.2.2.

We start with some generalities on $\mathbb{G}_{m,\infty}$-torsors. Given a line bundle L on an adic space X, the associated $\mathbb{G}_m$-torsor $L^\times$ is obtained by removing the zero section, and fits into an analytic-locally split sequence of adic spaces

$$\mathbb{G}_m \to L^\times \xrightarrow{\pi} X.$$

By locally split, we mean that there is an open cover $\{U_i\}$ of X such that there is an isomorphism $\pi^{-1}(U_i) \cong \mathbb{G}_m \times U_i$ identifying $\pi^{-1}(u)$ with $\mathbb{G}_m \times \{u\}$ for all opens U_i in the cover and $u \in U_i$.

Definition 6.3.1. *A $\mathbb{G}_{m,\infty}$-torsor on an adic space X is defined by replacing $\mathbb{G}_m$ in the above definition by the perfectoid cover $\mathbb{G}_{m,\infty} \sim \varprojlim_{[p]} \mathbb{G}_m$ constructed in Corollary 4.1.13.*

If X is perfectoid, then any $\mathbb{G}_{m,\infty}$-torsor on X is also perfectoid as the local splitting gives an open cover by perfectoid subspaces.

The map $[p] : \mathbb{G}_m \to \mathbb{G}_m$ fits into the commutative diagram

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r] \arrow[d, "{[p]}"] & L^\times \arrow[r] \arrow[d] & X \arrow[d, equal] \\
\mathbb{G}_m \arrow[r] & (L^\times)^{\otimes p} \arrow[r] & X.
\end{tikzcd}$$

On transition functions, the map $L^\times \to (L^\times)^{\otimes p}$ corresponds to the map $H^1(X, \mathscr{O}_X^\times) \xrightarrow{x \mapsto x^p} H^1(X, \mathscr{O}_X^\times)$. If we write $\mathscr{L}$ for the corresponding invertible sheaf, the induced map on global sections is

$$\bigoplus_{i\in\mathbb{Z}} H^0(X, \mathscr{L}^{\otimes pi}) \hookrightarrow \bigoplus_{i\in\mathbb{Z}} H^0(X, \mathscr{L}^{\otimes i}). \tag{6.12}$$

Now assume X_∞ is a perfectoid space, and let $(L_n)_{n\in\mathbb{N}}$ be a family of line bundles on X_∞ with $L_{n+1}^{\otimes p} = L_n$. We would like to use this data to build a $\mathbb{G}_{m,\infty}$-torsor $L_\infty^\times$ at the top of the diagram

$$\begin{tikzcd}
\mathbb{G}_{m,\infty} \arrow[r] & L_\infty^\times \arrow[r] & X_\infty \\
\vdots \arrow[d, "{[p]}"] & \vdots \arrow[d] & \vdots \arrow[d, equal] \\
\mathbb{G}_m \arrow[r] \arrow[d, "{[p]}"] & L_2^\times \arrow[r] \arrow[d] & X_\infty \arrow[d, equal] \\
\mathbb{G}_m \arrow[r] & L_1^\times \arrow[r] & X_\infty.
\end{tikzcd} \tag{6.13}$$

This can be done when all of the rows at finite level are compatibly locally split: if we can find an open cover $\{U_{i,\infty}\}$ of X_∞ over which all of the $L_i^\times$ are locally split. In this case, L_∞ is locally a product $\mathbb{G}_{m,\infty} \times U_\infty$. By Lemma 3.3.13, it is a perfectoid tilde-limit fitting into an analytic-locally split sequence of perfectoid spaces. More explicitly, we can extend a $\mathbb{G}_m$-torsor $L^\times$ on a perfectoid space X to a $\mathbb{G}_{m,\infty}$-torsor $L_\infty^\times$ if we can write a cocycle of transition functions (g_{ij}) of L for some open cover $\{U_{i,\infty}\}$ for which the elements $g_{ij} \in \Gamma(U_{i,\infty} \cap U_{j,\infty})$ all have a compatible sequence of pth power roots $g_{ij}, g_{ij}^{1/p}, g_{ij}^{1/p^2}, \ldots$, as we can then define L^{1/p^n} using the cocycle (g_{ij}^{1/p^n}). We can always do this in characteristic p as perfectoid $K^\flat$ algebras are perfect, giving the following:

Lemma 6.3.2. *If X'_∞ is a perfectoid space over a perfectoid field $K^\flat$ of characteristic p, then every $\mathbb{G}_{m,K^\flat}$-torsor $L'^\times$ extends to a unique $\mathbb{G}_{m,\infty,K^\flat}$-torsor $L'^\times_\infty$.*

Given a perfectoid space X_∞ with tilt $X_\infty^\flat$, and a suggestively named $\mathbb{G}_{m,\infty}$-torsor $L_\infty^{\flat,\times}$, we can untilt the sequence

$$\mathbb{G}_{m,\infty,K^\flat} \to L_\infty^{\flat,\times} \to X_\infty^\flat$$

to get a sequence

$$\mathbb{G}_{m,\infty,K} \to L_\infty^\times \to X_\infty.$$

The bottom sequence defines a $\mathbb{G}_{m,\infty}$-torsor $L_\infty^\times$ on X_∞ living over a $\mathbb{G}_m$-torsor L on X_∞. This can be seen either by untilting the local splitting of $L_\infty^{\flat;\times}$, or by noting that the sharp map on perfectoid $K^\flat$-algebras defined in Section 3.2 sends the cocycle (g_{ij}) defining L' to a cocycle $(g_{ij}^\sharp)$ defining a line bundle L on X_∞. This recovers the map $\theta_0 : \mathrm{Pic}(X_\infty^\flat) \to \mathrm{Pic}(X_\infty)$ constructed in [DH19, Section 5.1]. With this interpretation, we see that Dorfsman-Hopkins's “cohomological untilting" map on line bundles corresponds to literal untilting of the associated $\mathbb{G}_{m,\infty}$-torsors (see also [Sch13, Lemma 3.24]). He asks [DH19, Open Problem 5.14] when θ_0 is surjective. We don't know if this is true for our perfectoid spaces A_∞ (though see Remark 6.3.17), but we can show that the image of θ_0 contains all of the line bundles on A_∞ which are pulled back from finite level. This lets us transfer bundles from A to A'. We start with the good reduction case.

Proposition 6.3.3. *Let B be an abelian variety with good reduction, let N be a line bundle on B. Then the $\mathbb{G}_m$-torsor $q_1^*N^\times$ on B_∞ extends to a $\mathbb{G}_{m,\infty}$-torsor.*

Proof. We need to construct pth power roots of q_1^*N along with an open cover $\{U_{i,\infty}\}$ of B_∞ over which all of these line bundles are trivial. We first construct the pth power roots. By the theorem of the cube, for any integer n, we have

$$[n]^*N \cong N^{\frac{n^2+n}{2}} \otimes [-1]^*N^{\frac{n^2-n}{2}}$$

(see e.g. [Mil86, Corollary 6.6]). In particular, we see that if $p \neq 2$, $[p]^*N$ is the pth power of the line bundle $N^{\frac{p+1}{2}} \otimes [-1]^*N^{\frac{p-1}{2}}$. In this case, the pullback $q_1^*(N^{\frac{p+1}{2}} \otimes [-1]^*N^{\frac{p-1}{2}})$ is the desired pth root of $q_1^*(N)$. Iterating this process, the p^nth root of $q_1^*(N)$ comes from level n of the inverse system. If $p = 2$, $[4]^*N$ is the square of the line bundle $N^5 \otimes [-1]^*N^3$. The argument will therefore follow similarly in this case, but the 2^nth root now comes from level $2n$.

