Title
Every Number In Its Place : The Spatial Foundations Of Calculation And Conceptualization

Author
Marghetis, Tyler John Simons

Publication Date
2015

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA, SAN DIEGO

Every Number In Its Place:

The Spatial Foundations Of Calculation And Conceptualization

A dissertation submitted in partial satisfaction of the

requirements for the degree Doctor of Philosophy

in

Cognitive Science

by

Tyler John Simons Marghetis

Committee in charge:

Professor Rafael Núñez, Chair
Professor David Barner
Professor Benjamin K. Bergen
Professor Seana Coulson
Professor Rick Grush
Professor Edwin Hutchins
Professor Teenie Matlock

2015

The dissertation of Tyler John Simons Marghetis is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

2015

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

ACKNOWLEDGMENTS

Thanks to Rafael Núñez for convincing me that cognitive science offered the tools to answer my questions. And thanks to Benjamin Bergen, who has played an outsized role in my development as an experimentalist.

This dissertation is the child of its birthplace, shaped by interactions with inspiring mentors and collaborators, including my committee—David Barner, Benjamin Bergen, Seana Coulson, Rick Grush, Ed Hutchins, Rafael Núñez, Teenie Matlock—and many others, especially Lera Boroditsky, David Kirsh, Marta Kutas, and Jean Mandler. Thanks to you all.

My home for the last six years has been Núñez's Embodied Cognition Lab and Bergen's Language and Cognition Lab, and what a wonderful home it's been. For support, camaraderie, and inspiration, I'm also grateful to the members of Barner's Language and Development Lab, Boroditsky's Cognation, Coulson's Brain and Cognition Lab, Hutchins's Distributed Cognition and HCI Lab, and Sweetser's Gesture Group at UC Berkeley.

The writing of this essay was made less horrible by friends and colleagues at UCSD: Dan Burnston, Kensy Cooperrider, Mike Datko, Whitney Friedman, Crane Huang, Jasmeen Kanwal, Molly Kelton, Melanie McComsey, Greg Medders, Jake Olson, Nan Renner, Ben Sheredos, Katharine Tillman, Burcu Ürgen, Esther Walker, to name only a few.

My research would grind to a halt if not for the invisible labor of a team of undergraduate research assistants: Myrna Aboudiab, Natalie Allen, Richard Chen, Jordan Conway, Luke Eberle, Brittany Fitzgerald, Alec Gasperian, Gylmar Moreno, Chau Nguyen, Jeremiah Palmerston, Sarah Saturday, Kendall Youngstrom. Two among those deserve special recognition, having earned billing as co-authors on two of the chapters that make up this essay: Luke Eberle (Ch. 5) and Kendall Younstrom (Ch. 2).

In treating mathematics as a case study in meaning, power, regimentation, and the duality of agency and structure, I was guided by Bourdieu, Marx, Peirce, and Wittgenstein. Thanks, dead white guys.

Thanks to my family, for encouraging me to pursue all my wild fantasies—from being a professional magician to winning the Olympics—and supporting me along the way.

And most of all, thanks to Brock Dumville, for everything.

Some institutional acknowledgments:

Chapter 2, in full, has been submitted for publication, and appeared, in part, in the *Proceedings of the 36th Annual Conference of the Cognitive Science Society*. Marghetis, T.; Youngstrom, K., 2014. The dissertation author was the primary investigator and author.

Chapter 3, in full, is a reprint of the material as it appears in the *Quarterly Journal of Experimental Psychology*. Marghetis, T.; Núñez, R.; Bergen, B.K., 2014. The dissertation author was the primary investigator and author.

Chapter 5, in full, has been submitted for publication, and will appear, in part, in the *Proceedings of the 37th Annual Conference of the Cognitive Science Society*. Marghetis, T.; Eberle, L.; Bergen, B., 2015. The dissertation author was the primary investigator and author.

Thanks to the *Fonds de recherche sur la societé et la culture* (Quebec, Canada) and the *Robert J. Glushko and Pamela Samuelson Foundation,* who offered generous financial assistance.

VITA

2007 Bachelor of Science in Mathematics, Concordia University

2009 Master in the Teaching of Mathematics, Concordia University

2011 Visiting Student, University of California, Berkeley

2012 Master of Science in Cognitive Science, University of California, San Diego

2015 Doctor of Philosophy in Cognitive Science, University of California, San Diego

Fields of Study

Major Field: Cognitive Science

Studies in Mechanistic Biosemiotics
Professors Rafael Núñez, Benjamin K. Bergen, and Seana Coulson

Studies in the Reproduction of Cognitive Ecosystems
Professor Edwin Hutchins

Studies in Space, Time, and Number
Professors Rafael Núñez, David Barner, and Doug Nitz

ABSTRACT OF THE DISSERTATION

Every Number In Its Place:

The Spatial Foundations Of Calculation And Conceptualization

by

Tyler John Simons Marghetis

Doctor of Philosophy in Cognitive Science

University of California, San Diego, 2015

Professor Rafael Núñez, Chair

Mathematics involves thinking and communicating about the absent and abstract. Our primate brains and bodies, by contrast, evolved for the mundane exigencies of the concrete here-and-now. How, then, do we make sense of notions that lie beyond the reach of action and perception? Recent proposals suggest that mathematical cognition recycles neural systems specialized for processing space and action, assembled and coordinated by

cultural practices. *Every Number in its Place* explores this spatialization of arithmetic, the coupling of number and space in calculation, conceptualization, communication, and culture.

Inspired by 17th century debates about the reality of imaginary numbers, the five experiments of Chapter 2 demonstrate a novel phenomenon: a *sagittal number-line*. Thinking about negative and positive integers induces spatial dispositions to move backward and forward, respectively. I argue that these dispositions constitute, in part, our mathematical habitus, dispositions to act and think that reflect and reproduce our conceptual systems.

Chapter 3 describes the recruitment of space not just for isolated numbers but for calculation. During mental arithmetic, participants' hand movements revealed systematic spatial biases, as if calculation involves shifts in spatial attention along a mental number-line. This occurred even when the calculation was exact and symbolic, rather than approximate, lending support to proposals that arithmetic co-opts parietal circuits for spatial attention.

Mathematics requires not only rote calculation *with* numbers, but meaningful and reflexive reasoning *about* numbers. Combining observation and experiment, Chapter 4 analyses spontaneous gestures produced during mathematical reasoning to argue that we conceptualize arithmetic, in part, using a system of complementary spatial metaphors.

Chapter 5 investigates the contribution of the communicative body to perpetuating and propagating this spatial understanding. A series of lab- and internet-based experiments demonstrate that co-speech gesture shapes and spreads the mental number-line, a process I call "gestural contagion." Together, Chapters 3 and 4 foreground the body as a nexus for the cultural reproduction of mathematics, both *disciplined by* and *disciplining* abstract thought.

In sum, this essay is a case study of the production and reproduction of a conceptual system, of the relation between agency and structure, and of the origins of abstraction.

Chapter 1

Introduction

"[Space] provides a location for all things that come into being. […] [E]verything that exists must of necessity be somewhere, in some place and occupying some space, and that that which doesn't exist somewhere, whether on earth or in heaven, doesn't exist at all." – Plato, Timaeus.

1.1. "The Body of the Condemned"[1]

On a Thursday afternoon in October 2010, a college student stands before a half-dozen of his peers, agonizing, condemned to generate a proof or fail before their watchful eyes. They had assembled weekly throughout the academic quarter to learn how to really *do* mathematics. These were no remedial students. A self-selected group of high-achievers, they were training for the William Lowell Putnam Competition, an annual mathematics competition for the best college students across North American. The competition consists of a dozen problems. The maximum score is one hundred twenty. The median score? Zero, often. But today there is to be success. The student at the blackboard defends a critical step in his argument by coordinating speech, sketches, equations, and body movements (Figure 1.1). He finishes, pauses. The professor nods. And thus the proof is accepted as valid within *this* community of practice.

This brief segment of situated activity illustrates a central feature of mathematical practice, and cultural practices more generally: the coordination of sundry resources—

[1] Cf. Foucault (1995, ch. 1).

resources that span brain, body, and sociotechnical world—brought together into a more-or-less stable assemblage. Let us zoom in on one particular facet: the student's recruitment of space to reason and communicate about abstract mathematical entities.

As he works to prove the assigned theorem[2], the student spatializes numbers in three observable ways: by creating blackboard inscriptions; by speaking about the relations between numbers; and by gesturing. We shall refer to these as "sites" of spatialization. Over the course of his argument, these sites come in and out of coordination, sometimes coupled, sometimes independent, both entwined and autonomous.

First: the inscription. In an effort to decide how two inequalities might relate to each other, he decides to "graph them because I want to see what I'm doing" (25:07).[3] He sketches a standard Cartesian coordinate system (visible in the first panel of Figure 1.1). This coordinate system is the material manifestation of a stable graphical norm in which numbers are spatialized along two axes: positive numbers located to the right and top, negative numbers to the left and bottom. Within this coordinate system, he draws two curves to represent the possible values for "delta" and "epsilon," variables in his proof. This transient, chalky artifact spatializes numbers by associating them with specific locations according to stringent graphical norms. As a result, relative magnitude is legible as relative location.

Once created, his diagram becomes a target for speech and gesture, two more sites of spatialization. With his right hand, he traces the graph of possible values of delta, moving his hand from bottom-left to top-right (Figure 1.1, b). He then steps back to trace a similar

---

[2] **Theorem:** Let x_i, $i = 1, 2, \ldots, n$, be real numbers which add up to 0 and whose sum of squares is 1. What is the maximum value of $x_1x_2 + x_2x_3 + \ldots + x_nx_1$? I leave the solution to the reader.

[3] Numbers in (parentheses) indicate the start time of the utterance.

arc through the air, now decoupled from the blackboard and produced on a larger scale (Figure 1.1, c). With his left hand, finally, he traces a trajectory that mirrors the shape of the "epsilon" graph, though at a distance from the graph itself (Figure 1.1, d). His blackboard-coupled gesture laminates the static inscriptions with a dynamic trajectory, while his decoupled gestures spatialize numbers in a vertical plane within his gesture space.

Simultaneously, he describes the variables or their graphs as if they exhibited movement:

(1) Deltas will <u>go</u> like this. Epsilons will <u>go</u> like this. And then they'll <u>re-cross</u> somewhere <u>up</u> here. (26:04)

The underlined words assign motion to numbers that are, technically, motionless—a linguistic phenomenon known as abstract or fictive motion (Langacker, 1987; Talmy, 2000; Matlock, 2010). His gestures, meanwhile, enact an oriented trajectory, traced through the air, and thus animate the graphs themselves when his gestures are coupled to the blackboard, and the numerical deltas and epsilons when his gestures are decoupled. Both gesture and speech prompt his interlocutors to construe the numbers and graphs as dynamic entities, imposing a fictive motion on static mathematical entities (Núñez, 2006; Marghetis & Núñez, 2013, Núñez & Marghetis, *in press*).

Slightly later, he describes both a number from the theorem and the slope of the graphed function with language typically reserved for physical size:

(2) And we said a_i is <u>bigger</u>, so the slope of the deltas is <u>bigger</u>. (26:35)

Neither a_i nor the slope of the function are the kinds of things that can be literally "bigger" than anything. Perhaps they are *numerically greater*, an abstract relation. But by using "bigger" to describe the numerical relations, he associates abstract numerical magnitude with concrete spatial volume. Soon after, he describes the point of equality between delta and epsilon as occurring "up here," when of course numbers do not literally inhabit particular locations.

By the end of this brief scene, larger numbers are *higher* and *rightward* on the blackboard, *upward* and *rightward* within gesture space, and *bigger* in speech, while both his inscriptions and the numbers themselves are *moving* and *crossing*. And yet the deltas, as numbers, are not going anywhere; they are a static set of possible numerical values, not dynamically moving entities. His graphs are static buildups of chalk; they sit, inert, on the surface of the blackboard. Numbers are not literally higher, lower, bigger, or smaller. But through the temporally and semiotically coordinated use of speech, gesture, and inscription, he evoked numbers that have size and location, sets and inscriptions that have dynamic trajectories through space.

The student, therefore, spatializes number simultaneously in a variety of interconnected sites—speech, gesture, blackboard—that build on and constrain each other. Speech and gesture add motion to the static spatialization of the graph. The graph initially constrains the trajectory of his initial gesture, which is "environmentally-coupled" to the blackboard (Goodwin, 2007). And when he describes the point of equality between delta and epsilon as occurring "up here," his speech is constrained by the spatialization of gesture and inscription. His mathematical success depends in part on his ability to deploy these varied semiotic resources in concert, using space to explore and express numerical relations.

S1 (25:51 - 26:07):

(1) If the slope of the deltas is greater than the slope of the / epsilons [at **this** point,**]**

(2) then / there will be / [**some** /]

(3) [/ ***Deltas will go like** this.

(4) [**Epsilons will go like this.** And then they'll re- /]

(5) [cross **somewhere up** here. /]

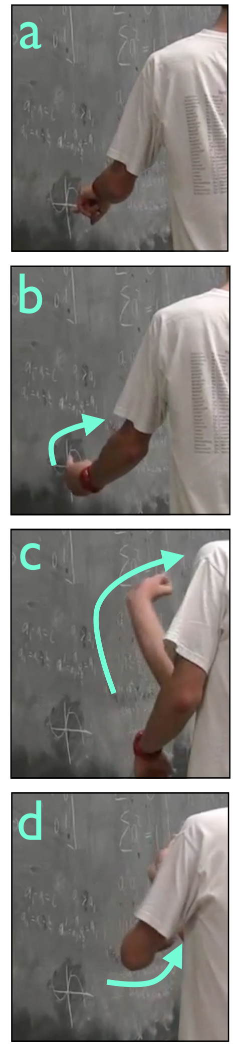


Figure 1.1. The situated activity of mathematics requires coordinating diverse resources, many of which spatialize number and arithmetic. The student first anchors his gesture to a graph of numerical relations (b), and then enacts the numerical relations on a larger scale, decoupled from the blackboard (c-d). In speech, / indicates a pause and * a self-interruption. Speech accompanying a gesture is enclosed in [square brackets], with the gesture stroke in bold and any holds underlined.

1.2. An ecosystem of arithmetic

"The dimension of time has been shattered; we cannot love or think except in fragments…." –Italo Calvino, "If on a Winter's Night a Traveler"

Throughout our mathematical lives, both expert and everyday, numbers and arithmetic are systematically and reliably coupled to aspects of space—that is, they are *spatialized*. An artifact, behavior, or process *spatializes number* whenever, implicitly or explicitly, it systematically associates some property of number—typically magnitude or order—with a property of space—typically extent or location. A number-line drawn on paper, for instance, spatializes number by associating numbers with locations, numerical order with spatial order, and numerical magnitude with the distance from the origin.

Spatialization recurs throughout the blackboard scene described above, and it is ubiquitous across diverse sites of signification. Numbers are spatialized in speech, where the language of space is systematically deployed to describe numerical relations (§1.1.1). They are also spatialized in artifacts and practices, where cultural norms regulate the use of space to represent number (§1.1.2). They are spatialized in spontaneous, communicative movements of the body, which recruit space in reliable ways (§1.1.3). And they are spatialized in the brain, where the processing of number and arithmetic is closely related to the processing of space, a link that manifests itself as spatial biases in rapid numerical behavior (§1.1.4).

As we shall see, the limits of this spatialization are yet to be fully understood—whether exact, symbolic calculation or advanced concepts like negative integers are spatialized in individual minds and brains (but see Chapters 2 and 3). Nor do we understand how various sites of spatialization come in and out of coupling with each other, and how this entwining of autonomous sites contributes to the reproduction of mathematics as a body of knowledge as well as an activity (but see Chapters 4 and 5). But there is now an extensive literature on the spatialization of number. In what follows, I briefly survey the evidence of spatialization in a variety of autonomous but interrelated sites: speech, things, practices, bodies, and brains. I argue that these diverse sites of spatialization are largely autonomous, relying on distinct mechanisms and operating on different timescales. But I also argue that they are so inextricably entwined as to constitute a stable assemblage within the cognitive ecosystem of arithmetic, a set of mutually constraining and sustaining cognitive resources that span brain, body, and sociotechnical world (cf., Hutchins, 2010).

1.2.1. First site of spatialization: Speech

English, like many languages, exhibits a systematic polysemy in which talk of numbers exploits a broad range of spatial language. This linguistic spatialization includes language typically used to express (i) locations in spatial frames of reference, (ii) topological relations like containment, and (iii) motion through space (cf., Levinson, 2003). Compare these pairs, in which the same linguistic constructions describe both space and number:[4]

(3) (a) Mount Everest is <u>higher</u> than Mount Logan. [spatial frame of reference]

(b) Two is <u>higher</u> than one. [numerical frame of reference]

(4) (a) The coffee is <u>in</u> the cup. [spatial containment]

(b) Five is <u>in</u> the interval between one and ten. [numerical containment]

(5) (a) The car is <u>on</u> the road from Tijuana to San Diego. [spatial contact]

---

[4] Throughout this section, <u>underlining</u> is used to highlight numerical uses of spatial language.

(b) The function is defined on the interval from one to ten. [numerical contact]

(6) (a) The bike route from Los Angeles to Santa Barbara goes past countless world-class beaches before reaching Santa Barbara. [fictive spatial motion]

(b) Counting from zero to one hundred goes past the first thirteen triangular numbers before reaching one hundred. [abstract or fictive numerical motion]

In (3b), for example, a construction that typically expresses a spatial relation is used to situate the magnitude of one number relative to the magnitude of another. The constructions in (4b) and (5b) typically express spatial containment and contact but here are used for numerical relations. And in (6b), a numerical process is construed as involving motion, so that numbers can be *passed* and *reached* as if they are landmarks along a trajectory (Langacker, 1987; Talmy, 2000; Matlock, 2010). Number talk, therefore, makes systematic use of a broad range of spatial language.

What kinds of spatial constructions are re-used, in particular, for numerical magnitude? The numerical senses of "bigger" and "smaller" associate numerical magnitude with spatial extent (area or volume), and this seems to be a reliable and productive association, across a wide variety of contexts:

(7) Charity "organizations now address a huge number of social and economic issues […]." (Huffington Post, "An Inaugural Shift," January 24, 2013)

(8) "There are 10^11 stars in the galaxy. That used to be a huge number. But it's only a hundred billion. It's less than the national deficit! We used to call them

astronomical numbers. Now we should call them economical numbers." (Richard P. Feynman)

In addition to spatial extent, English also associates numbers with spatial locations, such that numerical comparison can involve "higher," "lower," "highest," and "lowest" numbers. And various dynamic expressions describe numerical change in terms of spatial displacement, a form of fictive or abstract motion (Langacker, 1987; Talmy, 2000; Matlock, 2010). We can count *up* or *down,* depending on the *direction* of counting:

(9) Three is higher than two.

(10) My child can count up past one hundred; my parrot can count down from thirty to one.

(11) Even though the number [of jobs] is below the peak of 62,500 a decade ago, the industry has been rebounding from the deep recession of the past decade. (The Columbus Dispatch, "Columbus proves itself a powerful draw for financial companies," April 10, 2013)

Linguistic spatialization is often constrained by content, even when dealing with abstract referents that lack literal spatial dimensions:

(12) She made a big [?high] financial investment, of record size [*height].

(13) He has a higher [*bigger] score on the exam than me, but not the highest [*biggest] score in the class.

(14) The temperature kept going up [*getting bigger] until it reached a record high [?size].

We can quantify these distributional differences using the English corpus of Google N-grams, built from approximately 1,160,000 scanned books dating from 1600 to 2008 (Michel et al, 2011). Zooming in on the most recent decade covered by the corpus (1998-2008), we see that numbers are equally likely to be described in terms of height ("higher" or "lower") or volume ("bigger" or "smaller") ($t_{10} = -1.8$, $p = .09$). This was not true, however, across all quantitative domains (Figure 1.2). While an "investment" was twice as often described in volumetric rather than height-based terms ($t_{10} = -8.1$, $p << .001$), a score was forty-four times more often described in terms of height ($t_{10} = 19.8$, $p << .001$). The recruitment of size- or height-based language for number is constrained by the numbers' referents.

Contemporary English, therefore, uses two systems of spatial constructions to describe numerical magnitude: a volumetric system and a vertical location-based system (cf., Lakoff & Núñez, 2000). Moreover, this spatialization of numerical magnitude occurs within a much broader system of polysemy, involving language typically reserved for spatial frames of reference, topological relations, and motion. Notably, in (3) through (14), a non-spatial substitute would be impossible or highly marked.[5] In English, spatialization is nearly unavoidable when talking about numbers.

[5] Even the less obviously spatial *great* (e.g. "Five is greater than four.") is derived from the Old English *grēat*, meaning big.

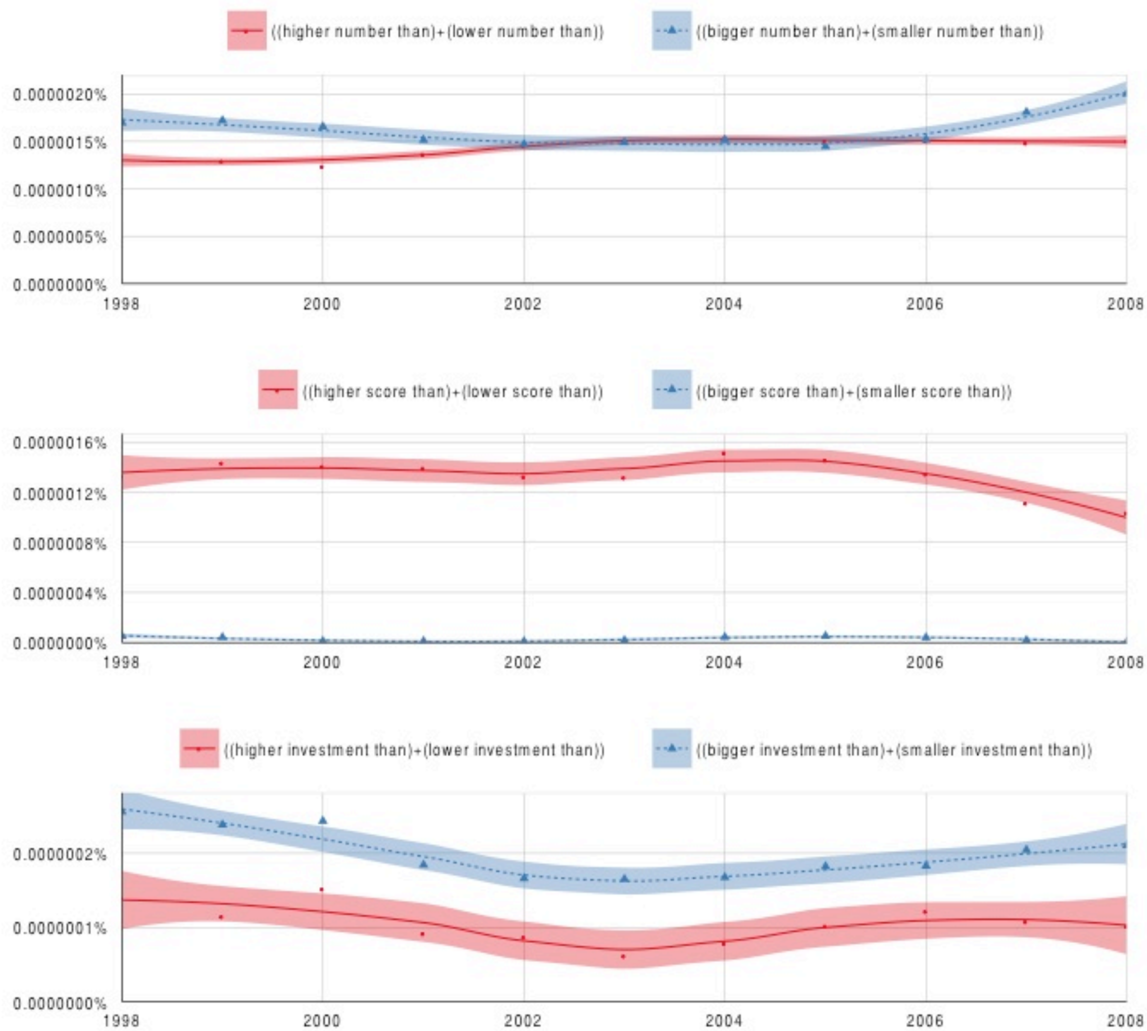


Figure 1.2. Relative frequency (vertical axis) of height- and size-based constructions over the last decade (horizontal axis). A number was equally likely to receive height- (i.e., higher, lower) or size-based (i.e., bigger, smaller) descriptions. But relative use of size- and height-based descriptions differed by quantitative domain. While "score" descriptions were almost entirely height-based, "investment" descriptions were most often size-based. Shaded regions indicate 95% confidence intervals.

1.2.2. Second site of spatialization: Artifacts and practices

Numerical magnitude is spatialized by a variety of cultural artifacts that embody stable graphical norms. Like in language, spatialization in artifacts often involves associating numerical magnitude with either spatial extent or location. The former is much more ancient. Ancient Babylonian diagrams associated numbers with spatial length and area (Høyrup, 2002; Robson, 2008); and coins, today and historically, have used smaller surface area for "smaller" denominations. Artifacts that map numbers to *locations,* by contrast, are a more recent arrival, entirely absent from the work of Descartes and unknown until the late 17th century (Núñez, 2010). Today such artifacts are ubiquitous. Elevator panels and calculators place linguistically "lower" and "higher" numbers in appropriately lower and higher locations[6]. The prototypical cultural artifact is probably the linear number-line, or its two-dimensional extension to the Cartesian plane, in which numbers are mapped to precise locations along the horizontal and the vertical axes.

Numbers are also mapped to locations by embodied practices for counting or calculating. Finger- and body-count routines associate numbers with locations relative to the body (Bender and Beller, 2012; Saxe, 2012; Wassmann & Dasen, 1994), while practices of reading and writing can associate numbers with the axis and direction of written language—left to right for European languages, right to left for Arabic, top right to bottom left for Mandarin Chinese in Taiwan. Cultural artifacts and practices, therefore, embody a set of cultural norms that systematically and reliably spatialize number.

---

[6] But note that this spatialization is inverted in most television remote controllers and telephone keypads, which arrange numerals from top-left to bottom-right, perhaps because these numerals have an arbitrary relation to their referent. Channel 4 is not numerically greater than Channel 3; phone numbers are not ordered by magnitude.

1.2.3. Third site of spatialization: Gesture

"What thrills and fascinates me about your body, then, is not any particular somatic feature in itself but the meaning that one or more of those features conveys to me." Halperin (2005:54)

When reasoning and communicating, we spontaneously move our bodies, especially our hands, in ways that are temporally and semantically coupled to speech and thought—that is, we gesture (Kendon, 2004; McNeil, 1992). Gesture is a powerful medium for spatializing the abstract (Cienki & Müller, 2008). The temporal gestures of native English speakers, for instance, associate past, future, present, and temporal sequences with locations along the sagittal (back-to-front) and transversal (left-to-right) axes. When talking about the future, native speakers of English will point forward or rightward; when talking about the past, backward or leftward (Cooperrider & Núñez, 2009; Casasanto & Jasmin, 2012).

A similar process of spatialization occurs for numbers. This often occurs when interacting with structure in the cultural world, such as graphs or other artifacts. Recall the first gesture in Figure 1.1, in which the student produced a gesture that traced the graph on the blackboard. In producing this environmentally-coupled gesture (Goodwin, 2007), the speaker literally embodies the spatialization encoded in the material structure, enacting overtly an otherwise implicit graphical norm. Other times, speakers will produce emblematic gestures to complement the spatialization in speech, accompanying the phrase "tiny number" with a "tiny" gesture, index finger and thumb pressed together (Winter, Perlman, & Winter, 2013). Gesture thus reproduces the spatialization found in artifacts and language.

But the gestural spatialization of number is far richer and more systematic, often occurring autonomously from other sites of spatialization (Núñez, 2006; Marghetis &

Núñez, 2013). When reasoning about numbers, for instance, speakers gesture spontaneously in ways that associate numbers with volumes or, alternatively, with locations along an imagined path. They can then build on this gestural spatialization to express more complex numerical relations and arithmetic operations. We return to these metaphorical gestures in Chapter 3, where we analyze them in detail and investigate the mechanisms responsible for their production.

1.2.4. Fourth site of spatialization: Brains and rapid behavior

In the 19th century, Galton (1880) described patients with number-space synesthesia, a syndrome in which numbers are systematically and reliably associated with locations in space to form ornate "number-forms." This mental and neural link between number and space is not restricted to exceptional individuals. During rapid judgments (e.g., deciding if 5 is odd), numerical magnitude interacts with both spatial extent and location, suggesting a cognitive and neural link between the two domains (Hubbard et al, 2005; Núñez & Marghetis, *in press*; Winter, Marghetis, and Matlock, 2015).

During rapid categorization of numbers (e.g. comparing magnitude or making odd/even judgments), behavioral responses often exhibit interactions between numerical magnitude and task-irrelevant spatial extent. Responses to numerals, for instance, are faster if their font sizes are congruent with their numerical magnitude (e.g. a small "2" and a larger "9") (Henik and Tzelgov, 1982; Pinel et al., 2004). This association is not limited two-dimensional area but appears also with one-dimensional length. When adults and children determine the midpoint of a line segment flanked by task-irrelevant dot arrays or numbers, judgments are biased toward the larger number or the dot array with the greater numerosity, as if the greater numerical quantity gave the impression of increased length on that side (de

Hevia and Spelke, 2008, 2009; Fischer, 2001). The association between spatial extent and numerical magnitude appears strikingly early in development—as early as a few hours of birth (de Hevia et al, 2014).

Numbers are also associated with egocentric *locations*. In the SNARC (Spatial-Numerical Association of Response Codes) effect, literate Western adults are faster to respond to lesser numbers in left space, but faster for greater numbers in right space (Dehaene et al, 1993). Similar effects have been found when feet are used in place of hands (Müller & Schwartz, 2007), and task-irrelevant numerals induce spatial biases in subsequent saccades (Fischer et al, 2003), ruling out the possibility that the SNARC effect is due merely to associations between numbers and specific effectors (e.g. between smaller numbers and the left hand). Numerical magnitude thus shapes subsequent spatial action, but spatial location and movement can also influence subsequent numerical decisions. During random number generation, higher numbers are more likely to be generated after upward movements or saccades, while lower numbers are more likely after downward movements or saccades (Hartmann, Grabherr, and Last, 2011; Loetscher et al, 2008). These associations between number and location, however, are highly variable across cultures and contexts, reversing in cultures that read from right-to-left (Shaki, Fischer, & Petrusic, 2009) and even changing rapidly as a result of number-space correlations in the local context (e.g., Fischer, Mills, and Shaki, 2010).

Similar spatial dispositions have been documented for numbers other than the simple counting numbers, though these results have been equivocal. Negative numbers, for instance, are sometimes associated with lateral locations as if located to the left of zero along an extended number-line, and other times processed just like their positive counterparts (i.e., their absolute value, |-3| = 3; Fischer, 2003; Ganor-Stern & Tzelgov, 2008; Ganor-Stern et al, 2010). And some authors have suggested that combinatorial operations like addition and subtraction may involve spatial processing (e.g., Hubbard et al, 2005; McCrink, Dehaene, and Dehaene-Lambertz, 2007), but this has not been tested directly (until recently; see Chapter 3). We return to these issues—the spatialization of advanced number concepts and of combinatorial operations—in the next two chapters.

These spatial dispositions during rapid numerical judgments are thought to reflect the recycling of, or close coupling with, neural systems specialized for processing space and action. Approximate numerical magnitude is processed within the intraparietal sulcus (IPS), in regions that overlap with or are adjacent to regions known to be responsible for controlling spatial attention and manual reaching (Hubbard et al, 2005). The neural representation of numerical magnitude, additionally, may tap into a domain-general representation of magnitude in posterior parietal cortex (Walsh, 2003; Winter, Marghetis, and Matlock, 2015). And mental arithmetic activates regions in the posterior superior parietal lobule that are responsible for orienting spatial attention—suggesting that these regions may be coopted to play a functional role in numerical calculation (Knops et al, 2009). Numbers, therefore, induce reliable spatial dispositions in behavior in virtue of the predictive computations of brain areas specialized for processing space and attention—and thus may acquire spatial content in much the same way as perceptual experience (Grush, 2007). The spatialization of numbers in rapid behavior, therefore, is the embodied manifestation of spatialization in the brain—that is the recycling of (or coupling with) neural systems specialized for processing spatial location and spatial extent.

1.3. Reproducing an assemblage of spatialization

Numbers are thus spatialized in diverse sites of conceptual activity: in speech, in cultural artifacts and practices, in spontaneous gesture, and in brain and rapid behavior. These sites associate numerical magnitude with spatial e*xtent* and with *location*, often preserving numerical properties such as order and the standard Euclidean metric (i.e., the "distance" between numbers n and m is $m - n$). The diversity of these sites of spatialization presents a mystery. Across speech, artifacts, practices, gesture, and thought, number and arithmetic are spatialized in similar ways, despite major differences between the media of spatialization: sound waves structured by language; objects and activities governed by cultural norms; hands guided by brain and world; and thought implemented by specialized neural circuitry. Why, then, are these spatializations so well aligned? How can we explain this unity within diversity?

This is a difficult question, a poorly studied one, and we can only give it passing attention here. But for now, two things: First, note that these sites of spatialization are largely *autonomous*, governed by their own laws, changing over different timescales, produced and reproduced by different mechanisms. The upshot of this is that the entire system must be studied ecologically, rather than treating "culture" or gestures as [only] the outward manifestations of skull-internal brain processing. But these sites of spatialization are also *entwined*, with bidirectional causal relations between sites, constraining and shaping each other on a variety of timescales. Many of these causal relations have yet to be studied empirically; Chapters 4 and 5 are steps in that direction. We address each of these points, briefly, below.

1.3.1. Autonomy of spatialization

The surface similarity between sites of spatialization—such as the reliable recruitment of extent and location—makes it easy to talk of *the* spatial representation of number. Indeed, a number of authors have suggested a common and privileged origin of all spatialization, typically situated within the brain. Conceptual Metaphor Theory, for instance, argues that various spatial regularities—in speech, behavior, and "culture"—are all manifestations of implicit "conceptual metaphors" in which the source domain of space is mapped to the target domain of number (Lakoff and Johnson, 1980; Lakoff, 2010). These cross-domain mappings are proposed to inhere in the brains of individuals (Lakoff, 2012), with diverse spatializations of number mere "surface manifestations" of neural mappings.

A similarly brain-centric account has been defended by a number of cognitive neuroscientists, who have argued that diverse spatializations have their origin in the "inherent spatial attributes" of number as processed in the brain (e.g., Treccani & Umilta, 2011). This spatial representation of number in the brain has been invoked to explain spatializations ranging from simple numerical reasoning to the existence of geometrical diagrams in ancient Babylonia (Izard et al, 2008). Some have gone even further, explaining cross-cultural regularities by arguing that the neural spatialization of number is innate (Dehaene et al, 2008; de Hevia et al, 2014; i.a.).

These approaches argue that diverse spatial phenomena—from linguistic polysemy to visual artifacts—are surface manifestations of a common spatialization in the brain. Lacking, however, has been explicit discussion of the precise mechanisms that would allow neural associations to shape, in one fell swoop, cultural practices, graphical norms, conventionalized metaphorical language, spontaneous gesture, and all the rest.

The sites of spatialization reviewed above are, in fact, governed by their own laws, change over different timescales, and are produced and reproduced by different mechanisms. In a word, they're autonomous. For instance, while there are similarities in the ways that language and rapid behavior associate numbers with spatial extent and location, the behavioral association between numbers and *horizontal* locations is entirely absent from speech. Greater numbers may be rightward during rapid categorization (e.g., Dehaene et al., 1993), but in language they're resolutely "larger" or "higher," never "rightward." And while gesture and artifacts often assign numbers to horizontal locations, co-produced language is limited to vertical locations (cf., Figure 1.1).

Even when different sites of spatialization have the possibility of alignment—they both associate numbers with length, for instance—they may not be aligned in practice. Recall that the student described numbers as "bigger" but used gesture to place them in locations along vertical and horizontal axis (Figure 1.1). The speaker *could* have gestured as if numbers had spatial extent, perhaps using a two-handed collecting gesture. But he didn't. Similarly, in using a finger-count routine, I might describe myself as "counting up" a pile of objects, even though standard North American count-routines spatialize numbers in horizontal or finger-based coordinates, not vertical.

Finally, various forms of spatialization differ in stability and in the timescale over which they emerge. Low-level behavioral associations between number and extent have been observed in human neonates just a few hours after birth (de Hevia et al, 2014), and may thus be innate and highly stable. Behavioral associations between number and *location*, by contrast, are highly flexible and sensitive to context and culture (Fischer, Mills, & Shaki, 2010; Núñez, 2011; Shaki, Fischer, and Petrusic, 2009; i.a.). Gestural spatialization is similarly flexible, sometimes expressing multiple, complementary construals during a single gesture unit, as we shall see in Chapter 4. The graphical norms that govern spatialization in artifacts, meanwhile, are stabilized within a culture by their material instantiations; classroom number-lines and elevator panels place numbers in canonical locations that are subject to strict conventions. And while speech is flexible and sensitive to context, the set of possible linguistic resources available to speech—words, constructions, etc.—is very stable and changes slowly over years, generations, or centuries. In English, for example, the numerical sense of "higher" is nearly one thousand years ago. The Oxford English Dictionary cites a non-spatial use of "higher" in an Old English homily dated to before 1225, in which a greater reward was described as a *heahere mede*, a "higher reward" (Higher, 2015). Sites of spatialization, therefore, operate on entirely different timescales: the numerical senses of "higher" and "lower" are stable across centuries; material artifacts preserve graphical norms for generations; individual spatial dispositions can shift within minutes.

Language, cultural artifacts, gesture, and behavior thus differ in the spatializations they make *available,* the spatializations they *actualize* at any particular time, and the *stability* of spatialization. This is because each site relies on distinct mechanisms. Spatialization in language is the product of various forces that drive semantic change (e.g., Traugott & Dasher, 2001). Once spatializing constructions emerge in a language, individuals within the linguistic community acquire them over the course of development—not because each individual is inventing them anew on the basis of their own individual conceptual metaphors, but because they are part of the lexicon. Material artifacts, meanwhile, are the products of conventionalized manufacturing processes. Gestures reflect a combination of cultural norms (Kita, 2009), imagistic processing in the brain (McNeill, 1992; Hostetter & Alibali, 2008), co-produced speech (Kita and Özyürek, 2003), and the surrounding environment (Goodwin, 2007). And behavioral spatializations are the product of spatializations in the brain, themselves the confluence of innate constraints and cultural experience (e.g., Hubbard et al, 2005; de Hevia et al, 2014). The upshot of these distinct mechanisms is that each site is relatively autonomous.

And yet different sites of spatialization *do* exhibit commonalities, a fact that requires explanation. How, then, to account for the undeniable coordination between the various sites of spatialization that co-exist within the cognitive ecosystem of arithmetic?

1.3.2. Entwined spatialization

If sites of spatialization are so autonomous, how then can we account for the coordinated spatialization of number across the cognitive ecosystem of arithmetic? This is a specific instance of an old puzzle. Leibniz (1696), for instance, imagines two clocks that exhibit impressive synchronization. “"Imagine two clocks or watches in perfect agreement as to the time. This may occur in one of three ways. The first consists in mutual influence; the second is to appoint a skillful workman to correct them and synchronize them at all times; the third is to construct these clocks with such art and precision that one can be assured of their subsequent agreement” (p. 548; quoted in Bourdieu, 1977, p. 80). Leibniz’s second solution is the one adopted by Conceptual Metaphor Theory and by cognitive neuroscientists who invoke a mechanism in the brain that is responsible for coordinating all instances of spatialization (even Babylonian diagrams!). His third solution is the one adopted by those who think that the spatialization of number is innate; no matter our experience, we are finely attuned from birth to the kinship of number and space.

But Leibniz's first solution suggests an account of our spatializing assemblages that decenters the individual and looks instead to mutual causality to explain the production and reproduction of coordination (see also Hutchins, 1995, 2010b; Latour, 2007). It is exactly this kind of causal entwining that maintains alignment and complementarity within the distributed assemblages that accomplish arithmetic. We saw some of this in the scene that began this chapter. The student's speech, gesture, and inscriptions were sometimes autonomous but often interacting. Any assemblage of spatializations is going to be dense with causal interrelations. Activating putative source domains can prime the production of related metaphorical language (Sato, Schafer, and Bergen, 2015); we suspect that a similar link exists between internal spatial processing and the production of spatial language for number. Inscriptions can constrain environmentally-coupled gestures—recall the scene in Figure 1.1—while spatializations that begin in gesture are sometimes materialized as inscriptions. Representational gestures reflect imagistic or spatial processing within the gesturer's brain (McNeill, 1992; Kita & Özyürek; Hostetter & Alibali, 2008; Marghetis & Bergen, 2014), and metaphorical gestures have been proposed to reflect the internal simulation of the source domain—a proposal that we test in Chapter 4. Conversely, gestures are known to have a causal impact on the internal mental processing of both the gesturer and any interlocutors, and in Chapter 5 we test whether this applies also to gesturers that systematically spatialized number and arithmetic. Within a situated assemblage of spatializations, diverse sites are brought into relations of alignment and complementarity through bidirectional, circular causal influences.