To find an open cover of B_∞ over which all of these line bundles are trivial, we pass to the special fiber. The argument is similar to that of Lemma 4.2.3, which handles the case of translation-invariant line bundles. By [Lüt16, Lemma 6.2.2], every line bundle N on B extends to a formal bundle $\overline{N}$ on the formal abelian scheme $\overline{B}/K^\circ$ and then reduces mod ϖ to a line bundle $\widetilde{N}$ on the abelian scheme $\widetilde{B}$ over (K°/ϖ).

On the special fiber, our tower becomes an inverse system of qcqs schemes over (K°/ϖ) with affine transition maps $\widetilde{B}_\infty = \varprojlim_{[p]} \widetilde{B}$. By [Sta20, Tag 0B8W], we have

$$\mathrm{Pic}(\widetilde{B}_\infty) = \varinjlim_{[p]^*} \mathrm{Pic}(\widetilde{B}).$$

In particular, any trivialization of the special fiber of some line bundle $(q_1^* N)^{1/p^n}$ is pulled back from a trivialization on some finite level. By smoothness, this trivialization lifts to a trivialization on the formal scheme, and therefore the corresponding rigid space as in [Lüt16, Proposition A.2.5]. It is therefore enough to trivialize all our pth roots on a cover of $\widetilde{B}_\infty$.

To do this, we work our way over to the tilt. If the degree of N is prime to p, then a theorem of Grothendieck (see [Oor71, Theorem 2.4.1] with the added input of the cotangent complex as in Section 5.1) shows that there is some lift of $\widetilde{B}$ to a formal abelian scheme $\overline{B'}$ over $K^{\flat\circ}$ such that $\widetilde{N}$ lifts to a line bundle $\overline{N'}$ on $\overline{B'}$. By Lemma 6.3.2, the generic fiber B' pulls back to a line bundle $q_0'^*(B')$ on B'_∞ which extends to a $\mathbb{G}_{m,\infty,K^\flat}$-torsor. This gives a cover trivializing all the pth roots of $q_0'^*(B')$, which reduces to a cover trivializing the pth roots of the special fiber as desired. $\square$

Example 6.3.4. When N is translation-invariant, the corresponding $\mathbb{G}_m$-torsor is a semi-abelian variety. The corresponding $\mathbb{G}_{m,\infty}$-torsor in this case is exactly the one constructed in Proposition 4.2.1.

Applying this Proposition to the Poincaré torsor $P^\times_{B\times B^\vee}$ on the abelian variety $B \times B^\vee$, we get our perfectoid cover.

Corollary 6.3.5. *There is a $\mathbb{G}_{m,\infty}$-torsor $P_\infty^\times \to (B \times B^\vee)_\infty \cong B_\infty \times B_\infty^\vee$ fitting into the commutative diagram of adic spaces*

$$\begin{tikzcd}
(\mathbb{G}_{m,K})_\infty \arrow[r] \arrow[d] & P_\infty^\times \arrow[r] \arrow[d] & B_\infty \times B_\infty^\vee \arrow[d, equal] \\
\mathbb{G}_{m,K} \arrow[r] \arrow[d, equal] & q_1^*(P_{B\times B^\vee}^\times) \arrow[r] \arrow[d] & B_\infty \times B_\infty^\vee \arrow[d, "q_1"] \\
\mathbb{G}_{m,K} \arrow[r] & P_{B\times B^\vee}^\times \arrow[r] & B \times B^\vee.
\end{tikzcd} \tag{6.14}$$

Corollary 6.3.6. *Fix a point $x_\infty^\vee \in B_\infty^\vee$, let $x^\vee := q_1(x_\infty^\vee) \in B^\vee$, so*

$$P_{x_\infty^\vee}^\times := P_\infty^\times \times_{B_\infty \times B_\infty^\vee} (B_\infty \times x_\infty^\vee)$$

is a $(\mathbb{G}_{m,K})_\infty$-torsor over B_∞. Then $P_{x_\infty^\vee}^\times \sim \varprojlim_{[p]} P_{B\times x^\vee}^\times$. The symmetric statement is true for $x_\infty \in B_\infty$, and these give compatible partial group laws which project down to the partial group laws on $P_{B\times B^\vee}^\times$.

Proof. Add the map $B \times x^\vee \to B \times B^\vee$ to (6.14), take fiber products everywhere, and notice that we recover precisely the diagram for $P_{x_\infty^\vee}$. $\square$

This construction tilts nicely. As B has good reduction, it extends to an abelian scheme $\overline{B}/K^\circ$ with special fiber $\widetilde{B}/(K^\circ/\varpi)$. Using the isomorphism $K^\circ/\varpi \cong K^{\flat\circ}/\varpi^\flat$, we lift $\widetilde{B}$ to a formal abelian scheme $\overline{B}'/K^{\flat\circ}$ with adic generic fiber $B'/K^\flat$. Then we have $B_\infty^\flat \cong (B')_\infty$ as the special fibers agree, and $(B^\vee)_\infty^\flat \cong (B'^\vee)_\infty$ for the same reason: taking duals commutes with taking the special fiber.

By [Lüt16, Theorem 6.1.1], the Poincaré bundle $P_{B\times B^\vee}$ extends to a formal line bundle on $\overline{B \times B^\vee}$ with the same special fiber as the Poincaré bundle $P_{B'\times B'^\vee}$ on $B' \times B'^\vee$. Let $P_\infty'^\times$ be the $(\mathbb{G}_{m,K^\flat})_\infty$-torsor over the Poincaré bundle on $B' \times B'^\vee$. Now returning to diagram (6.14), we have formal models for the spaces making up the outer columns and bottom row. As the bottom right square is a pullback and the top left square is a pushout, this is enough to give formal models for the middle column as well. As all of this commutes with taking the special fiber, we conclude:

Proposition 6.3.7. *There is an isomorphism* $(P_\infty^\times)^\flat \cong P_\infty'^\times$.

Restricting this isomorphism to the situation of Corollary 6.3.6 and noting that the fiber diagram of perfectoid spaces

$$\begin{tikzcd} P^\times_{x^\vee_\infty} \arrow[r] \arrow[d] & P^\times_\infty \arrow[d] \\ B_\infty \times x^\vee_\infty \arrow[r] & B_\infty \times B^\vee_\infty \end{tikzcd}$$

tilts to the diagram

$$\begin{tikzcd} (P^\times_{x^\vee_\infty})^\flat \arrow[r] \arrow[d] & P'^\times_\infty \arrow[d] \\ B'_\infty \times (x^\vee_\infty)^\flat \arrow[r] & B'_\infty \times B'^\vee_\infty, \end{tikzcd}$$

we obtain

Corollary 6.3.8. *For any point* $x_\infty^\vee \in B_\infty^\vee$*, there is an isomorphism* $(P^\times_{x^\vee_\infty})^\flat \cong P^\times_{(x^\vee_\infty)^\flat}$.

We have now constructed a perfectoid cover of the vertical map in (6.6) and showed that it tilts to an analogous map over $K^\flat$ .To complete the perfectoid version of this diagram we must lifting and tilting the bilinear form $\langle\cdot,\cdot\rangle$. This requires us to make choices, so we then check that A_∞ and $(A^\vee)_\infty$ are independent of these choices.

Fix a basis $m_1,\dots,m_r$ for M and $m_1^\vee,\dots,m_r^\vee$ for $M^\vee$. We first choose group homomorphisms $\phi_\infty : M \to B_\infty$ and $\phi_\infty^\vee : M^\vee \to B_\infty^\vee$ lifting ϕ and $\phi^\vee$ by picking any lift on our basis elements and then extending from there. Concretely, the element $\phi_\infty(m)$ corresponds to a sequence $(\phi(m), \frac{\phi(m)}{p}, \frac{\phi(m)}{p^2}, \dots)$ of elements of B where $[p](\frac{\phi(m)}{p^i}) = \frac{\phi(m)}{p^{i-1}}$ for all $i > 0$. We then choose a bilinear form $\langle\cdot,\cdot\rangle_\infty \to P_\infty^\times$ lifting $\langle\cdot,\cdot\rangle$ by picking any lift on the elements $m_i \times m_j^\vee$ which lies over $\phi_\infty(m_i) \times \phi_\infty^\vee(m_j^\vee)$, then extending bilinearly to all of $M \times M^\vee$ (using the compatibility of the partial group laws). This gives us the diagram

$$
\begin{tikzcd}
 & & P^{\times}_{\infty} \arrow[dl] \arrow[dd] \\
M \times M^{\vee} \arrow[urr, "\langle \cdot, \cdot \rangle_{\infty}"] \arrow[r, "\phi_{\infty} \times \phi_{\infty}^{\vee}"'] \arrow[dd, equal] & B_{\infty} \times B_{\infty}^{\vee} \arrow[dd] & \\
 & & P^{\times}_{B \times B^{\vee}} \arrow[dl] \\
M \times M^{\vee} \arrow[urr, "\langle \cdot, \cdot \rangle"] \arrow[r, "\phi \times \phi^{\vee}"'] & B \times B^{\vee} &
\end{tikzcd} \tag{6.15}
$$

Here the top triangle is a commutative diagram of perfectoid spaces, and does depend on our choice of lifts. However, we claim that no matter the choice, we can still extract the spaces E_∞, A_∞, and $A_\infty^\vee$ just from the top triangle alone.

Proposition 6.3.9. *We have*

$$E_\infty \cong P^{\times}_{\phi_\infty^\vee(m_1^\vee)} \times_{B_\infty} \cdots \times_{B_\infty} P^{\times}_{\phi_\infty^\vee(m_r^\vee)},$$

with notation as in Corollary 6.3.6.

Proof. This is the analogue of (6.3), and the proof builds off of this decomposition. The map $[p] : E \to E$ is induced by the maps $[p] : P^{\times}_{B \times \phi^\vee(m_i^\vee)} \to P^{\times}_{B \times \phi^\vee(m_i^\vee)}$, which live over $[p] : B \to B$. By Corollary 6.3.6, we have $P^{\times}_{\phi_\infty^\vee(m_i^\vee)} \sim \varprojlim_{[p]} P^{\times}_{B \times \phi^\vee(m_i^\vee)}$. Combining these, the proposition follows from uniqueness of perfectoid tilde-limits as in Lemma 3.3.4.

$\square$

The bilinear form $\langle \cdot, \cdot \rangle_\infty$ defines a group homomorphism $M \to E_\infty$ by sending $m \mapsto (\langle m, m_1^\vee \rangle_\infty, \ldots, \langle m, m_r^\vee \rangle_\infty)$. By the commutativity of (6.15), this lifts the homomorphism $M \to E$. In the language of [BGH$^+$18, Section 4], we have recovered a choice M_∞ of p-power roots in E of the lattice M, and our homomorphism is precisely the map $M_\infty \to E_\infty$ from [BGH$^+$18, Definition 4.2]. Continuing with the notation from [BGH$^+$18], we define $D_\infty := M_\infty \times_{\mathbb{Z}} \mathbb{Z}_p$.

Then by [BGH+18, Theorem 4.6.5], we have $A_\infty = D_\infty \times^{M_\infty} E_\infty$. By symmetry, the same bilinear form also defines $M^\vee \to E_\infty^\vee$, from which we can recover $A_\infty^\vee$. As the spaces A_∞ and $A_\infty^\vee$ can be determined purely from the initial abelian variety A, we conclude:

Proposition 6.3.10. *The spaces E_∞, $E_\infty^\vee$, A_∞, and $A_\infty^\vee$ are determined by the top triangle of (6.15), and do not depend on the choice of lift $\langle \cdot, \cdot \rangle_\infty$ of $\langle \cdot, \cdot \rangle$.*

Tilting the top triangle of (6.15) and applying Proposition 6.3.7 to fill in the bottom triangle, we get a diagram full of leading notation which will take some unpacking

$$\begin{tikzcd}
 & & P'^{\times}_\infty \arrow[dl] \arrow[dd] \\
M' \times M'^\vee \arrow[r, "\phi'_\infty \times \phi'^\vee_\infty"'] \arrow[urr, "\langle \cdot,\cdot \rangle'_\infty"] \arrow[dd, equal] & B'_\infty \times B'^\vee_\infty \arrow[dd] & \\
 & & P^{\times}_{B' \times B'^\vee} \arrow[dl] \\
M' \times M'^\vee \arrow[r, "\phi' \times \phi'^\vee"'] \arrow[urr, "\langle \cdot,\cdot \rangle'"] & B' \times B'^\vee . &
\end{tikzcd} \tag{6.16}$$

Here we have applied the isomorphism $B_\infty^\flat \cong B'_\infty$ which comes because the special fibers of B and B' agree, the analogous isomorphism for $B'^\vee$, and the isomorphism $P_\infty^\times \cong P'^\times_\infty$ from Proposition 6.3.7, which also gives the commutativity of the right vertical square. We are *defining* $M' \times M'^\vee$ to be the tilt of $M \times M^\vee$, and the morphisms filling out the top triangle are defined by the tilting correspondence. The morphisms in the bottom triangle are defined by composition of the morphisms in the top triangle with projections to the bottom. The map $\phi' \times \phi'^\vee$ is the composition of homomorphisms of adic groups, so it is itself a group homomorphism.

Lemma 6.3.11. *The map $\langle \cdot, \cdot \rangle' : M' \times M'^\vee \to P^\times_{B' \times B'^\vee}$ is a non-degenerate bilinear form.*

Proof. Bilinearity follows because tilting preserves the two partial group laws on P_∞. Non-degeneracy follows as in Lemma 5.2.1. $\square$

Applying [Lüt16, Proposition 6.1.8] to this bilinear form, we get a pair of abeloids $A' := E'/M'$ and $A'^{\vee} := E'^{\vee}/M'^{\vee}$. Our goal is to show that $A'_{\infty} \cong A^{\flat}_{\infty}$, which will imply by symmetry that $A'^{\vee}_{\infty} \cong (A'^{\vee}_{\infty})^{\flat}$. By Theorem 5.2.2, it is enough to show that $E'_{\infty} \cong E^{\flat}_{\infty}$ and that the map $M' \to E'_{\infty}$ induced by $\langle \cdot, \cdot \rangle'_{\infty}$ agrees with the tilt of the map $M \to E_{\infty}$ induced by $\langle \cdot, \cdot \rangle'_{\infty}$.