The moral of Leibniz's story is that coordination across space and time does not require a centralized locus of control. What makes these various phenomena hang together so nicely, therefore, is not a single, shared mechanism—an innate overlap between number and space (Walsh, 2003), or a neutrally-instantiated cross-domain mapping (Lakoff, 2012). Individual brains cannot be solely and uniquely responsible for shaping gesture, systematizing polysemy, maintaining graphical norms, and motivating rapid behavioral interactions between space and number. Instead, the unity of our cognitive ecosystems is a conspiracy (cf., Elman, 1999)—a conspiracy without plan or purpose—an emergent outcome of interdependent mechanisms in brain, body, and sociotechnical world. On this account, our explanatory target expands to include an entire assemblage of interacting and mutually constraining phenomena: gesture, polysemy, behavior, artifacts, practices. Our task, then, is to identify the mechanisms at work in each site of spatialization and the causal links between them.

1.4. Outline of the dissertation

> *"Of course, in one sense, mathematics is a branch of knowledge, but still it is also an activity." –Wittgenstein (1953/2009, p. 227)*

As both a branch of knowledge and an activity, mathematics is done not by brains inside individual skulls but by dynamic, distributed assemblages that are constantly produced and reproduced within larger cognitive ecosystems. The studies reported in this dissertation are a contribution to the study of the assemblage that accomplishes arithmetic, and in particular its spatialization of number and arithmetic. They address a series of puzzles: Is spatialization limited to the simplest of numerical activities? Does it extend beyond isolated numbers to combinatorial operations with numbers? How is arithmetic spatialized during reflexive, precise reasoning? How does spatialization in gesture relate to spatialization within the skull? And how does the spatialization of number perpetuate and propagate within communities, aligned between individuals and stabilized over time within an individual?

We respond to these puzzles over four substantive chapters. The first two investigate the spatialization of numerical concepts and capacities that are more complex than simply comparing whole numbers. Chapter 2 investigates the spatialization of an advanced numerical concept: the positive and negative integers. As we shall see, over five experiments, numerical magnitude is systematically associated with the *sagittal* axis, back-to-front. In Chapter 3, we demonstrate the spatialization not only of individual numbers but of arithmetic operations during exact, symbolic calculation. During mental arithmetic, calculation itself—addition or subtraction—induces subtle but reliable spatial biases. Calculating the solution to, e.g., 4 + 3 biases accompanying movements *rightward*, while calculating 6 – 2 biases motion *leftward*. Both chapters contribute to evidence that numerical reckoning—comparing and calculating—recycles neural systems that are specialized for processing action and space.

The next two chapters look at how schematized spatial relations are deployed during arithmetic reasoning, and explore how these are propagated and perpetuated within a community. Mathematics requires not only rote calculation *with* numbers, but meaningful and reflexive reasoning *about* numbers. Combining observation and experiment, Chapter 4 analyses spontaneous gestures produced during mathematical reasoning to argue that we conceptualize arithmetic, in part, using spatial metaphors. We document two systems of spatialization in gesture, and then investigate the proximal mechanisms that drive the production of these metaphorical gestures. Chapter 5 investigates the contribution of the communicative body to perpetuating and propagating this spatial understanding. A series of lab- and internet-based experiments demonstrate that co-speech gesture shapes and spreads the mental number-line, a process we dub "gestural contagion." Together, Chapters 4 and 5 foreground the body as a nexus for the cultural reproduction of mathematics, both *disciplined by* and *disciplining* abstract thought.

The focus throughout is on number and arithmetic. But my hope is that the studies assembled in this dissertation illustrate a more general phenomenon: the regimentation of abstract thought. I suspect the mechanisms that regiment mathematical thinking—including spatialization in speech, gesture, and thought—might account for the regimentation of belief more generally, whether religious, social, political, or scientific (cf., Bourdieu, 1977; Foucault, 1977; Marx, 1867/1976; Wittgenstein, 1953/2009). And even if readers think this dissertation does no such thing, at least by the end they'll know a tiny bit more about numerical cognition.

1.5. References

Bender, A., & Beller, S. (2012). Nature and culture of finger counting. *Cognition*, 124, 156-182.

Bourdieu, P. (1977). *Outline of a Theory of Practice*. Cambridge: Cambridge University Press.

Casasanto, D. & Jasmin, K. (2012). The Hands of Time: Temporal gestures in English speakers.*Cognitive Linguistics*, 23, 643 – 674.

Cienki, A. & Müller, C. (2008). Metaphor, gesture, and thought. In R. Gibbs (ed.), *The Cambridge Handbook of Metaphor and Thought* (pp. 483–501). Cambridge: Cambridge University Press.

Cooperrider, K., & Núñez, R. (2009). Across time, across the body: Transversal temporal gestures. *Gesture*, 9, 181-206.

de Hevia, M. D., Girelli, L., Bricolo, E., & Vallar, G. (2008). The representational space of numerical magnitude: Illusions of length. *The Quarterly Journal of Experimental Psychology*, 61, 1496-1514.

de Hevia M. D., Spelke E. S. (2009). Spontaneous mapping of number and space in adults and young children. Cognition 110, 198–207

de Hevia, M. D., Izard, V., Coubart, A., Spelke, E. S., & Streri, A. (2014). Representations of space, time, and number in neonates. *Proceedings of the National Academy of Sciences*, *111*(13), 4809-4813.

Dehaene, S., Bossini, S., & Giraux, P. (1993). The mental representation of parity and numerical magnitude. *Journal of Experimental Psychology: General*, 122, 371-396.

Dehaene, S., Izard, V., Spelke, E., & Pica, P. (2008). Log or linear? Distinct intuitions of the number scale in Western and Amazonian indigene cultures. S*cience*, 320, 1217-1220.

Elman, J.L. (1999). The emergence of language: A conspiracy theory. In B. MacWhinney (ed.), *The Emergence of Language* (pp. 1–28). Mahwah, NJ: Erlbaum.

Fischer, M. H. (2001). Number processing induces spatial performance biases. *Neurology*, 57, 822-826.

Fischer, M.H. (2003). Cognitive representation of negative numbers. *Psychological Science*, 14, 278-282.

Fischer, M. H., Castel, A. D., Dodd, M.D., & Pratt, J. (2003). Perceiving numbers causes spatial shifts of attention. *Nature Neuroscience,* 6, 555–556.

Fischer, M.H., Mills, R.A., & Shaki, S. (2010). How to cook a SNARC: Number placement in text rapidly changes spatial-numerical associations, *Brain and Cognition, 72,* 333-336.

Foucault, M. (1977). *Discipline and Punish: The Birth of the Prison.* New York: Vintage.

Galton, F., (1880). Visualised numerals. *Nature,* 21, 252–256.

Ganor-Stern, D., Pinhas, M., Kallai, A. & Tzelgov, J. (2010). Holistic representation of negative numbers is formed when needed for the task. *The Quarterly Journal of Experimental Psychology*, 63, 1969-1981.

Ganor-Stern, D. & Tzelgov, J. (2008). Negative numbers are generated in the mind. *Experimental Psychology*, *55*, 157.

Goodwin, C. (2007). Environmentally coupled gestures. In S. Duncan, J. Cassell, & E. Levy (Eds.), *Gesture and the Dynamic Dimensions of Language* (pp. 195–212). Philadelphia: John Benjamins.

Grush, R. (2007). Skill theory v2.0: Dispositions, emulation, and spatial perception. *Synthese*, 159, 389-416.

Hartmann, M., Grabherr, L., & Last, F. W. (2011). Moving along the mental number line: Interactions between whole-body motion and numerical cognition. *Journal of Experimental Psychology: Human Perception and Performance,* 38, 1416-1427

Hostetter, A. B., & Alibali, M. W. (2008). Visible embodiment: Gestures as simulated action. *Psychonomic Bulletin and Review*, 15, 495-514.

Henik, A., & Tzelgov, J. (1982). Is three greater than five: The relation between physical and semantic size in comparison tasks. *Memory & Cognition*, 10, 389–395.

Høyrup, J. (2002). *Lengths, widths, surfaces: a portrait of Old Babylonian algebra and its kin*. Springer Science & Business Media.

Hubbard, E.M., Piazza, M., Pinel, P., & Dehaene, S. (2005). Interactions between number and space in parietal cortex. *Nature Reviews Neuroscience* 6, 435–448.

Hutchins, E. (1995). *Cognition in the Wild.* Cambridge, MA: MIT Press.

Hutchins, E. (2010a). Cognitive ecology. *Topics in Cognitive Science*, 2, 705-715.

Hutchins, E. (2010b). Enculturating the supersized mind. *Philosophical Studies*, 152, 437-446.

Izard, V., Dehaene, S., Pica, P., & Spelke, E. (2008). Reading between the number lines (response). *Science*, 321, 1293-1294.

Kendon, A. (2004). *Gesture: Visible Action as Utterance*. Cambridge University Press.

Kita, S. (2009). Cross-cultural variation of speech-accompanying gesture: A review. *Language and Cognitive Processes*, 24, 145-167.

Kita, S., & Özyürek, A. (2003). What does cross-linguistic variation in semantic coordination of speech and gesture reveal? Evidence for an interface representation of spatial thinking and speaking. *Journal of Memory and language*, 48, 16-32.

Knops, A., Thirion, B., Hubbard, E. M., Michel, V., & Dehaene, S. (2009). Recruitment of an area involved in eye movements during mental arithmetic. *Science, 324*, 1583–1585.

Lakoff, G. (1993). The contemporary theory of metaphor. In Andrew Ortony (ed.), *Metaphor and Thought* (2nd edition). Cambridge: Cambridge University Press.

Lakoff, G. (2012). Explaining embodied cognition results. *Topics in Cognitive Science*, 4, 773-785.

Lakoff, G. & Johnson, M. (1980). Metaphors We Live By. University of Chicago Press.

Lakoff, G. & Núñez, R. (2000). Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being. Basic Books.

Langacker, R. W. (1987). *Foundations of Cognitive Grammar, Volume 1: Theoretical Prerequisites*. Stanford, CA: Stanford University Press.

Latour, B. (2007). *Reassembling the Social: An Introduction to Actor-Network-Theory*. Oxford University Press.

Leibniz, G. W. (1696/1866) *Second eclaircissement du systeme de la communication des substances*. In P. Janet (ed.), *Oeuvres philosophiques, vol 11*. Paris: de Lagrange.

Levinson, S. C. (2003). *Space in language and cognition: Explorations in cognitive diversity*. Cambridge University Press.

Loetscher, T., Schwarz, U., Schubiger, M., & Brugger, P. (2008). Head turns bias the brain's random number generator. *Current Biology*. 18, R60–R62

Marghetis, T., & Núñez, R. (2013). The motion behind the symbols: A vital role for dynamism in the conceptualization of limits and continuity in expert mathematics. *Topics in Cognitive Science, 5,* 299-316.

Matlock, T. (2010). Abstract motion is no longer abstract. *Language and Cognition*, 2, 243-260.

Marx, K. (1976). *Capital, Volume 1*. London: Penguin Books. (Original work published 1867)

McNeill, D. (1992). *Hand and Mind: What Gestures Reveal About Thought*. Chicago: Chicago University Press.

McCrink, K., Dehaene, S., & Dehaene-Lambertz, G. (2007). Moving along the number line: Operational momentum in nonsymbolic arithmetic. *Perception and Psychophysics, 69*, 1324-1333.

Michel, J. B., Shen, Y. K., Aiden, A. P., Veres, A., Gray, M. K., Pickett, J. P., ... & Aiden, E. L. (2011). Quantitative analysis of culture using millions of digitized books. *Science*, 331, 176-182.

Núñez, R. (2006). Do Real Numbers Really Move? Language, Thought, and Gesture: The Embodied Cognitive Foundations of Mathematics. Reprinted in R. Hersh (ed.), *18 Unconventional Essays on the Nature of Mathematics* (pp. 160-181). New York: Springer.

Núñez, R. (2011). No Innate Number Line in the Human Brain. *Journal of Cross-Cultural Psychology*, 45, 651-668.

Núñez, R., & Marghetis, T. (in press). Cognitive Linguistics and the Concept(s) of Number. In R. Cohen-Kadosh and K. Dowker (eds.), *Oxford Handbook of Numerical Cognition*. Oxford University Press.

Pinel, P., Piazza, M., Le Bihan, D., & Dehaene, S. (2004). Distributed and overlapping cerebral representations of number, size, and luminance during comparative judgements. *Neuron*, 41, 1-20.

Robson, E. (2008). *Mathematics in Ancient Irak*. Princeton, NJ: Princeton University Press.

Sato, M., Schafer, A. J., & Bergen, B. K. (2015). Metaphor priming in sentence production: Concrete pictures affect abstract language production. *Acta Psychologica*, 156, 136-142.

Saxe, G. B. (2012). *Cultural Development of Mathematical Ideas: Papua New Guinea Studies*. New York: Cambridge University Press.

Shaki, S., Fischer, M., & Petrusic W. (2009). Reading habits for both words and numbers contribute to the SNARC effect. *Psychonomic Bulletin & Review,* 16, 328-31.

Schwarz, W., & Müller, D. (2006) Spatial associations in number-related tasks: A comparison of manual and pedal responses. *Experimental Psychology*, 53, 4–15.

Talmy, L. (2000). *Towards a cognitive semantics, Volume I: Conceptual structuring systems*. Cambridge, MA: MIT Press.

Treccani, B., & Umilta, C., (2011). How to cook a SNARC? Space may be the critical ingredient, after all. *Brain and Cognition,* 75, 310-315.

Traugott, E. C., & Dasher, R. B. (2001). *Regularity in Semantic Change*. Cambridge University Press.

Walsh, V. (2003). A theory of magnitude: common cortical metrics of time, space and quantity. *TRENDS in Cognitive Sciences*, 7, 483-488.

Wassmann, J., & Dasen, P. R. (1994). Yupno number system and counting. *Journal of Cross-Cultural Psychology*, *25*, 78-94.

Winter, B., Marghetis, T, & Matlock, T. (2015). Of metaphors and magnitudes: Explaining cognitive interactions between space, time, and number. *Cortex,* 64, 209-224.

Wittgenstein, L (2009). *Philosophical Investigations*. (G. E. M. Anscombe, trans.). Wiley-Blackwell. (Original work published 1953)

Chapter 2

Pierced by the number-line: Integers induce embodied dispositions to move forward and backward

ABSTRACT

How does the human mind grasp entirely abstract concepts, such as time or number? One general strategy for dealing with abstraction is to ground our understanding in more concrete domains such as space and action. The positive whole numbers, for instance, are conceptualized as vertical and horizontal number-lines. What of more sophisticated concepts, such as negative integers? We investigated the possibility that negative and positive integers are associated with the back-to-front sagittal axis in the human mind. In four experiments, participants categorized numbers and responded by stepping forward and backward. We demonstrate a novel effect: a *sagittal number-line,* in which negative and positive integers induce spatial dispositions to move backward and forward, respectively (Exp. 1-4). These spatial dispositions appear to require the juxtaposition of both negative and positive integers (Exp. 1-3), reflect a systematic relation between the integers and space rather than a general categorical strategy (Exp. 3), and occur automatically (Exp. 4). These systematic spatial dispositions may constitute our mathematical habitus, habits of action and thought that reveal and enact our conceptual systems.

2.1. Introduction

While we share a limited numerical toolbox with other animals, formal education equips us with a suite of mathematical concepts and abilities that go far beyond this cognitive inheritance. Prominent among these are complex, abstract number concepts: negative, rational, irrational, imaginary. These concepts—shaped and elaborated by an ecosystem of axioms, notations, applications, and historical demands—outstrip the representations available to non-human animals, (Carey, 2009; Lakoff & Núñez, 2000; Núñez & Marghetis, *forthcoming*).

Despite their precision and abstractness—or perhaps *because* of these qualities—our number concepts are interwoven with more concrete practices, habits, and concepts (Barsalou, 1999; Kitcher, 1984; Lakoff & Núñez, 2000). This is exemplified by the link, within the human mind, between representations of number and those of space (for reviews, see Hubbard et al, 2005; Winter, Marghetis & Matlock, 2015). In particular, processing numbers induces a variety of systematic dispositions to act and think spatially. A now-classic finding is the "SNARC" effect: Literate Western adults are faster to respond to smaller numbers on the left, and faster to respond to larger numbers on the right, as if they represent the positive whole numbers along a left-to-right mental number-line (Dehaene et al, 1993). Negative integers, too, induce horizontal spatial dispositions, although these seem more fragile; task-demands influence whether negative numbers are located to the "left" of zero, in virtue of their relative magnitude, or mixed in with positive integers on the basis of their absolute value (e.g. Fischer, 2003; Ganor-Stern & Tzelgov, 2008; Ganor-Stern et al, 2010). In addition to the horizontal number-line, there are spatial-numerical associations along the vertical axis (Schwartz & Keuss, 2004; Hartmann, Grabherr, & Mass, 2012;

Holmes and Lourenco, 2012; Loetscher et al, 2010). There are even spatial dispositions associated with more complex tasks, including symbolic calculation (Knops et al, 2009; Marghetis et al, 2014) and algebraic manipulation (Goldstone, Landy, & Son, 2010). This system of spatial dispositions suggests that abstract mathematical concepts may build, in part, on evolutionarily older neural resources specialized for perception, action, and space (Barsalou, 1999; Dehaene & Cohen, 2007; Hubbard et al, 2005; Lakoff & Núñez, 2000).

Given the human mind's promiscuous spatialization of number, we are surprised that so little is known about spatial-numerical dispositions along an especially prominent axis: the sagittal axis, running through the body, back to front. The sagittal axis *is* associated with another abstract domain: Time (see Núñez & Cooperrider, 2013, for a review). In English, talking about time often involves language typically reserved for the sagittal spatial axis. We look *forward* to the future, think *back* to the past. Co-speech gestures reproduce this linguistic pattern (Casasanto & Jasmin, 2012), and temporal reasoning induces dispositions to move forward or backward (e.g., Miles, Nind, & Macrae, 2010). Language, gesture, and thought all enact a sagittal timeline.

What about number? In contemporary English, decreasing or increasing counting is most often described spatially as counting *down* or *up*, but also, though less frequently, as counting *backward* or *forward*. Historically, when mathematicians first grappled with the novel and perplexing concept of a negative integer, some invoked explicitly an analogy between integer arithmetic and forward and backward motion (Wallis, 1685: 265; see discussion in Fauconnier & Turner, 1998, Núñez, 2011). One might expect, therefore, that the sagittal axis would supply a natural model for representing the positive and negative integers, perhaps with the greater, positive integers associated with the space in front, and the lesser, negative integers associated with the space behind.

And yet there is little evidence that the contemporary adult mind ever represents number along the sagittal axis, as a back-to-front *sagittal number line* (SNL). Some suggestive evidence of an SNL comes from research on the so-called "vertical" SNARC, which often uses responses in near and far locations in front of the body rather than genuinely vertical low and high locations (e.g. Ito & Hatta, 2004; Gevers et al., 2006; Müller & Schwarz, 2007; Shaki & Fischer, 2012). The results of these "vertical" SNARC studies, therefore, might more accurately support an association with near-to-far *radial* space, since responses to smaller numbers are faster in near space, and responses to bigger numbers, in far space (Shaki & Fischer, 2012). This radial SNARC, however, has always been tested with both response buttons located in *front* of the body, thus confounding *location along the sagittal axis through the body* with *distance from the body*. Larger numbers may just prime responses that are farther away, regardless of axis.

A few studies have tried, explicitly, to find evidence of an unambiguously sagittal representation of number, with little success. In Hartmann et al (2012), Swiss participants generated significantly higher random numbers (1-30) while experiencing upward (*vs.* downward) or rightward (*vs.* leftward) motion; by contrast, there was no difference between forward and backward motion. Seno et al (2011), by contrast, reported that Japanese and Chinese participants generated significantly higher numbers while experiencing a visual illusion of *backward* movement, compared to an illusion of forward motion—a reversal of the way numbers are described in English, where counting *backward* involves smaller numbers, not larger. These results suggest that a sagittal representation of the positive whole numbers, if it exists, may be fragile or task-dependent.

Upon reflection, however, the sagittal axis is an unlikely model of the positive numbers: the body divides space categorically; positive numbers have no natural point at which they are divided, since they start at zero and continue indefinitely. The integers, on the other hand, share many structural properties with the sagittal axis: positive and negative are separated by zero; front and back, by one's body. Unlike the positive numbers, negative integers do not induce reliable spatial dispositions along the horizontal axis (e.g., Fischer, 2003; Ganor-Stern & Tzelgov, 2008; Ganor-Stern et al, 2010), suggesting that we use other resources to ground our understanding of negative numbers. Might we use the sagittal axis to conceptualize the positive and negative integers?

2.1.1. Current Study

Four experiments attempted to identify and characterize a back-to-front sagittal number-line for the integers. In each experiment, participants judged the magnitude (Exp. 1-3) or parity (Exp. 4) of visually-presented numbers and responded by stepping forward or backward in space, thus moving along the sagittal axis. The critical question was whether participants would spontaneously associate numbers with movement along this sagittal axis, as we would expect if participants were deploying an implicit SNL.

We foresaw a number of possible outcomes. First, there could be no systematic association between numbers and sagittal space; the spatialization of number might be exhausted by known horizontal, vertical, and radial dispositions. Second, the positive whole numbers and integers alike might prime forward and backward motion, perhaps due to so-called Polarity Correspondence. Since numbers and the sagittal axis are both dimensions with a clear orientation—a marked and an unmarked "pole"—processing both dimensions might be facilitated when their "polarity" is aligned (Proctor & Cho, 2006). Third, we might replicate the apparently "reversed" SNL reported by Seno et al (2011) for Chinese and Japanese adults. Finally, if there is a selective association between integers and sagittal space, rather than domain-general polarity correspondence, then we may find evidence of selective, systematic spatial dispositions: forward for positive numbers, backward for negative numbers, with zero mapped to the body.

In what follows, we begin by asking whether number processing induces dispositions to move forward or backward along the sagittal axis (i.e. a SNL), for either *positive whole numbers* or the *positive and negative integers* (Exp. 1a, 1b, 3). We then ask whether sagittal dispositions arise when processing negative integers alone, in isolation from the rest of the integers (Exp. 2). Finally, we investigate whether the SNL is automatic, emerging when the task does not require explicit processing of numerical magnitude (Exp. 4). We end by discussing the SNL in the context of other spatial dispositions, speculating on its origins, and exploring its relation to mathematical practice more generally.

2.2. Experiment 1a

2.2.1. Participants

Undergraduate students from UC San Diego (n = 32, M_{age} = 21, 22 women) participated in exchange for partial course credit. All procedures were approved by the ethics review board of UC San Diego. Sample size was determined in advance on the basis of previous experiments that used whole-body movements to study spatial-numerical biases (e.g. *n* = 24 and 36 in Hartmann et al, 2012).[1]

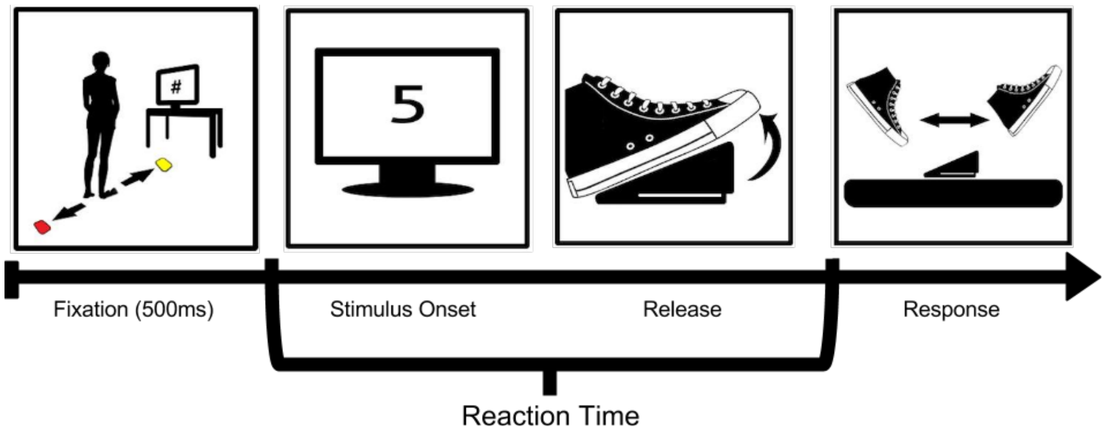


Figure 2.1. Procedure for Experiments 1-4. Participants made judgments on the basis of visually-presented numerals and responded by stepping forward toward yellow or backward toward red targets on the floor. Trials began with the pound sign (first panel). Reaction time was measured from stimulus onset (second panel) to foot-pedal release (third panel). Participants responded by moving forward or backward (fourth panel).

2.2.2. Design

In a fully within-subjects design, participants judged the relative magnitude of visually-presented single-digit numbers and responded by stepping forward or backward onto colored targets on the floor (Fig. 2.1, left panel). Targets were approximately two feet in front of (yellow) or behind (red) a central foot-pedal and were described to participants by their color, not location.

Participants completed relative magnitude judgments for two kinds of number: positive whole numbers (*Positive Only)* and positive and negative integers (*Integer)*. In the

[1] In what follows, "we report how we determined all sample sizes, all data exclusions, all manipulations, and all measures" (Simmons et al, 2012).

Integer condition, participants judged whether positive and negative integers from -9 to 9 (not including 0) were greater or less than 0. In the *Positive Only* condition, they judged whether whole numbers from 1 to 9 (not including 5) were greater or less than 5. The *Positive Only* condition was thus modeled after the classic SNARC paradigm (Dehaene et al, 1993), but with sagittal movements in place of lateralized buttons.

Participants completed two blocks for each number type. In one block, participants moved *forward* for *greater* numbers (i.e. greater than 0 or 5), and *backward* for *lesser* numbers; response assignment reversed in the other block; block order was assigned randomly. Number Type (*Positive Only* vs. *Integer*), response direction (*forward* vs. *backward*), and numerical magnitude (*greater* vs. *less than* the target) were fully crossed within-subjects. Number Type order (e.g. *Integer* first, *Positive Only* second) was counterbalanced between-subjects.

Trials began with an image of a shoe in the center of a computer monitor, located approximately four feet ahead of the participant, which prompted participants to depress and hold down a foot-pedal with their right foot. Once the foot-pedal was depressed, the pound sign ("#") appeared for 500ms, followed by a single-digit numeral. The numeral disappeared when participants lifted their foot to begin their response, or after 5000ms. Reaction time was measured, via the foot-pedal, from stimulus onset to the release of the foot pedal (Fig. 2.1). Participants were instructed to begin moving only after they had made their decision; trials were discarded in which participants changed direction after initiating their response. Response direction was recorded online by an experimenter in the room. Blocks began with 8 practice trials. In the *Positive Only* condition, each of the 8 possible numbers (1 to 9, excluding 5) were presented 10 times per block, in random order, for a total of 160 experimental trials over two blocks. In the *Integer* condition, each of the 18 possible numbers (-9 to 9, excluding 0) were presented 5 times per block, in random order, for a total of 180 experimental trials over two blocks. Participants were allowed to rest between number conditions. The entire experiment took approximately 30 minutes.

2.2.3. Results and Discussion

Four participants did not complete both tasks and were replaced before analysis. Overall accuracy was quite high (M = .97). Before analyzing response times, incorrect trials were removed, followed by trials with reaction times that were slower than three standard deviations above each participant's mean response time in each condition (<1% of trials). In addition, one participant was removed due to exceptionally low accuracy, answering only 73% of the trials correctly (accuracy > .92 for all other participants). Individual accuracy was not significantly correlated with reaction times, ruling out a speed-accuracy trade-off, t_{29} = 1.9, $p > .05$.

To analyze reaction times, we conducted a 2x2x2x2 mixed-design ANOVA, with Magnitude (*greater* vs. *less than*), Direction (*forward* vs. *backward*), and Number Type (*Integer* vs. *Positive Only*) as within-subjects factors, and Order (*Integer-first* vs. *Only-Positive-first*) as a between-subjects factor. Participants were faster overall in the *Integer* than the *Positive Only* conditions (M_{integer} = 441ms *vs.* $M_{\text{only-positive}}$= 466ms), $F_{(1,29)} = 4.2$, $p < .05$, $\eta_p^2 = .13$, perhaps because, in the Integer condition, the presence or absence of a minus sign made it easy to determine a number's magnitude relative to zero. Participants were also faster to respond to smaller than to larger numbers ($M_{<}$ = 448ms *vs.* $M_{>}$ = 458ms), $F_{(1,29)} = 12.0$, $p = .002$, $\eta_p^2 = .29$. And there was a marginally significant effect of direction, with faster responses backward (M_{backward} = 448ms *vs.* M_{forward} = 458ms), $F_{(1,29)} = 4.1$, $p = .053$, $\eta_p^2 = .12$.

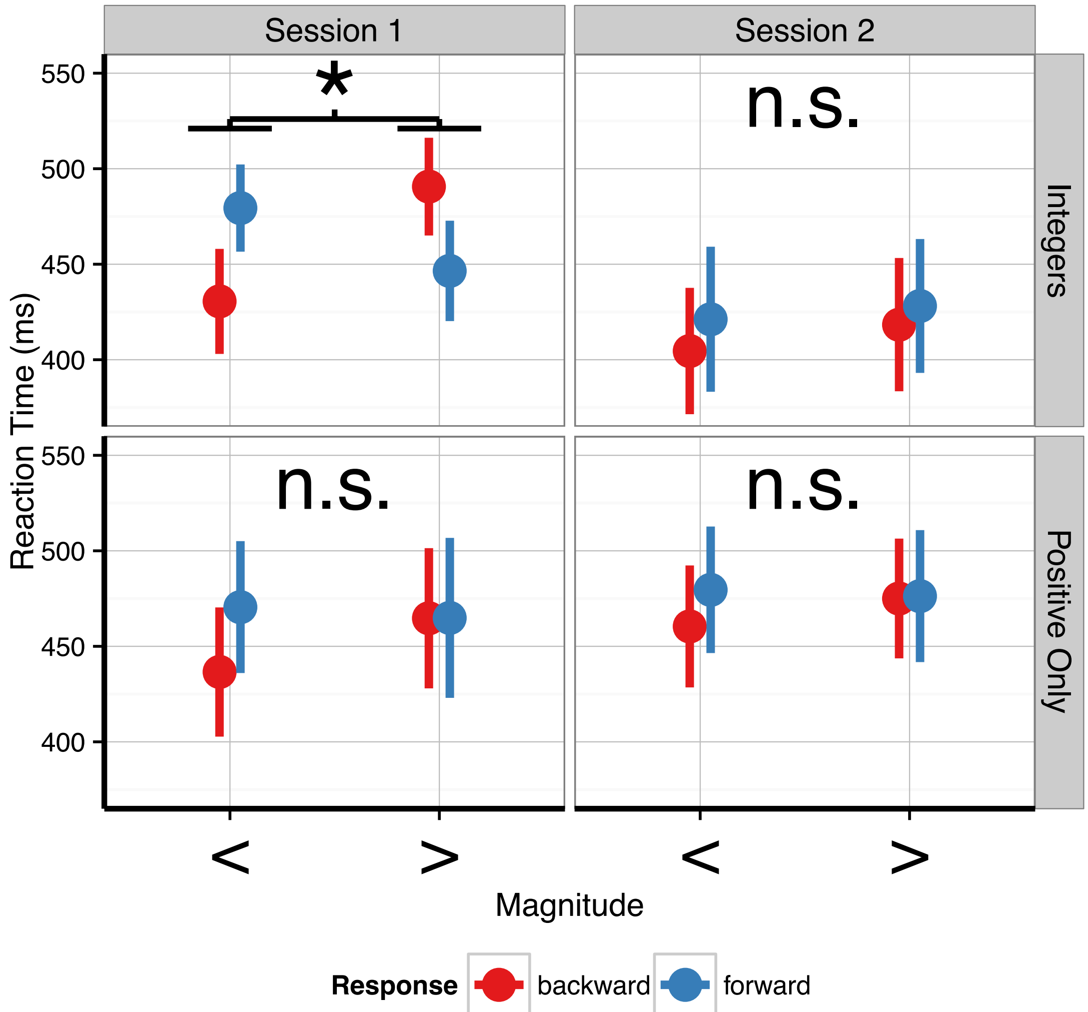


Figure 2.2. In Experiment 1, the relation between numerical magnitude and response direction differed by number condition (rows) and order (columns). When participants completed the Integer condition first (top left), there was a highly significant interaction between number and direction (p < .01). In no other case did response direction interact with numerical magnitude (ps > .3).

Only two other effects approached significance. There was a two-way interaction between Magnitude and Direction, $F_{(1,29)} = 4.4$, $p = .046$, $\eta_p^2 = .13$. Backward responses were faster for smaller than for larger numbers (M = 433ms *vs.* 462ms), while forward responses were faster for larger than for smaller numbers (M = 453ms *vs.* 463ms). This interaction, however, was complicated by an interaction between all four factors $F_{(1,29)} = 4.7$, $p = .039$, $\eta_p^2 = .14$. It was easy to see why: the two-way interaction between Magnitude and Direction was driven entirely by the Integer condition, and only when the Integer condition was completed first (Fig. 2.2). This was confirmed by four follow-up 2 (Magnitude) x 2 (Direction) ANOVAs, performed for each Number Type and Order. When the Integer condition was completed first, the two-way interaction between Magnitude and Direction was highly significant ($F_{(1,15)} = 9.9$, $p < 0.007$, $\eta_p^2 = .40$), with responses to negative numbers an average of 49ms faster when moving backward than forward ($t_{(15)} = -3.0$, $p = 0.01$) but responses to positive numbers an average of 44ms faster when moving forward ($t_{(15)} = -2.5$, $p = 0.02$; see Fig. 2.1a). By contrast, the interaction did not approach significance when the Integer condition was completed second, or for the Positive Only condition ever (all Fs < 1.1, ps > .3).

To further characterize this selective association between Magnitude and Direction, we performed a regression analysis of reaction times (Fig. 2.3), adapting an approach developed for the classic horizontal SNARC (Fias et al, 1996). Focusing on reaction times in the Integer condition, when it was performed first, we calculated, for each subject and number, the difference between backward and forward median response times (dRT). If dRT is positive, then backward responses were faster than forward responses for that number and subject; if dRT is negative, forward responses were faster. Then, for each subject, we regressed dRT onto numerical magnitude. The slopes of these linear regression lines index the orientation and intensity of the association: negative slopes indicate that smaller numbers are associated with the back, larger numbers with the front. As predicted, regression slopes were significantly less than zero ($M_\beta = -7.7$, $t_{(15)} = -3.4$, $p = .002$, one-tailed), and most participants had a negative slope (13/16, p = 0.02, binomial test). Participants in the Integer condition, therefore, associated negative numbers with the space behind them, positive numbers with the space ahead. By comparison, in the Positive Only condition, when it was completed first, regression slopes were not significantly less than zero (M_β = -4.7, $t_{(14)}$ = -0.8, p = .22, one-tailed). See Figure 2.3.

Experiment 1a thus demonstrated a novel effect, a *sagittal number line (SNL)*, in which negative numbers are associated with the space behind the body and positive numbers with the space in front. This effect, moreover, was restricted to the Integer condition; there was no evidence of systematic spatial dispositions during the Positive Only condition, which did not involve negative numbers. Note, in fact, that the stimuli in the Positive Only condition (1 to 9) were identical to the *greater* half of the stimuli in the Integer condition (i.e. integers greater than 0). The exact same numbers, therefore, were strongly associated with front space when they were processed in the context of negative numbers (Integer condition; Fig. 2.2, top-left) but had no associations—perhaps even a slight association with the rear—when processed on their own during the Positive Only condition (Fig. 2.2, bottom). The SNL, therefore, may be limited to contexts that juxtapose positive and negative integers.

The SNL in Experiment 1a was restricted to those participants who completed the Integer condition first. We suspect this was due to the length of the task. It took up to 45 minutes to complete all four blocks of trials, perhaps resulting in automatized and routine responses by the end of the experimental session. Our undergraduate student participants, it seems, no longer exhibit number-space associations after 45 minutes of this aerobic exercise, repeatedly stepping forwards and backwards. In all subsequent experiments, therefore, we manipulated the number condition *between* subjects, allowing us to limit time-on-task to less than half-an-hour.

It is possible, however, that this novel SNL effect may have been driven by demand characteristics. Given the spatial nature of the full-body response, participants may have inferred that the experiment was investigating associations between number and space, guessed the predicted direction of the association (i.e. negative-back, positive-front), and behaved accordingly. To rule out this deflationary account, Experiment 1b replicated directly the critical finding of Experiment 1a—an SNL for the positive and negative integers—but only in participants who were unaware of the experiment's purpose.

2.3. Experiment 1b

2.3.1. Participants

Undergraduate students from UC San Diego (n = 37, M_{age} = 21, 28 women), who did not participate in any of the other experiments, participated in exchange for partial course credit. Sample size was determined based on our expectation, from informal debriefing, that approximately half of participants would guess the experiment's purpose; we aimed for at least as many naive participants as there were participants who completed the Integer condition first in Exp. 1a (n = 16).

2.3.2. Design and Procedure

The design and procedure was identical to Experiment 1a, except participants completed only the Integer condition, followed by a funnel debrief questionnaire. Participants responded to the following questions, ordered by specificity:

1. "What do you think was the purpose of this experiment?"

2. "This experiment was about the mental representation of integers. What aspect of the mental representation of integers do you think we were investigating?"
3. "This experiment is investigating the existence of a "mental number line" for positive and negative integers. During the experiment, did you guess that this was the purpose of the experiment?"

We only included those participants who did not guess the experiment's purpose and made absolutely no mention of space or a number-line. (See Appendix A for the full questionnaire.) This stringent exclusion criteria left a subsample of 18 naive participants.

2.3.3. Results and Discussion

Overall, the results confirmed the findings of Experiment 1a. Accuracy was high ($M = .96$) and no participants were removed for low accuracy (all > .90). Before analyzing response times, incorrect trials were removed, followed by trials with reaction times that were slower than three standard deviations above each participant's mean response time in each condition (< 1% of trials). Individual accuracy was not significantly correlated with reaction times, ruling out a speed-accuracy trade-off, $t_{35} = 1.9$, $p > .05$.

We analyzed reaction times with a 2x2x2 mixed-design ANOVA, with Magnitude (*greater* vs. *less than* 0) and Direction (*forward* vs. *backward*) as within-subjects factors, and Debrief (*guessed* or *did not guess* the experiment's purpose) as a between-subjects factor. There was a main effect of Magnitude, $F_{(1,35)} = 12.7$, $p = .001$, $\eta_p^2 = .27$, with faster responses for numbers less than zero ($M_{<0} = 460$ms vs. $M_{<0} = 470$ms). The only other effect that approached significance was the predicted two-way interaction between Magnitude and Direction, $F_{(1,35)} = 4.2$, $p < .05$, $\eta_p^2 = .11$. Critically, this was unaffected by whether participants had inferred the experiment's purpose, $F_{(1,35)} = 0.45$, $p > .50$. To further characterize the SNL among naive participants, we conducted the same regression analysis from Experiment 1a (Fig. 2.3). Overall, slopes were significantly less than zero ($M_{\beta} = -3.35$, $t_{36} = -1.8$, $p = .039$, one-tailed), indicating a back-to-front SNL. Critically, this was true even among the subsample of participants who were naive to the experiment's purpose ($M_{\beta} = -4.9$, $t_{36} = -1.8$, $p = .049$, one-tailed).

In summary, participants associated positive and negative integers with the space in front and behind the body, respectively, even when they were naive to the experiment's purpose, thus ruling out an impact of demand characteristics. Experiment 1, therefore, suggests that participants represent the integers, positive and negative, along a sagittal number-line (SNL); there were no systematic spatial dispositions for positive numbers alone (Exp. 1a). The SNL, therefore, appears to reflect implicit and selective associations between sagittal space and the integers.