By the decomposition in Proposition 6.3.9, to see that $E'_{\infty} \cong E^{\flat}_{\infty}$ it is enough to see that

$$(P^{\times}_{\phi^{\vee}_{\infty}(m^{\vee})})^{\flat} \cong P'^{\times}_{\phi'^{\vee}_{\infty}(m^{\flat})},$$

but this is precisely the statement of Corollary 6.3.8. The statement about the bilinear forms follows because $h' : M' \to E'$ is determined by the r maps

$$\langle \cdot, \phi'^{\vee}(m'^{\vee}_i) \rangle'_{\infty} : M' \to P'^{\times}_{B'_{\infty} \times \phi'^{\vee}_{\infty}(m^{\vee}_i)},$$

which are by definition the tilts of the r analogous maps defining h in terms of $\langle \cdot, \cdot \rangle_{\infty}$. We therefore get the desired result:

Proposition 6.3.12. *Let A be an abeloid over a perfectoid field K with dual $A^{\vee}$, let A_{∞} and $(A^{\vee})_{\infty}$ be the corresponding perfectoid covers. Then there is an abeloid A' over $K^{\flat}$ with $A'_{\infty} \cong (A_{\infty})^{\flat}$ and $(A'^{\vee})_{\infty} \cong (A^{\vee})^{\flat}_{\infty}$.*

With this in hand, we tilt our line bundles.

Theorem 6.3.13. *Given a line bundle L on an abeloid A/K, there is some model $\mathfrak{A}_{\alpha}$ of A such that L extends to a formal line bundle $\mathfrak{L}_{\alpha}$ on $\mathfrak{A}_{\alpha}$. If the degree of L is prime to p, then after replacing K by a pro-p extension, there is some $A'/K^{\flat}$ as in Theorem 5.2.2 and line bundle L' extending to a formal line bundle $\mathfrak{L}'_{\alpha}$ such that the special fibers of the formal line bundles agree. The line bundle L is ample if and only if L' is.*

Proof. By Proposition 6.1.1, L is equivalent to a line bundle N on B, group homomorphism

$\sigma : M \to M^\vee$, and trivialization $r : M \to \pi^* N$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\phi} & B \\ \Big\downarrow{\scriptstyle\sigma} & & \Big\downarrow{\scriptstyle\varphi_N} \\ M^\vee & \xrightarrow{\phi^\vee} & B^\vee. \end{array} \tag{6.17}$$

commutes, and the equation

$$r(m_1+m_2)\otimes r(m_1)^{-1}\otimes r(m_2)^{-1} = \langle m_1, \sigma(m_2)\rangle \tag{6.18}$$

holds for all $m_1, m_2 \in M$.

By Proposition 6.2.1 and Proposition 6.2.3, L extends to some formal model $\mathfrak{A}_\alpha$, and is equivalent to a choice of formal model $\mathfrak{N}'$ of N and a continuous function f built from “tropical data" associated to σ and r. To construct the desired L', we use the data for L to construct analogous data N', σ', and r' for A'.

As we saw while proving Proposition 6.3.3, there is some lift of $\widetilde{B'}$ to a formal abelian scheme $\overline{B'}$ over $K^{\flat\circ}$ such that the line bundle $\overline{\mathfrak{N}}$ lifts to a line bundle $\mathfrak{N}'$ with generic fiber N'. The line bundle N' is ample if and only if its special fiber is. Using this B' when we construct A', we get the desired N'. We note that this step requires the condition on the degree of L (and is the only place where it is used).

Constructing A' and $A'^\vee$ simultaneously as in Proposition 6.3.12, we get isomorphisms $\flat : M \cong M'$ and $\flat : M' \cong M'^\vee$ which agree on the special fiber (that is, their actions on the special fiber agree). We use these isomorphisms to transfer σ to a group homomorphism $\sigma' : M' \to M'^\vee$ which agrees with σ on the special fiber. These isomorphisms therefore respect the resulting bilinear forms (6.10), so one is positive definite precisely when the other is. By the ampleness criteria in Proposition 6.1.1, we see that L is ample if and only if L' is.

To tilt the trivialization $r : M \to \phi^* N \cong M \times K$, we let $m'_1, \ldots, m'_r$ be the image in M' of the basis $m_1, \ldots, m_r$ of M under $\flat$. We define $r'(m'_i)$ to be any element in $\sharp^{-1}(r(m_i))$ - perhaps making a pro-p extension of K to ensure that such an element exists - then extend to all of M' using the rule (6.18). $\square$

The condition on the degree of L is somewhat annoying, but we believe it is unavoidable. For our perfectoid applications, the following lemma lets us get around the issue.

Lemma 6.3.14. *Let $f : A \to A_1$ be an isogeny of abelian varieties over K with p-power degree. Then the perfectoid covers A_∞ and $(A_1)_\infty$ constructed in Theorem 4.2.5 are isomorphic.*

Proof. Let $f^\vee$ be the dual isogeny, so we have $f^\vee \cdot f = [p^k]$ for some positive integer k. Then we have a commutative diagram:

$$\begin{tikzcd}
 & \cdots \arrow[r] & A_1 \arrow[rr, "{[p^k]}"] \arrow[dr, "f^\vee"] & & A_1 \arrow[dr, "f^\vee"] & \\
\cdots \arrow[r] & A \arrow[ur, "f"] \arrow[rr, "{[p^k]}"] & & A \arrow[ur, "f"] \arrow[rr, "{[p^k]}"] & & A.
\end{tikzcd}$$

This diagram induces morphisms $f_\infty : A_\infty \to (A_1)_\infty$ and $f_\infty^\vee : (A_1)_\infty \to A_\infty$ which compose to $[p^k] : A_\infty \to A_\infty$ and $[p^k] : (A_1)_\infty \to (A_1)_\infty$. As the latter maps are isomorphisms, we are done. $\square$

As a first application, we can upgrade Theorem 5.2.2 to work for abelian varieties, not just abeloids.

Theorem 6.3.15. *Let A be an abelian variety over an algebraically closed, non-archimedean field K. Let A_∞ be the perfectoid cover constructed in Theorem 4.2.5. After replacing K by a pro-p extension, there is an abelian variety A' over $K^\flat$ such that $(A_\infty)^\flat \cong (A')_\infty$.*

Proof. Proposition 6.3.13 proves this for abelian varieties with a polarization of degree prime to p. Fixing any polarization φ_A of A, we can quotient A by a p-Sylow subgroup of $\ker(\varphi_A)$, giving us an isogeny $A \to A/\ker(\varphi_A) =: A_1$ of degree p^k for some positive integer k. Then φ_A induces a polarization on A_1 of degree prime to p. By Lemma 6.3.14 we can prove the theorem after replacing A by A_1. $\square$

Corollary 6.3.16. *Given a line bundle L on an abelian variety A over K, after replacing K by a pro-p extension, there is a $(\mathbb{G}_{m,K})_\infty$ torsor $L_\infty^\times$ living over $q_1^*(L^\times)$.*

Proof. Now that we can transfer line bundles from A to A', the argument is essentially the same as in Proposition 6.3.3. We can again use the theorem of the cube to construct pth power roots at finite levels of the tower, and we can use Theorem 6.3.13 to find a system of line bundles over A' with isomorphic special fibers. On the tilt side, Lemma 6.3.2 gives a $\mathbb{G}_{m,\infty}^\flat$-torsor. Untilting, we get a $\mathbb{G}_{m,\infty}$ torsor with the correct special fiber, so it must be the one we are looking for. $\square$

Remark 6.3.17. Ben Heuer tells us that he can prove the stronger result that $\mathrm{Pic}(A_\infty) \cong \mathrm{Pic}(A_\infty^\flat)$, using that $H^i(A_\infty, \mathbb{Z}_p) = 0$ for $i > 0$. In particular, this gives an affirmative answer to [DH19, Open Problem 5.14] in this case.

This chapter contains material from joint work with Ben Heuer which is currently being prepared for submission for publication.

Chapter 7

Weight-monodromy for complete intersections in abelian varieties

We are now ready to prove our main result: the weight-monodromy conjecture for complete intersections in abelian varieties. In Section 7.1, we give a more detailed overview of the setup of the weight-monodromy conjecture. We remark that this section can be pretty much completely black-boxed: we will summarize what we need for our proof as we go. We then prove the theorem in Section 7.2.

7.1 Background on the weight-monodromy conjecture

Let X be a proper smooth d-dimensional variety over a local field k of residue characteristic p, let q be the cardinality of the residue field. Let ℓ be a prime not equal to p, let i be an integer between 0 and $2d$. Then the ith étale cohomology group $V := H^i_{\text{ét}}(X_{\overline{k}}, \overline{\mathbb{Q}}_\ell)$ is a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space with an action of the absolute Galois group $G_k := \mathrm{Gal}(\overline{k}/k)$ of k. This action defines a Galois representation $\rho : G_k \to \mathrm{GL}(V)$.

In this section, we explain how to define two filtrations on V. The monodromy filtration comes from the inertia subgroup I_k of G_k, and the weight filtration comes from the action of a lift of Frobenius from the residue field. The weight-monodromy conjecture is that these filtrations are the same.

The monodromy filtration comes out of the following result of Grothendieck.

Proposition 7.1.1. *Let $\rho : G_k \to \mathrm{GL}(V)$ be a finite-dimensional $\overline{\mathbb{Q}}_\ell$-representation. Then there is an open subgroup I_1 of the inertia subgroup I_k such that for all g in I_1, the matrix $\rho(g)$ is unipotent.*

Sketch. See the appendix of [ST68] for a full proof. After quotienting out by a finite subgroup of I_k, we can assume that the image of ρ is a pro-ℓ group. A Galois theoretic argument lets us focus on the pro-ℓ part of the tame inertia. This is

$$\mathrm{Gal}(k(\mu_{\ell^\infty})/k) \cong T_\ell(\mu) := \varprojlim \mu_{\ell^n}.$$

An element of $T_\ell(\mu)$ corresponds to a compatible sequence of ℓth power roots of unity: a sequence $(1, \lambda_\ell, \lambda_{\ell^2}, \dots)$ with $\lambda_{\ell^n}^\ell = \lambda_{\ell^{n-1}}$. The absolute Galois group G_k acts on an element of $T_\ell(\mu)$ by acting compatibly on each root of unity in the sequence. This action is represented by the *cyclotomic character*

$$\chi : G_k \to \mathbb{Z}_\ell^*,$$

where $\chi(g)$ is the unique unit in $\mathbb{Z}_\ell$ satisfying $g(\lambda_{\ell^i}) = \lambda_{\ell^i}^{\chi(g)}$ or all i.

The upshot is that after taking a finite extension of k (to kill a finite amount of prime-to-ℓ information), the representation $\rho : G_k \to \mathrm{GL}(V)$ factors through the ℓ-adic cyclotomic character $\chi : G_k \to \mathbb{Z}_\ell^\times$. It is therefore enough to check that the image of some element $\chi(g)$ in $\mathbb{Z}_\ell^\times$ is unipotent. As k is a local field, we can choose some g such that $\chi(g)$ is not a root of unity. For such elements, Grothendieck used the Galois action to show that $\log(\rho(g))$ is nilpotent, and therefore that $\rho(g)$ is unipotent. $\square$

We can now define the monodromy filtration using the nilpotent operator $N := \log(\rho(g))$ on V constructed in the course of the previous proof.

Definition 7.1.2. *Let $N : V \to V$ be a nilpotent operator on a finite-dimensional vector space. Then there is a unique finite increasing filtration Fil_j^N of V such that $N\,\mathrm{Fil}_j^N$ is contained in Fil_{j-2}^N and such that N^k induces an isomorphism $\mathrm{Gr}_{i+j}^N V \cong \mathrm{Gr}_{i-j}^N V$. Explicitly, we have*

$$\mathrm{Fil}_j^N V = \sum_{j_1 - j_2 = j} \ker N^{j_1+1} \cap \mathrm{Im}\, N^{j_2}.$$

This sweeps under the rug questions of uniqueness, well-definedness, and so on: see [Del80, Proposition 1.6.1] for details. We are also ignoring the Tate twist that the cyclotomic character carries. The important takeaway from this discussion is that the monodromy filtration depends only on the pro-ℓ part of the inertia in the Galois group. Taking pro-p extensions of our base field will therefore preserve the monodromy filtration.

To define the weight filtration, we fix an element $\Phi \in \mathrm{Gal}_k$ which maps to geometric Frobenius on the residue field. We use the matrix $\rho(\Phi) \in \mathrm{GL}(V)$ to decompose V into eigenspaces. Work of Rapoport-Zink, along with de Jong's alterations theorem, gives us control over the eigenvalues.

Proposition 7.1.3. *With notation as above, we have a decomposition*

$$V = \bigoplus_{j=0}^{2i} V_j,$$

where the eigenvalues of $\rho(\Phi)$ acting on V_j are Weil numbers of weight j. That is, they are algebraic numbers, and for any embedding of k into $\mathbb{C}$, they have absolute value $q^{j/2}$.

Sketch. See [RZ82] for details, or the nice expositions in [Ill94, Section 3] and [Sch11, Section 2]. We use de Jong's alterations theorem to reduce to the case where X has a semistable model. The special fiber of this model consists of finitely many smooth varieties, intersecting transversely.

By the Riemann hypothesis part of the Weil conjectures, the Frobenius eigenvalues on each irreducible component of the special fiber are all Weil numbers of weight i. If j components have non-empty intersection, it is a smooth variety with codimension j, so the Frobenius eigenvalues are Weil numbers of weight $i-j$. The Rapoport-Zink spectral sequence builds the actual cohomology of X out of these pieces, giving the desired decomposition. □

If we chose a different lift of Frobenius, we'd get a different decomposition of V. However, the weights and dimensions remain the same, making the following definition reasonable.

Definition 7.1.4. *With notation as in Proposition 7.1.3, the* weight filtration *on V is defined by letting the mth piece be*

$$W_m = \bigoplus_{j=0}^{m} V_j.$$

For any lift Φ of geometric Frobenius, we have $N\Phi = q\Phi N$. This implies that the monodromy operator interacts nicely with the weight filtration: we have $NW_m \subset W_{m-2}$. The weight-monodromy conjecture says more:

Conjecture 7.1.5 (Deligne, [Del71])**.** *For a proper smooth variety X over a local field k, the weight and monodromy filtrations on the ith ℓ-adic étale cohomology group of X are the same. Equivalently, the eigenvalues of any lift Φ of geometric Frobenius on any graded piece $\mathrm{gr}_j^N V$ of the monodromy filtration are Weil numbers of weight $i+j$.*

The equivalence of these statements is in [Del80, Section 1.6]. Deligne gave another interpretation of this conjecture in terms of the poles of L-functions associated to sheaves. He used this interpretation to prove the conjecture when k has characteristic p and X is defined over a curve. See [Del80, Section 1.8] for the proof, or [Sch12, Section 9] for a quick overview of the idea.