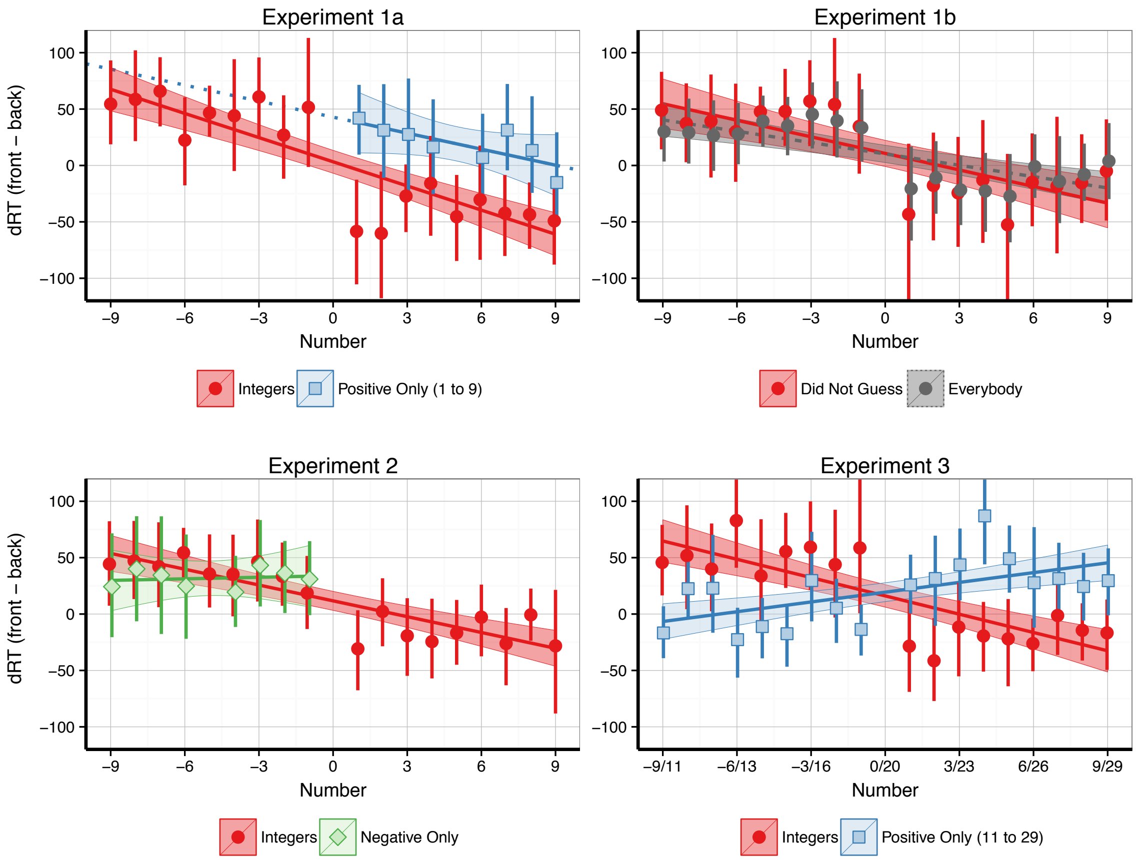


Figure 2.3. The sagittal number-line (SNL) when processing positive integers alone, negative integers alone, or both together ("Integer" condition). In the first session of Experiment 1a (top-left), there was a back-to-front SNL (i.e. negative regression slope) for Integers (red circles) but for positive numbers alone (blue squares). Experiment 1b (top-right) replicated this SNL among naïve participants (top-right). Experiment 2 (bottom-left) again found a back-to-front SNL for Integers (red circles) but not negative numbers alone (green diamonds). Experiment 3 compared the integers to ranged-matched positive numbers, and again found a back-to-front SNL in the Integer condition alone. Error lines and shaded regions indicate bootstrapped 95% confidence intervals.

2.4. Experiment 2

Experiments 2 and 3 were designed to rule out deflationary accounts of the back-to-front SNL. First, Experiment 2 addressed the possibility that the interaction between number and space in the *Integer* condition may have been due to the mere presence of negative integers rather than to a systematic association between integers and sagittal space (i.e. negative numbers with the space behind, positive numbers with the space in front). Perhaps the sagittal axis is deployed as a representational tool *whenever* negative numbers are involved, even when encountered on their own, isolated from the positive integers or zero. After all, the negative integers are less familiar than the positive integers, encountered later in school and less often in daily life, and might thus demand an increased reliance on more analog, spatial resources. Comparing -1 to -5, for instance, might be such an unfamiliar or difficult task that participants grasp at whatever representational resources are available within the experimental context—including sagittal movement. On this account, any task that involves negative numbers should induce sagittal dispositions; for instance, if negative integers were compared to -5, then perhaps numbers less than -5 would be associated with the back, and those greater than -5 with the front. Alternatively, if the SNL reflects the structural alignment of the integers—both positive and negative—and the sagittal axis, then there should be a stable association of negative numbers with rear space, positive numbers with front space, and zero with the body.

2.4.1. Participants

Undergraduate students from UC San Diego (n = 32, M_{age} = 21, 28 women), who did not participate in any of the other experiments, participated in exchange for partial course credit. Sample size was determined in advance, following Exp. 1a (n = 32).

2.4.2. Design and Procedure

Design and procedure were identical to Experiment 1a, except the *Positive Only* condition was replaced with an *Negative Only* condition, in which the stimuli ranged from -9 to -1 (instead of 1 to 9) and numbers were compared to -5 (instead of 5). The *Negative Only* condition, therefore, was matched to the *Positive Only* condition of Experiment 1a, except the task was performed with negative instead of positive numbers. Number Type (*Integer* vs. *Negative Only*) was counterbalanced between-subjects.

2.4.3. Results and Discussion

Overall accuracy was quite high (M = .96). Before analyzing response times, incorrect trials were removed, followed by trials with reaction times that were slower than three standard deviations above each participant's mean response time in each condition (<1% of trials). In addition, one participant was removed due to exceptionally low accuracy, answering only 65% of the trials correctly (accuracy $\geq$ .89 for all other participants). Individual accuracy was negatively correlated with reaction time, $t_{29} = 2.6$, $p = .014$, with reaction times faster on correct trials, ruling out a speed-accuracy trade-off.

Reaction times were analyzed with a 2 (Magnitude) x 2 (Direction) x 2 (Number Type) mixed-design ANOVA. Responses were faster once again for lesser compared to greater numbers, $F_{(1,29)} = 12.9$, $p = 0.001$, $\eta_p^2 = .31$. The only other significant effect was an interaction between Magnitude and Number Type, $F_{(1,29)} = 13.7$, $p = 0.001$, $\eta_p^2 = .32$, presumably driven by the different numerical stimuli in the *Positive Only* and *Integer* conditions. Against our predictions, the two-way interaction between Magnitude and Direction was not significant, $F_{(1,29)} = 1.6$, $p = 0.22$, nor was the three-way interaction with Number Type, $F_{(1,29)} = 1.6$, $p = 0.21$.

By contrast, the regression analysis of responses in both Number Type conditions *did* reveal spontaneous associations between integers and sagittal space in the *Integer* condition. For the *Integer* condition, as predicted, slopes were significantly less than zero ($\beta = -4.6$, $t_{16} =$

-2.25, p = .019, one-tailed), while slopes in the *Negative Only* condition were not significantly less than zero (β = 0.9, $t_{(15)}$ = 0.16, p = .56, one-tailed). Prompted by this regression analysis, we performed separate follow-up ANOVAs of both Number Type conditions. This confirmed the results of the regression analysis: Participants in the *Integer* condition did, indeed, associate numerical magnitude with response direction, as revealed by a significant interaction between Magnitude and Direction, $F_{(1,15)}$ = 4.9, p = 0.49, η_p^2 = .23, while those in the *Negative Only* condition did not, $F_{(1,14)}$ = 0.002, p > 0.9, η_p^2 < .001.

In summary, the mere presence of negative integers was insufficient to induce an association between numerical magnitude and sagittal space. Only when the task involved both positive and negative integers did participants associate numbers with the back-to-front sagittal axis, faster to respond backwards for negative than for positive integers (M = 448ms *vs.* 488ms), but faster to respond forwards for positive than for negative integers (M = 472ms *vs.* 488ms). The SNL, therefore, seems to require the juxtaposition of both negative and positive integers, so that participants can map the structure of the sagittal axis (origin at the body, space behind the body, and space in front of the body) to the structure of the integers (origin at zero, numbers less than zero, and numbers greater than zero).

2.5. Experiment 3

Experiment 3 addressed two remaining concerns with the results of Experiments 1 and 2. First, the numerical ranges of the *Positive Only* (Exp. 1) and *Negative Only* (Exp. 2) stimuli were less than the range of the *Integer* stimuli (from ±1 to ±9, *range* = 8 vs. -9 to +9, *range* = 18). Perhaps this increased range, and not the involvement of negative numbers, was responsible for the difference between the Integer and Positive Only conditions. Second, participants could succeed in the *Integer* condition simply by checking for the presence of a minus sign (e.g. -4 vs. 4), while both the *Positive Only* and *Negative Only* conditions required access to the magnitude represented by the numeral. This categorical strategy in the *Integer* condition may have induced associations between space and magnitude based solely on the dimensions' "polarity" (cf. Proctor & Cho, 2006).

To address these concerns, Experiment 3 modified the Positive Only condition so that the stimuli ranged from 11 to 29, judged relative to 20. These numbers have the same range as those in the Integer condition (i.e. 18), and their relative magnitude can be determined from the first digit alone (e.g. 11 *vs.* 21). If the interaction in Experiments 1 and 2 was an artifact of a categorical strategy or the larger numerical range, then we should find an interaction between magnitude and response direction in this modified *Positive Only* condition. If, on the other hand, the interaction reflected selective spatial dispositions for positive and negative integers—an *SNL*—then we should find the effect only when the task involves negative integers.

2.5.1. Participants

Undergraduate students from UC San Diego (n = 32, M_{age} = 21, 22 women), who did not participate in any of the other experiments, participated in exchange for partial course credit. Sample size was determined in advance, following Exp. 1a (n = 32).

2.5.2. Design and Procedure

In a between-subjects design, participants made magnitude judgments of either positive and negative integers (-9 to 9, *Integer*) or positive whole numbers (11 to 29, *Positive Only*). The procedure in the Integer condition was identical to Experiment 1. The *Positive Only* condition was modified in two ways: stimuli ranged from 11 to 29 (instead of 1 to 9) and numbers were compared to 20 (instead of 5). The tasks were therefore matched in two ways. First, participants could complete either task by attending only to the most leftward symbol ("1" or "2"; presence or absence of a negative sign), thus matching the tasks on the availability of a categorical strategy. Second, the stimuli had an identical range.

2.5.3. Results and Discussion

Overall accuracy was quite high (M = .96), and no participants were removed for poor accuracy. Before analyzing reaction times, incorrect trials were removed, followed by trials with reaction times that were slower than three standard deviations above each participant's mean response time in each condition (5% of trials). Individual accuracy was not significantly correlated with reaction times, , $t_{30} = 0.1$, $p > .9$, ruling out a speed-accuracy trade-off.

Reaction times were analyzed with a 2 (Magnitude) x 2 (Direction) x 2 (Number Type) mixed-design ANOVA. Once again, there was a main effect of Direction: backward responses (M = 441ms) were faster than forward responses (M = 458ms), $F_{(1,30)} = 21.3$, $p < .001$, $\eta_p^2 = .42$. The only other significant effect was a three-way interaction between Number Type, Magnitude, and Direction, $F_{(1,30)} = 9.72$, $p = 0.004$, $\eta_p^2 = .24$. Follow-up analyses of each Number Type, using 2 (Magnitude) x 2 (Direction) repeated-measures ANOVAs, revealed that this was driven by a significant two-way interaction in the *Integer* condition, $F_{(1,15)} = 5.82$, $p = .029$, $\eta_p^2 = .28$, which approached but did not reach significance in the *Positive Only* condition, $F_{(1,15)} = 4.45$, $p = .052$. When moving backwards in the *Integer* condition, participants were significantly faster to respond to negative than to positive integers (M = 442ms *vs.* 404ms), $t_{(15)} = -2.357$ $p = 0.02$; by contrast, when moving forwards they were faster to respond to positive than to negative integers (M = 422ms *vs.* 458ms), $t_{(15)}$ = 2.19, p = .04.

As in Experiment 1, we performed a regression analysis to further characterize the association between numerical magnitude and sagittal space (Fig. 2.3). For the *Integer* condition, as predicted, slopes were significantly less than zero (β = -5.4, $t_{(15)}$ = -2.26, p = .019, one-tailed), indicating a back-to-front SNL. In the *Positive Only* condition, by contrast, slopes were not significantly less than zero (β = 2.9, $t_{(15)}$ = 2.1, p = .97, one-tailed).

Experiment 3 thus replicated the main finding of Experiment 1—a back-to-front SNL—and ruled out a number of deflationary accounts of the difference between the Positive Only and Integer conditions. Since stimuli in both Number Type conditions had an identical range (-9 to 9 and 11 to 29), it is unlikely that the difference between the tasks was drive, for instance, by the Distance Effect. And since participants could complete both conditions by attending only to the leading symbol, it is unlikely that the effect in the Integer condition was due solely to a categorical strategy or Polarity Correspondence (Proctor & Cho, 2006).

In fact, the mean regression slope in the *Positive Only* condition, while not significantly different from zero ($t_{(15)}$ = 2.1, p > .05), was unexpectedly positive rather than negative. Polarity Correspondence (Proctor & Cho, 2006) actually predicts the opposite effect (i.e. negative slopes), since frontward responses and larger numbers are both unmarked. Larger numbers should therefore facilitate forward responses; we found the opposite effect (see Fig. 2.3). The positive slope in the *Positive Only* condition, on the other hand, is consistent with the results of Seno and colleagues (2011), who found, among Japanese and Chinese participants, an association between forward motion and smaller whole numbers, and backwards motion and larger whole numbers. Future research will need to determine whether there is a small but real association between smaller positive numbers and forwards motion.

In summary, Experiment 3 again replicated the finding that the integers are associated with the back-to-front sagittal axis—the SNL. When participants responded to matched positive numbers, by contrast, there was no evidence of a back-to-front SNL—if anything, there was a suggestion of front-to-back spatial dispositions. The back-to-front SNL for the integers, therefore, is not an artifact of a categorical strategy or of a larger numerical range.

2.6. Experiment 4

The classic SNARC effect is thought to reflect *automatic* activation of spatial information during number processing. Associations between number and horizontal space appear even if the task does not require participants to process numerical magnitude—for instance, deciding if numbers are even or odd (Dehaene et al, 1993). Is the SNL similarly automatic, or does it require explicit attention to magnitude?

To answer this question, Experiment 4 compared magnitude judgments, which require attention to numerical magnitude, with *parity* judgments (even *vs.* odd), which do not. If the SNL is automatic—activated implicitly whenever processing integers—then numerical magnitude should interact with spatial location during both Magnitude (*greater* vs. *less than* 0) and Parity (*even* vs. *odd*) tasks.

2.6.1. Participants

Undergraduate students from UC San Diego (n = 32, M_{age} = 20, 26 women), who did not participate in any of the other experiments, participated in exchange for partial course credit. Sample size was determined in advance, following Exp. 1a (n = 32).

2.6.2. Design and Procedure

Participants completed one of two tasks, with task assignment counterbalanced between-subjects. The *Magnitude* task was identical to the *Integer* condition in Experiments 1a, 1b, 2, and 3, and involved judging the magnitude of integers from -9 to 9, relative to 0. In the *Parity* task, participants determined the parity (even vs. odd) of integers from -9 to 9, not including zero. All other details of the design (number of trials, timing, etc.) were identical to the Magnitude task.

2.6.3. Results and Discussion

Accuracy was high and did not differ between tasks (both M = .96, $t_{(15)}$ = 0.07, p = .9). Once again, incorrect trials were removed, followed by trials with reaction times more than three standard deviations from the participant's mean response time in each condition (< 1% of trials). Individual accuracy was negatively correlated with reaction time, t_{29} = 2.3, p = .03, with reaction times faster on correct trials, ruling out a speed-accuracy trade-off.

To analyze reaction times, we conducted a 2x2x2 mixed-design ANOVA, with Magnitude and Direction as within-subjects factors and Task (*Magnitude* vs. *Parity*) as a between-subjects factor. There was again a significant main effect of Direction, $F_{(1,30)}$ = 10.2, p = .003, η_p^2 = .25, with backward responses 13.9ms faster than forward responses. The only other significant effect was the predicted two-way interaction between Magnitude and

Direction, $F_{(1,30)}$ = 4.66, p = .039, η_p^2 = .13. Backward responses were faster for negative integers ($M_{<0}$ = 475ms *vs.* $M_{>0}$ = 489ms), while forward responses were faster for positive integers ($M_{>0}$ = 504ms *vs.* $M_{<0}$ = 489ms). Crucially, there was no three-way interaction with Task, $F_{(1,30)}$ = 1.76, p = .20. Indeed, separate regression analyses for each task found that the mean slope was less than zero in both tasks (Magnitude: β = -3.5; Parity: β = -1.0), these slopes did not differ from each other (t_{30} = -1.2, p = .25), and the number of participants with negative slopes did not differ between tasks (12/16 *vs.* 11/16, p > .9, Fisher's Exact). Separate analyses of the slopes for each Task, however, revealed that they were only significantly less than zero for the Magnitude task (Magnitude: t_{15} = -1.8, p = .04; Parity: t_{15} = -1.0, p = .17; one-tailed). Processing negative and positive integers, therefore, prompts spatial dispositions along the sagittal axis, although explicit magnitude processing may amplify these dispositions.

2.7. General Discussion

Are we pierced by a number-line that runs through our bodies, back-to-front? Four experiments suggest that the answer is *yes*. We established—and then replicated four times—a novel effect: Negative numbers are associated with the space behind the body and positive numbers with the space in front—a back-to-front sagittal number-line (SNL). This reliable disposition suggests a systematic link between the integers and space, with the integers mapped, back-to-front, with the sagittal axis, divided by the body at zero. These sagittal spatial dispositions, moreover, were specific to the integers as a whole. We failed to find evidence of a back-to-front SNL for positive (Exp. 1a, 3) or negative (Exp. 2) numbers in isolation, ruling out the possibility that the experimental setup was sufficient to prompt an association between stimuli and sagittal locations. The SNL appeared even when the task did not require explicit processing of numerical magnitude, although the effect was largest when the task required explicit processing (Exp. 4). These results suggest that the SNL is systematic, specific, and automatic.

There was also a recurring main effect of stepping direction, with consistently faster responses *backward*. This was likely due to the experimental set-up (e.g. the angle of the foot-pedal), though we cannot exclude the possibility that American undergraduate students have an innate or acquired aversion to numbers, prompting them to recoil backwards in terror.

We turn now to a number of outstanding issues. First, how might these spatial dispositions relate to other spatial-numerical associations? Second, where might such dispositions come from, and what impact might they have? Third, how might the SNL relate to mathematical activity more generally, with its motley practices, striking precision, singular abstraction?

2.7.1. Relating the SNL to other spatial dispositions

A few studies have investigated previously whether negative integers induce spatial dispositions, but only along the horizontal axis. The results have been mixed, with negative integers sometimes associated with locations to the left of zero along an extended left-to-right mental number-line (Fischer, 2003), other times associated with space on the basis of their absolute rather than relative value (e.g. 9 and -9 both associated with the right, since $|-9| = 9$; Ganor-Stern & Tzelgov, 2008). These horizontal dispositions appear to depend on the experimental task (Ganor-Stern et al, 2010). As far as we know, nobody has investigated whether negative integers, like their positive counterparts (e.g. Schwartz & Keuss, 2004), induce vertical dispositions. However, a variety of cultural artifacts map the positive and negative integers to a vertical axis; Cartesian graphs and cold-climate thermometers are the most obvious examples. We suspect these conventions are enough to induce, in the human mind, implicit associations between the negative integers and lower space.

These multiple spatial dispositions for number—horizontal, vertical, and now sagittal—evoke the multiple spatial construals that have been documented for the domain of Time (e.g. Núñez & Cooperrider, 2013). In "external" spatial representations of time, the spatial axis does not include the body—such as when temporal sequences are conceptualized as a left-to-right path in front of the speaker. All previously documented mental number lines are of this type; they involve paths outside the body, whether left-to-right, right-to-left, or bottom-to-top. In "internal" spatial representations, by contrast, the body is part of the representation. Time, for instance, can be conceptualized as running from back to front, with "now" co-located with the body. Unlike previous number-lines, the SNL appears to be of this type, with the body mapped to zero and thus dividing positive from negative. Núñez and Cooperrider (2103) suggest that, for time, external representations may require extensive cultural scaffolding, while internal representations may appear more spontaneously. On this point, number likely differs from time. Even an internal representation like the SNL is likely to require extensive cultural support, at least for the initial acquisition of the integer concept.

Moreover, the three domains of space, time, and number are tightly intertwined in both world and mind (Winter, Marghetis, & Matlock, 2015). The SNL may further entangle number and time, perhaps linking past to negative, future to positive. Indeed, abstract concepts in general often induce systematic dispositions to act and think spatially (e.g. valence, Casasanto, 2009). This web of spatial dispositions may bind concepts across diverse domains, establishing relations among relations.

However, given the critical role of inscription within mathematical practice, a sagittal representation of number comes with challenges. The 17th century mathematician Wallis (1685), for instance, was the first to explicitly state an analogy between the integers and the sagittal axis (Fauconnier & Turner, 1998; Núñez, 2011). But when it came time to represent this analogy on paper, as a diagram, he was forced to follow the constraints of writing and reading practices and thus transpose back-to-front sagittal motion to the left-to-right horizontal axis, so that locations "ahead" were drawn to the right on the page, and those "behind" were drawn to the left. In so doing, Wallis took an internal, immersive spatialization of number and transformed it into an external representation, with the reader situated outside the axis of motion. This illustrates one of the limitations of the sagittal axis: As a result of the front-back asymmetry of our reach and perception, the sagittal axis resists being used for writing. This may explain, in part, the absence of material artifacts that embody the SNL—unlike horizontal number-lines, which are commonplace.

2.7.2. Origins of the SNL

What is the origin of the SNL? The SNL could derive from conventional expressions that use the language of sagittal space to describe numerical relations (e.g. "count backward or forward"); a similar process may account for some aspects of the spatial representation of time (Boroditsky, 2001). We think this is unlikely, however, given the rarity of such expressions, and the fact that the SNL appears only when processing both negative and positive integers.

Another possibility is that the SNL is the behavioral manifestation of an implicit conceptual metaphor. "Conceptual metaphors" are sets of mappings between conceptual domains (e.g. Space, Time, Arithmetic) in which the inferential structure of a (typically more concrete) source domain is used to structure a (typically more abstract) target domain (Lakoff & Johnson, 1980). In their account of the metaphorical nature of mathematical thought, Lakoff and Núñez (2000) suggest that we conceptualize arithmetic using our embodied experience of *motion along a path*: numbers are locations along the path; addition is movement away from an origin; zero is the origin; larger numbers are farther from the origin. Wallis's (1685) analogy between arithmetic and sagittal motion, discussed above, may have been an explicit statement of this implicit conceptual metaphor—and may even have marked its historical origin (Núñez, 2011; cf. Fauconnier & Turner, 1998). Might this metaphor be responsible for the SNL? This is unlikely, for two reasons. First, on Lakoff and Núñez's proposal, this metaphor is used first to conceptualize the positive whole numbers (2000, p. 72), and then used to motivate the extension of the number domain to include negatives (p. 73). Therefore, if the conceptual metaphor were to manifest itself behaviorally (e.g. as spatial dispositions), then this should happen, first and foremost, for the positive whole numbers. And yet we found no evidence of an SNL for isolated positive integers. Second, one of the core insights of conceptual metaphor theory is that *inference* and *reasoning* are structured by metaphor (Lakoff & Johnson, 1980). In fact, it is this focus on inference and reasoning that distinguishes conceptual metaphor theory from competing accounts of cross-domain interactions (Winter, Marghetis, & Matlock, 2015). The SNL, by contrast, involves low-level, unconscious, implicit spatial dispositions. While we suspect these simple dispositions may play a critical role in helping us make sense of negative integers and may even play a functional role in simple tasks like magnitude comparison, they are nevertheless far from the realm of inference and reasoning. While Lakoff and Núñez's proposed path-based metaphor may relate to the SNL in other ways—for instance, both the metaphor and the SNL may spring from a common origin—we doubt the SNL is simply a behavioral manifestation of the underlying metaphor.

Perhaps the SNL is the product of analogical structure-mapping (Gentner, 1983). The sagittal axis and the integers share considerable structure: a single dimension; a privileged reference point; transitive relations between elements. Indeed, Clark (1973) presaged the current results when he observed, in his seminal discussion of the semantics of space and time, that "since everything in front of the vertical plane is easily perceptible and everything behind it is not, the forward direction can be considered the positive perceptual direction, and backward the negative one, where *positive* its taken in its natural sense to mean the presence of something, and *negative*, the absence" (emphasis in the original). Clark's insight is that structural similarity between sagittal space and other domains may facilitate analogical mappings between the sagittal axis and those other domains.

Indeed, the sagittal axis is striking in its asymmetry, and spatial dispositions typically reflect asymmetries in our bodies and experience. The vertical axis is oriented in virtue of physiological and gravitational asymmetries, and this oriented axis may become associated with numerical magnitude by experiential correlations between *more* and *up* (Lakoff & Núñez, 2000). And while our bodies are bilaterally symmetric, a lifetime of enculturation into spatial practices shapes horizontal dispositions; hypothesized influences on the orientation of the horizontal mental number-line, for instance, include writing direction (Shaki, Fischer, & Petrusic, 2009), body-counts (Fischer, 2008, Beller & Bender, 2012), and gesture (Marghetis, Eberle, and Bergen, *submitted*). The human mind, therefore, appears sensitive to spatial asymmetries, capitalizing on them to build up a system of spatial dispositions. Unlike the horizontal axis, which depends on cultural experience for its orientation, the orientation of the sagittal axis may flow more directly from our species-typical physiology and its accompanying perceptual experiences. It may be this physiological asymmetry that makes the axis salient as a representational tool more generally—but only for domains, like time and the integers, that share its divided structure (Clark, 1973).

This is not to suggest that we should expect the SNL to be universal, found in the same form in every culture. For starters, quite obviously, negative numbers are far from universal. The concept is a recent and highly technical development, responding to specialized constraints and demands within mathematical practice (cf. Núñez, 2011). Second, the intuition that numbers can be mapped to spatial locations (rather than to spatial *lengths*) is highly variable across cultures—and may not even exist in some (Núñez, Cooperrider, and Wassmann, 2012). Perhaps most critically, even when individuals within a culture possess both the concept of a negative number and the intuition that numbers can be mapped to spatial locations—necessary prerequisites for the SNL—there may be variability in the SNL's orientation. Our species-typical physiology, with its sagittal asymmetries, might seem sufficient to induce "natural" ways of mapping sagittal space to other domains (Clark, 1973). But within the constraints of our shared embodiment, cultural variability can nevertheless flourish. While Western cultures typically associate the future with the space in front of the body and the past with the space behind—what might seem a "natural" mapping—the Aymara of the Andes reverse this sagittal time-line (i.e. past-front, future-back; cf. Núñez and Cooperrider, 2013). Sagittal number-lines, when they exist within a culture, may exhibit similar variability. This may explain why Seno and colleagues (2011) report that, for Chinese and Japanese adults, forward motion was associated with smaller numbers and backward motion with larger numbers.

2.7.3. Implications for embodied cognition and mathematical competence

The SNL has theoretical implications both for grounded or embodied cognition and for our understanding of mathematical competence. Theories of grounded or embodied cognition argue that higher cognition co-opts neural subsystems specialized for interaction with the world—that is, for space, action, and perception (e.g., Barsalou, 1999). Mental arithmetic, for instance, may recycle parietal brain circuits adapted for the control of spatial attention (Knops et al, 2009), even when the calculation requires exact, symbolic quantities (Marghetis et al, 2014). A recurring worry about grounded approaches, however, is whether sensorimotor systems contribute to the conceptual representation of entirely abstract concepts—concepts that lack concrete, real-world instantiations. The current results affirm that even the most abstract, technical concepts—such as the negative integers—may rely on subsystems specialized for space and action. On the other hand, theories of embodiment often tell a developmental story in which abstract concepts are built up progressively from more concrete experiences and concepts (Lakoff & Johnson, 1980; Lakoff & Núñez, 2000). Evidence of spatial dispositions along the horizontal axis for arithmetic, for instance, may reflect concrete experiences with cultural artifacts that map numbers to a physical left-to-right line (e.g. physical number-lines in school). Our finding of an SNL for the integers, but not the positive whole numbers, illustrates that the grounding of an abstract concept (e.g. the integers) may be independent from the grounding of its conceptual antecedents (e.g. the positive whole numbers). Moreover, the lack of an obvious experiential origin for the association between the integers and the sagittal axis highlights non-experiential origins of sensorimotor grounding, including analogical reasoning.

How might the SNL impact mathematical competence? In general, spatial dispositions may scaffold the acquisition of number concepts, supporting children's early sense-making (Núñez & Marghetis, *forthcoming*) or may supply a subjective, gut-feeling "quality of quantity" that complements more exact calculations (Marghetis et al, 2014). The SNL might therefore be a productive target for educational intervention, similar to successful interventions that have targeted the association between whole numbers and horizontal space (Siegler & Ramani, 2009). There is evidence, however, that *external* spatial representations of number (e.g. the horizontal mental number-line) play a negligible role during adult numerical cognition (e.g. Cipora & Nuerk, 2013), perhaps because they are displaced by horizontal dispositions associated with more complex mathematical skills (e.g. exact symbolic arithmetic, Knops et al, 2009, Marghetis et al, 2014; algebraic manipulation, Goldstone et al, 2010). Conversely, if complex arithmetic and algebraic skills are not associated with the *sagittal* axis, then the acquisition and mastery of these skills may not interfere with sagittal spatial construals like the SNL. Unlike other spatial-numerical associations, the SNL might thus continue to contribute to support advanced mathematical competence.

Indeed, mathematical practice involves more than thinking about isolated numbers. For starters, numbers are manipulated by calculation and symbolic manipulation, which themselves induce spatial dispositions (e.g., Knops et al, 2009; Marghetis et al, 2014; Goldstone et al, 2010). And while most research in numerical cognition has focused on small positive numbers, advanced mathematics goes beyond simple whole numbers (Núñez & Marghetis, 2014). Massively large numbers, complex numbers, discontinuous functions—these are central to science, technology, engineering, and mathematics. Even when working with these rarified, abstract concepts, however, mathematical experts deploy dynamic, spatial intuitions to supplement their technical reasoning (Marghetis & Núñez, 2013). It remains to be seen how the SNL interacts with these varied aspects of mathematical activity.

In conclusion, we investigated whether the integer concept, while shaped considerably by enculturation into complex socio-technical practices, nevertheless builds on more concrete, embodied intuitions. In particular, we pursued the possibility that the sagittal asymmetry of the human body, and resulting asymmetries in experience along the sagittal axis, may be mapped, in the human mind, to the asymmetry between negative and positive integers. Four experiments established that processing negative integers induces spontaneous spatial dispositions to move backward, and positive numbers, to move forward—a *sagittal number line* (SNL). The cognitive processing of abstract integers, therefore, appears to be grounded in action, prompting systematic dispositions to act. We may even be tempted to say about the integers what Bourdieu said about honor: that they are "nothing other than the cultivated disposition, inscribed in the body schema and the schemes of thought," which he calls the *habitus* (Bourdieu, 1977:15). Of course, this goes too far. Our understanding of the integers outstrips the embodied dispositions we have internalized from a lifetime of experience; our understanding depends on and is partially constituted by notational systems, axioms, diagrammatic practices, an entire sociotechnical ecosystem. But these spatial dispositions—part of our mathematical habitus—may nevertheless play a central role in enacting our mathematical conceptual systems.

2.8. Acknowledgments

We are indebted to Ben Bergen and Rafael Núñez for guidance, Esther Walker for help with experimental set-up, the *CL community for feedback, and audience members at the 36th Annual Conference of the Cognitive Science Society for insightful questions and suggestions. Ben Bergen, Rafael Núñez, and Esther Walker offered careful comments on an earlier draft. All remaining failings are those of the authors alone.

Chapter 2, in full, has been submitted for publication, and appeared, in part, in the *Proceedings of the 36th Annual Conference of the Cognitive Science Society*. Marghetis, T.; Youngstrom, K., 2014. The dissertation author was the primary investigator and author.

2.9. References

Barsalou, L. W. (1999). Perceptual symbol systems. *Behavioral and Brain Sciences*, 22, 577-660.

Bender, A., & Beller, S. (2012). Nature and culture of finger counting: Diversity and representational effects of an embodied cognitive tool. *Cognition*, *124*, 156-182.

Boroditsky, L. (2001). Does language shape thought? English and Mandarin speakers' conceptions of time. *Cognitive Psychology*, 43, 1-22.

Bourdieu, P. (1977). *Outline of a Theory of Practice*. Cambridge: Cambridge University Press.

Carey, S. (2009). *The Origin of Concepts*. Oxford: Oxford University Press.

Casasanto, D. (2009). Embodiment of Abstract Concepts: Good and bad in right- and left-handers. *Journal of Experimental Psychology: General,* 138, 351-367.

Casasanto, D., & Jasmin, K. (2012). The hands of time: temporal gestures in English speakers. *Cognitive Linguistics*, 23, 643-674.

Cipora, K., & Nuerk, H. C. (2013). Is the SNARC effect related to the level of mathematics? *Quarterly Journal of Experimental Psychology,* 66, 1974-1991.

Clark, H. H. (1973). Space, time, semantics, and the child. In T. Moore (Ed.), Cognitive development and the acquisition of language (pp. 27e63). New York: Academic Press.

Dehaene, S., Bossini, S., & Giraux, P. (1993). The mental representation of parity and numerical magnitude. *Journal of Experimental Psychology: General*, 122, 371-396.

Dehaene, S., & Cohen, L. (2007). Cultural recycling of cortical maps. *Neuron,* 56, 384–98.

Fauconnier, G., & Turner, M. (1998). Conceptual Integration Networks. *Cognitive Science*, 22, 133-87.

Fias, W., Brysbaert, M., Geypens, F., & d'Ydewalle, G. (1996). The importance of magnitude information in numerical processing: Evidence from the SNARC effect. *Mathematical Cognition,* 2, 95–110.

Fischer, M.H. (2003). Cognitive representation of negative numbers. *Psychological Science*, 14, 278-282.

Fischer, M. H. (2008). Finger counting habits modulate spatial-numerical associations. *Cortex*, *44*, 386-392.

Ganor-Stern, D., Pinhas, M., Kallai, A. & Tzelgov, J. (2010). Holistic representation of negative numbers is formed when needed for the task. *The Quarterly Journal of Experimental Psychology*, 63, 1969-1981.

Ganor-Stern, D. & Tzelgov, J. (2008). Negative numbers are generated in the mind. *Experimental Psychology*, *55*, 157.

Gevers, W., Lammertyn, J., Notebaert, W., Verguts, T., & Fias, W. (2006). Automatic response activation of implicit spatial information: Evidence from the SNARC effect. *Acta Psychologica,* 122, 221-233.

Gentner, D. (1983). Structure-mapping: A theoretical framework for analogy. *Cognitive Science,* 7, 155-170.

Goldstone, R. L., Landy, D., & Son, J. Y. (2010). The education of perception. *TopiCS, 2*, 265-284.

Hartmann, M., Grabherr, L., & Last, F. W. (2012). Moving along the mental number line: Interactions between whole-body motion and numerical cognition. Journal of Experimental Psychology: Human Perception and Performance, 38, 1416-1427.

Holmes, K. J., & Lourenco, S. F. (2012). Orienting numbers in mental space: Horizontal organization trumps vertical. *Quarterly Journal of Experimental Psychology*, *65*, 1044-1051

Hubbard, E.M., Piazza, M., Pinel, P., & Dehaene, S. (2005). Interactions between number and space in parietal cortex. *Nature Reviews Neuroscience* 6, 435–448.

Ito, Y., & Hatta, T. (2004). Spatial structure of quantitative representation of numbers: evidence from the SNARC effect. *Memory & Cognition*, 32, 662-673.

Kitcher, P. (1984). *The Nature of Mathematical Knowledge*. Oxford: Oxford University Press.

Knops, A, Thirion, B, Hubbard, E, Michel, V, & Dehaene, S (2009). Recruitment of an area involved in eye movements during mental arithmetic. *Science*, *324*, 1583.

Lakoff, G., & Johnson, M. (1980). *Metaphors we live by*. Chicago: University of Chicago Press.

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from*. New York: Basic Books.

Loetscher, T., Bockisch, C., Nicholls, M. E. R., & Brugger, P. (2010). Eye position predicts what number you have in mind. *Current Biology*, 20, R264-R265.

Marghetis, T., Núñez, R, & Bergen, B. (2014). Doing arithmetic by hand: Hand movements during exact arithmetic reveal systematic, dynamic spatial processing. *Quarterly Journal of Experimental Psychology,* 67, 1579-1596.

Marghetis, T., & Núñez, R. (2013). The motion behind the symbols: A vital role for dynamism in the conceptualization of limits and continuity in expert mathematics. *Topics in Cognitive Science*, 5, 299-316.

Miles, L. K., Nind, L. K., & Macrae, C. N. (2010b). Moving through time. *Psychological Science,* 21, 222–223.

Müller, D., & Schwarz, W. (2007). Is there an internal association of numbers to hands? The task set influences the nature of the SNARC effect. *Memory & Cognition*, 35, 1151-1161.

Núñez, R. (2011). No Innate Number Line in the Human Brain. *Journal of Cross-Cultural Psychology*, 45, 651-668.

Núñez, R., & Cooperrider, K. (2013). The tangle of space and time in human cognition. *Trends in Cognitive Sciences*, 17, 220-229.

Núñez, R., Cooperrider, K., & Wassmann, J. (2012). Number Concepts without Number Lines in an Indigenous Group of Papua New Guinea. *PLoS ONE* 7: e35662. doi:10.1371/journal.pone.0035662

Núñez, R., & Marghetis, T. (*forthcoming*). Cognitive Linguistics and the Concept(s) of Number. In R. Cohen-Kadosh and K. Dowker (eds.), *Oxford Handbook of Numerical Cognition.* Oxford University Press.

Proctor, R.W., & Cho, Y.S., (2006). Polarity Correspondence: A general principle for performance of speeded binary classification tasks. *Psychological Bulletin, 132,* 416-442.

Schwarz, W., & Keus, I. M. (2004). Moving the eyes along the mental number line: Comparing SNARC effects with saccadic and manual responses. *Perception and Psychophysics*, 66, 651-64.

Seno, T., Taya, S., Ito, H., Sunaga, S. (2011). The mental number line in depth revealed by vection. *Perception,* 40, 1241-44

Shaki, S., & Fischer, M. H. (2012). Multiple spatial mappings in numerical cognition. *Journal of Experimental Psychology: Human Perception and Performance*, *38*, 804.

Shaki, S., Fischer, M., & Petrusic W. (2009). Reading habits for both words and numbers contribute to the SNARC effect. *Psychonomic Bulletin & Review,* 16, 328-31.

Siegler, R., & Ramani, G. (2009). Playing linear number board games—but not circular ones—improves low-income preschoolers' numerical understanding. *Journal of Educational Psychology*, 101, 545.

Simmons, J. P., Nelson, L. D., & Simonsohn, U. (2012). A 21 word solution. *Dialogue, The Official Newsletter of the Society for Personality and Social Psychology*, 26, 4–7.

Wallis, J. (1685). *A Treatise of Algebra.* London: John Playford.

Winter, B., Marghetis, T, & Matlock, T. (2015). Of metaphors and magnitudes: Explaining cognitive interactions between space, time, and number. *Cortex,* 64, 209-224.

Appendix: Debrief from Experiment 1b

We include, below, the full list of questions that were asked in Experiment 1b's debrief. Possible responses to multiple-choice questions are in parentheses. Questions were presented on a computer, in the following order, one question at a time. No other questions were asked (cf. Simmons et al, 2012). Questions 1, 2, and 7 were designed specifically to detect whether participants had surmised the experiment's purpose; for the purposes of Experiment 1b, participants were "naive" if they did not mention space in their responses to questions 1 and 2 and responded negatively to question 7.

1. What do you think was the purpose of this experiment? (free response)
2. This experiment was about the mental representation of integers. What aspect of the mental representation of integers do you think we were investigating? (free response)
3. Overall, do you think you were significantly faster to move forward, move backward, or about the same speed in either direction? (faster when moving forward; faster when moving backward; no difference)
4. Overall, do you think you were significantly faster to respond to positive integers, negative integers, or about the same speed for both? (faster for positive integers; faster for negative integers; no difference)
5. When you were responding to positive integers, do you think you were significantly faster to move forward, move backward, or about the same speed in either direction? (faster when moving forward; faster when moving backward; no difference)

6. When you were responding to negative integers, do you think you were significantly faster to move forward, move backward, or about the same speed in either direction? (faster when moving forward; faster when moving backward; no difference)
7. This experiment is investigating the existence of a "mental number line" for positive and negative integers. During the experiment, did you guess that this was the purpose of the experiment? (yes; no)
8. Given that the purpose of the experiment was to investigate the existence of a mental number line, what kind of number-line do you predict that we will find? (back-to-front: negative numbers behind and positive numbers in front; front-to-back: negative numbers in front and positive numbers behind; no number-line: negative and positive numbers are both abstract concepts so they should not have a spatial representation in the mind; I don’t know)

Chapter 3

Doing arithmetic by hand: Hand movements during exact arithmetic reveal systematic, dynamic spatial processing

ABSTRACT

Mathematics requires precise inferences about abstract objects inaccessible to perception. How is this possible? One proposal is that mathematical reasoning, while concerned with entirely abstract objects, nevertheless relies on neural resources specialized for interacting with the world—in other words, mathematics may be grounded in spatial or sensorimotor systems. Mental arithmetic, for instance, could involve shifts in spatial attention along a mental "number-line," the product of cultural artifacts and practices that systematically spatialize number and arithmetic. Here, we investigate this hypothesized spatial processing during exact, symbolic arithmetic (e.g. 4 + 3 = 7). Participants added and subtracted single-digit numbers and selected the exact solution from responses in the top corners of a computer monitor. While they made their selections using a computer mouse, we recorded the movement of their hand as indexed by the streaming x, y coordinates of the computer mouse cursor. As predicted, hand movements during addition and subtraction were systematically deflected toward the right and the left, respectively, as if calculation involved simultaneously simulating motion along a left-to-right mental number-line. This spatial-arithmetical bias, moreover, was distinct from—but correlated with—individuals' spatial-numerical biases (i.e. SNARC effect). These results are the first evidence that exact, symbolic arithmetic prompts systematic spatial processing associated with mental calculation.

We discuss the possibility that mathematical calculation relies, in part, on an integrated system of spatial processes.

3.1 Introduction

Mathematics exemplifies some of the most remarkable properties of human cognition: exact yet abstract, mediated by notations and diagrams, and accompanied by a compelling sense of certainty. And yet mathematics itself is such a recent cultural innovation that the neural resources responsible for mathematical thought could not have evolved specifically for that purpose. This article explores the possibility that mathematical thought, and arithmetic calculation in particular, relies on neural resources that are specialized for processing space (e.g. Dehaene and Cohen, 2007; Anderson, 2010). On this account, mathematical cognition involves mapping mathematical entities to space, a space which then affords reasoning and reflection (Lakoff and Núñez, 2000; Núñez and Marghetis, in press). We may recycle the brain's spatial prowess to navigate the abstract mathematical world.

The last two decades have generated an abundance of evidence that human numerical cognition does, indeed, interact with spatial processing. During a variety of simple tasks, numerical magnitude has been found to be associated with spatial length (de Hevia and Spelke, 2009), area (Tzelgov, Meyer, and Henik, 1992), and locations along horizontal (Dehaene et al, 1993) and vertical (Schwarz and Keus, 2004; Ito and Hatta, 2004) axes. These effects exist across response modalities: Thinking about numbers induces spatial biases in subsequent manual responses (Dehaene et al, 1993), covert attention (Fischer et al, 2003), eye movements (Fischer et al, 2004; Schwarz & Keus, 2004), and grip aperture (Lindemann et al, 2007). Spatial attention, conversely, systematically influences random number generation (e.g. Loetscher et al, 2008, 2010). And linguistically, talk about numbers is loaded with spatial language: we count *up* to arrive at *bigger* or *higher* numbers, but count *down* to *smaller* or *lower* numbers (Lakoff and Núñez, 2000). There is evidence, therefore, of bidirectional interactions between numerical cognition and spatial processing.

In particular, systematic associations between numerical magnitude and spatial location—along vertical or horizontal axes—are often referred to as a "mental number-line." The specific direction of the horizontal mental number line (e.g. left-to-right) is thought to emerge from rich cognitive ecosystems of cultural practices and artifacts, including reading (Shaki, Fischer, and Petrusic, 2009), finger-counter (Fischer, 2008), and physical number-lines (Núñez, 2011).

But mature mathematical competence far outstrips basic numerical abilities like number comparison. A bedrock of mathematics is the ability to manipulate and combine numbers, performing calculations to produce exact solutions. Might exact, symbolic arithmetic also rely on basic spatial resources, further elaborating a foundation of spatial-numerical associations?

Recent research raises the tantalizing possibility that this may be the case. McCrink and colleagues (2007) reported that adults systematically over- and under-estimated the results of approximate addition and subtraction, respectively—the so-called "Operational Momentum" effect (hereafter *OM*). This effect has since been replicated (Pinhas and Fischer, 2008; Knops et al, 2009a, 2009b; McCrink and Wynn, 2009). A leading explanation of OM ascribes the effect to concurrent spatial processing (McCrink et al, 2007). On this account, mental calculation involves associating numbers with locations along a mental number-line and then shifting spatial attention along that line—a form of simulated or abstract motion (cf. Langacker, 1987). The observed over- and under-estimation is due to the momentum of this simulated motion, a momentum that propels the thinker past the correct response: toward greater numbers in the case of addition, and toward lesser numbers in the case of subtraction. We'll refer to this as the **Spatial Account** of OM (McCrink et al, 2007; Hubbard et al, 2005).

In support of the Spatial Account, Knops and colleagues (2009a) reported that a machine learning classifier that had been trained to distinguish right and left saccades on the basis of fMRI data from the posterior superior parietal lobule (PSPL) was able to generalize spontaneously to approximate arithmetic, successfully distinguishing addition from subtraction. This suggests that approximate arithmetic and spatial attention, at the very least, involve similar, overlapping neural activity in the PSPL.

This Spatial Account is appealing on theoretical grounds. For starters, it explains over- and under-estimation during arithmetic (i.e. OM) by appealing to known interactions between numerical magnitude and space, thus implicating spatial-numerical interactions in arithmetical calculation. This raises the possibility that simple spatial processing might play a functional role during more complex mathematical capacities like symbolic calculation (Hubbard et al, 2005).

In so doing, the Spatial Account offers an explanation of how a relatively recent cultural innovation like symbolic calculation could emerge, in part, from evolutionarily older cortical foundations (Dehaene and Cohen, 2007), shaped and assembled by cultural practices and artifacts like external number-lines (Núñez, 2011). By connecting arithmetic to spatial processing, the Spatial Account thus situates arithmetic within the broader frameworks of Grounded Cognition (Barsalou, 1999, 2008), Embodied Cognition (Lakoff and Núñez, 2000), and various forms of Neural Reuse (Anderson, 2010; Gallese and Lakoff, 2005;

Hurley, 2008; Dehaene and Cohen, 2007). These frameworks argue that higher cognition, including capacities like mathematical reasoning or language comprehension, may rely on neural resources that evolved in response to entirely different evolutionary pressures—namely, the constraints and demands of interacting with the external world via perception and action. This re-deployment of sensorimotor neural resources during higher cognition is sometimes referred to as simulation (Barsalou, 1999). To borrow an example from language comprehension: understanding language about motion, whether literal ("I gave him the butter.") or figurative ("I gave him an idea."), may rely on the same neural machinery that subserves the perception and execution of real-world motion (e.g. Glenberg and Kaschak, 2002; Kaschak et al, 2005; Saygin et al, 2010; Matlock, 2004; Glenberg et al, 2008). Similar proposals for arithmetic date back at least to Hubbard and colleagues (2005), who noted that "the parietal mechanisms that are thought to support spatial transformations might also be ideally suited to supporting arithmetic transformations" (p. 445). By situating arithmetic within the frameworks of Grounded Cognition, Embodied Cognition, or Neural Re-use, the Spatial Account thus offers an explanation of how a historically recent, human-specific capacity like symbolic arithmetic might have emerged from neural resources in our evolved cognitive toolbox—as part of a larger cultural-cognitive ecosystem, of course. The Spatial Account, therefore, supplies a mechanistic proposal for how neural resources specialized for space might be responsible for parts of mathematical calculation.

3.1.1 Non-spatial accounts of Operational Momentum

However, there are compelling non-spatial alternative explanations of known Operational Momentum effects. One possibility is that over- and under-estimation during mental arithmetic is due to a logarithmically-compressed representation of numerical magnitude. Children's early representations of number seem to be compressed logarithmically, with smaller numbers allocated more representational resources than larger numbers (Siegler and Opfer, 2003). Human adults continue to exhibit a logarithmic representation of approximate, non-symbolic numerical magnitude under certain circumstances (e.g. when responding non-spatially, Núñez, Doan, and Nikoulina, 2011). And non-human primates represent non-symbolic numerosities using neural codes with logarithmically-compressed "receptive fields" for numerosity (Nieder and Miller, 2003; Dehaene, 2003). On this account, the systematic over- and under-estimation of addition and subtraction is due to small errors induced by these logarithmically-compressed approximate magnitudes. Adding 40 and 8, for instance, may involve transducing these exact numbers to logarithmically-compressed approximate magnitudes (e.g. $\log_2(40) + \log_2(8) \approx 8.32$), and then trying to transduce this back to an approximate number ($2^{8.32} \approx 69 > 48$), a process which can overestimate the result of the addition. A corresponding bias emerges for subtraction (e.g. $\log_2(40) - \log_2(8) \approx 2.32$, $2^{2.32} \approx 5.4 < 32$). Following Knops et al (2013), we shall refer to this as the **Compression Account** of OM (Chen and Verguts, 2012).

A second non-spatial explanation ascribes OM to a heuristic that, simply stated, assumes addition will always produce a larger number, and subtraction, a smaller number (McCrink and Wynn, 2009). This is a reasonable assumption under most circumstances; arithmetic involving negative numbers is a notable exception. Applying this heuristic, crucially, would make a reasoner more likely to accept larger solutions from a list of options when adding but more likely to accept smaller solutions when subtracting. This proposal is bolstered by the existence of OM in infants as young as nine months old (McCrink and Wynn, 2009; but see Knops et al, 2013), presumably too early for them to have acquired any systematic associations between numbers and lateral locations. Following Knops et al (2013), we shall refer to this as the **Heuristic Account** of OM (McCrink and Wynn, 2009).

These non-spatial alternatives can explain the systematic biases in arithmetic that are characteristic of OM without invoking spatial processing of any sort. Of course, these alternatives are not in opposition to each other, and it is entirely possible that each proposed mechanism makes its own contribution to observed over- and under-estimation during arithmetic (e.g. the computational model of OM in Chen and Verguts, 2012, involves both spatial and logarithmically-compressed representations of number). But an immediate consequence of these viable alternatives is that the mere existence of over- and under-estimation is insufficient on its own to implicate space in mental arithmetic. Any putative evidence in favor of the Spatial Account will need to adjudicate between genuinely spatial accounts of OM and these non-spatial alternatives.

3.1.2 Existing evidence for the Spatial Account

Besides intuitive plausibility, then, what evidence do we currently have in favor of the Spatial Account? Very little, in fact. Previous studies of spatial biases during arithmetic have not distinguished between spatial-numerical and genuinely spatial-arithmetical biases, or they have only found spatial biases for non-symbolic or approximate calculation. Pinhas and Fischer (2008), for instance, had participants respond to single-digit symbolic arithmetic problems by pointing to locations along a number-line on a computer touchscreen. They found that the magnitude and location of participants' responses were systematically biased by the arithmetic operation: rightward towards larger numbers for addition, and leftward towards smaller numbers for subtraction. This demonstrates that mental arithmetic can induce biases in the way we interact with external numerical artifacts (i.e. a number-line displayed on a screen). However, since the experiment involved an explicit, built-in mapping between numerical magnitudes and response locations (e.g. larger numbers were more rightward along the visually-displayed number-line), rightward and leftward deflection was thus confounded with over- and under-estimation, respectively. In other words, the observed deflection may have been the spatial manifestation of numerical over- and under-estimation during approximate calculation—perhaps due to logarithmic compression or a simple heuristic—rather than genuinely *spatial* biases.

When spatial biases *have* been demonstrated unequivocally, they have only been reliable for approximate arithmetic using analog, non-symbolic number representations. Knops, Viarouge, and Dehaene (2009) had participants solve approximate arithmetic problems, involving the addition or subtraction of symbolic (Arabic numerals) or non-symbolic (sets of dots) representations of numbers. Participants had to select the best response from options displayed in a circle on a computer monitor. As predicted, participants selectively over- and under-estimated the result of approximate addition and subtraction, respectively, replicating McCrink et al (2007). Crucially, they also found that participants were more likely to choose a response on the right of the screen after addition, and on the left after subtraction—an effect they dubbed the Spatial-Operation Association of Responses (SOAR). However, this SOAR effect was only reliable for non-symbolic sets of dots; across two experiments, the effect was non-significant for symbolic representations (Arabic numerals). We know of no evidence, therefore, that unequivocally demonstrates spatial biases during *symbolic* approximate arithmetic.

As far as we know, moreover, there have been no studies of OM or spatial biases during *exact* calculation, in many respects a crucial test-case for embodied or grounded accounts of mathematical thought. The precise, highly constrained reasoning required for exact calculation may be more amenable to "amodal" or symbolic approaches than to sensorimotor or grounded approaches, since spatial simulation seems to lack the necessary precision and abstraction (Mahon and Caramazza, 2008; Dove, 2009). The solution to 7+2 is exactly 9, after all, not approximately 9, and this remains true regardless of whether we are dealing with diamonds, dragons, or decimal numbers. For these reasons, evidence of spatial processing during exact calculation is necessary if the Spatial Account is going to scale up to advanced mathematics, beyond basic capacities for approximation.

3.1.3 The current study

At present, therefore, there is no unequivocal evidence of spatial biases during symbolic calculation; previous studies have either confounded spatial effects with other non-spatial sources of over- and under-estimation or have only found reliable effects with analog, non-symbolic stimuli. Existing research, moreover, has been limited to approximate arithmetic, so there is currently no evidence of spatial biases during exact calculation. To address these limitations of previous work with respect to the current question, we tested the Spatial Account of OM during exact, symbolic arithmetic, using the dynamics of motor activity during mental calculation to look for systematic spatial perturbations associated with arithmetic operations.

In particular, we turned to computer mouse-tracking, a methodology in which hand movements—as indexed by the streaming *x, y* coordinates of the computer mouse cursor—are recorded during real-time reasoning and decision making (e.g. Spivey, Grosjean, and Knoblich, 2007). These continuous hand trajectories are ideally suited for investigating the temporal dynamics of cognition, and have been used to study the real-time processing of language, categorization, and even race and gender (for a review, see Freeman et al, 2011), and continuous measures of hand movements have been used previously to study numerical cognition (Dotan and Dehaene, 2013; Song and Nakayama, 2008). More recently, computer mouse-tracking has been used to test grounded theories of abstract thought. Miles and colleagues (2010) recorded hand movements while participants decided whether generic events were in the past or the future. In line with previous research showing that literate Westerners represent time on a left-to-right mental time-line, they found that hand movements were deflected to the left when reasoning about past events, and to the right when reasoning about future events. This methodology is sensitive to subtle perturbations in the spatial and temporal dynamics of hand trajectories and can therefore reveal sensorimotor or spatial processing during higher cognition, unlike typical offline measures used in cognitive psychology that only capture the discrete outcomes of cognition (Spivey, 2007).

As a direct test of spatial-arithmetical biases during exact, symbolic calculation, we had participants solve arithmetic problems while using a computer mouse to select their response. We reasoned as follows. If mental calculation involves dynamic shifts in attention along a spatial representation of number—the Spatial Account—then exact arithmetic should systematically influence the spatial trajectory of concurrent motor activity (Barsalou, 2008). For our American participants, this implies that adding and subtracting should induce spatial deflections not only along a left-to-right conceptual number-line but also in ongoing interactions with the world. We thus hypothesized that, if the Spatial Account is correct, the trajectory of participants' hands should be systematically deflected in the direction of simulated motion: to the right during addition and to the left during subtraction (the SOAR effect). By contrast, since response location was independent of solution magnitude, neither the Compression nor the Heuristic Accounts predict any systematic influences of mental arithmetic on concurrent hand movements.

3.2 Method

3.2.1 Participants

Undergraduate students (n = 44, 14 males, mean age 21.4) from the University of California, San Diego, completed the experiment in return for partial course credit. All experimental procedures were approved by the university's Institutional Review Board.

3.2.2 Materials

On each trial, participants were presented with an arithmetic problem (e.g. 6 + 2) and had to select the correct solution from two options (e.g. 8 or 9), one of which was always correct (see Procedure below). Arithmetic problems were generated according to the following criteria. All problems involved the addition or subtraction of single-digit numbers and had a single-digit result. Paired addition and subtraction problems were created with the same first and second terms (e.g. 3+1=4 and 3-1=2), and with the second term ranging from 0 to 3, inclusive. Since the incorrect distractor response was always one higher or lower than the correct solution, we restricted the problems to those with correct solutions between 1 and 8 so the distractor responses were also single digit numbers. This produced a list of 32 problems, 16 each for addition and subtraction. Each of these problems then generated two items: one where the distractor response was higher than the correct solution, and another where it was lower. All told, therefore, there were 64 items, half of which involved addition, with addition and subtraction items matched for the first and second terms (see Appendix A).

3.2.3 Procedure

The experiment consisted of two blocks of 128 trials presented in a random order. Each of the 64 items appeared twice during each block, each time with the correct answer in a different location. The trial structure is illustrated in Figure 3.1. Trials began by displaying the two response options in the top right and left corners of a computer monitor (474mm wide x 296mm high). These response options were displayed for 1000ms to allow participants sufficient time to familiarize themselves with the response locations. After this 1000ms familiarization period, a button marked “START” appeared in the bottom center of the screen, which participants could then click to display the arithmetic problem. The arithmetic problem appeared sequentially in the center of the screen: the first term (e.g. “5”) appeared for 500ms, followed by the operation (e.g. “+”) for 500ms, followed by the second term (e.g. “2”) for 500ms (see Fig. 3.1). As soon as the second term appeared, the computer mouse became responsive to participants’ hand movements, allowing participants to begin moving the cursor toward the upper response buttons. In order to encourage hand movements during mental calculation, participants were instructed to begin moving the cursor as soon as the second term appeared, and received a warning message if it took them longer than 1000ms to initiate a response.

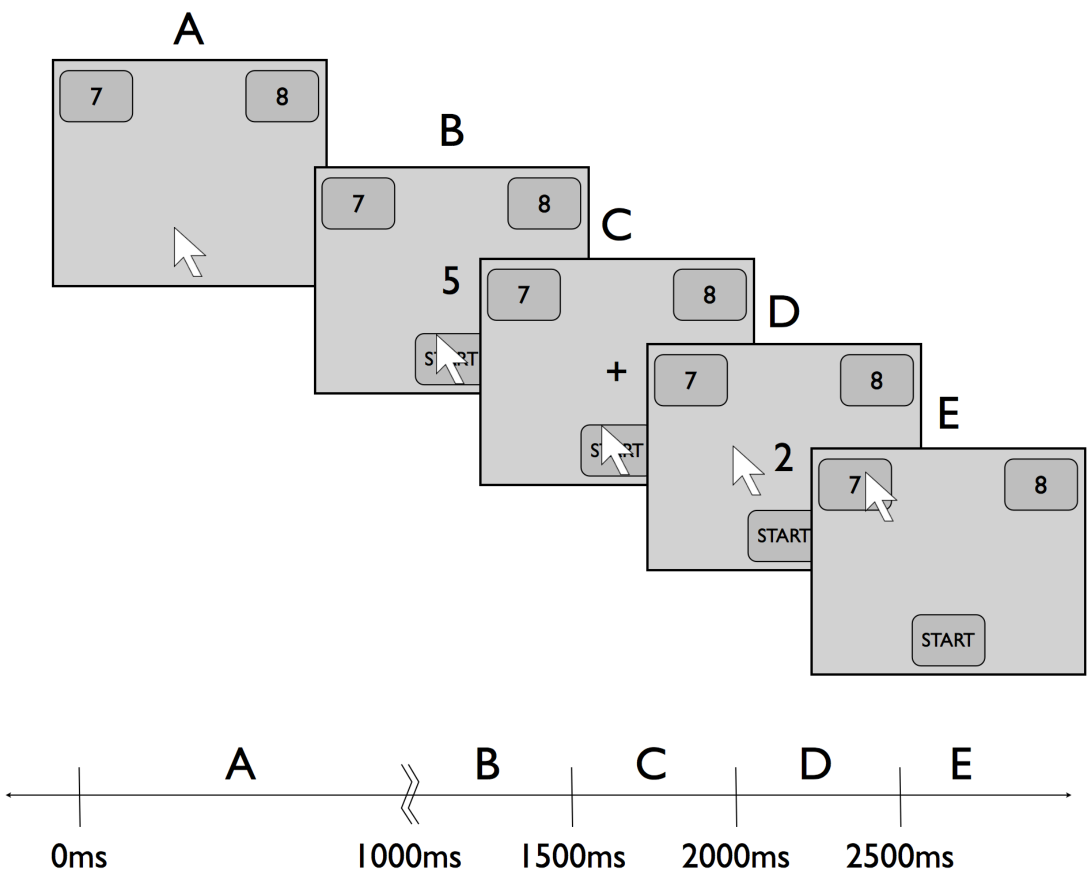


Figure 3.1. Timeline of each trial. Participants had 1000ms to familiarize themselves with the possible solutions (A), after which they could press the START button to begin the trial. They were then presented sequentially with the arithmetic problem (B, C, D), but only able to move the cursor toward their response after the onset of the second term (D). Reaction times were measured from the onset of the second term.

3.2.4 Data collection and pre-processing

We used Mousetracker software (Freeman & Ambady, 2010) to record the streaming *x*- and *y*-coordinates of the computer mouse-cursor, which served as an index of participants' hand movements. The mouse was a Dell Optical USB Scroll Mouse (model XN966), and the cursor location was sampled at approximately 70Hz by Mousetracker. Before analysis, all trajectories are rescaled to a 1.5 x 2 standard coordinate space, with the top-left of the screen at (-1, 1.5) and the bottom-right at (1,0), and remapped rightward. Trajectories were time-normalized to 101 time-steps using linear interpolation, in order that we could average across the full length of trials that varied in duration. All statistical analyses were performed using R statistical software (R Development Core Team, 2008).

3.3 Results

Accuracy was quite high (M = 98.99%, SE = 0.18), and no participants were removed due to low accuracy. We first conducted a 2x2 repeated-measures ANOVA of mean accuracy, with SOAR-Congruency and Arithmetic Operation (addition, subtraction) as within-subjects factors. SOAR-Congruency was defined as the match between the arithmetic operation and the response direction: congruent addition trials were those where the correct answer was on the right; congruent subtraction trials where those were the correct solution was on the left. There were no significant effects on accuracy (all ps > .3). Incorrect trials (n=114) were removed for all further analyses.

We used two measures to characterize the curvature of these hand trajectories: Maximum Deviation (MD) and Area Under the Curve (AUC) (Freeman & Ambady, 2010). A trajectory's Maximum Deviation is the maximum distance it reaches from a hypothetical "perfect" trajectory, that is, a straight line from the start button to the correct response. Area Under the Curve is the area bordered by the actual trajectory and this perfect, straight trajectory. These two measures were highly correlated (r=.89) but reflect slightly different spatial properties of a trajectory: MD captures the extremes of deflection but is blind to the trajectory as a whole; AUC captures average deflection over the course of the entire trajectory but is less sensitive to sudden, acute deviations. We therefore report analyses of both measures, even though in this study they produced nearly identical results (with slightly larger effect sizes for MD).

While computer-mouse trajectories are typically fluid, they sometimes involve highly aberrant or discontinuous movements due to hardware error (e.g. mouse-sticking), initial errors that are corrected mid-response, or other anomalies. To exclude these highly aberrant hand trajectories in an objective manner, we removed trials where the initiation time, reaction time, Maximum Deviation, or Area Under the Curve (AUC) was more than 3 standard deviations away from each subject's mean (4.4% of trials). No other trials were removed.

3.3.1 Spatial deflection

To investigate the spatial deflection of hand trajectories, we analyzed MD and AUC using 2x2 repeated-measures ANOVAs, by subjects and by items. SOAR-Congruency was a within-subjects and within-items factor, while Arithmetic Operation (addition, subtraction) was within-subjects but between-items.

The only significant effect was the main effect of SOAR-Congruency (see Fig. 3.2). Hand trajectories on incongruent trials had a significantly larger Maximum Deviation than on congruent trials (*M*=0.202, *SE*=0.02; *M*=.178, *SE*=0.02), both by subjects ($F(1,43)=8.01$, $p=0.007$, $\eta_p^2=0.16$) and by items ($F(1, 62)=10.7$, $p=0.002$, $\eta_p^2=0.15$). Similarly, incongruent trials had a significantly larger Area Under the Curve than congruent trials (*M*=0.345, *SE*=0.04; *M*=.309, *SE*=0.04), by subjects ($F(1,43)=4.61$, $p=0.038$, $\eta_p^2=0.10$) and items ($F(1,62)=5.92$, $p=0.002$, $\eta_p^2=0.09$). Thus, hand trajectories were reliably deflected in the predicted direction: to the right for addition, and to the left for subtraction.

3.3.2 Relation between spatial biases for magnitude and arithmetic operation

Since addition and subtraction of the same terms will produce results that are on average higher and lower, respectively, we conducted additional analyses to tease apart the observed spatial-arithmetical biases from possible spatial biases associated with the magnitude of the problems' solutions. Did spatial-arithmetical biases (i.e. the SOAR effect) make a contribution above and beyond any effect of the solution's magnitude—that is, a SNARC effect driven by the solution? To answer this question, we modeled MD and AUC as functions of both SNARC- and SOAR-congruency. Since all numbers were between 1 and 9, we assumed that any spontaneous SNARC effect would associate solutions less than 5 with left space, and solutions greater than 5 with right space (Dehaene et al, 1993). We thus began by removing trials where the solution was 5, since 5 was the midpoint of the range of numbers used in the experiment (1-9) and thus associated with neither left nor right space. Next, we constructed mixed-effects models of MD and AUC with SNARC-Congruency and SOAR-Congruency as fixed effects, Subject and Solution as random effects, and by-Subject and by-Solution random slopes for SNARC-congruency and SOAR-Congruency (Barr et al, 2013). Visual inspection of residual plots did not reveal any obvious deviations from homoscedasticity or normality. To test the influence of SOAR-Congruency, these full models were then compared to reduced models that were identical except they lacked a fixed-effect of SOAR-congruency (i.e. with only the fixed effect of SNARC-congruency).

Even after controlling for the congruency between the solution's magnitude and its location, there was a significant effect of SOAR-congruency on hand movements. The full models with SOAR-congruency fit the data significantly better than the reduced models, (MD: $\chi^2(1) = 5.12$, $p = .02$; AUC: $\chi^2(1) = 4.06$, $p = .04$), demonstrating that SOAR-incongruent trials were significantly deflected compared to SOAR-congruent trials, above and beyond any deflection due to final solution magnitude. According to the full model, a mismatch between arithmetic operation and response direction increased MD by 0.028 +/- 0.012 (standard errors) and AUC by 0.040 +/- 0.020 (standard errors). Therefore, the incongruency of arithmetic operation and response direction produced a reliable deflection of hand trajectories, and this deflection was in addition to any spatial deflection associated with the solution (i.e. a SNARC effect of the solution).

Next, we asked whether individuals' spatial-arithmetical biases were related to the size of their SNARC effects. To measure the size of each participant's SNARC effect, we adapted the regression method of Fias et al (1996). We first calculated "dMD" and "dAUC," the difference in mean MD and AUC between left and right responses for each possible numerical solution[1]. These are thus measures of the left-side advantage for each solution magnitude: positive values of dMD and dAUC indicate that responses for that numerical solution were deflected leftward, while negative values indicate that responses for that numerical solution were deflected rightward. Next, for each participant, we regressed both dMD and dAUC onto solution magnitude. The slope of this regression line is an index of participants' SNARC effect: more negative values of β are evidence of a larger SNARC effect, since they indicate that rightward responses are increasingly favored as magnitude

---

[1] To illustrate: If an individual's mean MD for calculations with a solution of 3 was 0.35 for rightward responses and 0.3 for leftward responses, then their dMD for 3 would be .05, the difference of 0.35 and 0.3.

increases. To measure the size of each individual's SOAR effect, we computed the Standardized Mean Difference (SMD) between mean MD and AUC on SOAR-Congruent and SOAR-Incongruent trials. A negative SMD, therefore, indicates the presence of a SOAR effect: increased deflection on SOAR-Incongruent trials compared to SOAR-Congruent trials. For both measures, therefore, more *negative* values indicate a larger canonical effect (following Fias et al, 1996).

First, we checked that these measures did, indeed, capture reliable spatial biases associated with the solution's numerical magnitude and the arithmetic operation. Overall, the slopes of the SNARC linear regressions were significantly less than zero (MD: $M_\beta = -0.015$, $t(43) = -3.05$, $p = 0.004$; AUC: $M_\beta = -0.027$, $t(43) = -2.92$, $p = 0.005$), confirming the presence of a SNARC effect associated with the solutions. Moreover, whether calculated with MD or AUC, thirty out of 44 participants (68%) had negative regression slopes, evidence of a canonical SNARC, in line with previous studies that find a canonical SNARC effect in ~70% of participants (e.g. Cipora and Nuerk, 2013). This is a significantly higher proportion than expected by chance ($p = 0.01$, one-tailed binomial test). Similarly, individuals' *SMD*s differed significantly from zero (MD: $M_{SMD} = -0.074$, $t(43) = -3.27$, $p = 0.002$; AUC: $M_{SMD} = -0.063$, $t(43) = -2.88$, $p = 0.006$), and 28 out of 44 participants had negative values of SMD when calculated with MD, evidence of a canonical SOAR effect ($p = 0.04$, one-tailed binomial test; for AUC: 27/44, p=0.09). These measures thus successfully indexed individuals' SNARC (β) and SOAR (SMD).

Next, we looked at individual differences in the relation between the SNARC and SOAR effects (see Fig. 3.3). As predicted, a linear regression analysis found that the size of an individual's SNARC effect was significantly predictive of their SOAR effect (MD: β =

3.22, $t(42) = 6.33$, $p < 0.001$; AUC: $\beta = 1.54$, $t(42) = 5.37$, $p < 0.001$), and that the SNARC effect explained a significant amount of the variance in the SOAR effect (MD: $r^2 = 0.49$, $F(1,42) = 40.05$, $p < 0.001$; AUC: $r^2 = 0.41$, $F(1,42) = 28.82$, $p < 0.001$). To further confirm this coupling of numerical and arithmetical spatial biases, we used two separate k-means cluster analyses to sort individuals into three groups based on the size of their SOAR and SNARC effects, corresponding roughly to standard, reversed, and no effect (cf. Beecham, Reeve, & Wilson, 2009). We then looked at whether these clusters were independent. They were not: The presence or absence of a SOAR effect differed by the presence or absence of a SNARC effect ($p < 0.001$ for both AUC and MD, Fisher's Exact Test). Inspection of these clusters revealed that a majority of participants (MD: 30 out of 44; AUC: 25 out of 44) were in corresponding clusters for SNARC and SOAR: either showing a standard effect for both SNARC and SOAR, a reversed effect for both, or no effect for both.

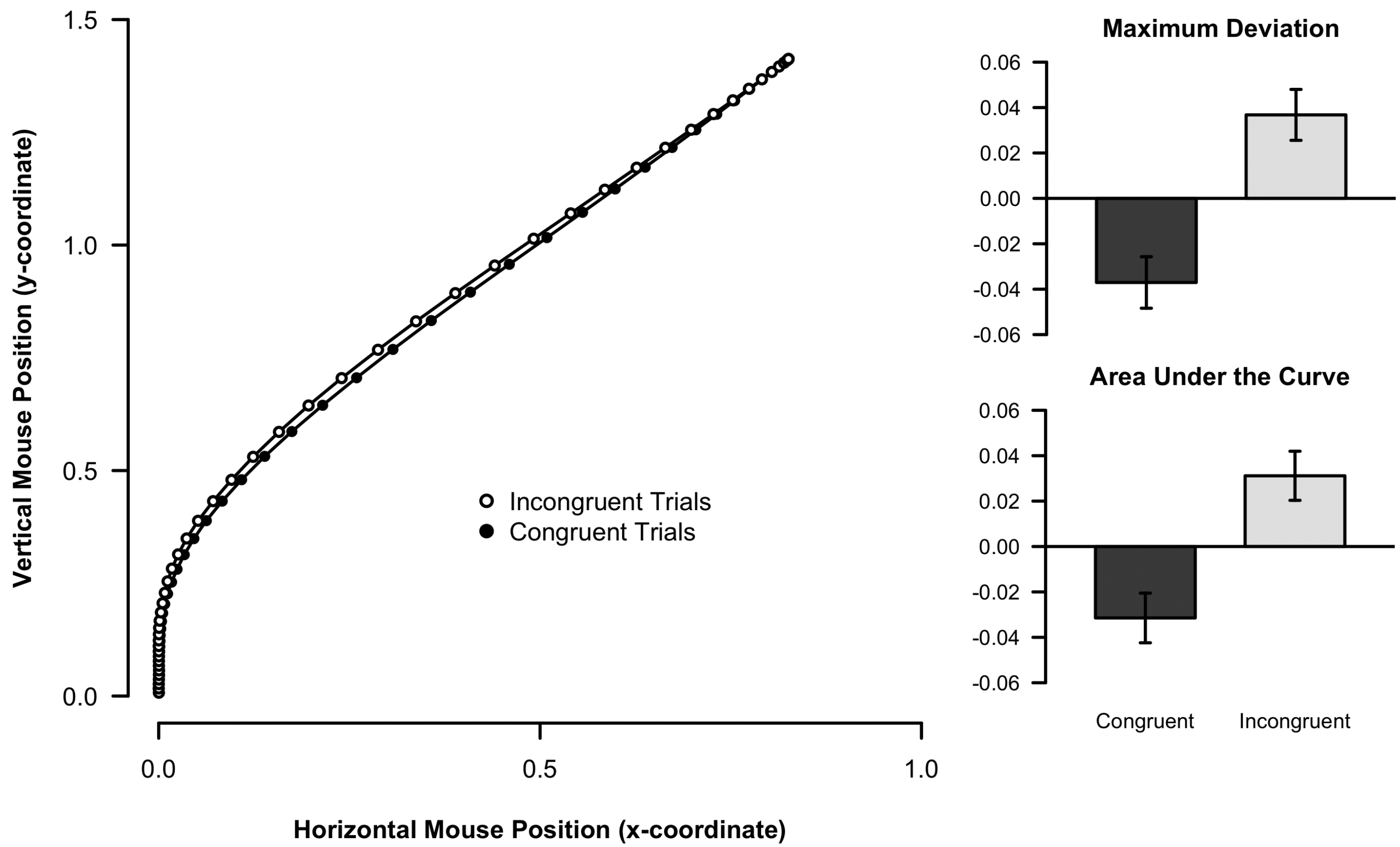


Figure 3.2. Spatial deflection of incongruent trajectories. Mean hand trajectories on trials where the motion was incongruent or congruent with the arithmetic operation, remapped rightwards for comparison (left panel). Circles indicate even time-steps from 0 to 100. SOAR-incongruent trajectories were reliably deflected in the opposite direction, as indicated by significantly larger Maximum Deviation (top right) and Area Under the Curve (bottom right). MD and AUC were normalized by subject before plotting. Error bars show Standard Error of the Mean.

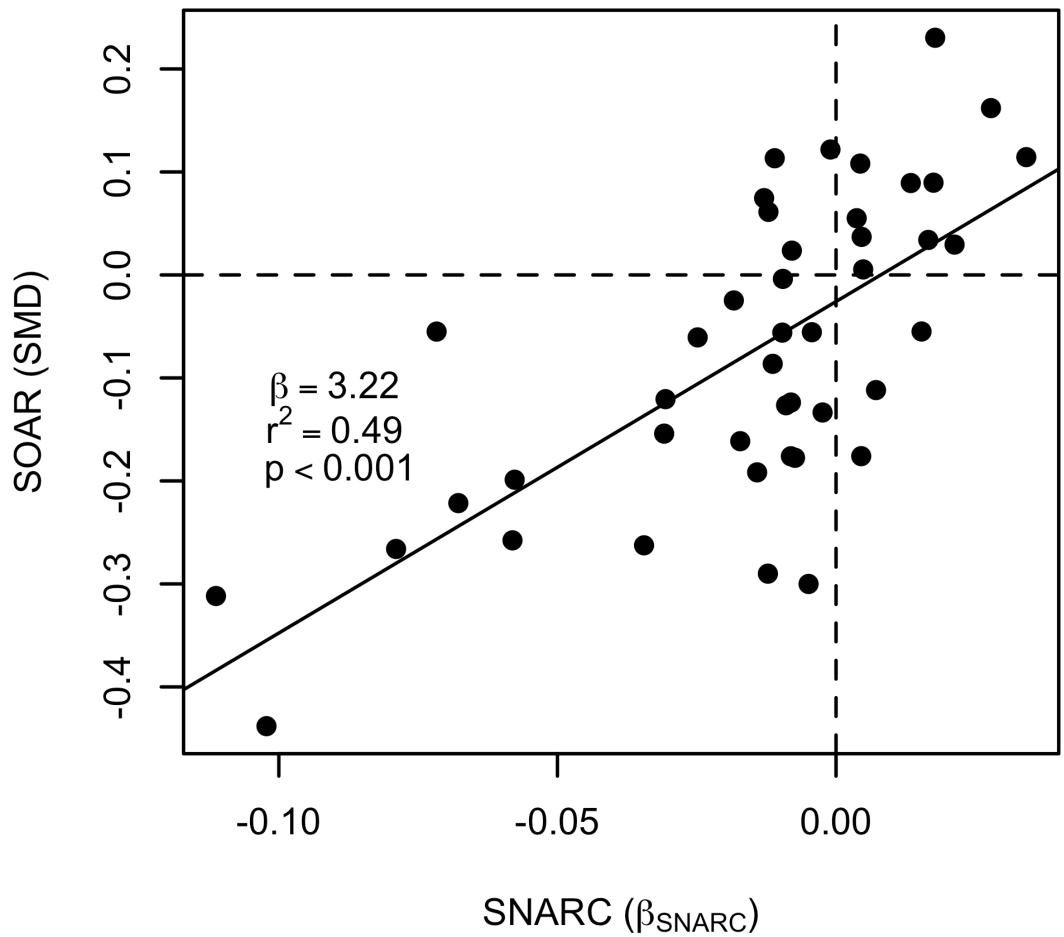


Figure 3.3. Relations between SNARC and SOAR. Individuals' SNARC effect (horizontal axis) and SOAR effect (vertical axis), calculated on the basis of MD. For both axes, negative values indicate larger canonical effects (following Fias et al, 1996). The size of an individual's SNARC effect was significantly predictive of their SOAR effect. The solid line shows the least squares regression of SOAR onto SNARC. Points below the horizontal doted line indicate participants with a canonical SOAR effect; those to the left of the vertical doted line indicate a canonical SNARC effect.

In sum, an individual's sensitivity to the congruency between the location of a response button and the magnitude of the response (i.e. SNARC effect) was coupled to their sensitivity to the congruency between arithmetic operation and the direction of motion (i.e. SOAR effect). This was true despite the fact that the SOAR effect was distinct from the SNARC-like effect of the final solution's magnitude. The spatial deflection of hand trajectories due to the arithmetic operation was therefore distinct from, but related to, any spatial deflection due to numerical magnitude.

3.3.3 Timecourse of spatial processing

Tracking the real-time trajectory of the hand in motion allows us to evaluate not only the global properties of the response, but also the dynamic timecourse of spatial deflection. To do so, we conducted a series of pairwise t-tests of the mean x-coordinates of SOAR-Congruent and SOAR-Incongruent trajectories at each normalized time-step, using an α-level of .05. To correct for multiple comparisons, we conducted a bootstrap simulation ($n=1000$) to estimate the number of significant t-tests that would be expected by chance alone (Dale, Kehoe, and Spivey, 2007). This simulation revealed that random variability alone should have produced significant differences at 11 or more consecutive time-steps only 1.6% of the time; and at 12 or more consecutive time-steps, only 0.7% of the time. We therefore settled on eleven consecutive significant time-steps as a threshold for statistical significance, assuring a false positive rate of $p < 0.05$.

Pairwise t-tests comparing the horizontal deflection of SOAR-Incongruent to SOAR-Congruent trajectories first reached statistical significance halfway through the trial—on average, 734ms after the onset of the second term—and remained significant until 75% through the trajectory (Fig. 3.4A). Congruent and incongruent trajectories, therefore, differed significantly at 25 consecutive time-steps, a highly significant divergence ($p < 0.001$).

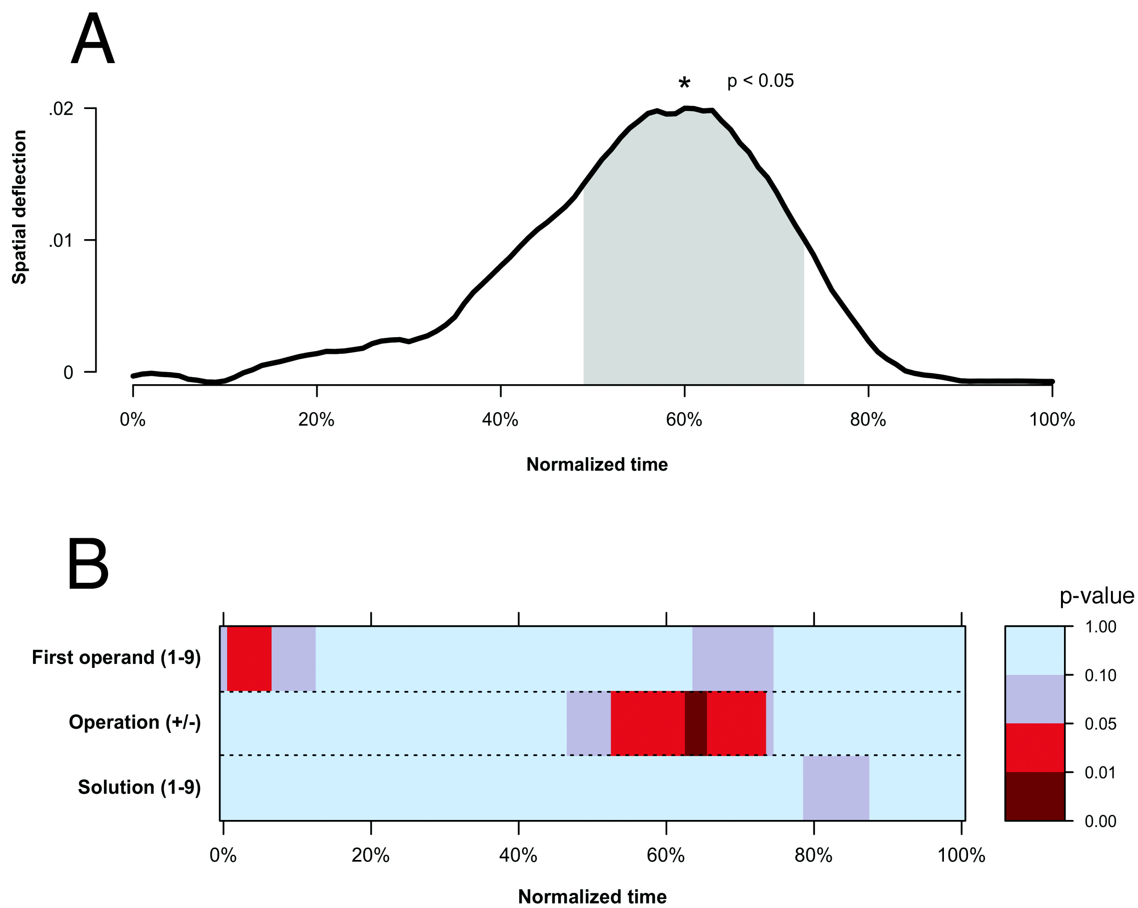


Figure 3.4. Timecourse of spatial perturbations. (A) Timecourse of the spatial attraction due to arithmetic operation. Normalized time is plotted along the horizontal axis, from start (0%) to end (100%) of the trial. The horizontal distance between congruent and incongruent trajectories is plotted on the vertical axis. The grey area indicates the period during which this spatial deflection reached significance. (B) Hand trajectories revealed a cascade of distinct spatial influences. Color indicates the statistical significance of each problem component at each time-point; corresponding *p*-values are indicated in the legend at right. There was an early influence of the first number, deflecting hand trajectories toward the canonical side of egocentric space (left for small, right for large numbers). Halfway through the trajectory, the arithmetic operation began to affect concurrent manual action. The final solution had a marginal influence toward the end of the trial.