7.2 Weight-monodromy for complete intersections in abelian varieties

Scholze's approach to weight-monodromy is summarized in the following:

Proposition 7.2.1. *Let Y be a d-dimensional, geometrically connected, proper smooth variety over a local field k of residue characteristic p, fix a prime $\ell \neq p$. Let K be a perfectoid pro-p extension of k, let G be the absolute Galois group of K. To prove the weight-monodromy conjecture for Y, it is enough to construct a d-dimensional, geometrically connected, proper smooth variety Z over $K^\flat$ which can be defined over some local field contained in $K^\flat$, along with maps*

$$f_i^* : H^i(Y_{\mathbb{C}_p,ét}, \overline{\mathbb{Q}}_\ell) \to H^i(Z_{\mathbb{C}_p^\flat,ét}, \overline{\mathbb{Q}}_\ell)$$

for $i \in \{0, \dots, 2d\}$ which are G-equivariant, compatible with cup product, and such that f_{2d}^* is an isomorphism.

Proof. We first check that the maps f_i^* are all injective. To see this, we work with the diagram

$$\begin{array}{ccccc}
H^i(Y_{\mathbb{C}_p,\text{ét}}, \overline{\mathbb{Q}}_\ell) \times H^{2d-i}(Y_{\mathbb{C}_p,\text{ét}}, \overline{\mathbb{Q}}_\ell) & \xrightarrow{\cup} & H^{2d}(Y_{\mathbb{C}_p,\text{ét}}, \overline{\mathbb{Q}}_\ell) & \xrightarrow{\cong} & \overline{\mathbb{Q}}_\ell \\
\big\downarrow f_i^* \times f_{2d-i}^* & & \big\downarrow f_{2d}^* & & \big\downarrow \cong \\
H^i(Z_{\mathbb{C}_p^\flat,\text{ét}}, \overline{\mathbb{Q}}_\ell) \times H^{2d-i}(Z_{\mathbb{C}_p^\flat,\text{ét}}, \overline{\mathbb{Q}}_\ell) & \xrightarrow{\cup} & H^{2d}(Z_{\mathbb{C}_p^\flat,\text{ét}}, \overline{\mathbb{Q}}_\ell) & \xrightarrow{\cong} & \overline{\mathbb{Q}}_\ell.
\end{array}$$

By Poincaré duality, the rows give perfect pairings between H^i and H^{2d-i} for every i. The left square commutes because the f_i^* are assumed to be compatible with cup product. The right square is all isomorphisms because f_{2d}^* is assumed to be an isomorphism, and because Y and Z are geometrically connected.

Let $c \in H^i(Y_{\mathbb{C}_p,\text{ét}}, \overline{\mathbb{Q}}_\ell)$ be a class with $f_i^*(c) = 0 \in H^i(Z_{\mathbb{C}_p^\flat,\text{ét}}, \overline{\mathbb{Q}}_\ell)$. As every class in $H^{2d-i}(Z_{\mathbb{C}_p^\flat,\text{ét}}, \overline{\mathbb{Q}}_\ell)$ pairs to 0 with 0, commutativity of the diagram tells us that every class in $H^{2d-i}(Y_{\mathbb{C}_p,\text{ét}}, \overline{\mathbb{Q}}_\ell)$ must pair to 0 with c. As the top row is a perfect pairing, c must therefore be 0, so f_i^* is injective as desired.

By G-equivariance, the map f_i^* respects the monodromy filtrations on $H^i(Y_{\mathbb{C}_p,\text{ét}},\overline{\mathbb{Q}}_\ell)$ and $H^i(Z_{\mathbb{C}_p^\flat,\text{ét}},\overline{\mathbb{Q}}_\ell)$. By injectivity, the jth graded piece of $H^i(Y_{\mathbb{C}_p,\text{ét}},\overline{\mathbb{Q}}_\ell)$ is a sub-vector space of the jth graded piece of $H^i(Z_{\mathbb{C}_p^\flat,\text{ét}},\overline{\mathbb{Q}}_\ell)$. Every Frobenius eigenvalue for Y is therefore a Frobenius eigenvalue for Z. As Z is proper, smooth, and can be defined over a characteristic p local field, Deligne proved weight-monodromy for Z. That is, every Frobenius eigenvalue of Z is a Weil number of degree $i+j$. The same is therefore true for Y and weight-monodromy holds. □

Specializing to our case of interest, let Y be a compete intersection in an abelian variety A which satisfies the properties of Proposition 7.2.1. We must construct a variety $Z \subset A'$ and maps f_i^* which also satisfy the properties of Proposition 7.2.1. To construct Z, we use the following approximation result: our analogue of Scholze's approximation result [Sch12, Proposition 8.7].

Proposition 7.2.2. *Let $Y \subset A$ be a hypersurface, let $\tilde{Y}$ be a small open neighborhood of Y, where we are considering Y and A as adic spaces. Let $q_1 : |(A_\infty)^\flat| \to |A|$ be the map of topological spaces induced by tilting and $\sim$, let $q_1' : |(A_\infty)^\flat| \to |A'|$ be the map induced by $\sim$. Then there exists a hypersurface $Z' \subset A'$ such that $q_1'^{-1}(Z') \subset q_1^{-1}(\tilde{Y})$. Equivalently, there exists some open $\tilde{Y}' \subset A'$ with $q_1'^{-1}(\tilde{Y}') = q_1^{-1}(\tilde{Y})$ and a hypersurface $Z' \subset \tilde{Y}'$.*

Proof. There is some line bundle L on A and an element $f \in H^0(A,L)$ with zero locus Y. In Corollary 6.3.16, we constructed a corresponding $(\mathbb{G}_{m,K})_\infty$-torsor L_∞ over A_∞ which fits into the diagram

$$\begin{tikzcd}
1 \arrow[r] & (\mathbb{G}_{m,K})_\infty \arrow[r] \arrow[d] & L_\infty \arrow[r] \arrow[d] & A_\infty \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{G}_{m,K} \arrow[r] & L \arrow[r] & A \arrow[r] & 1
\end{tikzcd} \tag{7.1}$$

where all the down arrows are projections to the bottoms of the corresponding tilde-limits. In Theorem 6.3.13, we built L_∞ as the adic generic fiber of a limit of formal schemes - pullbacks of bundles $\mathfrak{L}_{\alpha/p^n}$ along $q_{n+1} : \mathfrak{A}_{\alpha/p^n} \to \mathfrak{A}_\infty$. By Lemma 2.3.10, we have

$$\Gamma(L_\infty) = \left(\varinjlim \Gamma(q_{n+1}^* \mathfrak{L}_{\alpha/p^n})\right)^\wedge \otimes_{K^\circ} K.$$

Each term $q_{n+1}^* \mathfrak{L}_{\alpha/p^n}$ is itself an inverse limit of formal schemes

$$q_{n+1}^* \mathfrak{L}_{\alpha/p^n} \cong \varprojlim_{[\mathfrak{p}]_{n+m}} \mathfrak{L}_{\alpha/p^{n+m}}$$

and therefore satisfies the analogous density condition. We conclude that global sections of the $\mathbb{G}_{m,\infty}$-torsor L_∞ can be approximated by functions coming from finite level. Concretely, we take the limit of the formula (6.12) to see that $\Gamma(L_\infty^\times)$ is the completion of

$$\bigoplus_{i/p^n \in \mathbb{Z}[1/p]} H^0(A_\infty, q_{n+1}^*(\mathscr{L}^{\otimes i}))$$

and that $H^0(A_\infty, q_{n+1}^*(\mathscr{L}^{\otimes i}))$ is the completion of

$$\varinjlim_{m \in \mathbb{N}} H^0(A, [p^{n+m}]^*(\mathscr{L}^{\otimes i})).$$

Scholze gives an analogous description for line bundles on toric varieties in [Sch12, Proposition 8.7].