As an exploratory analysis, we next looked at the timecourse of spatial deflections due to various sub-parts of the arithmetic problems: the magnitude of the first number, the arithmetic operation, and the magnitude of the final solution. When calculating “6 - 2 = 4,” for instance, at what point is motor activity influenced by the facts that the first term is greater than 5, that the operation is subtraction, or that the final solution is less than 5? To answer this, we analyzed the mean horizontal position (x-coordinate) at each time-point using a repeated measures ANOVA with the following three factors: SNARC-congruency associated with the first term; SOAR-Congruency associated with the arithmetic operation; and SNARC-congruency associated with the final solution[2].

In accord with the Spatial Account, we found a cascade of spatial perturbations (Fig. 3.4B). Recall that participants were able to begin moving the cursor as soon as the second term appeared on the screen. By the time participants could start moving, therefore, they had already seen the first term for a full second. In line with this, there was a very early effect of the relative magnitude of the first term, deflecting the trajectory toward the corner that was congruent with the term's magnitude (left for numbers less than 5, right for numbers greater than 5). This influence was already marginally significant at the first time point, and lasted for the first 13% of the trajectory. Next, halfway through the trajectory, the effect of SOAR-congruency kicked in, deflecting the trajectory in the direction congruent with the arithmetic operation. This influence of arithmetic operation lasted from 47% to 74% of the trajectory. For the last part of this period, there was again a marginal influence of the first term's magnitude. Finally, towards the very end of the trajectory, there was a marginal influence of the final solution's magnitude, from 79% to 87% of the trajectory. Participants' hand trajectories, therefore, revealed a cascade of distinct, sequential spatial influences: starting

---

[2] We did not include a factor for the second term because it only ranged from 0 to 3.

with the first term, an anchor of sorts for the calculation; followed by the arithmetic operation; and finally, the solution (Fig. 3.4B). Although these analyses, unlike the previous timecourse analyses, are not corrected for multiple comparisons, they may capture subtle contributions of early and late spatial-numerical associations, in coordination with spatial-arithmetic associations. Since the current experiment used a design in which the first operand, the arithmetic operation, and the second operand were presented in order, it remains to be seen whether the same cascade of spatial influences appears when the entire problem is presented simultaneously rather than sequentially.

3.4 Discussion

During mental addition and subtraction, participants' hand movements were deflected dynamically to the right and left (Fig. 3.2, 3.4B), respectively, suggesting that both mental arithmetic and motor control rely on shared resources for controlling spatial attention. This was true despite the fact that the calculation was exact and symbolic, rather than approximate or non-symbolic. While these results do not contradict the Compression or Heuristic accounts of Operational Momentum (e.g. McCrink and Wynn, 2009; Chen and Verguts, 2012), the observed spatial-arithmetical biases are neither explained nor predicted by these non-spatial alternatives. Correct responses were controlled for location (left, right) and relative magnitude (greater or lesser of the responses), so spatial biases were not due to initial over- or under-estimation. Spatial-arithmetical biases, moreover, contributed above and beyond biases associated with the final solution, reinforcing their distinctly *arithmetical* character. We thus observed for the first time that calculation—even when exact and symbolic—is associated unequivocally with systematic spatial biases.

3.4.1 Could exact calculation rely on an integrated system of spatial resources?

We turn now to an outstanding question: What might this spatial processing actually *do* during calculation? After all, spatial-arithmetical and spatial-numerical associations may be epiphenomenal; spatial processing could be entirely downstream from the cognitive work of calculation. This is a general concern about research conducted under the umbrella of Grounded Cognition. If thinking about dogs, for instance, prompts visual imagery of dogs, this might be due to spreading activation from abstract "dog" concepts to associated visual percepts, without visual processing contributing to conceptual representation (Mahon and Caramazza, 2008). Both spatial-arithmetical and spatial-numerical biases, similarly, could reflect simple associations between distinct neural circuits responsible for calculation and for spatial attention.

What, then, are some plausible contributions to mental calculation of systematic spatial processing? We briefly consider three: computing the exact or approximate solution; supplying intuitions that complement and possibly constrain rote, algorithmic strategies; and scaffolding the learning of arithmetic.

First, spatial processing may help determine the solution of a calculation. This is the heart of the Spatial Account: numbers are mapped to locations along a mental number-line, and then arithmetic is computed by simulating movement along that number-line. Biases in spatial processing would thus produce the systematic over- and under-estimation that characterizes Operational Momentum (McCrink et al, 2007). But to make this functional contribution, diverse spatial resources need to be integrated appropriately. Recall that spatial processing during arithmetic is thought to rely on the posterior superior parietal lobule (PSPL; Knops et al, 2009a); interactions between number and space, by contrast, are thought to occur within the intraparietal sulcus (IPS; Dehaene et al, 2003; Hubbard et al, 2005). These neural circuits need to be coordinated in at least two ways: in the way they recruit space and in their timecourse. First, they need to recruit space in a coordinated fashion, with arithmetic-related shifts in spatial attention aligned with spatial representations of number (e.g. associating right-space with both large numbers and addition). Given that nearly a third of participants typically show no or reversed SNARC effects (e.g. Cipora and Nuerk, 2013), these spatial associations should sometimes be reversed (i.e. associating right-space with both small numbers and subtraction). Second, these spatial resources must coordinate temporally: first associating the initial operand with a location and then deploying more posterior neural resources to shift attention. In short, the neural resources responsible for spatial-arithmetic and spatial-numerical associations must form an integrated system, coordinated both in the way they recruit space and in their timecourse.

There were hints that these spatial-arithmetic biases were, indeed, part of an integrated spatial system for processing both numerical magnitude and arithmetic. For starters, we found evidence that calculation was accompanied by a cascade of spatial perturbations (Fig. 3.4B), due initially to the first term, then to the arithmetic operation, and finally to the solution—although this may have been a product of the experiment's sequential design. Spatial biases associated with numerical magnitude and arithmetic, moreover, were distinct but coupled: individuals' spatial-numerical biases reliably predicted the size and direction of their spatial-arithmetical biases, and more than two-thirds of participants exhibited spatial-arithmetical biases that were coordinated with their spatial-numerical biases (e.g. they associated both subtraction and smaller numbers with the left; Fig. 3.3). If this coordination is necessary for the spatial system to play a functional role in calculation, then we should see improved performance among individuals with coordinated spatial-arithmetical and spatial-numerical biases—that is, individuals should perform better on calculation tasks if they have the same spatial association (either left or right) for both larger numbers and addition. Suggestively, there was a trend toward better performance among such participants. They made fewer errors (M=2.3 vs. M=3.2), responded faster (M=1454ms vs. M=1494ms), and produced trajectories with less deflection (MD=0.18 vs. MD=0.21), although none of these differences were statistically significant (all ps > .2). In short, mental arithmetic prompted a series of coordinated but distinct spatial deflections, unfolding over time throughout the process of calculation. The origin of this coordination is an open question. Spatial-numerical and spatial-arithmetical biases may have a common origin—perhaps a general predilection to associate abstract notions with space, or experience with cultural artifacts that associate both numbers and arithmetic with space (e.g. number-lines). Alternatively, one spatial association may build on the other, so that, for instance, spatial-arithmetical biases may derive from pre-existing, culturally-shaped spatial-numerical biases. The coordination of SOAR and SNARC—its source and implications—is ripe for investigation.

A second potential functional role for spatial processing is to supply intuitions that complement rote, algorithmic calculation. To re-purpose a military aphorism, "quantity has a quality all its own." Correct calculations often just *feel* right—and spatial intuitions are a good candidate for the source of this quality of quantity. In the case of incorrectly recalled arithmetic facts or algorithmic errors (e.g. "operation errors" like 20 x 3 = 23, where multiplication is confused for addition; Campbell, 1994), the subjective "quality of quantity" can flag these errors if the solution violates our spatial intuitions (i.e. 20 x 3 should be considerably greater than 23!). In this way, spatial processing may provide an intuitive check on rote or algorithmic calculation, supplying a rough sense of expected magnitude against which the algorithmically-derived solution can be compared. Individuals who deploy spatial processing during symbolic calculation should thus be insulated against gross errors due to the misapplication of an algorithm.

Third and finally, spatial processing may support initial learning during development, supplying a spatial scaffold for the acquisition of abstract arithmetical concepts and procedures (Núñez and Marghetis, in press). Early spatial skills are highly predictive of long-term mathematical success (for review, see Mix and Cheng, 2012). This correlation, moreover, is mediated by the ability to map numbers to a physical number-line in a linear fashion (Gunderson et al, 2012), and game-based intervention studies with children have found that training this linear number-space mapping improves number estimation and calculation (Siegler and Ramani, 2009). Conversely, a failure to deploy spatial resources may contribute to Mathematics Learning Disability (e.g. Geary, 1993). Additionally, spatial processing may give meaning and value to otherwise meaningless calculations, improving children's affective relation to mathematics and increasing the likelihood they'll gravitate towards Science, Technology, Engineering, and Mathematics (STEM) fields.

3.4.2 Beyond simple calculation

An integrated spatial system, therefore, may contribute in a variety of ways to calculation. But as mathematical expertise develops, this system may be re-tooled for new purposes. Goldstone, Landy, and Son (2010) argued that solving equations relies on perceptual systems "rigged up" for symbol manipulation (see also Schneider et al, 2012). On their proposal, solving equations involves a visuospatial simulation of moving terms from one side of the equation to the other. In support of this, they report that the ability to solve equations was selectively impaired when concurrently viewing incongruent motion (e.g. rightward motion when a term is to be "moved" leftward). What's more, this effect was strongest in participants with more mathematical training; mathematical expertise was associated with more, not less, use of a visuospatial strategy. This suggests one reason why Cipora and Nuerk (2013) failed to find a relation between the SNARC effect and performance on an equation verification task: Verifying equations might require the use of the spatial system to simulate the motion of the equation's terms (as proposed by Goldstone et al, 2010) rather than to represent numerical magnitude and arithmetic, as manifest in SNARC and SOAR effects[3]. Furthermore, when mathematics PhD students collaborate on proofs, they complement their technical, non-spatial language with gestures that express dynamic, spatial reasoning (Marghetis and Núñez, 2013), confirming Hadamard's classic claim that expert mathematicians rely on spatial or sensorimotor intuitions (1954). This suggests a productive way to think about the relation between space and mathematics: different mathematical activities (e.g. calculation vs. equation verification) may require distinct assemblies of spatial resources, recruited and coordinated by cultural practices. Calculation may rely on spatial-numerical representations coupled with shifts in attention;

---

[3] Giaquinto (2007) distinguishes between *syntactic* and *semantic* manipulation of symbols, which may relate to the use of space to simulate *movement* of the terms rather than to ground the calculation in *meaningful* spatial intuitions.

algebra may use similar resources, rigged up differently to support the internal manipulation of external inscriptions.

More generally, the present study contributes to a growing body of evidence that abstract thought in general—and mathematical cognition in particular—is tightly and dynamically coupled to perception and action (Barsalou, 2008; Lakoff and Núñez, 2000; Spivey, 2007). This entangling of body and mind is often manifest in the hands. We have shown here, for instance, that hand movements reflect the spatial character of addition and subtraction, adding to the literature on how hand trajectories can reveal the dynamics of thought (Freeman et al, 2011). But the hands take place of prominence even when they are not directly called upon by the task. Situated mathematical practice requires the hands to interact with external artifacts—equations, diagrams, computers. And during communication, manual gestures reflect speakers' sensorimotor or spatial simulations (e.g. Hostetter & Alibali, 2008) and also shape the simulations of both listener and speaker (e.g. Wu and Coulson, 2007; Alibali et al, 2011; for review, see Marghetis and Bergen, *in press*). This is particularly true in mathematics, where gesture reveals spatial conceptualizations of abstract concepts in calculus (Marghetis and Núñez, 2013; Núñez, 2006; Marghetis, Edwards, and Núñez, in press) and arithmetic (Marghetis, in preparation; Núñez and Marghetis, in press) and can even give the gesturer entirely new ideas (Goldin-Meadow et al, 2009). One possible account of these varied online interactions between body and mind is that evolutionarily-older neural resources (Anderson, 2010; Dehaene and Cohen, 2007), recruited and regimented by cultural practices and artifacts (Hutchins, 2008; Núñez, 2011), are redeployed during advanced cognitive activities like mathematics, thus grounding abstract thought in action and space.

3.5 Conclusions

Converging evidence suggests that mathematics builds upon a foundation of spatial skills (Mix and Cheng, 2012; Núñez and Marghetis, in press). Here we demonstrated, for the first time, that *exact, symbolic* calculation is accompanied by systematic spatial processing. The arithmetic operation influenced the spatio-temporal dynamics of participants' concurrent motor activity while they were engaged in exact arithmetic. We argued that this reflected the deployment of a coordinated system of spatial resources, co-opted to run a mental simulation of abstract motion along a spatial representation of number. Spatial processing may play a number of roles, from helping compute the outcome of a calculation, to supplying meaning during mathematical development. This spatial processing during arithmetic, moreover, is an instance of a more general strategy in which we associate abstract objects with spatial locations and then take advantage of our evolved spatial skills to support reasoning. Learning and doing mathematics may involve navigating metaphorical spaces.

3.6 Acknowledgments

Thanks to Brock Hazen Dumville, Esther Walker, and Bodo Winter for helpful discussions, and to two anonymous reviewers for productive criticism. We are grateful to the undergraduate research assistants who helped with data collection: Luke Eberle, Alec Gasperian, Chau Nguyen, and Kendall Youngstrom. TM was supported by a doctoral fellowship from the Fonds de recherche sur la société et la culture (Québec, Canada) and a Glushko Fellowship.

Chapter 3, in full, is a reprint of the material as it appears in the *Quarterly Journal of Experimental Psychology*. Marghetis, T.; Núñez, R.; Bergen, B.K., 2014. The dissertation author was the primary investigator and author.

3.7 References

Alibali, M. W., Spencer, R. C., Knox, L., & Kita, S. (2011). Spontaneous Gestures Influence Strategy Choices in Problem Solving. *Psychological Science,* 22, 1138-1144.

Anderson, M. L. (2010). Neural reuse: A fundamental organizational principle of the brain. *Behavioral and Brain Sciences* 33, 245–313

Barr, D.J., Levy, R., Scheepers, C., & Tily, H.J. (2013). Random effects structure for confirmatory hypothesis testing: Keep it maximal. *Journal of Memory and Language,* 68, 255-278.

Barsalou, L.W. (1999). Perceptual symbol systems. *Behavioral and Brain Sciences,* 22, 577-660.

Barsalou, L.W. (2008). Grounded cognition. *Annual Review of Psychology, 59,* 617-645

Beecham, R., Reeve, R.A., & Wilson, S.J. (2009). Spatial representations are specific to different domains of knowledge. *PlosOne, 4,* 1-5.

Campbell, J. I. (1994). Architectures for numerical cognition. *Cognition*, 53, 1-44.

Chen, Q., and Verguts, T. (2012). Spatial intuition in elementary arithmetic: a neurocomputational account. *PLoS ONE* 7:e31180.

Cipora, K., & Nuerk, H. C. (2013). Is the SNARC effect related to the level of mathematics? No systematic relationship observed despite more power, more repetitions, and more direct assessment of arithmetic skill. *The Quarterly Journal of Experimental Psychology*, (ahead-of-print), 1-18.

Dale, R., Kehoe, C., & Spivey, M. J. (2007). Graded motor responses in the time course of categorizing atypical exemplars. *Memory & Cognition*, 35, 15-28.

de Hevia, M. D., & Spelke, E. S. (2009). Spontaneous mapping of number and space in adults and young children. *Cognition*, *110*(2), 198-207.

Dehaene, S. (2003). The neural basis of the Weber–Fechner law: a logarithmic mental number line. *Trends in cognitive sciences*, *7*(4), 145-147.

Dehaene, S., Bossini, S. & Giraux, P., (1993). The mental representation of parity and number magnitude. *Journal of Experimental Psychology: General, 3,* 371-396.

Dehaene, S., and Cohen, L. (2007). Cultural Recycling of Cortical Maps. *Neuron,* 56, 384-398.

Dehaene, S., Piazza, M., Pinel, P., & Cohen, L., (2003). Three parietal circuits for number processing, *Cognitive Neuropsychology, 20,* 487-506.

Dotan, D., & Dehaene, S. (2013). How do we convert a number into a finger trajectory? *Cognition*, 129, 512-529.

Dove G. (2009). Beyond perceptual symbols: A call for representational pluralism. *Cognition,* 110, 412–431.

Fias, W., Brysbaert, M., Geypens, F., & d'Ydewalle, G. (1996). The importance of magnitude information in numerical processing: Evidence from the SNARC effect. *Mathematical Cognition*, 2(1), 95–110.

Fischer, M. H. (2008). Finger counting habits modulate spatial-numerical associations. *Cortex*, *44*(4), 386-392.

Fischer, M. H., Castel, A. D., Dodd, M.D., & Pratt, J. (2003). Perceiving numbers causes spatial shifts of attention. *Nature Neuroscience,* 6, 555–556.

Fischer, M. H., Warlop, N., Hill, R. L., & Fias, W. (2004). Oculomotor bias induced by number perception. *Experimental Psychology*, 51(2), 91-97.

Freeman, J. B., and Ambady, N. (2010). MouseTracker: software for studying real-time mental processing using a computer mouse-tracking method. *Behavior Research Methods*, 42, 226–241.

Freeman, J., Dale, R., & Farmer, T. (2011). Hand in Motion Reveals Mind in Motion. *Frontiers in Psychology*, *2*, 59.

Gallese, V. & Lakoff, G. (2005) The brain's concepts: The role of the sensory-motor system in conceptual knowledge. *Cognitive Neuropsychology,* 22, 455 – 79.

Geary, D. C. (1993). Mathematical disabilities: cognitive, neuropsychological, and genetic components. *Psychological bulletin*, *114*(2), 345.

Giaquinto, M. (2007). *Visual thinking in mathematics*. Oxford: Oxford University Press.

Glenberg, A. M., & Kaschak, M. P. (2002). Grounding language in action. *Psychonomic Bulletin & Review*, 9 (3), 558-565.

Glenberg, A. M., Sato, M., & Cattaneo, L. (2008). Use-induced motor plasticity affects the processing of abstract and concrete language. *Current Biology,* 18, R290-R291.

Goldin-Meadow, S., Cook, S. W., & Mitchell, Z. A. (2009). Gesturing gives children new ideas about math. *Psychological Science*, *20*(3), 267-272.

Goldstone, R., Landy, D., & Son, J.Y. (2010). The education of perception. *Topics in Cognitive Science*, *2*, 265-284.

Gunderson, E. A., Ramirez, G., Beilock, S. L., & Levine, S. C. (2012). The relation between spatial skill and early number knowledge: The role of the linear number line. *Developmental psychology*, *48*(5), 1229.

Hadamard, J. (1954). The Psychology of Invention in the Mathematical Field. New York: Dover.

Hostetter, A. B., & Alibali, M. W. (2008). Visible embodiment: Gestures as simulated action. *Psychonomic Bulletin and Review*, 15, 495-514.

Hubbard, E.M., Piazza, M., Pinel, P., & Dehaene, S., (2005). Interactions between number and space in parietal cortex, *Nature Reviews Neuroscience, 6,* 435-448.

Hurley, S. L. (2008) The shared circuits model (SCM): How control, mirroring, and simulation can enable imitation, deliberation, and mindreading. *Behavioral and Brain Sciences*, 31, 1–58.

Hutchins, E. (2008). The role of cultural practices in the emergence of modern human intelligence. *Philosophical Transactions of the Royal Society B: Biological Sciences*, *363*(1499), 2011-2019.

Ito, Y., & Hatta, T. (2004). Spatial structure of quantitative representation of numbers: Evidence from the SNARC effect. *Memory & Cognition*, 32, 662–673

Kaschak, M. P., Madden, C. J., Therriault, D. J., Yaxley, R. H., Aveyard, M. E., Blanchard, A. A., & Zwaan, R. A. (2005). Perception of motion affects language processing. *Cognition*, 94, B79–B89.

Knops, A., Thirion, B., Hubbard, E. M., Michel, V., & Dehaene, S. (2009a). Recruitment of an area involved in eye movements during mental arithmetic. *Science, 324*, 1583–1585.

Knops, A., Viarouge, A., & Dehaene, S. (2009b). Dynamic representations underlying symbolic and nonsymbolic calculation: Evidence from the operational momentum effect. *Attention, Perception, & Psychophysics*, 71, 803-821.

Knops, A., Zitzmann, S., & McCrink, K. (2013). Examining the presence and determinants of operational momentum in childhood. *Frontiers in Psychology*, *4*, 325.

Lakoff G., & Núñez, R. (2000). Where Mathematics Comes From: How the Embodied Mind Brings Mathematics Into Being. New York: Basic Books.

Langacker, R. W. (1987). *Foundations of Cognitive Grammar, Vol, 1: Theoretical Prerequisites*. Stanford, CA: Stanford University Press.

Lindemann, O., Abolafia, J. M., Girardi, G., & Bekkering, H. (2007). Getting a grip on numbers: numerical magnitude priming in object grasping. *Journal of Experimental Psychology: Human Perception and Performance*, *33*(6), 1400.

Loetscher, T., Bockisch, C., Nicholls, M. E. R., & Brugger, P. (2010). Eye position predicts what number you have in mind. *Current Biology*. 20, R264–R265.

Loetscher, T., Schwarz, U., Schubiger, M., & Brugger, P. (2008). Head turns bias the brain's random number generator. *Current Biology*. 18, R60–R62.

Mahon, B.Z., & Caramazza, A. (2008). A critical look at the Embodied Cognition Hypothesis and a new proposal for grounding conceptual content. Journal of Physiology - Paris, 102, 59-70.

Marghetis, T., & Bergen, B. (*in press*). Embodied meaning, inside and out: The coupling of gesture and mental simulation. In Cornelia Müller, Alan Cienki, Ellen Fricke, Silva H. Ladewig, David McNeill & Sedinha Tessendorf (Eds.), *Body-Language-Communication*. New York: Mouton de Gruyter.

Marghetis, T. (*in preparation*). Bigger, higher, and both: Blended space in mathematical gesture.

Marghetis, T., Edwards, L., & Núñez, R (*in press*). More than mere handwaving: Gesture and embodiment in expert mathematical proof. In L. Edwards, F. Ferrara, and D. Moore-Russo (Eds.), *Emerging Perspectives on Gesture and Embodiment in Mathematics*. Charlotte, NC: IAP-Information Age Publishing.

Marghetis, T. & Núñez, R. (2013). The motion behind the symbols: A vital role for dynamism in the conceptualization of limits and continuity in expert mathematics. *Topics in Cognitive Science*, 5, 299-316.

Matlock, T. (2004). Fictive motion as cognitive simulation. *Memory & Cognition*, 32, 1389-1400.

McCrink, K., Dehaene, S., & Dehaene-Lambertz, G. (2007). Moving along the number line: Operational momentum in nonsymbolic arithmetic. *Perception and Psychophysics, 69*, 1324-1333.

McCrink, K., & Wynn, K. (2009). Operational momentum in large-number addition and subtraction by 9-month-olds. *Journal of Experimental Child Psychology*, *103*(4), 400-408.

Miles, L. K., Betka, E., Pendry, L. F., & Macrae, C. N. (2010). Mapping temporal constructs: actions reveal that time is a place. *The Quarterly Journal of Experimental Psychology*, *63*(11), 2113-2119.

Mix, K. S., & Cheng, Y. L. (2012). The relation between space and math: developmental and educational implications. *Advances in child development and behavior*, *42*, 197.

Nieder, A., & Miller, E. K. (2003). Coding of cognitive magnitude: Compressed scaling of numerical information in the primate prefrontal cortex. *Neuron*, *37*(1), 149-157.

Núñez, R. (2006). Do Real Numbers Really Move? Language, Thought, and Gesture: The Embodied Cognitive Foundations of Mathematics. Reprinted in R. Hersh (Ed.), *18 Unconventional Essays on the Nature of Mathematics* (pp. 160-181). New York: Springer.

Núñez, R. (2011). No Innate Number Line in the Human Brain. *Journal of Cross-Cultural Psychology*, 45, 651-668.

Núñez, R., Doan, D, Nikoulina, A. (2011). Squeezing, Striking, and Vocalizing: Is Number Representation Fundamentally Spatial? *Cognition*, 120, 225-235.

Núñez, R., & Marghetis, T. (*in press*). Cognitive Linguistics and the Concept(s) of Number. In R. Cohen-Kadosh and K. Dowker (eds.), *Oxford Handbook of Numerical Cognition.* Oxford University Press.

Pinhas, M., & Fischer, M. H. (2008). Mental movements without magnitude? A study of spatial biases in symbolic arithmetic. *Cognition, 109*, 408–415.

Saygin, A.P., McCullough, S., Alac, M., & Emmorey, K. (2010). Modulation of BOLD response in motion sensitive lateral temporal cortex by real and fictive motion sentences. *Journal of Cognitive Neuroscience*, 22 (11), 2480-90.

Schneider, E., Maruyama, M., Dehaene, S., & Sigman, M. (2012). Eye gaze reveals a fast, parallel extraction of the syntax of arithmetic formulas. *Cognition*, 125, 475-490.

Schwarz, W., & Keus, I.M., (2004). Moving the eyes along the mental number line: Comparing SNARC effects with saccadic and manual responses. *Perception & Psychophysics*, 66, 651–664

Shaki, S., Fischer, M. H., & Petrusic, W. M. (2009). Reading habits for both words and numbers contribute to the SNARC effect. *Psychonomic Bulletin & Review*, *16*, 328-331.

Siegler, R. S., & Opfer, J. E. (2003). The Development of Numerical Estimation Evidence for Multiple Representations of Numerical Quantity. *Psychological Science*, *14*(3), 237-250.

Siegler, R. S., & Ramani, G. B. (2009). Playing linear number board games—but not circular ones—improves low-income preschoolers' numerical understanding. *Journal of Educational Psychology*, *101*(3), 545.

Song, J.-H., & Nakayama, K. (2008). Numeric comparison in a visually-guided manual reaching task. *Cognition*, 106, 994-1003.

Spivey, M. (2007). *The continuity of mind*. New York: Oxford University Press.

Spivey, M. J., Grosjean, M., & Knoblich, G. (2005). Continuous attraction toward phonological competitors. *Proceedings of the National Academy of Sciences of the United States of America*, *102*, 10393-10398.

Tzelgov, J., Meyer, J., & Henik, A. (1992). Automatic and intentional processing of numerical information. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *18*(1), 166.

Wu, Y.C. & Coulson, S. (2007). How iconic gestures enhance communication: An ERP study. *Brain & Language*, 101, 234-245.

Appendix: List of arithmetic problems

First number	Operation	Second Number	Solution (addition / subtraction)
3	+/-	0	3 / 3
3	+/-	1	4 / 2
3	+/-	2	5 / 1
4	+/-	0	4 / 4
4	+/-	1	5 / 3
4	+/-	2	6 / 2
4	+/-	3	7 / 1
5	+/-	0	5 / 5
5	+/-	1	6 / 4
5	+/-	2	7 / 3
5	+/-	3	8 / 2
6	+/-	0	6 / 6
6	+/-	1	7 / 5
6	+/-	2	8 / 4
7	+/-	0	7 / 7
7	+/-	1	8 / 6

Chapter 4

Does abstract mathematical reasoning involve spatial metaphors?

ABSTRACT

We often rely on analogies or metaphors to ground our reasoning about one domain in our understanding of another, but little is known about the role of metaphorical thought in a paragon of abstract thought: mathematics. There are hints, however, that our understanding of arithmetic may rely on spatial metaphors: in speech, descriptions of number often rely on spatial constructions (e.g., "bigger" or "higher" numbers); during rapid comparison, numerical magnitude induces biases to respond spatially. But mathematics is more than naming numbers and making approximate comparisons. Does careful, reflexive mathematical reasoning involve spatial metaphors? To address this question, we combined observation and experiment to analyze spontaneous co-speech gesture, which served as a window on real-time, dynamic reasoning. Gestures produced during mathematical reasoning revealed two complementary gestural systems: *Path* gestures in which arithmetic was construed metaphorically as *motion along a path*; *Collection* gestures in which it was construed as the *manipulation of collections of objects* (Studies 1-2). Performing path- or collection-related mental imagery primed the production of one gesture system over the other (Study 2), suggesting that metaphorical gestures are not merely conventionalized ways of communicating but reflect the real-time deployment of spatial models. We conclude that metaphorical gestures are both private and public: private, since they reflect speakers' internal spatial simulation; public, as a shared semiotic resource.

4.1. Introduction

Humans have a singular ability to reason about the absent and the abstract: far-flung friends and possible worlds, futures and pasts, fractions and prime numbers. While these concepts denote entities that are inaccessible to our perceptual apparatus—either in practice or in principle—they are grounded nevertheless in rich, layered dispositions to move, act, feel, and think about the concrete (Bourdieu, 1977; Lakoff and Johnson, 1980). Mathematics is exemplary in this regard. Numbers and simple calculations induce systematic dispositions to react spatially: larger numbers and addition induce rightward shifts in attention, while smaller numbers and subtraction prompt leftward shifts (Dehaene et al, 1993; McCrink et al, 2007; Knops et al, 2009; Marghetis et al, 2014). Spatial biases for number and arithmetic have been localized to posterior regions of parietal cortex that control manual grasping and spatial attention (Hubbard et al, 2005; Knops et al, 2009). These findings suggest that the neural processing of number and arithmetic recycles more "embodied" neural resources that are specialized for processing action and space (Anderson, 2010; Barsalou, 1999; Dehaene and Cohen, 2007; Walsh, 2003; Winter, Marghetis, and Matlock, 2015).

But our mathematical competence outstrips simple skills like rote numerical comparison or calculation. Not only can we *do* arithmetic, but we can *think about* arithmetic, reflecting on numbers as abstract entities and reasoning about their abstract properties and relations. This reflexive capacity may distinguish human mathematical competence from the simple numerical capacities we share with non-human animals. Human infants and non-human primates can perform simple approximate "addition," predicting the approximate numerosity of a collection of objects formed by combining two other collections (Barth et al, 2006; Flombaum, Junge, and Hauser, 2005). But for babies and macaques, mathematics ends with such reckoning or numerical estimation. By contrast, mature human practices of counting, adding, and subtracting are imbued with meaning and interpreted in virtue of rich, structured conceptual systems (Sfard, 2009; Lakoff and Núñez, 2000). We calculate, but we also *conceptualize*; we reckon, but we also *reason*. Human mathematical reasoning is complex and meaningful, a far cry from rote calculation or meaningless symbol manipulation.

Does precise, reflexive mathematical reasoning rely on space? There are hints that it may, most obviously in linguistic and graphical practice. Many languages, English included, use spatial language to describe numerical properties and relations (Lakoff and Núñez, 2000). These linguistic metaphors for number are systematic, productive, and old. Numbers are "higher" in the absence of literal height, "bigger" in the absence of literal size. We might describe a utility bill as "sky high" if it were greater than expected, creatively using height to describe numerical magnitude. And the Oxford English Dictionary dates the non-spatial sense of "higher" to the 13th century, if not earlier, quoting an Old English homily in which a greater reward was described as a *heahere mede*, a "higher reward" (Higher, 2015). Judging from conventional language, at least, number and space are tightly coupled.

The contribution of space is also apparent in graphical representations, such as Cartesian graphs, in which strict norms dictate how number should be spatialized. Mathematical concepts are thus spatialized in notations and diagrams, both by educators in teaching but also by experts in practice (Giaquinto, 2007). These contemporary artifacts and practices, however, are the residues of a long history of conceptual innovation, and their use today requires no more than a superficial appreciation for the coupling of space and number. Similarly, conventional language can be an unreliable index of individual thought, suggesting conceptual structure that does not exist, hiding structure that does (Casasanto, 2009).

More compelling evidence comes from observational and historical case studies of expert scientists and mathematicians, where metaphor and analogy appear to have played a key role in major discoveries and debates. The venerable physicist James Clerk Maxwell, for instance, relied on a physical analogy to derive his electromagnetic field equations (Nersessian, 1992, 2008). Similarly, the historical acceptance of imaginary numbers by mathematicians may have depended on integrating conceptions of space and number into a more complex, blended conceptualization (Fauconnier and Turner, 1998). Today, mathematical experts—graduate students in a mathematics department—spontaneously make use of implicit spatial models when confronted with a novel, non-trivial proof (Marghetis and Núñez, 2013). Spatial construals have sometimes led to discord rather than discovery. For decades, the mathematician Augustin Cauchy defended his "proof" of a theorem about continuous functions (1821, 1853), despite unanimous agreement among his colleagues that the theorem was false (Lakatos, 1978; Kitcher, 1984). We have argued that the locus of their disagreement was the implicit conceptualization of number on which either camp relied, with Cauchy deploying an idiosyncratic spatial construal that differed from his colleagues' (Marghetis and Núñez, 2013). Throughout history and today, therefore, expert discovery and disagreement have been driven by spatial metaphors and analogies, which supply intuitions to help make sense of the complex or the abstract (Gentner, 2002; Fauconnier and Turner, 2002; Nersessian, 2008).

Besides these case studies of exceptional insight, however, there is little empirical evidence that everyday mathematical reasoning and understanding involve spatial metaphors or analogies (Núñez and Marghetis, *in press*). Perhaps this is unsurprising: in the canonical mathematical encounter, a solitary individual manipulates symbolic equations—hardly evocative of metaphorical or analogical thought. How, indeed, might space contribute to mature conceptualizations of notions as abstract as the integers and arithmetic?

4.1.1. Spatial metaphors for abstract arithmetic

The most extensive proposal for the role of spatial models in the conceptualization of mathematics is due to Lakoff and Núñez (2000). On their account, which builds on Conceptual Metaphor Theory (Lakoff and Johnson, 1980), much of mathematical reasoning is metaphoric, in the sense that the inferential structure of mathematics is imported from the inferential structure of concrete "source" domains. They argue that our conceptualization of the integers and arithmetic, for instance, is built metaphorically out of recurring, shared patterns of embodied activity like *collecting objects* and *moving along a path.* Critically, these concrete source domains are highly spatial and embodied, and thus depend on the kinds of reasoning at which we excel: spatial reasoning and predicting the outcome of embodied interaction with the world. By conceptualizing the "target" domain of arithmetic in terms of these source domains, we are able to draw on a shared set of embodied, spatial intuitions.

For instance, they propose that arithmetic is conceptualized metaphorically as *object collection,* mapping entities and inferences from the activity of collecting objects to the domain of arithmetic (Lakoff and Núñez, 2000, p. 55; see Table 4.1). Conceptualizing arithmetic as object collection allows us to understand numbers as collections, numerical magnitude as collection size, addition as the combination of collections. The domain of arithmetic also inherits the inferential relations of the source domain. The fact that addition is commutative (e.g., $3 + 5 = 5 + 3$), for instance, is guaranteed by the fact that the outcome of combining multiple collections is the same regardless of the order in which they are combined.

Lakoff and Núñez (2000) argue that the inferential structure of arithmetic is the outcome of not *one* but *multiple*, complementary metaphors. They suggest that arithmetic may be conceptualized as *motion along a path* (p. 72; see Table 4.1). By mapping the inferential structure of linear motion onto the domain of number, we can understand numbers as locations or displacements along that path, numerical magnitude as distance from the origin, and arithmetic as motion away from or toward an origin. And since transitivity holds for locations along a linear path, it holds for relative numerical magnitude:

Locations: if B is farther than A, and C is farther than B, then C is farther than A

Integers: if $B > A$, and $C > B$, then $C > A$

On their proposal, this conceptual metaphor motivates the linear spatial representations that are ubiquitous in contemporary Western visual culture: rulers, graphs, physical number-lines on the walls of classrooms.

The deployment of these conceptual metaphors need not manifest itself as the conscious invocation of a concrete analogy ("well, if you imagine numbers as collections…") or the physical creation of a model (e.g., drawing a visual number-line). Rather, metaphorical reasoning likely involves implicit, embodied simulations of the source domain (Gibbs, 2006). Reasoning about arithmetic might rely on mentally simulating actions and elements within the source domain—implicitly simulating the combination of distinct collections, for instance, and the resulting collection. Indeed, reasoning about concrete actions and objects is known to activate neural systems specialized for processing perception, action, and space (Barsalou, 1999, 2008; Gallese & Lakoff, 2005), systems that may contain predictive models that can not only recapitulate but actively predict the sensorimotor effects of hypothetical actions and events (cf., Grush, 2007). Conceptual mappings between source and target domains may yoke abstract reasoning to spatial or sensorimotor simulation.

A proof of a theorem in elementary number theory, therefore, could draw on either of these complementary conceptual metaphors in order to ground inferences to more basic, shared intuitions of space and action. Consider the following claim: *The sum of an odd and an even number is always odd.* A typical proof might start by breaking down the even and odd number into smaller components, and then recombining them in such a way that the result is demonstrably odd. But depending on how arithmetic is conceptualized, this process of numerical decomposition and recombination can be understood quite differently. If one were to conceptualize arithmetic as *object collection,* one might reason about an even number by simulating the separation of one collection into two collections of equal size. If, on the other hand, one were to conceptualize arithmetic as motion along a path, one might simulate a location along a path, a location reached by making two equal displacements from the origin. These spatial or embodied simulations could then generate insights into the integers.

4.1.2. Internal simulation, external gesture

There is relatively little evidence, however, that basic arithmetic is indeed conceptualized metaphorically, let alone that reasoning about arithmetic and the integers involves simulating the elements and relations in the relevant source domain. One way to gain insight into real-time, dynamic reasoning is to look at *spontaneous gesture*, meaningful movements of the body—especially the hands—produced while thinking or talking (Kendon, 2004; Goldin-Meadow, 2005; McNeill, 1992).

Table 4.1. Complementary conceptual metaphors for arithmetic: Object Collection and Motion Along a Path (adapted from Lakoff and Núñez, 2000)

Source: Object Collection		*Target*: Arithmetic
collection of objects	→	number
collection size	→	numerical magnitude
bigger [/smaller]	→	greater [/lesser]
combining collections	→	addition
removing a collection from larger collection	→	subtraction

Source: Motion Along a Path		*Target*: Arithmetic
location on path	→	number
distance from origin	→	numerical magnitude
further from [/closer to] origin	→	greater [/lesser]
motion away from origin	→	addition
motion toward origin	→	subtraction

According to a number of processing models of gesture production, representational gestures are the outward manifestation of internal imagistic representations (e.g., embodied simulation) generated while formulating the message to be communicated (Hostetter and Alibali, 2008; Kita and Özyürek, 2003; McNeill 1992). For instance, according to the *Gesture as Simulated Action* framework (hereafter GSA; Hostetter and Alibali, 2008), the production of representational gestures is driven, in part, by embodied simulation in sensorimotor brain areas. Their proposal is especially clear for concrete referents. When speakers are formulating a message about a concrete action or event, they rely on sensorimotor brain areas to simulate spatial, perceptual, and motoric features (Barsalou, 1999, 2007; Gallese & Lakoff, 2005). If neural activity in sensorimotor brain areas—especially areas responsible for action—surpasses a threshold, then internal simulation spills out as external gesture. Thinking about hammering a nail, for instance, might involve a visuospatial simulation of the hammer's trajectory or motor simulation of the manual action of swinging the hammer. According to GSA, were this internal simulation to surpass a threshold of activation, then the internal neural activity might spill out as one-handed "hammering" gesture. This is precisely what recent studies have reported. For example, people produce more representational gestures during tasks that require more spatial simulation (Sassenberg & Van Der Meer, 2010) or when recollecting past events that involved manual action (Hostetter and Alibali, 2010). At least for reasoning about concrete actions or events, external gestures are closely coupled to internal mental simulations (for review, see Marghetis and Bergen, 2014).

But what about metaphorical gestures? The GSA framework makes a clear prediction: "We contend that metaphoric gestures arise from perceptual and motor simulations of spatial image schemas on which metaphors are based" (Hostetter and Alibali, 2008, p. 504). On this account, therefore, pointing rightward when talking about the future (Cooperrider and Núñez, 2009; Casasanto and Jasmin, 2012) is not merely a conventionalized communicative strategy, but the outward manifestation of the internal simulation of a schematic, spatial representation of time, in which the past is to the left and the future to the right along a lateral timeline. If the conceptualization of arithmetic is similarly metaphorical, then this should be reflected in spontaneous gestures produced during mathematical reasoning.