By Corollary 6.3.16, there is a corresponding diagram over $K^\flat$

$$\begin{tikzcd}
1 \arrow[r] & (\mathbb{G}_{m,K^\flat})_\infty \arrow[r] \arrow[d] & (L'^\times)_\infty \arrow[r] \arrow[d] & (A')_\infty \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{G}_{m,K^\flat} \arrow[r] & L'^\times \arrow[r] & A' \arrow[r] & 1
\end{tikzcd} \tag{7.2}$$

where the top row of (7.2) is the tilt of the top row of (7.1).

We can assume that $\tilde{Y} = \{x \in A : |f(x)| \leq \varepsilon\}$ for some ε, where the absolute value comes from the integral model for L defined in Proposition 6.2.1. Pulling back $\tilde{Y}$ and f along q_1, we get an open set in A_∞ defined as an ε-neighborhood of $q_1^*(f)$, which we can view as an element of $H^0(L_\infty^\times)$. Using [Bha17, Proposition 9.2.7], which builds off of [Sch12, Lemma 6.5], we can approximate the vanishing locus of $q_1^*(f)$ by an element $g^\sharp$ in the image of $\sharp : H^0(L_\infty'^\times) \to H^0(L_\infty^\times)$. By our density statements, we can tweak g so that it is the pullback of some $g \in H^0(A', [p^m]^*(\mathscr{L}'^\times)^{\otimes i})$ from some finite level of the tower, and so that it is defined over k. Then $q_1^{-1}(\tilde{Y})$ is an ε-neighborhood of the vanishing locus of g. Taking a big enough power of g, we can assume that it comes from the bottom copy of A' on the tower. The vanishing locus Z' of g is the desired hypersurface. $\square$

As in [Sch12, Corollary 8.8], we can intersect hypersurfaces to get the analogous result when Y is a complete intersection in A. We get an open $\tilde{Y}' \subset A'$ such that $q_1'^{-1}(\tilde{Y}') = q_1^{-1}(\tilde{Y})$ as topological spaces in A'_∞, and a closed variety $Z' \subset A'$ of dimension d. We can choose Z' to be defined over a local field in $K^\flat$, and take a component to ensure that it is geometrically connected. We emphasize that even though we used the theory of adic spaces to construct Z', it is a variety over a local field.

We now make some reductions. By [Hub98, Theorem 3.6a], the relevant cohomology groups can be computed on the associated adic spaces. We therefore analytify everything from now on, working strictly with adic spaces. By [Hub96, Theorem 3.8.1], there is an open neighborhood $\tilde{Y}$ of Y in A such that Y and Y' have the same cohomology. By de Jong's alterations theorem [dJ96], we can choose a projective smooth alteration $Z \to Z'$. The induced maps on cohomology are injective, giving $H^i(Z'_{\mathbb{C}_p^\flat,\text{ét}}, \overline{\mathbb{Q}}_\ell)$ as a direct summand of $H^i(Z_{\mathbb{C}_p^\flat,\text{ét}}, \overline{\mathbb{Q}}_\ell)$. It is therefore enough to construct maps from the cohomology of Y to the cohomology of Z.

The final difficulty, compared with Scholze's proof for toric varieties, is that the map $[p] : A' \to A'$ does not induce an isomorphism of étale topoi. We therefore have no analogue of Scholze's projection map $\pi : \mathbb{P}^n_{K^\flat} \to \mathbb{P}^n_K$ (defined on étale topoi and topological spaces, and more generally defined for toric varieties). To get around this, our cohomology computations must run through the perfectoid covers more explicitly.

The tilting equivalence, along with [Sch12, Theorem 7.17] gives isomorphisms of étale topoi

$$\varprojlim_{[p]} (A'_{K^\flat})^{\sim}_{\text{ét}} \cong (A^\flat_\infty)^{\sim}_{\text{ét}} \cong \varprojlim_{[p]} (A_K)^{\sim}_{\text{ét}}.$$

By [Sch12, Theorem 7.16], these inverse limits commute with taking fiber products along étale maps to A_K or $A'_{K^\flat}$. We therefore get the following diagram of étale topoi

$$\begin{tikzcd}
 & (A^\flat_\infty)^{\sim}_{\text{ét}} \arrow[dl, "q'_1"'] \arrow[dr, "q_1"] & \\
(A'_{\mathbb{C}_p^\flat})^{\sim}_{\text{ét}} & (q_1^{-1}(\tilde{Y})_{\mathbb{C}_p^\flat})^{\sim}_{\text{ét}} \arrow[u] \arrow[dl] \arrow[dr] & (A_{\mathbb{C}_p})^{\sim}_{\text{ét}} \\
(\tilde{Y}'_{\mathbb{C}_p^\flat})^{\sim}_{\text{ét}} \arrow[u] & & (\tilde{Y}_{\mathbb{C}_p})^{\sim}_{\text{ét}} \arrow[u] \\
(Z_{\mathbb{C}_p^\flat})^{\sim}_{\text{ét}} \arrow[u] & & (Y_{\mathbb{C}_p})^{\sim}_{\text{ét}}. \arrow[u, "\cong"]
\end{tikzcd}$$

Here the parallelograms are fiber diagrams, and the diagonal maps are projection maps to the bottom of the tilde-limits constructed by iterating the maps $[p]$ for A' and A.

We first construct the maps f_i^* from this diagram, then check that it satisfies the desired properties. In the above diagram, pullback induces isomorphisms

$$H^i(\tilde{Y}_{\mathbb{C}_p,\text{ét}}, \mathbb{Z}/\ell^m\mathbb{Z}) \to H^i(Y_{\mathbb{C}_p,\text{ét}}, \mathbb{Z}/\ell^m\mathbb{Z})$$

for all i and m, and maps

$$H^i(\tilde{Y}_{\mathbb{C}_p,\text{ét}}, \mathbb{Z}/\ell^m\mathbb{Z}) \to H^i(q_1^{-1}(\tilde{Y})_{\mathbb{C}_p^\flat,\text{ét}}, \mathbb{Z}/\ell^m\mathbb{Z}),$$

$$H^i(\tilde{Y}'_{\mathbb{C}_p^\flat,\text{ét}}, \mathbb{Z}/\ell^m\mathbb{Z}) \to H^i(Z_{\mathbb{C}_p^\flat,\text{ét}}, \mathbb{Z}/\ell^m\mathbb{Z}),$$

It remains to construct a map

$$H^i(q_1^{-1}(\tilde{Y})_{\mathbb{C}_p^\flat}, \mathbb{Z}/\ell^m\mathbb{Z}) \to H^i(\tilde{Y}'_{\mathbb{C}_p^\flat,\text{ét}}, \mathbb{Z}/\ell^m\mathbb{Z})$$

To do this, we factor the morphism $[p] : A' \to A'$ as

$$A' \xrightarrow{\varphi} A'/A'[p]^0 \xrightarrow{\psi} A'.$$

Here $A'[p]^0$ is the identity component of the finite group scheme $A'[p]$. The morphism φ is radical and induces a homeomorphism of topological spaces, so it induces an isomorphism of étale topoi as in [Sch12, Corollary 7.19]. The morphism ψ is finite étale.

Restricting these morphisms to

$$[p]^{-1}(\tilde{Y}'_{\mathbb{C}_p^\flat}) \xrightarrow{\varphi} \psi^{-1}(\tilde{Y}'_{\mathbb{C}_p^\flat}) \xrightarrow{\psi} \tilde{Y}'_{\mathbb{C}_p^\flat},$$

we again have a radical map followed by a finite étale map. As the radical map induces an isomorphism of étale topoi, we get isomorphisms

$$H^i([p]^{-1}(\tilde{Y})_{\mathbb{C}_p^\flat}, \mathbb{Z}/\ell^m\mathbb{Z}) \xrightarrow{\cong} H^i(\psi^{-1}(\tilde{Y}_{\mathbb{C}_p^\flat}), \mathbb{Z}/\ell^m\mathbb{Z})$$

for all i and m. As ψ is finite étale, the trace map [Sta20, Tag 03SH] gives maps

$$H^i(\psi^{-1}(\tilde{Y}_{\mathbb{C}_p^\flat}), \mathbb{Z}/\ell^m\mathbb{Z}) \to H^i(\tilde{Y}_{\mathbb{C}_p^\flat}, \mathbb{Z}/\ell^m\mathbb{Z})$$

for all i and m. These maps are induced by the adjunction maps $f_*f^*(\mathbb{Z}/\ell^m\mathbb{Z}) \to \mathbb{Z}/\ell^m\mathbb{Z}$ of sheaves on $\tilde{Y}_{\mathbb{C}_p^\flat}$, and are defined as the composition

$$H^i(\psi^{-1}(\tilde{Y}_{\mathbb{C}_p^\flat}), \mathbb{Z}/\ell^m\mathbb{Z}) \xrightarrow{\cong} H^i(\tilde{Y}_{\mathbb{C}_p^\flat}, f_*f^*(\mathbb{Z}/\ell^m\mathbb{Z})) \to H^i(\tilde{Y}_{\mathbb{C}_p^\flat}, \mathbb{Z}/\ell^m\mathbb{Z}).$$

Composing these, we get a map

$$H^i([p]^{-1}(\tilde{Y})_{\mathbb{C}_p^\flat}, \mathbb{Z}/\ell^m\mathbb{Z}) \to H^i(\tilde{Y}_{\mathbb{C}_p^\flat}, \mathbb{Z}/\ell^m\mathbb{Z}).$$

Iterating this process for the inverse system defining $q_1^{-1}(\tilde{Y})_{\mathbb{C}_p^\flat} \sim \varprojlim_n [p^n]^{-1}(\tilde{Y}'_{\mathbb{C}_p})$, we get the desired map

$$H^i(q_1^{-1}(\tilde{Y})_{\mathbb{C}_p^\flat}, \mathbb{Z}/\ell^m\mathbb{Z}) \to H^i(\tilde{Y}'_{\mathbb{C}_p^\flat,\text{ét}}, \mathbb{Z}/\ell^m\mathbb{Z}).$$

All of the maps are induced by a combination of pullback along morphisms of adic spaces and (in the case of trace) homomorphisms of étale sheaves. By functoriality of these constructions, the compositions $H^i(Y_{\mathbb{C}_p,\text{ét}}, \mathbb{Z}/\ell^m\mathbb{Z}) \to H^i(Z_{\mathbb{C}_p^\flat,\text{ét}}, \mathbb{Z}/\ell^m\mathbb{Z})$ are G-equivariant and compatible with cup product. Taking the inverse limit and tensoring with $\overline{\mathbb{Q}}_\ell$, the same is true of the maps $f_i^* : H^i(Y_{\mathbb{C}_p,\text{ét}}, \overline{\mathbb{Q}}_\ell) \to H^i(Z_{\mathbb{C}_p^\flat,\text{ét}}, \overline{\mathbb{Q}}_\ell)$. There is only one condition left to check for Proposition 7.2.1 to apply.

Lemma 7.2.3. *The map $f_{2d}^* : H^{2d}(Y_{\mathbb{C}_p,ét}, \overline{\mathbb{Q}}_\ell) \to H^{2d}(Z_{\mathbb{C}_p^\flat,ét}, \overline{\mathbb{Q}}_\ell)$ is an isomorphism.*

Proof. Both cohomology groups are isomorphic to $\overline{\mathbb{Q}}_\ell$, so it is enough to show that the map is non-zero. It is enough to check this when the coefficients are $\mathbb{Z}/\ell^m\mathbb{Z}$, then pass to the inverse limit. In this situation, the above constructions give the following commutative diagram.

$$\begin{tikzcd}
 & H^{2d}(A^\flat_\infty, \mathbb{Z}/\ell^m\mathbb{Z}) \arrow[dl, "tr"'] \arrow[d] & \\
H^{2d}(A'_{\mathbb{C}_p^\flat}, \mathbb{Z}/\ell^m\mathbb{Z}) \arrow[d] & H^{2d}(q_1^{-1}(\tilde{Y})_{\mathbb{C}_p^\flat}, \mathbb{Z}/\ell^m\mathbb{Z}) \arrow[dl, "tr"'] & H^{2d}(A_{\mathbb{C}_p}, \mathbb{Z}/\ell^m\mathbb{Z}) \arrow[ul] \arrow[d] \\
H^{2d}(\tilde{Y}'_{\mathbb{C}_p^\flat}, \mathbb{Z}/\ell^m\mathbb{Z}) \arrow[d] & & H^{2d}(\tilde{Y}_{\mathbb{C}_p}, \mathbb{Z}/\ell^m\mathbb{Z}) \arrow[ul] \\
H^{2d}(Z_{\mathbb{C}_p^\flat}, \mathbb{Z}/\ell^m\mathbb{Z}) & & H^{2d}(Y_{\mathbb{C}_p}, \mathbb{Z}/\ell^m\mathbb{Z}) \arrow[u, "\cong"]
\end{tikzcd}$$

The argument is now essentially the same as [Sch12, Lemma 9.8], with a slightly bigger diagram. As multiplication by p induces an isomorphism on all ℓ-adic cohomology groups of an abelian variety, the two diagonal maps at the top of the diagram are isomorphisms. It therefore suffices to show that the map $H^{2d}(A'_{\mathbb{C}_p^\flat}, \mathbb{Z}/\ell^m\mathbb{Z}) \to H^{2d}(Z_{\mathbb{C}_p^\flat}, \mathbb{Z}/\ell^m\mathbb{Z})$ is non-zero. The dth power of the first Chern class of an ample line bundle on A' will have non-zero image, so we are done.

$\square$

This concludes the proof of Theorem 1.2.2.

Remark 7.2.4. The words "set-theoretic" in the statement of both Scholze's theorem and our version make it difficult to know what varieties the theorems apply to. For most varieties, it is unknown if it is possible to write them in this fashion. Ignoring the words set-theoretic then, most proper smooth complete intersections Y in abelian varieties are not also proper smooth complete intersections in toric varieties. We learned the following argument from the MathOverflow answer [Pol]. When $\dim(Y) \geq 3$, the Lefschetz hyperplane theorem says that $\pi_1(Y_{\mathbb{C}_p}) = \pi_1(A_{\mathbb{C}_p})$. If Y were also a complete intersection in a proper smooth toric variety X, we would also have $\pi_1(Y_{\mathbb{C}_p}) = \pi_1(X_{\mathbb{C}_p})$. But $\pi_1(X_{\mathbb{C}_p})$ is trivial and $\pi_1(A_{\mathbb{C}_p})$ is not, giving a contradiction.

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