Unlike the established and growing literature on metaphorical gestures for time (e.g. Núñez and Sweetser, 2006; Cooperrider and Núñez, 2009; Núñez et al, 2012; Casasanto and Jasmin, 2012; among others), there are few studies of how basic concepts of number and arithmetic become spatialized in gesture (but see Núñez and Marghetis, in press; Winter, Perlman, and Matlock, 2013). The work of Goldin-Meadow and colleagues on children's algebra solutions, for instance, deals primarily with children's procedural strategies, not their conceptualization of algebra and arithmetic (cf., Goldin-Meadow and Wagner, 2005; Goldin-Meadow, 2005). We know of no studies, moreover, that have investigated whether metaphorical gestures—like concrete representational gestures—reflect embodied simulation.

4.1.3. Current Studies

We combined observation and experimentation to investigate the spatialization of arithmetic in thought and gesture. The goal of Study 1 was to document and describe the metaphorical representation of number and arithmetic in spontaneous co-speech gesture. Using a semi-controlled interview, we elicited spontaneous gestures while participants reasoned about arithmetic. This leaves open the question of whether these gestures merely reflect conventionalized ways of communicating with the hands, or—as proposed by the GSA framework—they reflect gesturers' real-time, dynamic simulations of motion and action. Study 2 thus investigated the proximal cause of these metaphorical gestures for arithmetic, using a priming paradigm that manipulated participants' spatial imagery.

We reasoned as follows. If participants were deploying spatial models to support their mathematical reasoning, then this should be reflected systematically in their spontaneous gesture—even if there was no sign of metaphorical thought in their speech. Specifically, if Lakoff and Núñez (2000) are correct, then we should be able to identify two recurring, complementary gestural systems, corresponding to the *Path* and *Object Collection* metaphors. Moreover, if these gestures are the outward manifestation of the "perceptual and motor simulations of spatial image schemas on which metaphors are based" (Hostetter and Alibali, 2008, p. 504), then priming the spatial simulation of one source-domain or the other should encourage the deployment of the associated conceptual metaphor and thus increase the prevalence of associated gestures. Specifically, simulating motion along a path should prime path-based metaphorical gestures, while simulating the combination of collections of concrete objects should prime collection-based metaphorical gestures. Alternatively, if these gestures are merely a conventionalized communicative strategy, more akin to gestural emblems like the "ok" sign, then mental simulation should have no effect on subsequent metaphorical gestures.

4.2. Study 1: Does reasoning about arithmetic involve spatial metaphors?

4.2.1. Participants

Volunteers (n = 14) from the UCSD subject pool participated in return for partial course credit. All procedures were approved by the university's Institutional Review Board.

4.2.2. Design

In order to facilitate mathematical reasoning, participants first read the proof of a simple mathematical theorem: *The sum of an odd number and an even number is always odd* (Appendix A). For instance, 5 is odd, 2 is even, 5 + 2 = 7, and 7 is odd. Before and after reading this proof, participants completed a brief mental imagery task, intended to discourage participants from merely memorizing the proof's text and later repeating it verbatim. In this mental imagery task, participants were shown an image of four animals, told to memorize the animals and their location, and then, after a brief delay, asked to recall the identity of an animal in a particular location (e.g. "Was there a pig in the top right?").

Participants were then brought to a different room and asked to reason aloud while they responded to a series of questions about arithmetic. Responses were audio- and video-recorded. Throughout this stage, the experimenter followed a structured script and never gestured. Participants were first asked to explain *why* this theorem is always true, in their own words. If participants hesitated, the experimenter gave them scripted encouragement (e.g. "Do you remember the proof you just read? It was related to this. Does that help?"). Participants were then asked to explain a related theorem: *The sum of two odd numbers is always even.* This second theorem can be proved using an argument similar to the one used to prove the first theorem, but requires some novel insights.

4.2.3. Results

Participants did not use explicit analogies between numbers and concrete objects. Nobody said, for instance, "Well, if you think of numbers as collections of beads…," or, "Imagine a left-to-right number-line." By contrast, participants' gestures used space in systematic and recurring ways to represent number and arithmetic. To foreshadow our results, in one system of gestures, arithmetic was construed as a process of *object collection*, of grouping, combining, and otherwise manipulating discrete entities. In the other, arithmetic was construed as *motion along a path*, typically along a horizontal axis.

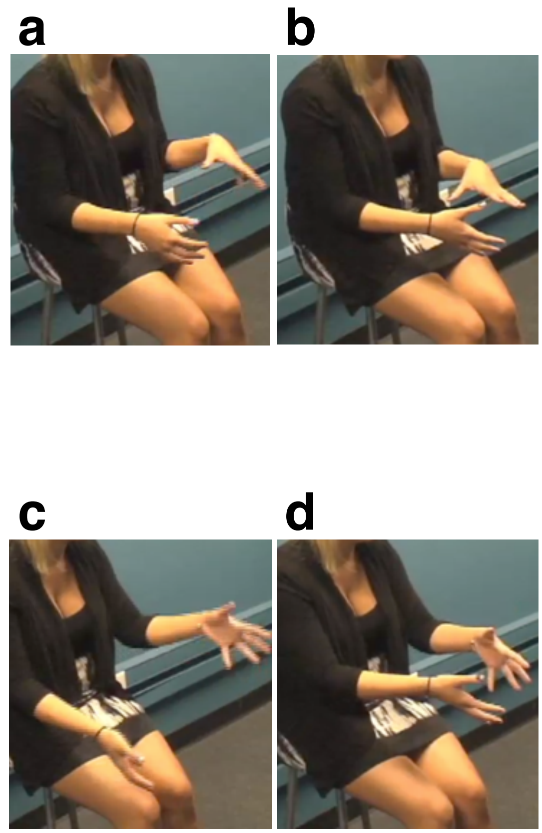
4.2.3.1. Collection Gestures

The Collection system of gestures made systematic use of spatial extent to represent numerical magnitude—either the size of a single grasping handshape or the volume bounded by a bimanual gesture—and combination and separation to represent arithmetic. In its canonical form, when expressing addition, the Collection gesture consisted of both hands moving inward toward the center of gesture space, with the hands shaped as if grasping, pinching, or holding. The impression was of two objects or collections, held within the hands, being brought together and combined. The Collection system of gestures thus involves mappings between arithmetic and space that parallel those that constitute the conceptual metaphor in which arithmetic is conceptualized as *object collection* (Lakoff and Núñez, 2000).

Figure 4.1 illustrates a sequence of typical Collection gestures. The speaker is describing a series of calculations: first adding one plus one to get two, and then adding that even result to another even number. Both calculations are accompanied by a similar gesture. She begins by shaping her hands as if each is grasping or holding some imagined object, held apart to represent the two numbers being added. As she describes adding those addends, she rapidly moves both hands inward toward each other (a to b; c to d), thus using inward motion to represent the *process* of addition. To represent the *product* of the addition—the sum itself—she maintains a post-stroke hold in which both hands are held slightly apart, demarcating a region of space that stands in for the sum (b, d). The outcome of the second addition is larger than the outcome of the first, and, accordingly, her hands carve up a larger volume of space during the post-stroke hold after the second Collection gesture (compare b to d).

This use of relative volume to represent relative magnitude is illustrated nicely in Figure 4.2. The speaker is describing a sequence of three calculations: first, adding two indeterminate even numbers; second, adding one and one; third, adding these first two sums. She accompanies each of these calculations with a bimanual gesture in which the hands begin by demarcating separate spatial regions and then come together to indicate a larger, combined space. Thus, once again, the process of addition is represented by an inward-directed motion enacting the combination of two objects or collections, while the product of addition is represented by a post-stroke hold that demarcates a region of space. We can determine the relative magnitude of these three sums from the speaker's speech: the first is an indeterminate even number; the second is two (1 + 1), the smallest even positive integer, and thus less than or equal to the first sum; the third, final number is larger than either of the first two, since it is their sum. And, as we might expect if relative volume stands in for relative magnitude, the volumes demarcated by the three post-stroke holds respect this relative ordering of the three numerical magnitudes (b, d, f).

Within the Collection system, gestures recruit extent, not location, to represent numerical magnitude. As a result, the exact same number can be in different locations. This was true in the first stroke of the gesture phrase depicted in Figure 4.1 (a, b), and again in the second gesture from Figure 4.2 (c, d). In both cases, the two hands represent the same number (i.e., one). Another example is illustrated in Figure 4.3. The speaker states explicitly that she is adding a number to itself ("same numbers added up"). To represent these equal addends, she pulls her hands apart to form two equal grasping handshapes (a, b), using volume rather than location to express numerical magnitude and thus representing the same number in different locations. She completes the gesture phrase by bringing her hands together, representing the addition of the equal addends (b, c). Looking over the entire gesture phrase, the equal addends were located to the left and right, while their larger sum was located between the addends. Within the Collection gesture, therefore, there is no systematic relation between location and numerical magnitude.



(1) You have one from each number left over and [**adding** that together] is an even number.

(2) So an even number [**plus** an even number] is an even number.

Figure 4.1. A gesture unit consisting of canonical Collection gestures, each representing the sum of two numbers. The speaker first adds 1 + 1, accompanying her speech with a bimanual inward gesture (a-b) to represent the process of addition. She uses the post-stroke hold to represent the sum—an even number—keeping her hands slightly apart to demarcate a small volume. She next adds this number to yet another even number (c-d), again representing the addition process with an inward directed movement and using the post-stroke hold to represent the sum. The second sum is greater than the first, and, correspondingly, the volume contained by the co-produced gesture is larger in (d) than in (b). Speech accompanying a gesture is enclosed in [square brackets], with the gesture stroke in bold and any holds underlined (McNeill, 1992).

(1) You have [**the two even** <u>numbers that go together to make another even number</u>]

(2) And then [the two **ones that** <u>go together to make an even number</u>]

(3) And then you add [**your** <u>new two even numbers</u>] to make another even number

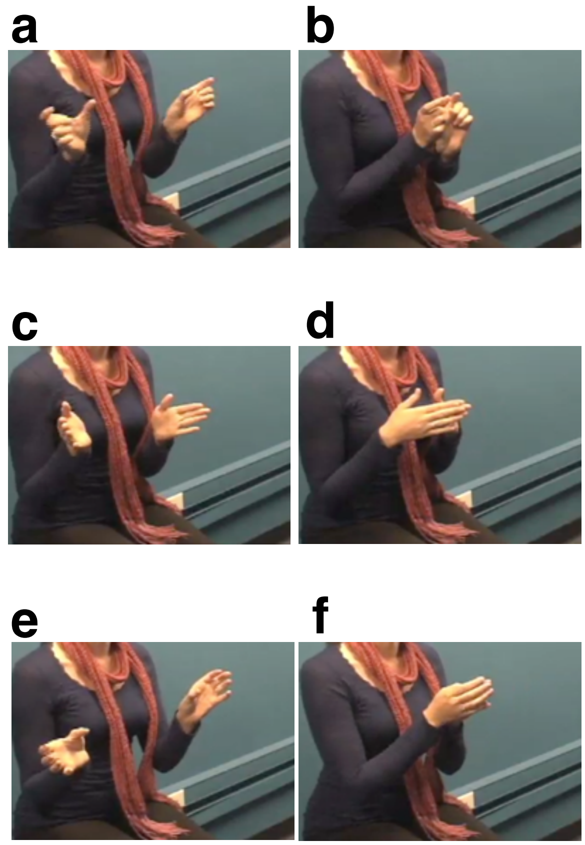


Figure 4.2. A sequence of Collection gestures, with handshapes marking relative magnitude. The first gesture (a-b) represents each number with "pincer" handshape, with thumb and forefinger extended to demarcate a region of space. To represent the addition of these two numbers, the hands are brought together and fingers overlaid. The second gesture (c-d) uses extended fingers to indicate punctate locations, and then brings the fingertips together to represent the addition of one and one. The final gesture (e-f) represents the addition of the two preceding sums. The handshape indicating this final sum demarcates a larger space than in either (b) or (d). Relative volume thus represents the relative magnitude of all three sums.

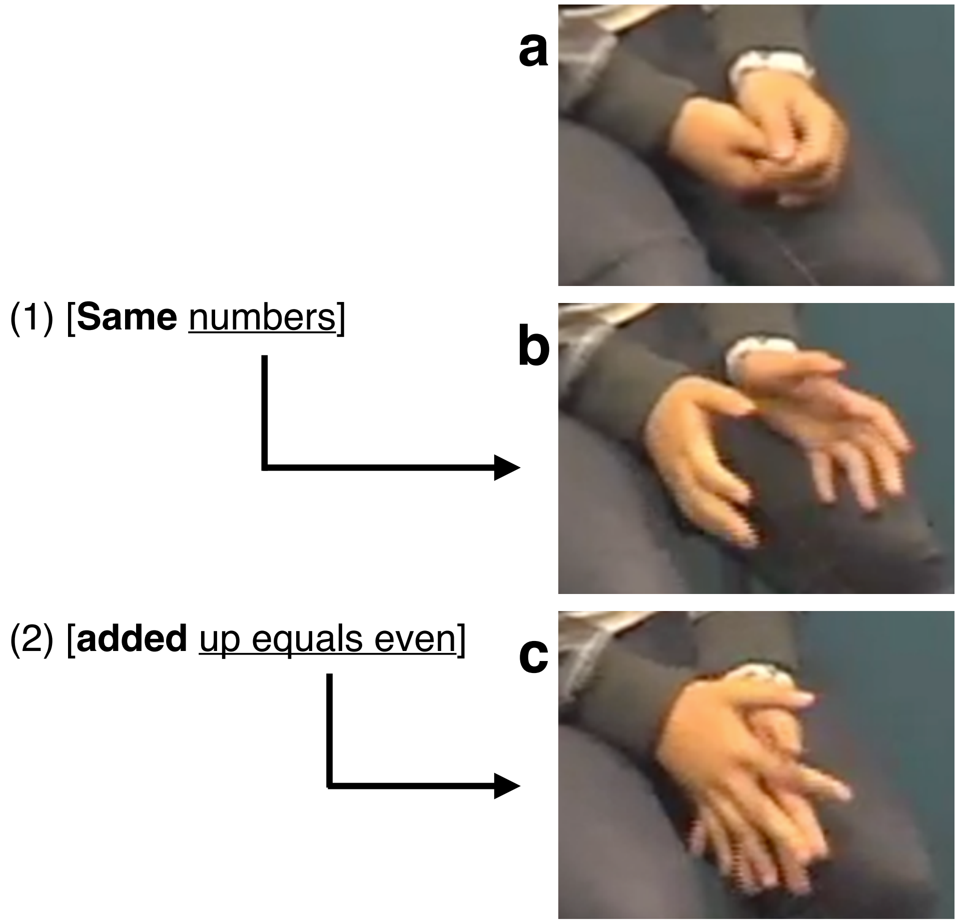


Figure 4.3. Within the Collection system, equal numbers can be in two places at once. Both hands began by pulling back from rest (a-b), with matched grasping handshapes representing equal addends. These handshapes were maintained during the post-stroke hold, as she explicitly stated she was summing the "same numbers." Both hands were then brought together (b-c) to represent their addition. Even though the addends were identical, they were located in different locations; numerical magnitude was expressed by volume, not location.

4.2.3.2. Path Gestures

The Path system of gestures represented numerical magnitude by location along the left-to-right transversal axis, and represented arithmetic by movement along that axis. In its canonical form, when expressing addition, the Path gesture consisted of a single hand, with index finger extended, tracing a rightward path from one location (the first addend) to another (the sum), with distance traveled representing the magnitude of the second addend. This system of Path gestures thus appears to be a gestural analog of the **MOTION ALONG A PATH** conceptual metaphor (Lakoff and Núñez, 2000).

For instance, in the gesture phrase depicted on the right of Figure 4.4, the speaker's index finger traces a path from left to right as he describes the addition of three numbers, pausing for a brief pre- or post-stroke hold to indicate each stage of the calculation. His first gesture associates the first addend, the variable *b*, with a location near his left leg, thus anchoring the rest of the gestures in the sequence (a). As numbers are added, he slides his finger along a transversal path, with each new addition accompanied by a rightward stroke (b, c). These Path gestures were sometimes enacted on a small scale, barely noticeable to the interlocutor, as in the second gesture phrase depicted in Figure 4.5. As the speaker says "three," she points to a location to the left of her midline, anchoring the number to that location. As she adds, twice, to this initial addend, she produces two rapid rightward strokes, each co-timed with the word "plus" and the addend ("three," "one").

Path gestures used the precise *direction* of motion to distinguish between addition and subtraction, or between increase and decrease more generally. In Figure 4.5, for example, the speaker is observing that an even number will always be one greater than and one less than an odd number. As she notes that it will be one greater than an odd number ("it would either be one up"), she moves her left hand, with all fingers extended, in a smooth trajectory rightward, from slightly left to slightly right of her midline. She then notes that it could be one less than an odd number, co-produced with a leftward sweep of her hand. Notably, this was one of the few times that a participant used spatial language to describe numerical relations ("one up"), though this speech used a different axis—the vertical axis—than the transversal axis deployed in gesture. Path gestures thus locate larger numbers to the right, lesser numbers to the left.

While the first example in Figure 4.4 uses locations to indicate the relative magnitude of variables (b vs. $b+b+1$), the gestures in the second example map specific numbers to particular locations. Thus, while the Path gesture system recruits spatial locations to individuate numbers, this is relative to the particular numbers being discussed and the specific locations indicated in the rest of that particular gesture excursion. One speaker's location for "twelve" might be another's "twelve billion." These gestures, therefore, are a species of abstract deixis (McNeill et al, 1993). Abstract deixis refers to the phenomenon where speakers point to empty space in order to refer to an absent referent or discourse element that was introduced earlier and associated with that space. For instance, a speaker might point to the left when introducing a character into her narrative, and then point back to that location to refer to that character throughout the rest of her narrative. Unlike abstract deixis to concrete discourse elements—in which a concrete referent is associated idiosyncratically and transiently with a particular location—Path gestures anchor an entire domain to the speaker's gesture space. Thus, in deploying a Path gesture, the speaker overlays the numerical continuum on an imaged left-to-right spatial path—laminating the local gesture space with the abstract space of numbers (cf., Haviland, 1996). The speaker can then refer to other locations along that imagined path to invoke numbers greater or lesser than the initial number.

Both Path and Collection gestures can individuate numbers, distinguishing between those that are equal and those that are not, but they differ in how they recruit space to do so. A Collection gesture might use both hands to represent two equal addends by using identical handshapes, thus associating identical numbers with different locations but equal volumes. By contrast, Path gestures individuate numbers by their location, so the same number cannot be in two locations; if the hand moves to a new location, then it represents a new number.

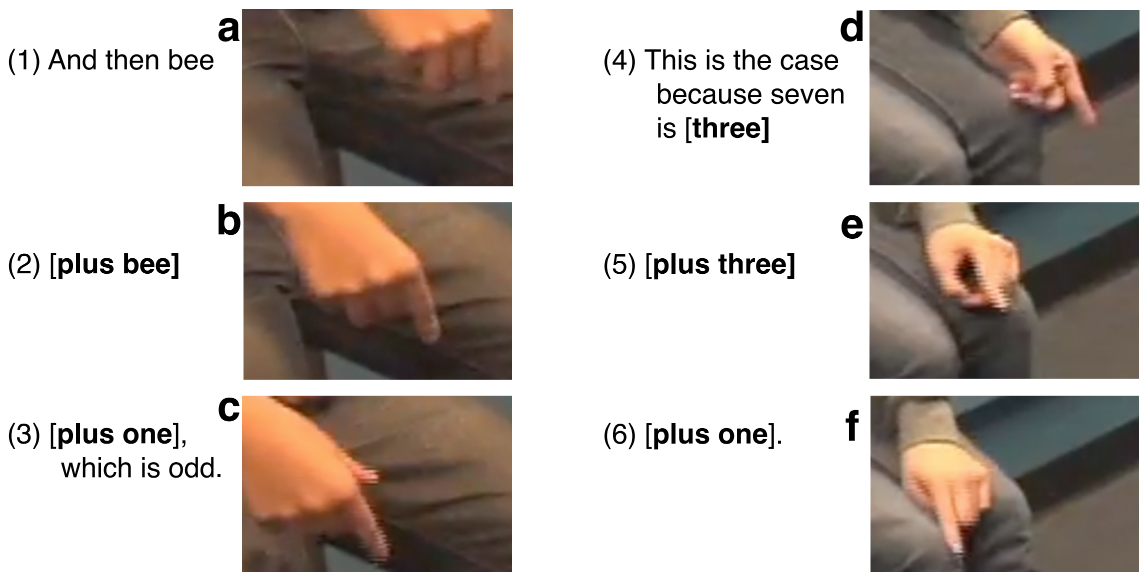


Figure 4.4. Canonical Path gestures produced by two different participants in Study 1: unimanual, pointing handshape, rightward strokes for addition. Each sum is accompanied by a rightward displacement, so the finger points to more rightward locations for larger numbers. In the gesture phrase on the left, the speaker begins by indexing a location near his left thigh for the variable *b*, then a location more rightward for *b+b*, and more rightward again for *b+b+1*. In the gesture phrase on the right, the speaker begins by indexing a location to the left for *3* (d), then more rightward for *3+3*(e), and more rightward again for *3+3+1* (f).

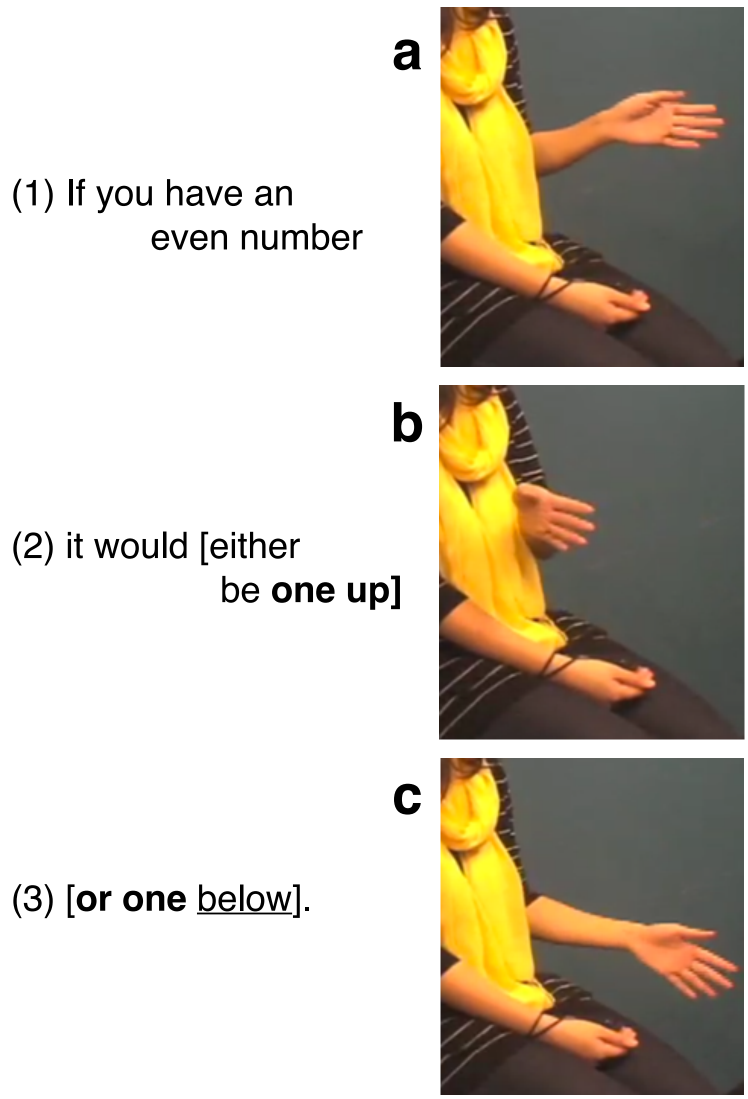


Figure 4.5. Path gestures use *direction* of motion to indicate the orientation of change or relative magnitude. Here, the speaker gestures rightward to indicate a larger number ("one up"), and leftward to indicate a smaller number ("one below"). While language relies on the vertical axis ("up," "below"), Path gestures rely primarily on the horizontal axis—never used in speech to describe numerical relations, in neither English nor any attested spoken language.

4.2.4. Discussion

Participants' spontaneous gestures spatialized arithmetic in two complementary ways. One system of gestures represented numbers with spatial volume and expressed addition as the combination of objects. The other represented numbers with lateral location and expressed addition as rightward motion along the transversal axis. These two gesture systems mirrored conceptual metaphors that have been proposed to structure our conceptualization of arithmetic (Lakoff and Núñez, 2000), namely, conceptualizing arithmetic as object collection or as motion along a path. Participants' gesture production, therefore, suggests that these spatial construals of arithmetic have cognitive reality, deployed spontaneously by participants as they make sense of abstract mathematical relations.

Critically, Path and Collection gestures were not isolated gestures with idiosyncratic, conventionalized meanings—like the "ok" sign, or an extended index finger to represent "one"—but holistic *gesture systems*. Meaningless in isolation, numerical gestures were meaningful in virtue of their relation to other elements, potential and actualized, of the gestural system. Adapting a distinction from Saussure (1917/1986), we can say that a particular gesture acquired meaning in relation to two dimensions of contrast: the alternative gestures that could have been produced in its stead (paradigm), and the preceding and following gestures (syntagm). Recall the Collection gestures depicted in Figure 4.2. The handshapes in (f) represent larger quantity only because they carve out a larger space than the handshapes in (d) or (b), not because of a stable mapping between those specific handshapes and particular quantities. The particular location indexed by the Path gesture in Figure 4.4-e, similarly, represents the number six because of its placement relative to the other gestures in that particular sequence of gestures—right of a gesture used to represent three, left of a gesture used to represent seven. These gestures are part of a system that allows for the productive and flexible representation of number and arithmetic.

Both systems—Path and Collection—spatialized number and arithmetic in ways that are either absent entirely from speech or appear only in an impoverished, limited fashion. Take Collection gestures. While participants *could* have referred to numbers as “bigger” or “smaller,” in actual fact they seldom accompanied their Collection gestures with such spatial language. Moreover, Collection gestures often represented simultaneously different facets of a calculation, impossible in the linearized speech stream. The first gesture of Figure 4.2 simultaneously represents the magnitude of the addends (using the two handshapes), the type of arithmetic operation (using inward-directed motion), and the larger magnitude of the sum (using the volume demarcated by both hands in the post-stroke hold). Speech, by contrast, needs to separate these facets in order to encode them in a linear stream. The contrast with speech is even more stark for Path gestures. While English does allow numerical magnitude to be described along the vertical axis (“higher or lower numbers”), Path gestures rely primarily on the horizontal axis—never used in speech to describe numerical relations, in neither English nor any attested spoken language. Indeed, both Path and Collection gestures offer holistic representations of complex arithmetic relations—the magnitude of multiple addends, the operation used to combine them, etc.. Speech, by contrast, typically makes only targeted use of space to describe a single number[1] (e.g., “tiny number”). The spatial representation of arithmetic in gesture, therefore, is distinct from and autonomous of spatialization in language.

[1] These targeted uses of spatial language are often accompanied by “quotable” gestures, such as a “tiny” gesture with index and thumb extended and touching (Winter, Perlman, and Matlock, 2013).

The system of Path gestures, in particular, is reminiscent of the system of transversal temporal gestures, which represent temporal order along the transversal axis (cf. Cooperrider and Núñez, 2009), or the multiple timelines that are grammaticalized in some sign languages, including American Sign Language (Winston, 1989; Emmorey, 2001) and Danish Sign Language (Engberg-Pedersen, 1993). These gestural and sign language timelines express relative temporal order by pointing to locations along a left-to-right transversal axis, with earlier events located to the left, and later events to the right. Within a particular gesture excursion, therefore, specific times or temporal periods are individuated by their locations in transversal space. Similarly, within a particular excursion of the hands, Path gestures individuate numbers by pointing to locations along a left-to-right transversal axis.

While the systematicity, productiveness, and autonomy of these two gestural systems suggests that mathematical reasoning involves complementary conceptual metaphors, an observational study cannot tell us about the proximal mechanisms driving their production. It is entirely plausible that these metaphorical gestures are merely acquired, conventionalized ways of representing arithmetic concepts during conversation— static mappings between abstract concepts (e.g., addition) and specific gestures (e.g., bimanual Collection gesture). These gestures could be the sedimentation of conceptual practice—learned in school or acquired from others—and not actually reflect real-time processes of conceptualization. In fact, humans are known to acquire gestures through social learning, picking up the gestures by observing the gestures of others in their community (Halina, Rossano, and Tomasello, 2013).

Study 2, therefore, used a priming paradigm to test the claim of the GSA framework, that metaphoric gestures reflect internal sensorimotor and spatial simulation of the metaphoric source domain. If co-speech metaphoric gesture does, in fact, reflect a spatial conceptualization of arithmetic, then spatial simulation should affect subsequent metaphoric gesture: simulating motion along a transversal path should prime Path gestures; simulating the manipulation of collections of concrete objects should prime Collection gestures. If, on the other hand, these gestures are merely acquired conventions, then mental imagery should have no systematic effect on subsequent metaphorical gesture.

4.3. Study 2: Do metaphorical gestures reflect internal mental simulation?

4.3.1. Participants

Volunteers (n = 18) from the UCSD undergraduate subject pool participated in return for partial course credit. All procedures were approved by the university's Institutional Review Board.

4.3.2. Design and Procedure

In a between-subjects design, participants completed a mental imagery activity in one of two conditions, followed by a mathematical reasoning task. They were told the study was investigating the relation between concentration and abstract reasoning.

4.3.2.1. Mental Imagery

One group of participants had to memorize a picture of a horizontal wire, with locations along the wire distinguished by color (*Path-based Imagery* condition). The other group of participants memorized a picture of three colored plates, each containing a small collection of beads (*Collection-based Imagery* condition). Once they had memorized the picture, participants were given written instructions to imagine moving the beads from one location to another, without visual access to the picture (Appendix B). In the Collection-based

Imagery condition, participants were asked to imagine moving all the beads from one plate to another (e.g. "Imagine moving the beads around between the plates, described by their color […]: From red to blue."). Instructions in the Path-based Imagery condition were identical, except the word "plate" was replaced with "wire location," so participants were instructed to imagine sliding a bead between colored locations along a horizontal path. To encourage compliance, participants then answered a simple comprehension question: "After all your imagined manipulations, are there any beads in the green plate [/wire location]?"

In order to facilitate their subsequent mathematical reasoning, participants then read the same theorem from Study 1 ("*The sum of an odd number and an even number is always odd.")* and two proofs of the theorem (Appendix A). Two proofs were mathematically equivalent but used slightly different phrasing to avoid biasing path-based or collection-based construals of arithmetic. For instance, whereas one proof described an arithmetic sum as "combining three and three together," the other described the sum as "starting at three and then adding three more." These proofs were matched in number of words (283 vs. 302), number of lines (17), and logical organization. Each participant read both proofs; order of presentation was counterbalanced to eliminate order effects or any potential bias. Between reading the proofs, participants performed a second round of mental imagery, in the same condition. Thus, the only difference between conditions was the type of mental imagery: either path-related (bead on a wire) or collection-related (collections of beads).

4.3.2.2. Mathematical Reasoning

Participants were then led to another room, where they were asked to explain in their own words *why* the sum of an odd number and an even number is always odd—that is, explain the mathematical theorem for which they had just read a proof. To encourage expressiveness, they were asked to phrase their explanation as if they were speaking to "an intelligent high school student." Their speech and gesture were video-recorded.

4.3.3. Speech and Gesture Coding

Coders were blind to participants' mental imagery condition. Participants' speech was transcribed verbatim, including pauses and disfluencies. Gesture was coded using a three-step process. First, participants' speech was searched for talk about arithmetic—operationalized as any mention of numbers, addition, subtraction, multiplication, or division—and, if accompanied by gesture, the video was segmented by gesture stroke. This produced a corpus of gestures co-produced with talk of number or arithmetic. Second, each stroke in this gesture corpus was coded for handedness, morphology (i.e., handshape), and kinematics (i.e., motion trajectory of the stroke) (see Table 4.2). Handedness was coded as *one-* or *two-handed*, depending on whether one or both hands were involved in the stroke. Morphology was coded as *pointing* (e.g., extended index finger), *grasping* (e.g., pinching or holding), or *other* if the handshape fell into neither category. Kinematics was coded as *inward* (e.g. both hands moving toward each other), *transversal* (e.g., left-to-right), or *other* if the gesture stroke's motion fell into neither category.

Third, annotations for all three features (handedness, handshape, morphology) were converted to numerical scores. A feature was converted to -1 if it was path-like (i.e., one-handed, pointing handshape, inward-directed motion), to +1 if it was collection-like (e.g., two-handed, grasping handshape, transversal motion), and to 0 otherwise. For each gesture, these three scores were then summed to generate a composite score that indexed how closely the gesture resembled a canonical Path or Collection gesture. On this scale, canonical Collection gestures received a score of +3 and canonical Path gestures received a score of -3, while gestures with a mix of features (e.g., a two-handed, grasping gesture with left-to-right transversal kinematics) received scores ranging from -2 to +2.

To assess reliability, a second coder independently coded each gesture's morphology and kinematics. There was good agreement: Coders' composite gesture scores were highly correlated ($r = .84$, $t_{104} = 15.7$, $p << .001$), as were individual scores for morphology ($r = .70$, $t_{104} = 10.1$, $p << .001$) and kinematics ($r = .44$, $t_{104} = 5.0$, $p << .001$). We used the first coder's annotations for all subsequent analyses, though the results were unchanged—and in some cases more significant—with the second coder's.

Table 4.2. Features coded in Study 2. Collection-like features contributed one point to each gesture's composite score; path-like features subtracted one point.

Feature:	Handedness	Morphology	Kinematics
Collection gesture	two-handed	grasping	inward-directed
Path gesture	one-handed	pointing	along traversal axis

4.3.4. Results and Discussion

We begin with quantitative and qualitative descriptions of participants' mathematical gestures, replicating and extending the findings of Study 1. We then answer the critical question, *Does mental imagery have a causal impact on subsequent metaphorical gesture?* All statistical analyses were performed in R (*R Development Core Team*, 2010). Three participants gestured very little overall and never while discussing arithmetic; they were removed from further analysis.

4.3.4.1. How did participants gesture about arithmetic?

Participants produced gestures with a range of features (Figure 4.6, top panel), but these were clustered nevertheless around two distinct types: one involving two-handed collecting motions, the other involving one-handed pointing along a transversal path. The majority of one-handed gestures had pointing morphology (57%), while the majority of two-handed gestures had grasping morphology (59%), as we would predict if participants were producing Path or Collection gestures, respectively (χ^2 test of independence, $\chi^2_{(2)} = 30.2$, $p << .001$). Moreover, morphology and handedness combined with kinematics in reliable, meaningful clusters. Using k-means, an unsupervised machine learning algorithm for clustering multidimensional data, we categorized gestures into two maximally distinct clusters on the basis of morphology, handedness, and kinematics. Recall that more negative scores indicate more path-like features, while more positive scores indicate more collection-like features. One cluster of gestures consisted of one-handed gestures with pointing morphology ($M_{\text{morphology}} = -0.34$) and mostly transversal motion ($M_{\text{kinematics}} = -0.53$), all typical of Path gestures. By contrast, the other cluster consisted of two-handed gestures that were more collection-like in morphology ($M_{\text{morphology}} = 0.5$) and motion ($M_{\text{kinematics}} = -0.22$), all *t*s > -2.2, *p*s < .03. Therefore, while handedness, morphology, and kinematics are, in principle, independent dimensions along which gestures could vary, they patterned in predictable ways, clustering in line with the gesture systems identified in Study 1. Path and Collection gestures carve the world of spontaneous numerical gestures at its natural joints.

For instance, Figure 4.7 shows a gesture in the first cluster, with all the features of a canonical Collection gesture: bimanual, grasping morphology, inward-directed motion. The participant's hands are shaped as if grasping or holding some imagined object or collection.

Maintaining this handshape, she produces a downward stroke with each hand, indicating the numbers to be added (a-b). She then brings her hands together to indicate their addition, saying, "You can combine the one from the even number and the one from the odd number to create one [number]." Note that this gesture, as an exemplary Collection gesture, illustrates the recruitment of the transverse axis to distinguish terms in an arithmetic operation—not to represent their magnitude, like we see in the Path system.

Or consider Figure 4.7. The participant is explaining that an odd number can always be decomposed into two equal numbers, with one left over. She begins by describing the addition of the two equal numbers (a-c), rapidly bringing her hands together. Both hands begin in an open grasping handshape, and close into a pinching shape as they approach each other (b). The gestural result of this addition is a brief hold, indicating the sum of the two numbers (c). Next she states that, "you just add one" to this partial sum, while simultaneously opening up and displacing her hands slightly to the right (d). Notice that two equal addends ("the same numbers") are enacted by nearly identical handshapes, while the addition of one more ("add one") is accompanied by an expansion in the volume of her grip. Morphology mirrors magnitude. Furthermore, since the *same* number was represented by different hands, and therefore anchored to different locations, we see again that Collection gestures divorce magnitude from location, instead representing numerical magnitude with spatial extent—paralleling the conceptual mapping between magnitude and volume proposed by Lakoff and Núñez (2000).

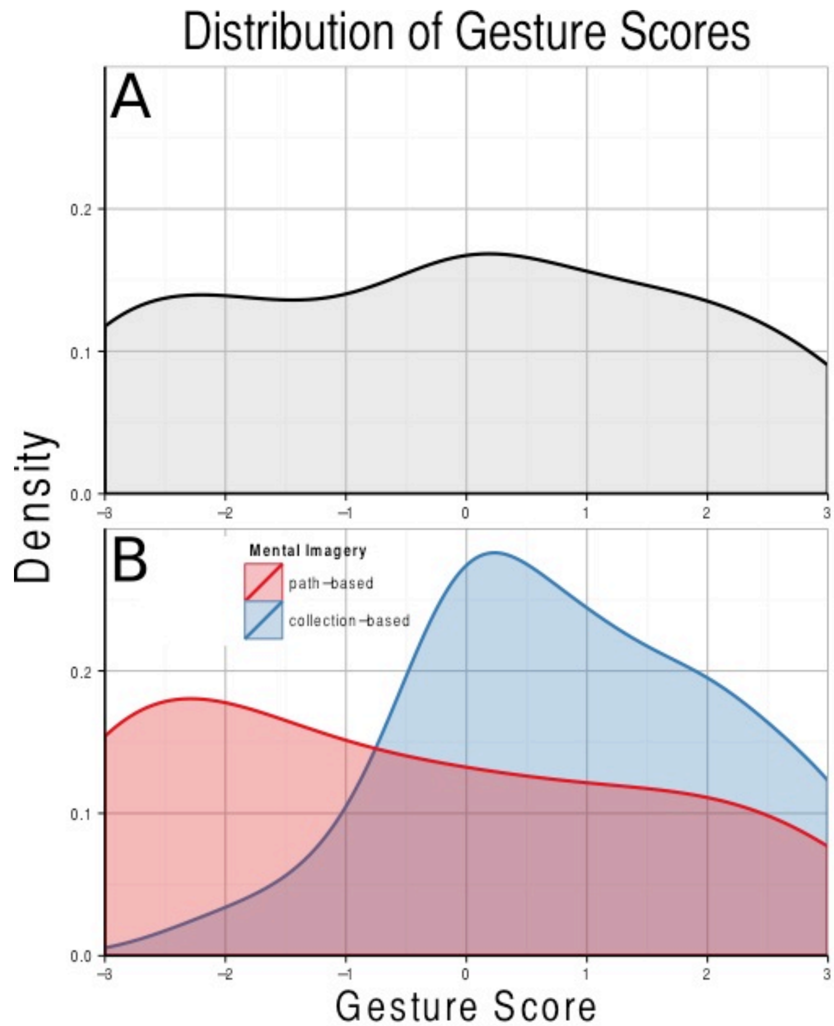


Figure 4.6. Distribution of gesture features (Study 2). Gesture score along the x-axis indicates whether gestures were more "path-" (negative) or "collection-like" (positive), combining morphology, handedness, and kinematics. (A) Overall, gestures had both path- and collection-like features. (B) The distribution of gesture features, however, differed by imagery condition. Gestures were primarily path-like after path-based imagery (red), and primarily collection-like after collection-based imagery (blue). Plots show Gaussian kernel density estimates (Silverman, 1986).

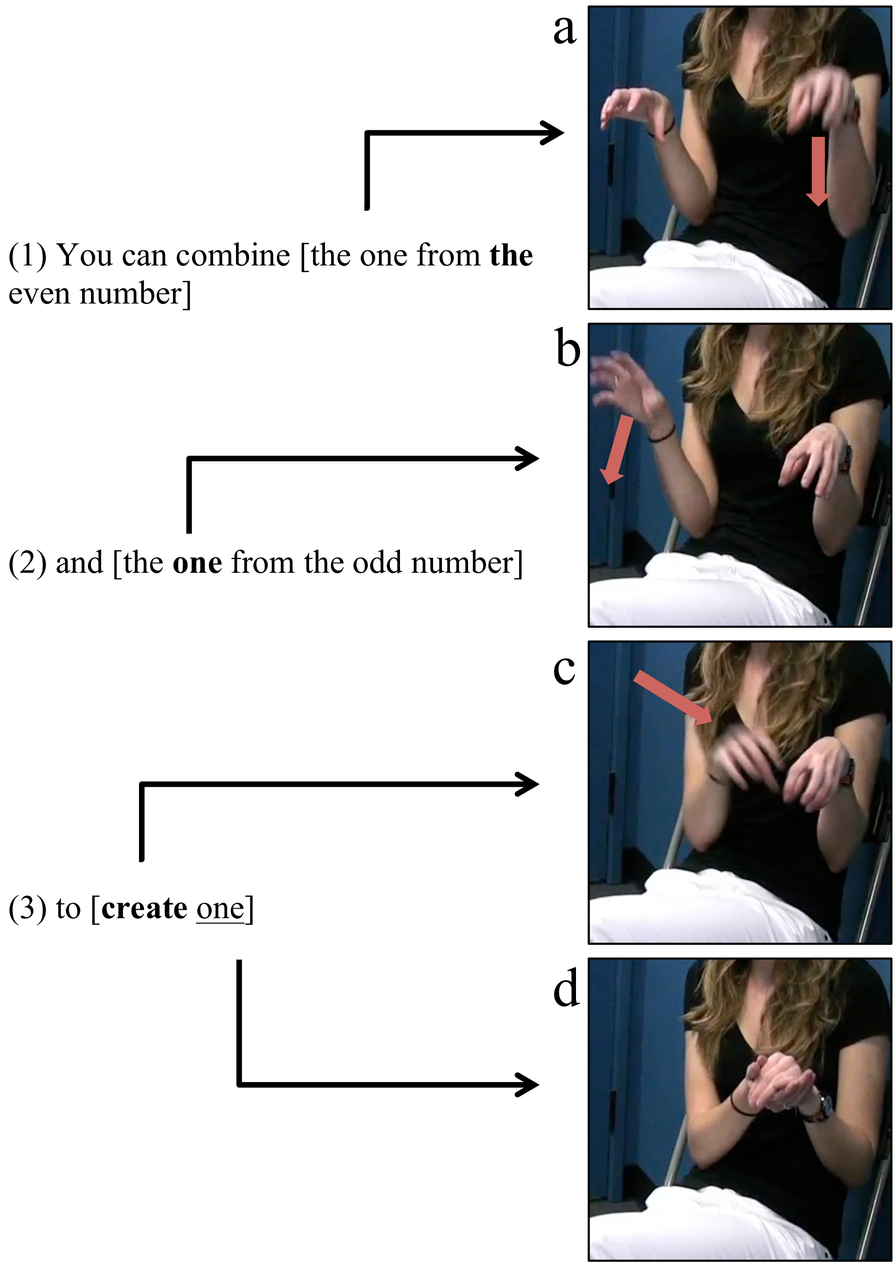


Figure 4.7. Collection gesture produced after collection-based imagery (Study 2). Instead of representing relative magnitude, horizontal location distinguishes terms (a, b). The sum (c, d) is located between the two addends, rather than farther to the right, where it would be in the Path gesture system.

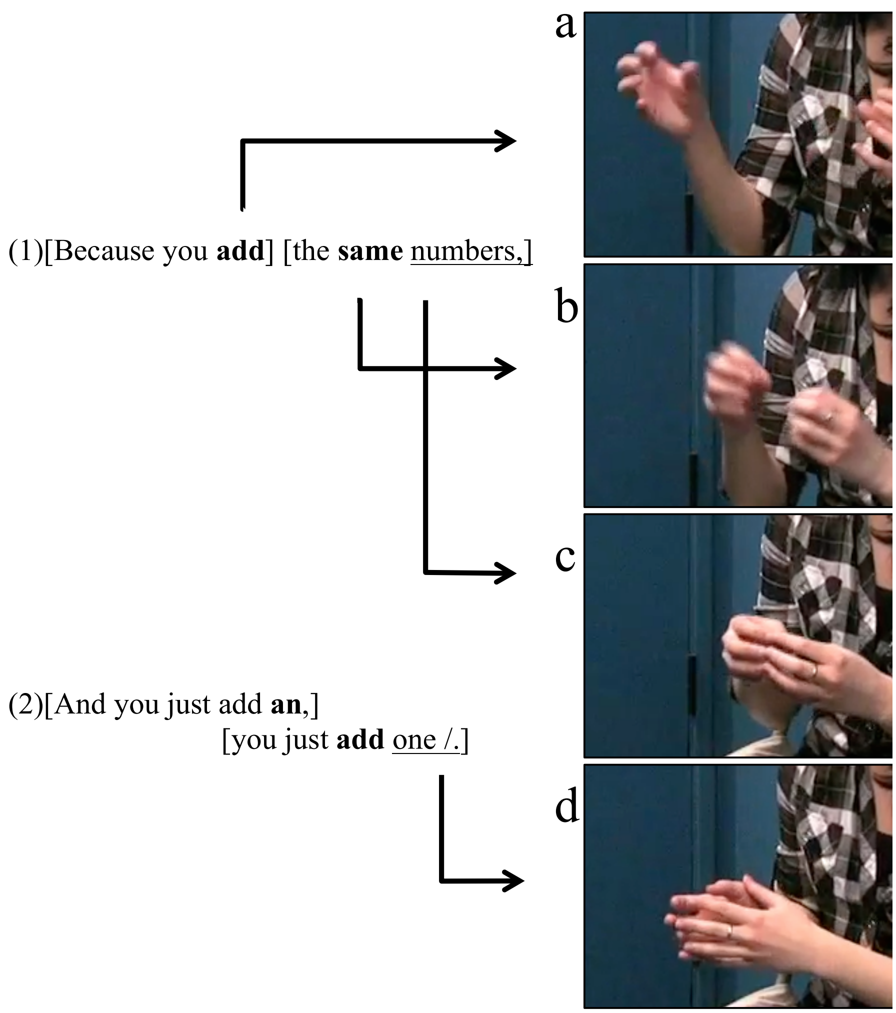


Figure 4.8. Collection gesture recruiting volume to represent numerical magnitude (Study 2). The first stroke (a-c) is prototypical: bimanual grasping handshapes, inward movement to represent arithmetic. The second (d) represents addition by expanding the volume between the hands, as if numerical magnitude were volume.

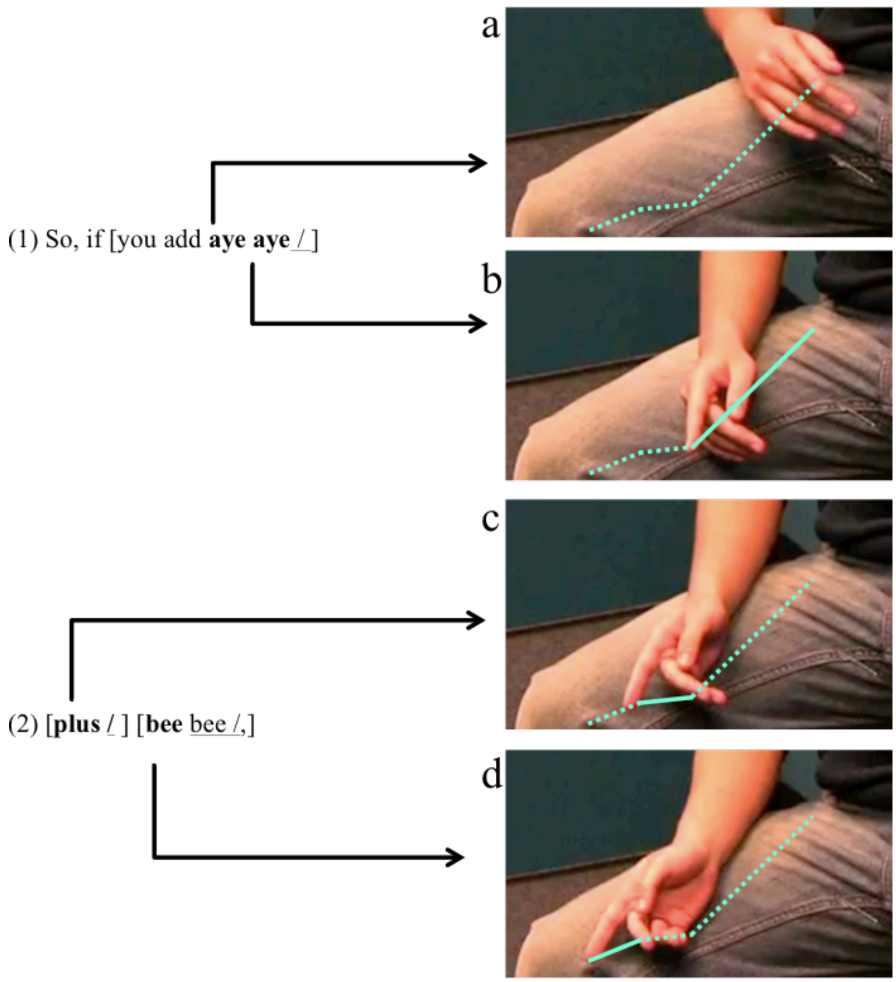


Figure 4.9. A canonical Path gesture after path-based imagery. Describing the sum of four terms (*a*+*a*+*b*+*b*), he produces rightward strokes for each partial sum. He represents the first sum with a large displacement (a, b), and the second two equal sums with smaller but equal displacements (c, d).

By contrast, the gesture in Figure 4.9, from the second cluster, has all the features of a canonical Path gesture: single-handed, pointing morphology, transversal motion. The participant traces his hand along the transversal axis in a canonical pointing handshape, coupled to speech about the addition of five terms. The addition of each term is accompanied by a rightward stroke. The participant begins by adding "*a* plus *a*," accompanied by a large diagonal displacement. He next adds a constant *b,* and then adds *b* again. Each addition of *b* is accompanied by a small rightward displacement of his hand, again with canonical pointing morphology. These two displacements are nearly identical in length, evoking the common magnitude of both operations. Linear displacement represents relative magnitude.

What of gestures that lay between the poles of canonical Path and Collection gestures? These were often hybrid gestures, combining features of both gestural systems. It is here that participants performed some of the most creative gestural enactments of their abstract mathematical reasoning. Consider the gesture phrase in Figure 4.10, produced by a speaker who is explaining why a sum of the form "$a + a + b + b + 1$" is always odd. The speaker begins with a series of standard Path gestures, producing a short rightward stroke for every partial sum (e.g., "$a + a$"). But when it comes to adding the final term—adding one—he rapidly changes both handshape and trajectory. From a typical pointing handshape with extended index and middle fingers, indexing the even sum of the first four terms, he retracts his hand briefly while extending his thumb to create a grasping handshape. As he talks of adding one ("And then you have the one left over."), he gestures as if adding a small object to the location of the even sum. Note that, in the Path system, numerical magnitude is represented by spatial location; addition is a displacement rightward along the number line, not an operation performed in a single location. In the Collection system, by contrast, numbers are collections of objects, not locations; moving a collection from one location to another does nothing to change its cardinality. This gesture violates the internal logic of either system. And yet, by selectively projecting elements from both systems and composing them together in a new blended system (Fauconnier and Turner, 2002; Parrill and Sweetser, 2004), the speaker is able to represent numbers as both locations and collections simultaneously, within a single stroke.

4.3.4.2. Did mental imagery shape subsequent metaphorical gestures?

In the face of this diversity of gestures, the critical question is whether participants' metaphorical gestures for arithmetic were shaped by the content of their mental simulation of space and action. As predicted, mental imagery had a reliable and systematic influence on subsequent gesture (Figure 4.6, bottom panel). Recall that negative gesture scores indicate more path-like gestures, while positive scores indicate more collection-like gestures. Overall, participants who had performed path-based mental imagery produced gestures that were more path-like (composite score, M = -0.68, SE = 0.47), while those who had performed collection-based imagery produced gestures that were more collection-like (M = 0.67, SE = 0.39), a significant influence of mental imagery on metaphorical gesture ($t_{12.954}$ = 2.21, p = .045; Figure 4.11, left panel).

To further investigate this effect of mental imagery, we modeled each gesture's composite score using a linear mixed-effects model, with Mental Imagery as a fixed effect, and random intercepts and slopes by Participant. Here and in all subsequent models, we dummy-coded Mental Imagery (i.e., path-based imagery = 0; collection-based imagery = 1) to facilitate interpretation of the model coefficients, and we always used the maximal converging random effects structure[2] (Barr et al, 2013), with random intercepts and slopes for Mental Imagery condition. As predicted, there was a significant effect of mental imagery (β = 1.5, $t_{8.1}$ = 2.5, p = .039), such that collection-based mental imagery prompted gestures that were more Collection-like. This full model, moreover, was significantly better than a reduced model without mental imagery condition, $\chi_{(1)}$ = 5.5489, p = .018, confirming the causal influence of mental imagery on gesture.

As a further check, we looked at whether mental imagery also influenced whether subsequent gestures were classified as path-like or collection-like, according to the unsupervised k-means algorithm described above (see Figure 4.11, right panel). A mixed-logit model (Jaeger, 2008) with Mental Imagery as a fixed effect and Participant as a random effect found that, as predicted, participants were far more likely to produce collection-like gestures after collection-based imagery, compared to path-based imagery (β = 9.98, z = 2.4, p = .025). Path-based imagery prompted gestures that were highly likely to be categorized as path-like (predicted probability: 64%), while those produced after collection-based imagery were almost always categorized as collection-like (predicted probability: 99%). Thus, simulating space and action had a systematic influence on subsequent metaphorical gestures about arithmetic.

---

[2] For both morphology and handedness, the model with a fully maximal random effects structure converged, with correlated intercepts and slopes. A model with this random effects structure did *not* converge for kinematics, so, following the recommendations of Barr et al (2013), we used a random effects structure with uncorrelated intercepts and slopes.

(1) [Or aye plus **bee** /]
a b

(2) [plus / **aye]** [plus **bee** /]
c d

(3) [has to be an **even** number /]
e

(4) [And then / you have the
f

one **left** over /]
g

(5) [So it's an **odd** number.]
h i

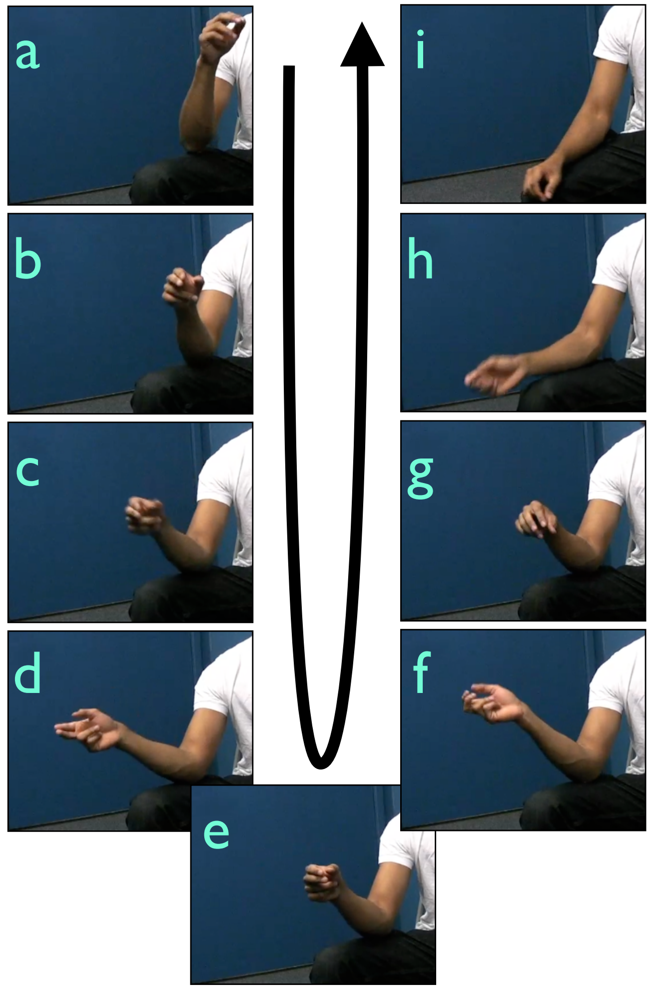


Figure 4.10. A hybrid gesture phrase that blends Path and Collection construals. The speaker is describing a complex sum ($a + a + b + b + 1$). He begins with a series of standard Path gestures (a-d): one-handed, with canonical pointing handshapes, and each rightward stroke corresponding to a sum. However, when the speaker describes adding one (e-g), he adopts the grasping handshape and inward-directed movement of a Collection gesture, "placing" the unit (g) at the location where he had placed the outcome of the first three operations (e).

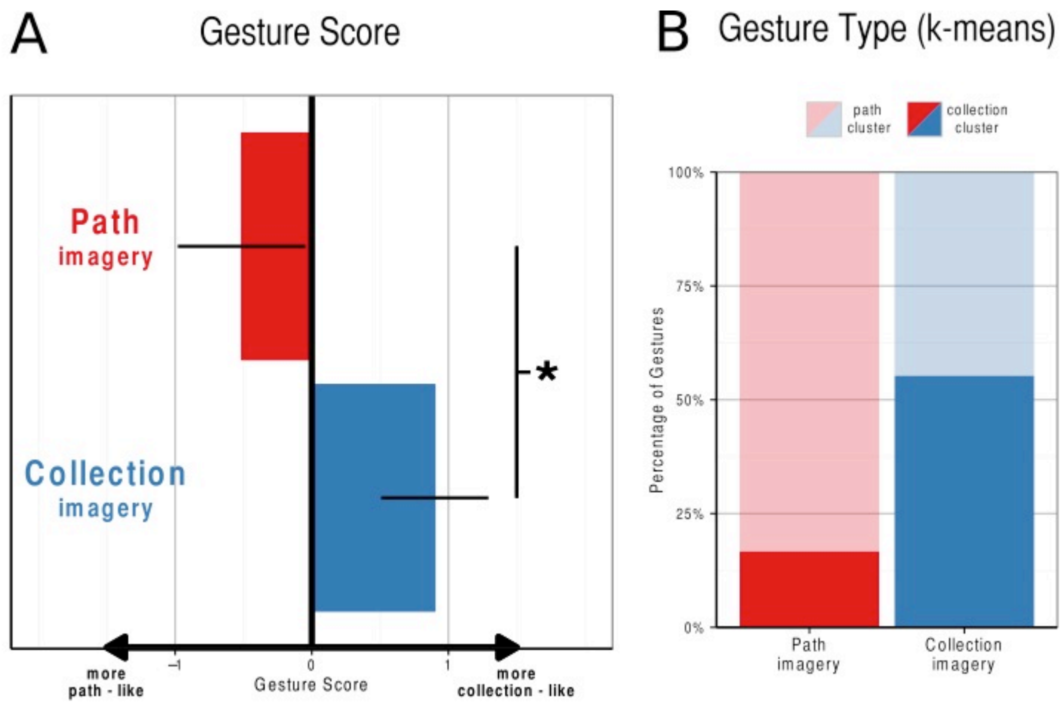


Figure 4.11. Effect of mental imagery on metaphorical gesture (Study 2). (A) The horizontal axis indicates whether gestures were more "path-like" or "collection-like," a composite score that combines morphology, handedness, and kinematics. After path- or collection-related mental imagery, participants produced metaphorical gestures that were more Path- and Collection-like, respectively. Error lines = SEM. (B) Based on morphology, handedness, and kinematics, gestures were clustered into two categories using k-means, an unsupervised machine learning technique for partitioning multidimensional data. After path-based imagery, most gestures were in the Path cluster (light red); after collection-based imagery, most were in the Collection cluster (dark blue).

Finally, we performed a series of follow-up analyses to determine the influence of mental imagery on each individual feature: handedness, morphology, and kinematics. We modeled each gesture feature using a mixed-logit model with Mental Imagery condition as a fixed effect and Participant as a random effect, allowing us to model each feature as a categorical variable (e.g., *grasping* vs. *pointing* morphology). A model of gestures with grasping or pointing morphology revealed a significant influence of mental imagery on gesture morphology ($\beta = 13.3$, $z = 2.5$, $p = .012$). Path-based imagery prompted gestures that were highly likely to have congruent *pointing* morphology (predicted probability: 63%), while those produced after collection-based imagery almost always had congruent *grasping* morphology (predicted probability: 99%). This full model of morphology was significantly better than a reduced model without mental imagery ($\chi_{(1)} = 5.3$, $p = .021$), confirming the causal influence of spatial imagery on gesture morphology. Similarly, the model of handedness found that performing collection-based rather than path-based imagery significantly increased the probability of producing two-handed gestures ($\beta = 10.0$, $z = 2.2$, $p = .025$). After path-based imagery, most gestures were one-handed (predicted probability: 64%), congruent with a canonical Path gesture; after collection-based imagery, most gestures were two-handed (predicted probability: 99%), congruent with a canonical Collection gesture. This full model of handedness was significantly better than a reduced model without mental imagery ($\chi_{(1)} = 5.3$, $p = .021$), confirming the causal influence of spatial imagery on gesture handedness. By contrast, although gestures *were* more likely to involve transversal motion after path-based imagery (predicted probability: 85%) than after collection-based imagery (predicted probability: 72%), there was no evidence that this was due to anything other than chance ($\beta = 0.8$, $z = 0.8$, $p > .35$; $\chi_{(1)} = 62$, $p > .4$). Inspection of a few gestures from the Collection cluster suggested an explanation: Many involved "collecting" movements that occurred primarily along the transversal axis, such as the gesture depicted in Figure 4.7-c. Thus, the influence of imagery on subsequent metaphorical gesture seems to have been driven primarily by form—the number and shape of the hands—rather than the precise motion through space.

4.3.4.3. Is the influence of imagery on gesture mediated by speech?

Is the influence of imagery on gesture the result of a direct influence of internal simulation on external metaphorical gesture, or is the link between simulation and gesture mediated by speech? Indeed, on a deflationary account, mental imagery may have changed the semantic content of speech, and this change in speech might have been responsible for the change in gesture. For instance, simulating the manipulation of collections could have primed participants to *talk* about numbers as if they were collections; or perhaps simulating sliding a bead along a wire primed participants to describe arithmetic, explicitly, in terms of spatial paths. To investigate this deflationary account, we analyzed the transcripts of participants' speech for systematic differences between mental imagery conditions. We searched the transcripts for any explicit spatial language, used metaphorically or otherwise. There was no evidence that participants used spatial language at all, whether literally or metaphorically, despite the ubiquity of metaphorical gestures that used space to represent number and arithmetic. The words "collect," "path," "move," "slide," "high," "low," "big," or their derivatives were never used. The words "smaller" and "combine"—possibly related to a Collection construal—were each used by a lone participant, but both had performed path- rather than collection-based imagery. And while the word "together" was used by a few participants (e.g., "adding even numbers together"), it was used by participants in both imagery conditions—and in fact was used by more participants in the path-based imagery condition, but only a collection-based construal of addition involves bringing numbers "together." We thus found no evidence that differences in metaphorical gesture were the product of systematic differences in explicit uses of spatial language.

Perhaps the influence of mental imagery on speech is more subtle, involving nuanced shifts in semantic content rather than differences in explicit spatial language. To test this possibility, we used an unsupervised machine learning technique, Latent Dirichlet Allocation (LDA), to categorize participants' speech on the basis of their latent semantic content (Griffiths, Steyvers, and Tenenbaum, 2007). On the basis of the "bag of words" in a series of texts—in this case, the participants' transcribed speech—LDA creates a generative model of the latent semantic topics in the corpus and calculates the probability that each text is discussing a particular topic. We decided *a priori* to fit the model to two topics, since participants were exposed to one of two mental imagery conditions. After removing punctuation, disfluencies, and standard stop words (e.g. "the," "is," "which"), we ran 1000 iterations of LDA and selected the model with the lowest perplexity, a measure of model fit. LDA thus assigned to each text (i.e., speech transcript) a posterior probability that it discussed one of the semantic topics rather than the other, an index of the semantic content of each participant's speech.

Did the latent semantic content of participants' speech differ systematically by imagery condition? No. The best LDA model assigned participants' speech to topics in a way that did not differ by imagery condition ($t_{10.43} = 0.5$, $p > .6$). On average, the posterior probability of discussing the first latent topic after collection-based imagery was 66%, while the posterior probability of discussing that topic after path-based imagery was 52%. In addition, we added these posterior probabilities as a fixed effect to the mixed-effects model of composite gesture scores, thus accounting for subtle differences in the semantic content of speech. This model was still significantly better than a reduced model without Mental Imagery, $\chi_{(1)} = 4.71$, $p = .03$, confirming the direct influence of mental simulation on metaphorical gesture, unmediated by the content of accompanying speech.

In sum, Study 2 found that the internal simulation of action and space has a systematic influence on subsequent metaphorical gesture, as predicted by the GSA framework (Hostetter and Alibali, 2008). Imagining the combination of concrete collections of objects led participants to gesture metaphorically as if arithmetic were a form of abstract object collection; imagining a concrete object sliding along a path led participants to gesture metaphorically as if arithmetic were motion along a path. This was true despite the absence of metaphorical speech; indeed, participants' speech did not differ systematically as a function of mental imagery, nor did the semantic content of speech mediate the effect of spatial imagery on metaphorical gesture. Not only do people gesture spontaneously as if arithmetic is object collection or, alternatively, motion along a path, but this external spatialization is shaped by the internal mental simulation of action and space.

4.4. General Discussion

Our goal was to determine whether mathematical reasoning—a paragon of abstraction—deploys spatial metaphors to make sense of number and arithmetic. We used spontaneous co-speech gesture as an index of real-time, dynamic reasoning. We reasoned that, if the conceptualization of arithmetic involves mental simulating the metaphorical source domain (e.g., motion along a path), then this should manifest itself externally as metaphorical gesture (Hostetter and Alibali, 2008). This is, indeed, what we found. In each study, participants deployed two systems of metaphorical gesture that recruited space in systematic and complementary ways: *Path* gestures represented numbers as locations and arithmetic as motion along a path. *Collection* gestures represented numerical magnitude as spatial volume and arithmetic as the manipulation of collections of objects.

The results of Study 2, moreover, shed light on *why* people gesture metaphorically—that is, on the proximal causes of metaphorical gesture. While a number of studies suggest that internal simulation drives the production of concrete representational gestures (Marghetis and Bergen, 2014), we do not know of any that have tested this proposal for metaphorical gestures. In Study 2, imagining collections of concrete objects primed the production of Collection gestures, while imagining motion along a path primed Path gestures. This link between simulation and gesture, moreover, was not mediated by speech, which was unaffected by the mental imagery manipulation. These results suggest that the production of metaphorical gestures is driven, in part, by implicit simulation of source domains (cf., Hostetter and Alibali, 2008)—in this case, object collection or linear motion. Thus, like gestures with concrete referents, metaphorical gestures are doubly spatial: they use external space to represent abstract entities and relations among them; and they reflect internal spatial processing, the activation of source domains and their constitutive images schemas.

4.4.1. Flexible, dynamic, and hybrid thought

One striking feature of the spatial construal of arithmetic was its flexibility and creativity. Participants did not necessarily stick to a single gestural system, with some producing both Path and Collection gestures over the course of their explanations. The finding, moreover, that metaphorical gesture was shaped by mental imagery (Study 2) further suggests that participants were able to adopt either construal. The availability of both metaphors created the opportunity to creatively integrate both construals into a new, blended conceptualization (Coulson, 2001; Fauconnier and Turner, 1998, 2002; Parrill and Sweetser, 2004). The hybrid gesture depicted in Figure 4.10, for instance, is meaningless in either gesture system alone, since it combines both location- and collection-based representations of numerical magnitude within a single gesture. The gesture's semantics requires the conceptual integration of both metaphors into a novel, blended construal in which numbers are simultaneously locations and collections.

Hybrid gestures of this sort are reminiscent of three other gestural phenomena in which multiple spaces or concepts are combined. First, in *laminated* gestures, multiple spaces are brought into alignment in order to communicate about spatial relations that are displaced or operate on different scales (Haviland, 1996). When describing at noon the first time you saw a sunrise, for instance, you might point eastward, thus "laminating" an allocentric spatial relation from the past onto the gesture space of the current communicative encounter. Second, *speech-gesture mismatches* communicate different but related information in speech and gesture (Goldin-Meadow, 2005). When children explain how they solved a mathematics problem, they sometimes express one problem solving strategy in speech and an entirely different one in gesture (Goldin-Meadow and Wagner, 2005). And third, in *mixed* metaphorical gestures, a single gesture reflects the concurrent activation of multiple metaphors, such as when temporal gestures combine the left-right transversal axis with the back-to-front sagittal axis to produce a diagonal trajectory (Walker and Cooperrider, *in press*). In all three cases, however, the coordinated spaces or concepts remain distinct; the elements are overlaid or juxtaposed but not integrated into a structure. Hybrid gestures, on the other hand, reflect the selective projection of elements from both conceptualizations, blended together to create a novel but coherent construal. The semantics of these hybrid gestures is thus more complex than the mere co-activation of two construals. Indeed, hybrid gestures are thus an opportunity to study processes of conceptual blending (Coulson, 2001; Fauconnier and Turner, 2002), though a full analysis must await another venue.

4.4.2. An ecosystem of arithmetic

The metaphorical gestures investigated in these studies are just a facet of the broader cognitive ecosystem of arithmetic (cf., Hutchins, 2010). As Wittgenstein put it, "Of course, in one sense, mathematics is a branch of knowledge, but still it is also an activity" (1953/2009, p. 227), and this activity depends on the skillful coordination of a distributed set of resources, from specialized neural systems (e.g., for spatial and sensorimotor simulation) to extracranial resources like notations, diagrams, and cultural practices. It is these resources and their interrelations which constitute the cognitive ecosystem of arithmetic.

Within distributed mathematical activity, speech, body, and thought interact in a complex web of mutual causality, with each influencing and being influenced by the others. Priming a metaphorical source domain, for example, can trigger the subsequent production of related metaphorical language (Sato, Schafer, and Bergen, 2015); describing numbers as "bigger" or "higher" may thus reflect the activation of the source domains of size or vertical location. During mental calculation, internal processing spills out of the skull and into the body as subtle spatial biases (Knops et al, 2009; Marghetis et al, 2014). And gesture is coupled not only to transient internal processing (Study 2) but also to stable structure in the external cultural world: Interpreting and interacting with external artifacts like Cartesian graphs involves coupling spontaneous gesture to the material structure of the artifact (cf., Goodwin, 2007). Tracing the graph of a function with one's index finger, for instance, may yield gestures that resemble the Path gestures analyzed here. Understanding the larger ecosystem of arithmetic thought, therefore, will require tracing the relations of mutual causality between private conceptualization within individual brains and the public structures of gesture, speech, and material artifacts.

In particular, the current results establish an influence of internal simulation on external metaphorical gesture. But little is known about the other direction of influence—of metaphorical gesture on individual, intracranial conceptualization. Indeed, if it had this effect, gesture would make an excellent candidate for a mechanism to align private understandings of individuals within a community of practice. Ongoing studies are investigating the possibility that metaphorical gestures shape the conceptualization of both the gesturers themselves and their observers. This process of "gestural contagion" may perpetuate and propagate abstract understandings within a community, contributing both to cross-cultural variability and to within-cultural agreement in abstract thought.

While elements of the cognitive ecosystem shape each other through relations of mutual influence, they also maintain a degree of autonomy. Speech and gesture, for instance, deploy space in distinct but complementary ways, with gesture recruiting the transversal axis (e.g., rightward gesture for a larger number, Figure 4.5) and speech recruiting the vertical (e.g., "higher number"). While graphs that use the Cartesian coordinate system recruit location to represent numbers, they do not use motion to represent the processes of arithmetic; a standard graph, after all, is a static representation. These resources, moreover, are subject to constraints that operate over entirely different timescales. Metaphorical language is stable over centuries, with the numerical sense of "higher" emerging eight hundred years ago (Higher, 2015); graphical norms are acquired in childhood and stable within a culture (Tversky, Kuglemass, and Winter, 1991), and new technologies are beholden to these norms; but gesture is highly sensitive to context, adaptable over the course of an explanation and even within a single utterance (e.g., Figure 4.10). A complete understanding of this mathematical activity requires accounting for the entangling of the diverse resources within this cognitive ecosystem, but also their relative autonomy.

In conclusion, our results suggest that, despite its abstractness, mathematical reasoning recruits conceptual metaphors to ground understanding in concrete, embodied domains, and that the mental simulation of these source domains manifests itself external as metaphorical gesture. *Collection* gestures used bimanual grasping handshapes to associate numbers with bounded regions. *Path* gestures, by contrast, used a single hand to trace a path and place numbers along that path. We suspect it is these metaphors—with their rich structure and ties to action and experience—that guide reasoning during mathematical activity, rather than the formal axioms or definitions that have been invented by mathematicians to characterize arithmetic (e.g., Frege, 1884/1960; Dedekind, 1888/1996). In the words of American artist Richard Serra, "I consider space to be a material." Serra was talking about art installations, but the same could be said about the human conceptual edifice. The capacity for reflexive, systematic, abstract reasoning may depend on our singular ability to build complex understandings on a foundation of space and action.

4.5. Acknowledgments

For helpful feedback, we thank K. Cooperrider, S. Coulson, S. Golden-Meadow, R. Hendricks, E. Hutchins, M. Kutas, J. Mandler, J. Olsen, N. Renner, J. Radford-Doyle, E. Sweetser and the *Berkeley Gesture Group,* and E. Walker. Earlier incarnations of this research were presented at *Conceptual Structure, Discourse, & Language 11* (2012); *Researching and Applying Metaphor 9* (2012); the *4th Meeting of the UK Cognitive Linguistics Association* (2012); and the *5th Conference for the International Society for Gesture Studies* (2012). Industrious undergraduate research assistants helped with data collection and gesture coding: Anthony Chan, Richard Chen, Luke Eberle, Brittany Fitzgerald, Jeremiah Palmerston, and Sarah Saturday. TM is grateful for doctoral fellowships from FQRSC (Québec, Canada) and the Robert J. Glushko and Pamela Samuelson Foundation.

4.6. References

Anderson, M. L. (2010). Neural reuse: A fundamental organizational principle of the brain. *Behavioral and Brain Sciences,* 33, 245–313

Barr, D. J., Levy, R., Scheepers, C., & Tily, H. J. (2013). Random effects structure for confirmatory hypothesis testing: Keep it maximal. *Journal of Memory and Language*, 68, 255-278.

Barsalou, L. W. (1999). Perceptual symbol systems. *Behavioral and Brain Sciences*, 22, 577-660.

Barsalou, L.W. (2008). Grounded cognition. *Annual Review of Psychology, 59,* 617-645

Barth, H., La Mont, K., Lipton, J., Dehaene, S., Kanwisher, N., & Spelke, E. (2006). Non-symbolic arithmetic in adults and young children. *Cognition*, 98, 199-222.

Bourdieu, P. (1977). *Outline of a Theory of Practice*. Cambridge: Cambridge University Press.

Casasanto, D. (2009). When is a Linguistic Metaphor a Conceptual Metaphor? In V. Evans & S. Pourcel (Eds.), *New Directions in Cognitive Linguistics*. Amsterdam: John Benjamins.

Casasanto, D. & Jasmin, K. (2012). The Hands of Time: Temporal gestures in English speakers.*Cognitive Linguistics*, 23, 643 – 674.

Cauchy, A. L. (1821). Cours d'analyse de l'École royale polytechnique. Paris: Debures. In Oeuvres complètes, Series 2, Vol. 3. Paris: Gauthier-Villars, 1897.

Cauchy, A. L. (1853). Note sur les series convergentes don't les divers termes sont des fonctions continues d'une variable rélle ou imaginaire, entre des limites données. In *Oeuvres complètes, Series 1*, Vol. 12, pp. 30-36. Paris: Gauthier-Villars, 1900.

Cooperrider, K., & Núñez, R. (2009). Across time, across the body: Transversal temporal gestures. *Gesture*, 9, 181-206.

Coulson, S. (2001). *Semantic leaps: Frame-shifting and conceptual blending in meaning construction*. Cambridge University Press.

de Saussure, F. (1986). *Course in General Linguistics* (3rd ed.). (R. Harris, trans.). Chicago: Open Court Publishing Company. (Original work published 1917)

Dedekind, R. (1996). Was sind und was sollen die Zahlen? (What are and what should the numbers be?). In W. B. Ewald (ed.), *From Kant to Hilbert: A Source Book in the Foundations of Mathematics* (pp. 787–832). Oxford University Press. (Original work published 1888)

Dehaene, S., Bossini, S., & Giraux, P. (1993). The mental representation of parity and numerical magnitude. *Journal of Experimental Psychology: General*, 122, 371-396.

Dehaene, S., & Cohen, L. (2007). Cultural Recycling of Cortical Maps. *Neuron,* 56, 384-398.

Emmorey, K. (2001). Space on hand: The exploitation of signing space to illustrate abstract thought. In M. Gattis (Ed.), *Spatial Schemas and Abstract Thought* (pp. 147-174). Cambridge: MIT Press

Engberg-Pedersen, E. (1993). *Space in Danish Sign Language: The semantics and morphosyntax of the use of space in a visual language*. SIGNUM-Press.

Fauconnier, G. & Turner, M. (1998). Conceptual Integration Networks. *Cognitive Science*, 22, 133-187.

Fauconnier, G. & Turner, M. (2002). *The Way We Think: Conceptual Blending and the Mind's Hidden Complexities*. Basic Books, New York.

Flombaum J.I., Junge J.A., Hauser M.D. (2005). Rhesus monkeys (Macaca mulatta) spontaneously compute addition operations over large numbers. *Cognition,* 97, 315–325.

Frege, G. (1960). The Foundations of Arithmetic. (J. Austin, trans.). New York: Harper. (Original work published 1884).

Gallese, V. & Lakoff, G. (2005) The brain's concepts: The role of the sensory-motor system in conceptual knowledge. *Cognitive Neuropsychology,* 22, 455 – 79.

Gentner, D. (2002). Analogy in scientific discovery: The case of Johannes Kepler. In L. Magnani & N. J. Nersessian (Eds.), *Model-based reasoning: Science, technology, values* (pp. 21-39). New York: Kluwer Academic.

Giaquinto, M. (2007). *Visual thinking in mathematics: An epistemological study*. Oxford University Press.

Gibbs, R. (2006). Metaphor interpretation as embodied simulation. *Mind & Language,* 21, 434–458.

Griffiths, T. L., Steyvers, M., Tenenbaum, J. B. (2007). Topics in semantic representation. *Psychological Review,* 114, 211-244.

Grush, R. (2007). Skill theory v2.0: Dispositions, emulation, and spatial perception. *Synthese*, 159, 389-416.

Goldin-Meadow, S. (2005). *Hearing gesture: How our hands help us think*. Harvard University Press.

Goldin-Meadow, S. & Wagner, S. (2005). How our hands help us learn. *TRENDS in Cognitive Sciences*, 9, 234–241.

Goodwin, C. (2007). Environmentally coupled gestures. In S. Duncan, J. Cassell, & E. Levy (Eds.), *Gesture and the Dynamic Dimensions of Language* (pp. 195–212). Philadelphia: John Benjamins.

Halina, M., Rossano, F., & Tomasello, M. (2013). The ontogenetic ritualization of bonobo gestures. *Animal Cognition*, 16, 653-666.

Halperin, D. M. (2005). Love's Irony: Six Remarks on Platonic Eros. In S. Bartsch & T. Bartscherer (eds.), *Erotikon: Essays on Eros, Ancient and Modern* (48-58). Chicago: University of Chicago Press.

Haviland, J. B. (1996). Projections, transpositions, and relativity. In J. J. Gumperz & S. C. Levinson (eds.), *Rethinking Linguistic Relativity*. (pp. 271-323). Cambridge: Cambridge University Press.

Higher. (2015). In *OED Online*. Oxford University Press. Retrieved from: http://www.oed.com/view/Entry/86873

Hostetter, A. B., & Alibali, M. W. (2008). Visible embodiment: Gestures as simulated action. *Psychonomic Bulletin & Review*, 15, 495-514.

Hostetter, A.B., & Alibali, M.W. (2010) Language, gesture, action! A test of the Gesture as Simulated Action framework. *Journal of Memory and Language,* 63: 245–257.

Hubbard, E.M., Piazza, M., Pinel, P., & Dehaene, S. (2005). Interactions between number and space in parietal cortex. *Nature Reviews Neuroscience* 6, 435–448.

Hutchins, E. (2010): Cognitive Ecology. *Topics in Cognitive Science*, 2, 705-15.

Jaeger, T. F. (2008). Categorical data analysis: Away from ANOVAs (transformation or not) and towards logit mixed models. Journal of Memory and Language, 59, 434–446.

Kendon, A. (2004). *Gesture: Visible Action as Utterance*. Cambridge University Press.

Kitcher, P. (1984). *The Nature of Mathematical Knowledge*. Oxford: Oxford University Press.

Kita, S., & Özyürek, A. (2003). What does cross-linguistic variation in semantic coordination of speech and gesture reveal? Evidence for an interface representation of spatial thinking and speaking. *Journal of Memory and language*, 48, 16-32.

Knops, A., Thirion, B., Hubbard, E. M., Michel, V., & Dehaene, S. (2009). Recruitment of an area involved in eye movements during mental arithmetic. *Science, 324*, 1583–1585.

Lakatos, I. (1978). Cauchy and the continuum: The significance of non-standard analysis for the history and philosophy of mathematics. Mathematical Intelligencer, 1, 151-161.

Lakoff, G. & Johnson, M. (1980). Metaphors We Live By. University of Chicago Press.

Lakoff, G. & Núñez, R. (2000). Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being. Basic Books.

Marghetis, T., & Bergen, B. (2014). Embodied meaning, inside and out: The coupling of gesture and mental simulation. In C. Müller, A. Cienki, E. Fricke, S. H. Ladewig, D. McNeill & Sedinha Tessendorf (Eds.) *Body-Language-Communication: An International Handbook on Multimodality in Human Interaction*. New York: Mouton de Gruyter.

Marghetis, T. & Núñez, R. (2013). The motion behind the symbols: A vital role for dynamism in the conceptualization of limits and continuity in expert mathematics. *Topics in Cognitive Science*, 5, 299-316.

Marghetis, T., Núñez, R, & Bergen, B. (2014). Doing arithmetic by hand: Hand movements during exact arithmetic reveal systematic, dynamic spatial processing. *Quarterly Journal of Experimental Psychology,* 67, 1579-1596.

McCrink, K., Dehaene, S., & Dehaene-Lambertz, G. (2007). Moving along the number line: Operational momentum in nonsymbolic arithmetic. *Perception and Psychophysics, 69*, 1324-1333.

McNeill, D. (1992). *Hand and Mind: What Gestures Reveal About Thought.* Chicago: Chicago University Press.

McNeill, D., Cassell, J., & Levy, E.T. (1993). Abstract deixis. *Semiotica*, 95, 5-19.

Núñez, R, Cooperrider, K., Doan, D, & Wassmann, J. (2012). Contours of time: Topographic construals of past, present, and future in the Yupno Valley of Papua New Guinea. *Cognition*, 124, 25-35.

Mix, K. S., & Cheng, Y. L. (2012). Space and math: The developmental and educational implications. In J. B. Benson (Ed.), *Advances in child development and behavior* (Vol. 42). New York, NY: Elsevier.

Nersessian, N. (1992). How do scientists think? Capturing the dynamics of conceptual change. In. R. N. Giere (ed.), *Cognitive Models of Science*. Minneapolis: University of Minnesota Press.

Núñez, R., & Sweetser, E. (2006). With the Future Behind Them : Convergent Evidence From Aymara Language and Gesture in the Crosslinguistic Comparison of Spatial Construals of Time. *Cognitive Science*, 30, 401-450.

Núñez, R., & Marghetis, T. (in press). Cognitive Linguistics and the Concept(s) of Number. In R. Cohen-Kadosh and K. Dowker (eds.), *Oxford Handbook of Numerical Cognition*. Oxford University Press.

Nersessian, N. (2008). *Creating Scientific Concepts*. Cambridge, MA: MIT Press.

Parrill, F., & Sweetser, E. (2004). What we mean by meaning. *Gesture*, 4(2), 197–219.

Sassenberg, U., & Van Der Meer, E. (2010). Do we really gesture more when it is more difficult? *Cognitive Science,* 34, 643–664.

Sato, M., Schafer, A. J., & Bergen, B. K. (2015). Metaphor priming in sentence production: Concrete pictures affect abstract language production. *Acta Psychologica*, 156, 136-142.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge University Press.

Silverman, B. W. (1986) *Density Estimation*. London: Chapman and Hall.

Tversky, B., Kugelmass, S., & Winter, A. (1991). Cross-cultural and developmental trends in graphic productions. *Cognitive Psychology,* 23, 515–557.

Walker, Esther and Kensy Cooperrider (*in press*). The continuity of metaphor: Evidence from temporal gestures. *Cognitive Science*.

Walsh, V. (2003). A theory of magnitude: common cortical metrics of time, space and quantity. *TRENDS in Cognitive Sciences*, 7, 483-488.

Winston, E. (1989). Timelines in ASL. Paper presented at *The Deaf Way*, Washington, DC.

Winter, B., Marghetis, T, & Matlock, T. (2015). Of metaphors and magnitudes: Explaining cognitive interactions between space, time, and number. *Cortex,* 64, 209-224.

Winter, B., Perlman, M., & Matlock, T. (2013). Using space to talk and gesture about numbers: Evidence from the TV News Archive. *Gesture*, 13, 377-408.

Wittgenstein, L (2009). *Philosophical Investigations.* (G. E. M. Anscombe, trans.). Wiley-Blackwell. (Original work published 1953)

Appendix A: Generic proof that the sum of an even and an odd number is odd.

A number is even whenever it is the result of adding two equal numbers. For example, 6 is even, and it is the result of adding 3 and 3, since 6=3+3. A number is odd whenever it is the result of adding two equal numbers, and then adding one more. For example, 9 is odd, and is the result of adding 4 and 4, and then adding one, since 9=4+4+1.

We want to add an odd number to an even number. Now, the even number is the result of adding two equal numbers. Let's call these numbers "*a*," so the even number is: *a+a*. Also, the odd number is the result of adding two equal numbers, and then adding one. Let's call these numbers "*b*," so the odd number is: *b+b+1*.

To add the even number to the odd number, we can split up the even and odd numbers into these smaller numbers. So add one "a" to one "b," and then add the other "a" to the other "*b*." We will still have one left over for the odd number, of course. This gives us *a+b*, and another *a+b*. Adding these two equal numbers will give us an even number: *a+b + a+b*. But we have one left over for the odd number – so we need to add that left-over one: *a+b + a+b + 1*. But now we've added the same number twice (*a+b*), and then added one more. So this is an odd number.

Therefore, adding an even number to an odd number always gives an odd number.

Appendix B: Sample instructions for the Mental Imagery task

Imagine the bead[s] [in their colored plates/ on the colored wire]. Following the instructions below, imagine moving the bead[s] around between the [plates/locations], described by their color. After each step, imagine the bead[s] in [their new plate/its new location].

1. From blue to red.
2. From red to green.
3. From green to red.
4. From red to blue.

Chapter 5

The mental number-line spreads by gestural contagion

ABSTRACT

Mathematical expertise builds on a foundation of space—especially the ability to map exact numbers to linear space. This "mental number-line" is known to vary cross-culturally, but there is debate about the mechanisms responsible for its cultural elaboration. We investigated the role of co-speech gesture, a ubiquitous cultural activity, in stabilizing and entrenching the mental number-line within a community. Imitating culture-specific gestures systematically shaped gesturers' mental number-line. Moreover, gestures were used spontaneously to infer speakers' spatial understanding of number, and merely observing these gestures was sufficient to shape the observer's own mental number-line. These findings establish co-speech gesture as one mechanism for propagating and perpetuating the number-line.

5.1 Introduction

From calculus to the complex plane, mathematics is rife with links between number and space. This is reflected in the human mind (Hubbard et al, 2005; Lakoff & Núñez, 2000; Winter, Marghetis, & Matlock, 2015). In many cultures, for instance, people can conceptualize exact numbers as locations along a horizontal path (Dehaene et al, 1993; Dehaene et al, 2008; Lakoff & Núñez, 2000), known as a mental number-line (MNL). The MNL has been argued to contribute to diverse mathematical abilities, including the mental representation of number (Zorzi, Priftis, and Umiltà, 2002; Opfer, Thompson, and Furlong, 2010), arithmetic (Knops et al, 2009; Marghetis, Núñez, and Bergen, 2014), and understanding complex concepts like imaginary numbers (Lakoff & Núñez, 2000; Marghetis and Youngstrom 2014; Núñez and Marghetis, *in press*). Moreover, it figures prominently in debates about the origin of abstract concepts in the human mind, since there is evidence that it emerges from a mix of innate biases and cultural influences (Dehaene et al, 20008; Shaki et al, 2009; de Hevia et al, 2014; Núñez, Cooperrider, and Wassman, 2012; Rugani et al, 2015). For instance, human neonates associate approximate numerical magnitude with spatial length (de Hevia et al, 2014), an early disposition that may support the acquisition of more precise mappings between exact numbers and spatial locations (i.e. the MNL). These early dispositions are elaborated considerably by cultural experience, with cross-cultural variability in the MNL's orientation (Dehaene et al, 1993; Shaki et al, 2009), whether the number-space mapping is linear or logarithmic (Dehaene et al, 2008), and even whether the MNL exists at all (Núñez et al, 2012). For instance, while Western adults typically exhibit a left-to-right MNL, Arabic-speaking Palestinians exhibit a right-to-left MNL (Shaki et al, 2009).

How culture-specific aspects of the MNL propagate and perpetuate within communities is poorly understood. Language is one possible mechanism. Many languages, like English, place numbers in vertical space (e.g. "high [/low] number"). But language can't be the whole story. There are no known uses of horizontal spatial language or distinctively linear versus logarithmic language to refer to number. In neither English nor Arabic, for instance, are numbers described using the words for *left* and *right*. Other proposed mechanisms include writing direction (Dehaene et al, 1993; Shaki et al, 2009), finger-counting routines (Beller and Bender, 2012; Fischer, 2008), experience with technical artifacts (Núñez et al, 2012; Siegler and Ramani, 2009), and formal education in topics like measurement (Dehaene et al, 2008; Núñez et al, 2012). There is correlational evidence in favor of each proposed mechanism, but determining distinct causal contributions has proven challenging, in part because the mechanisms are correlated with one another and other cultural variables.

One cultural activity that has not been considered in this debate is co-speech gesture, communicative bodily movements produced spontaneously by speakers in all cultures (McNeill, 1992). This may be because—compared to more stable aspects of culture like artifacts or writing—gesture is transient and thus less likely to be noticed or, when noticed, harder to measure. But there are reasons to suspect that gesture might play a critical role in propagating and perpetuating the MNL. Both novices and experts gesture when talking about mathematics, and these gestures can reveal spatial intuitions that are absent from speech (Goldin-Meadow and Beilock, 2010; Marghetis and Núñez, 2013). Moreover, culture-specific associations between number and space emerge in children as young as four years old (Opfer et al, 2010; Hoffman et al, 2013), which means that cultural influences on the MNL begin before formal education, literacy, or mastery of artifacts like physical number-lines. But not before gesture starts to shape development (Rowe and Goldin-Meadow, 2009). Gestures about number, in particular, appear early: Children as young as two-years-old and their caregivers produce numerical gestures spontaneously during play (Lee et al, *in press*).

Critically, cross-cultural differences in the conceptualization of abstract concepts often covary with differences in gesture. Americans, for instance, think and talk about the future as ahead of them, and also point forward when talking about the future, while the Aymara people of the Andes place the future behind them in language, thought, and gesture (Núñez and Cooperrider, 2013). Numerical gestures similarly vary cross-culturally. The Oksapmin people of Papua New Guinea indicate exact numbers by pointing to locations along a body-based path that runs hand-to-hand (e.g., right thumb for one, left ear for sixteen), though individuals differ in the orientation of this system (i.e., left-to-right or right-to-left) (Saxe, 2014). By contrast, when Americans talk about arithmetic, they gesture spontaneously in ways that reflect complementary spatial conceptualizations of number: as if numbers are *collections* of objects or, alternatively, as if numbers are locations along a left-to-right horizontal *path* (Fig. 5.1A) (Núñez and Marghetis, *in press*). Given the structural similarity between "Path" gestures and the MNL (e.g., both involve mapping numbers to locations along a path), these gestures may reflect gesturers' path-based understanding of number, that is, their MNL.

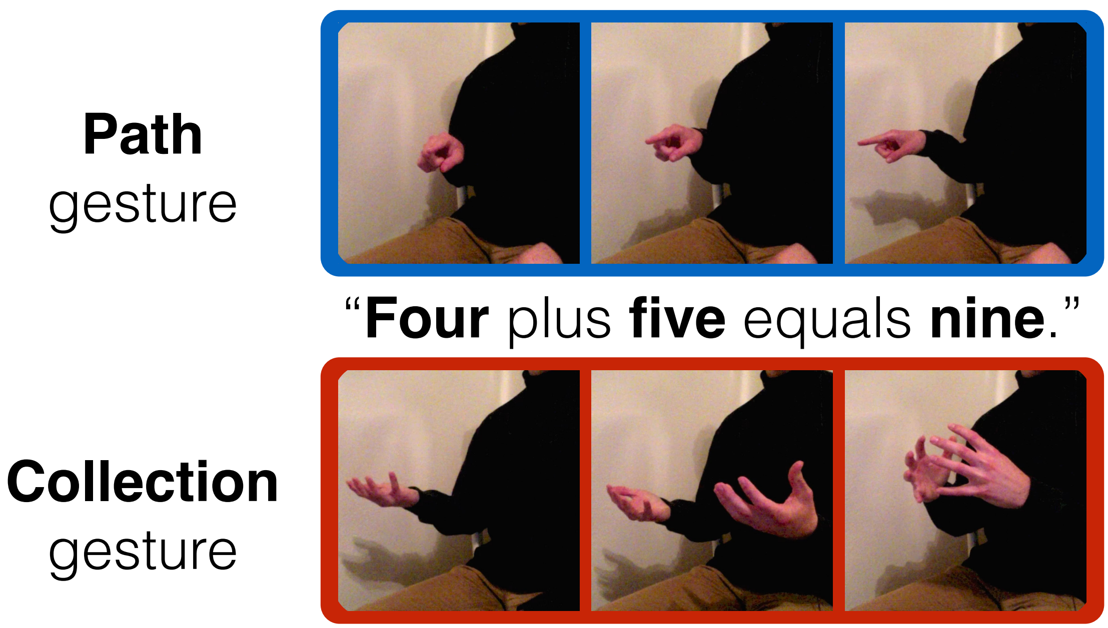


Figure 5.1. When Americans talk about number, they gesture spontaneously as if numbers are either locations along a path or collections of objects (Núñez and Marghetis, in press). We created pairs of videos (n = 8; see Movies S1-2) that had identical speech but different gestures: Path (top) or Collection (bottom). Gestures were modeled after naturally occurring co-speech numerical gestures. The same video stimuli were used in all experiments. Here, the speaker produces a gesture for each addend and their sum; boldface indicates lexical affiliates.

Could these Path gestures not only reflect but actively shape the MNL? Along with other primates, humans imitate and learn from others' actions (Tomasello, 2014), but humans may be unique in acquiring gesture through social learning (Halina, Rossano, and Tomasello, 2013). Gesture systems are, among other things, repositories of culture-specific understandings of abstract concepts (Núñez and Cooperrider, 2013; Levinson, 2003). The spread of gestures and their associated meanings may thus disseminate abstract concepts within human communities (cf., Sperber, 1996), a process we call "gestural contagion." In several experiments, we asked whether gestural contagion contributes to propagating and perpetuating the MNL.

5.2 Results

Reproducing gesture shapes the MNL: Since gestures, acquired through imitation, can shape the gesturer's own mental representations (Goldin-Meadow and Beilock, 2010), Study 1 investigated whether imitating culture-specific gestures might shape one's own MNL. Native-English-speaking adults viewed videos of a man stating mathematical facts (e.g., "four plus five is nine") while he produced semantically-related numerical gestures. Critically, we manipulated the kind of gestures he produced—either *Path* or *Collection* (Fig. 5.1A), assigned randomly between subjects—while keeping speech identical. After each video, participants were required to reproduce the speaker's speech and gesture. Following this gesture imitation task, participants completed a standard number comparison task designed to measure spontaneous associations between numbers and horizontal space, an implicit measure of the MNL. On each trial, participants indicated whether a number (1 to 9) was greater or less than 5 and responded by pressing a left or right button. A left-to-right MNL is indexed by faster responses to smaller numbers on the left and faster responses to larger numbers on the right (the "SNARC" effect; Dehaene et al, 1993). If observing and imitating numerical gestures shapes gesturers' own conceptualization of number, then participants should exhibit a more pronounced left-to-right MNL after reproducing left-to-right Path gestures.

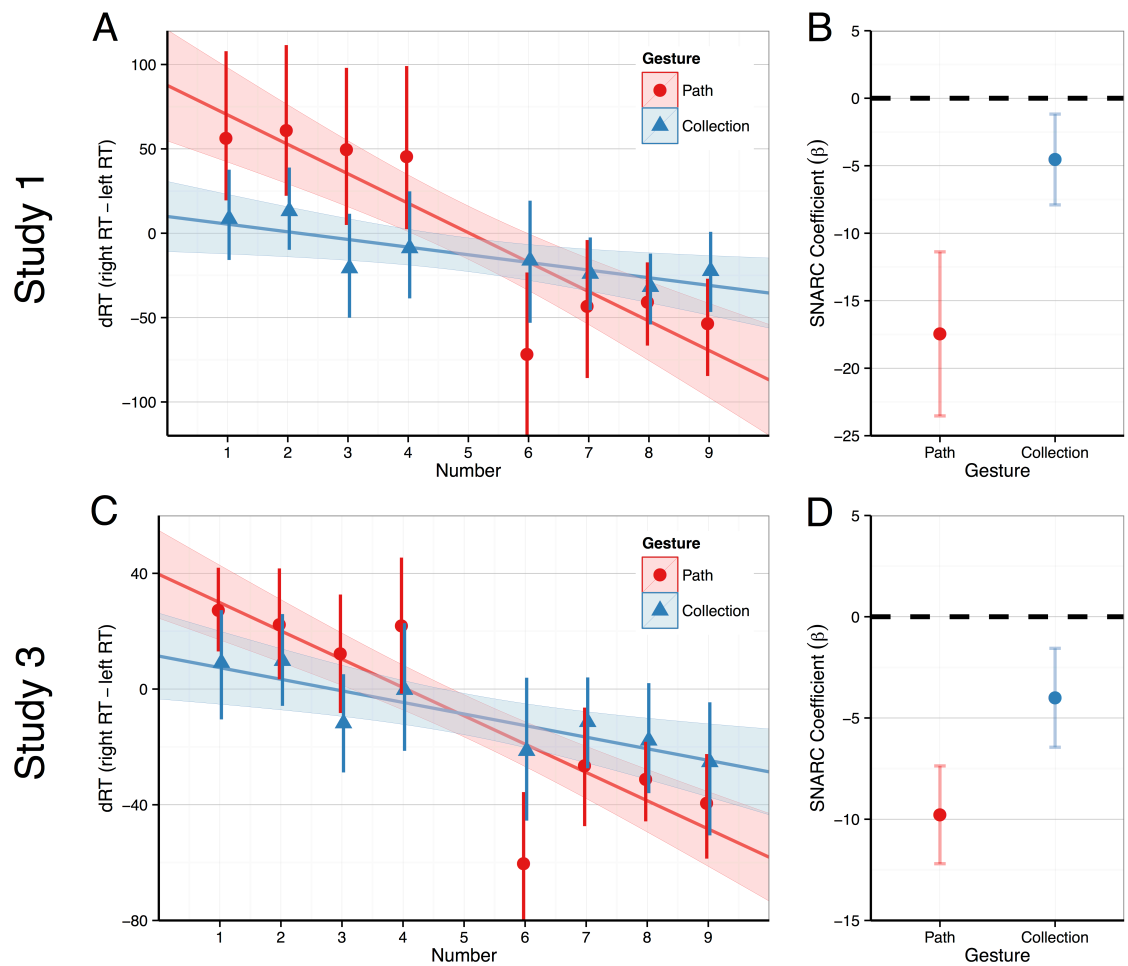


Figure 5.2. Effect of gesture on the MNL, as indexed by the SNARC effect, in Studies 1 (n = 50, top) and 3 (n = 122, bottom). (A, C) In both studies, there was evidence overall of a left-to-right MNL (i.e. negative regression coefficient), but this was significantly more pronounced in the Path condition. Error lines and shaded regions indicate bootstrapped 95% confidence intervals. (B, D) In both studies, participants' MNL, as indexed by SNARC regression coefficients (± SEM), was more pronounced in the Path gesture condition.

Overall, participants exhibited a left-to-right MNL ($F_{(3,138)}$ = 7.4, P = 0.0001), but this was modulated, as predicted, by the type of gesture they had observed and reproduced ($F_{(3,138)}$ = 3.26, P = 0.023). To quantify this effect, we calculated, for each participant and each number, the difference between mean left- and right-sided reaction times (dRT), and then regressed dRT against numerical magnitude. The magnitude of the regression slope ("SNARC coefficient," β) indicates the strength of the number-space association; the slope's sign indicates the association's orientation (negative slopes indicate a left-to-right MNL). Participants in both conditions showed evidence of a canonical left-to-right MNL (β_{path} = -17.5, t_{190} = -6.0, P << 0.001; $\beta_{collection}$ = -4.5, t_{190} = -2.5, P = 0.015; Fig. 5.2A), but, as predicted, the MNL was far more pronounced after observing and reproducing Path gestures (t_{46} = -1.9, P = 0.03, one-tailed; Fig. 5.2B). Imitating culture-specific gestures shaped gesturers' MNL: Gesturing as though numbers were locations along a path caused participants to conceptualize numbers accordingly.

Gesture shapes interpretations of the gesturers understanding: Study 2 investigated whether merely *observing* gestures, rather than imitating them, could propagate spatial understandings of number within a community. Since humans excel at inferring conspecifics' intentional states (e.g., Tomasello, 2014), observers might use a speaker's gestures to discern that speaker's number understanding. We tested this possibility in an online experiment. Native English-speaking adults, recruited from across the United States, viewed the same videos from Study 1, with gesture type (*Path* vs. *Collection*) assigned randomly between-subjects. Without mentioning gesture, we then asked participants to describe the speaker's number understanding. To determine the "gist" of these descriptions, we used an unsupervised machine learning technique—Latent Dirichlet Allocation (LDA)—

to extract two latent topics from the words used in the descriptions (Griffiths, Steyvers, and Tenenbaum, 2007).

This model extracted latent topics that reflected alternative spatial conceptualizations of number. One of the latent topics extracted by the model was associated with terms like "part," "whole," and "together," and appeared to capture a *collection-based* understanding of number (e.g., "the sum [...] as a whole and the numbers [...] as parts of that whole," "numbers as groups of things"). The other topic was associated with terms like "left" and "right," and appeared to capture a *path-based* understanding (e.g. "he sees them going from left to right," "an imaginary number line in his head").

Critically, even though gesture had not been mentioned in any instructions, the gist of participants' descriptions was shaped by the speaker's gesture (Fig. 5.3A). As a measure of gist, we used the mean-centered posterior probability that the description dealt with the path-based (vs. collection-based) topic. A positive value of this measure thus indicates that the description was more path-related than average, compared to the rest of the descriptions; a negative value indicates that the description was more collection-related than average.

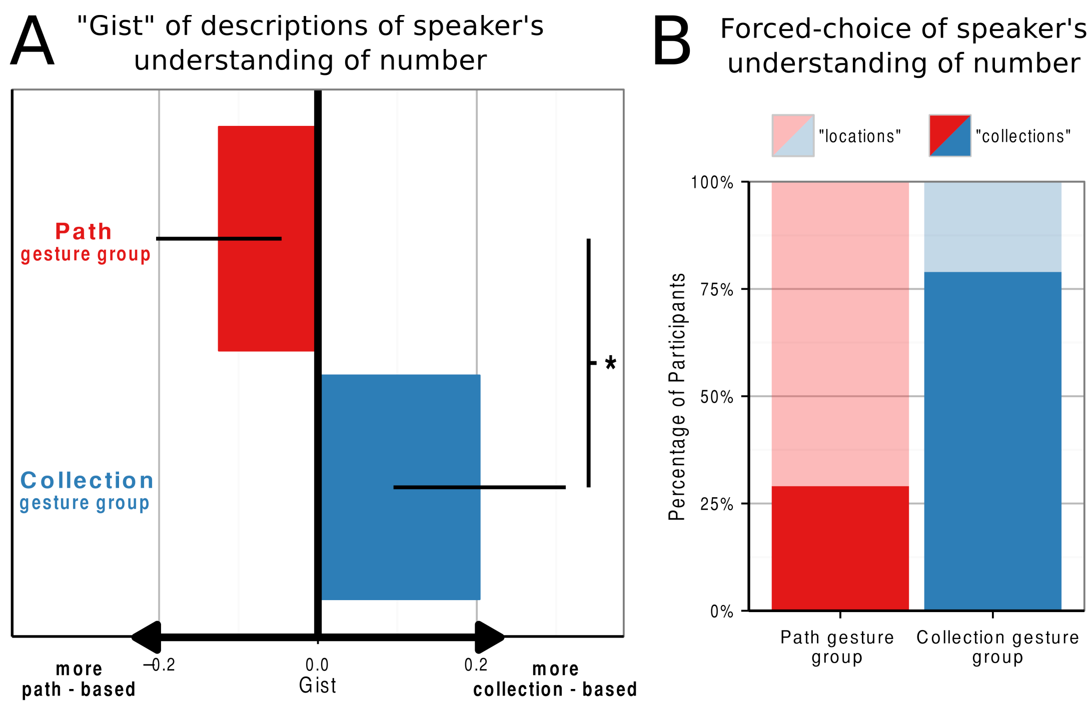


Figure 5.3. Gesture shaped observers' interpretation of speaker's conceptualization of number (Study 2, n = 50). (A) Participants spontaneously incorporated information from the speaker's gesture into their descriptions of his conceptualization (P < 0.01). Positive values of gist indicate more collection-based descriptions; negative values, more path-based descriptions. Error lines indicate SEM. (B) When forced to decide whether the speaker conceptualized numbers as "locations along a path" or "collections of objects," most participants chose the conceptualization that aligned with his gesture (P < 0.001).

There was a significant effect of gesture on participants' interpretation of the gesturer's conceptualization (P < 0.01, Mann-Whitney). If the speaker used Path gestures, descriptions of his understanding were more path-based overall (M = -0.12) and most participants (74%) gave a *path*-based description; if he used Collection gestures, descriptions were more collection-based (M = 0.20) and most participants (58%) gave *collection*-based descriptions. Indeed, when we asked participants outright whether the speaker's conceptualization was best characterized in terms of "locations along a path" or "collections of objects," their responses were shaped by his gesture (P < 0.001, Fisher's exact; Fig. 5.3B), with most participants (71%) responding that he understood numbers as "locations along a path" if he had produced *Path* gestures (P = 0.03, binomial test), and most (80%) responding that he understood numbers as "collections of objects" if he had produced *Collection* gestures (P = 0.01). Numerical gestures were thus meaningful for naïve observers, who spontaneously relied on them to infer the speaker's spatial conceptualization of number.

Gesture observation shapes the observer's MNL: Intersubjective coordination of thinking is a cornerstone of human culture. Study 3 thus investigated whether merely observing gestures not only sways observers' inferences about the *speaker's* understanding (as found in Study 2) but also shapes observers' *own* MNL. As in Study 1, participants were exposed to the same prerecorded Path and Collection gestures, with one change: while half of participants had to reproduce both the speech and gesture of the videorecorded speaker, the other half reproduced his speech only and thus merely observed the speaker's gestures. Participants' subsequent behavior on the number comparison task revealed that the overall left-to-right MNL ($F_{(3,339)}$ = 12.5, P << 0.001) was once again influenced by whether participants were exposed to Path or Collection gestures ($F_{(3,339)}$ = 2.8, P = 0.038). Critically, this was unaffected by whether participants had reproduced rather than merely observed the gestures (all Fs < 1.72, all Ps > 0.19). Further regression analyses confirmed a left-to-right MNL in both gesture conditions (β_{path} = -10.2, t_{502} = -7.2, P << 0.001; $\beta_{collection}$ = -4.0, t_{446} = -3.0, P < 0.01; Fig. 5.2C), along with a significant impact of gesture, such that participants in the Path condition had a more pronounced left-to-right MNL than in the Collection condition (t_{115} = -1.8, P = 0.038, one-tailed; Fig. 5.2D). Moreover, a linear mixed-effects model across Studies 1 and 3 confirmed the causal influence of gesture on the MNL (P = 0.016), unmodulated by whether gestures were reproduced or observed (P = 0.68; Table 5.1).

Table 5.1. Influences on the mental number-line in Studies 1 and 3. There was evidence overall of a left-to-right mental number-line (i.e. negative regression slope), but this was significantly more pronounced after exposure to Path gestures.

Predictor of SNARC effect	Coefficient	SEM	$P(>\|t\|)$
Gesture (Path vs. Collection)	–0.372	0.15	.02
Reproduction (vs. *Observation*)	0.188	0.16	.24
Gesture x Reproduction	–0.134	0.32	.68
Intercept	0.000	0.07	.99
No. of observations (groups)	165 (2)		
Log-likelihood	–231.33		

5.3 Discussion

Previous research has found considerable cross-cultural variability in the mental number-line, often attributed to differences in writing practices, finger-counting, or formal education. Our findings suggest that co-speech gestures also play a causal role in propagating and perpetuating the MNL. Imitating culture-specific numerical gestures impacted the gesturer's MNL; observing those gestures helped the observer infer the speaker's spatial understanding of number and influenced the observers' own MNL, even when unmediated by gesture imitation. In humans, therefore, action imitation and interpretation appear to propagate not just culture-specific behaviors, as previously established by work on the social learning of action (e.g., Tomasello, 2014), but also culture-specific conceptualizations of abstract ideas (cf. Sperber, 1996).

By taking advantage of within-culture variability in the gestural representation of number (Fig. 5.1), we were able to experimentally manipulate one aspect of culture while controlling for others, such as literacy, language, or formal education. These other factors, however, may also shape the spatial conceptualization of number, with multiple mechanisms operating over disparate timescales to reproduce an interpersonally-shared MNL. Artifacts like graphs and practices like literacy, for instance, are enduring cultural influences that can stabilize the MNL on an historical timescale (Dehaene et al, 2008; Shaki et al, 2009). The specific contribution of gesture may derive from its combination of flexibility and conventionality. Spatial-numerical associations, while stable at the population-level, are highly flexible within individuals (Siegler and Ramani, 2009; Fischer, Mills, and Shaki, 2010). Gesture may regiment individuals' flexible conceptualizations, aligning numerical intuitions within a community to maintain socially coordinated thinking. It remains to be seen whether gestural contagion could spread the MNL to communities that lack the concept entirely (e.g., Núñez et al, 2012) or reverse the MNL in communities where it already exists (cf., Fischer, 2008; Shaki et al, 2009).

If non-human primates acquire complex behaviors but not gestures through social learning (Halina et al, 2013), gestural contagion may be a uniquely human mechanism for cultural transmission, particularly of space-related domains. Cultures differ in how they talk and think about abstract concepts like time, social relations, and even space itself, and these culture-specific understandings are often expressed in gesture (Enfield, 2005; Núñez and Cooperrider, 2013; Le Guen, 2011; Levinson, 2003). Thus, differences in multimodal communication may not only reflect but actively drive cross-cultural differences in abstract thought, including but not limited to the MNL (cf., Le Guen, 2011). Across a variety of conceptual domains, cultural knowledge may be propagated and entrenched through gestural contagion.

5.4 Methods

We report how we determined all sample sizes, all data exclusions, all manipulations, and all measures (Simmons, Nelson, and Simonsohn, 2012)

5.4.1 Participants

In Studies 1 and 3, native-English-speaking adults from UC San Diego participated in exchange for partial course credit (Study 1: $n = 50$, $M_{age} = 21$ years, 37 women; Study 3: $n = 122$, $M_{age} = 21$ years, 91 women). In Study 2, native-English speaking adults located in the USA ($n = 50$), recruited from Amazon Mechanical Turk (AMT), participated in exchange for payment. All procedures were approved by UC San Diego's Institutional Review Board.

Sample sizes were determined in advance. In Study 1, sample size was determined on the basis of similar studies on the plasticity of the SNARC effect, e.g. $n = 44$ in Fischer et al (2010). In Study 2, sample size was determined on the basis of similar studies on the effect of gesture on comprehension, e.g. $n = 44$ in Exp. 2 of Kelly, Ozyurek, and Maris (2010). In Study 3, in which there was the additional factor of reproducing *vs.* observing gesture, an *a priori* power analysis found that 116 participants would have sufficient power ($1\text{-}\beta > 0.95$) to replicate an effect of similar size to the one found in Study 1; we therefore determined in advance to collect a sample size of 124 subjects, or as close as possible before the end of the academic year.

No participants were removed from Study 2. In Studies 1 and 3, participants were removed for exceptionally poor performance (< 80% accuracy; $n = 2$ in Study 1; $n = 4$ in

Study 3). Accuracy was high among remaining participants (Study 1: M = 94.8%, 95% CI [93.6, 96.1]; Study 3: M = 94.3%, 95% CI [93.9, 94.6]).

5.4.2 Materials

We created sixteen brief video clips, two for each of eight mathematical facts (e.g. 4 + 3 = 7; see Table 5.S1). In each video clip, a man—depicted from the neck down—was heard stating a mathematical fact (e.g. "Four plus three equals seven.") and accompanied his speech with gestures modeled after spontaneous gestures attested during mathematical discourse among American adults (Núñez & Marghetis, *to appear*). *Path* gestures represent numbers by pointing to locations along a horizontal axis in front of the speaker—smaller numbers to the left, larger numbers to the right (Fig. 5.1A). *Collection* gestures represent numbers as volumes in space, using either single-handed grasping gestures or, for larger numbers, two-handed gestures that delimit relatively larger regions (Fig. 5.1B). These two kinds of gesture thus represent complementary ways of conceptualizing number spatially (Lakoff and Núñez, 2000; Núñez and Marghetis, *in press*). To create the videos, we first audio-recorded the man stating the eight mathematical facts. Then, for each recorded fact, we made two video-recordings: one in which the man produced naturalistic Path gestures in time with the pre-recorded speech, and another in which he produced naturalistic Collection gestures. These two video-recordings were then combined with the pre-recorded audio to create eight pairs of video files; paired videos thus had identical audio but contrasted in co-speech gesture (i.e. Path vs. Collection).

5.4.3 Design of Study 1 and 3

In a between-subjects design, participants completed two tasks: an initial Gesture Imitation task in which they were exposed to either Path or Collection gestures, followed by a standard Number Comparison task designed to measure associations between numbers and lateral space, i.e. the SNARC effect (Dehaene et al, 1993).

Gesture Imitation task: Participants viewed either Path or Collection video clips (see *Materials*, above). The type of gesture (*Path* vs. *Collection*) was manipulated between-subjects and assigned randomly. In each trial, the experimenter played a video clip once and then asked the participant to reproduce exactly the speech, or speech and gesture, depending on the study. In Study 1, participants had to reproduce the clips' speech and gesture. In Study 3, they had to reproduce either both speech and gesture (*Reproduce*) or only speech (*Observe*), manipulated between-subjects and assigned randomly. Participants were given the opportunity to re-watch each video. One block consisted of viewing and reproducing all eight Path or Collection videos; participants completed four blocks, for a total of thirty-two trials.

Number Comparison task: Participants judged whether positive integers—from one to nine, inclusive—were greater or less than 5. The task was implemented in E-Prime (Psychology Software Tools, Pittsburgh, PA, USA). Each trial began with a fixation cross in the center of a computer monitor, replaced after 1000ms by an Arabic numeral between 1 and 9 (excluding 5). Participants then had up to 3000ms to respond by pressing one of two buttons on a serial response box: either the leftmost button with their left index finger, or the rightmost button with their right index finger. Participants completed two blocks, each of which began with eight practice trials (one for each numeral) followed by eighty experimental trials (ten for each numeral); trial order was randomized within blocks. Critically, the stimulus-response mapping between numerical magnitude and spatial response changed between blocks (e.g. left-side response for numbers less than five); block order was counterbalanced between subjects. If a participant had a canonical left-to-right mental number-line, therefore, they would be faster to categorize smaller numbers when responding on the left, and larger numbers, on the right. We measured accuracy and reaction time; no other measures were recorded.

5.4.4 Design of Study 2

Participants began by viewing all eight video clips, with gesture (*Path* vs. *Collection*) assigned randomly between-subjects. They were told simply to view the videos and that they would be asked questions afterward. Participants then responded to two questions about the speaker's understanding of number. First, they were asked to describe, in a few written sentences, the speaker's "understanding of number and arithmetic." Second, they were asked whether the speaker's understanding was best described as "numbers are like locations along a path" or "numbers are like collections of objects." Up to this point, gesture was never mentioned. Then, as a manipulation check, participants were asked whether they had paid attention to the speaker's gestures (every participant responded at least "maybe a little") and were played two video clips and asked whether or not they recognized them (every participant was correct on either one or both of these clips). They then supplied demographic information (gender, age, ZIP code, education, primary occupation, languages spoken). No other measures were collected.

5.4.5 Analyses of Study 1 and 3

Before analyzing reaction times, we removed incorrect responses (Study 1: 5.2% of trials; Study 3: 5.7%), followed by responses that were either faster than 275ms or slower than three standard deviations above the participant's condition mean (Study 1: 2.3% of trials; Study 3: 4.0%). To model individual SNARC regression coefficients across both experiments, we used a linear mixed-effects model with maximal converging random effects structure (Barr et al, 2013): uncorrelated random intercepts and slopes for both factors and their interaction. Models were fit using restricted maximum likelihood; p-values for parameter estimates were calculated using Satterthwate's approximations; model fit was evaluated using a Likelihood Ratio test. Before analysis, SNARC regression coefficients were standardized for each experiment to control for possible differences in the participant population.

5.4.6 Analysis of Study 2

We used a Latent Dirichlet Allocation (LDA) topic model to model the gist of participants' descriptions. LDA is an unsupervised machine learning technique that models the words used in a set of documents (in this case, participants' descriptions) using a generative model based on latent topics (Griffiths et al, 2007). We decided *a priori* to fit the model to two topics, since participants were exposed to two ways of gesturing about number. We first removed punctuation, numbers, and standard stop words (e.g. "the," "is," "which"), and then ran 1000 iterations of LDA and selected the model with the lowest perplexity, a measure of model fit.

5.5 Acknowledgments

Thanks to Natalie Allen and Myrna Aboudiab for help with data collection, to Lera Boroditsky for advice, and to Kensy Cooperrider, Esther Walker, David Barner, and Rose Hendricks for comments on an early draft.

Chapter 5, in full, has been submitted for publication, and will appear, in part, in the *Proceedings of the 37th Annual Conference of the Cognitive Science Society*. Marghetis, T.; Eberle, L.; Bergen, B., 2015. The dissertation author was the primary investigator and author.

5.6 References

Barr, D.J., Levy, R., Scheepers, C., & Tily, H.J. (2013). Random effects structure for confirmatory hypothesis testing: Keep it maximal. *Journal of Memory and Language,* 68, 255-278.

Bender, A., & Beller, S. (2012) Nature and culture of finger counting. *Cognition,* 124, 156-182.

de Hevia, M. D., Izard, V., Coubart, A., Spelke, E. S., & Streri, A. (2014). Representations of space, time, and number in neonates. *Proceedings of the National Academy of Science,* 111, 4809-4813.

Dehaene, S., Bossini, S., & Giraux, P. (1993). The mental representation of parity and number magnitude. *Journal of Experimental Psychology: General,* 122, 371-396.

Dehaene, S., Izard, V., Spelke, E., & Pica, P. (2008). Log or linear? Distinct intuitions of the number scale in Western and Amazonian indigene cultures. *Science,* 320, 1217-1220.

Enfield, N. (2005). The body as a cognitive artifact in kinship representations. *Current Anthropology,* 46, 51-81.

Fias, W., Brysbaert, M., Geypens, F., & d'Ydewalle, G. (1996). The importance of magnitude information in numerical processing: Evidence from the SNARC effect. *Mathematical Cognition,* 2, 95–110.

Fischer, M. H. (2008). Finger counting habits modulate spatial-numerical associations. *Cortex,* 44, 386-392.

Fischer, M. H., Mills, R. A., & Shaki, S. (2010). How to cook a SNARC: Number placement in text rapidly changes spatial–numerical associations. *Brain and Cognition,* 72, 333-336.

Goldin-Meadow, S., & Beilock, S. L. (2010) Action's influence on thought: The case of gesture. *Perspectives on Psychological Science,* 5, 664-674.

Griffiths, T. L., Steyvers, M., & Tenenbaum, J. B. (2007). Topics in semantic representation. *Psychological Review,* 114, 211-244.

Halina, M., Rossano, F., & Tomasello, M. (2013) The ontogenetic ritualization of bonobo gestures. *Animal Cognition*, 16, 653-666.

Hoffman, D., Hornung, C., Martin, R., & Schiltz, C. (2013). Developing number–space associations: SNARC effects using a color discrimination task in 5-year-olds. *Journal of Experimental Child Psychology*, 116, 775-791.

Hubbard, E. M., Piazza, M., Pinel, P., & Dehaene, S. (2005). Interactions between number and space in parietal cortex. *Nature Reviews Neuroscience*, 6, 435-448.

Kelly, S. D., Ozyurek, A., & Maris, E. (2010). Two Sides of the Same Coin: Speech and Gesture Mutually Interact to Enhance Comprehension. *Psychological Science,* 21, 260–267.

Knops, A., Thirion, B., Hubbard, E., Michel, V. & Dehaene, S. (2009). Recruitment of an area involved in eye movement during mental arithmetic. *Science*, 324, 1583-1585.

Lakoff, G., & Núñez, R. (2000). *Where Mathematics Comes From.* New York: Basic Books.

Lee, J., Kotsopoulos, D., Tumber, A., & Makosz, A. (in press). Gesturing about number sense. *Journal of Early Child Research.*

Le Guen, O. (2011). Speech and Gesture in Spatial Language and Cognition Among the Yucatec Mayas. *Cognitive Science,* 35, 905-938.

Levinson, S. (2003). *Space in Language and Cognition.* New York: Cambridge University Press.

Marghetis, T., & Núñez, R. (2013). The motion behind the symbols: A vital role for dynamism in the conceptualization of limits and continuity in expert mathematics. *Topics in Cognitive Science,* 5, 299-316.

Marghetis, T., Núñez, R., & Bergen, B. K. (2014) Doing arithmetic by hand: Hand movements during exact arithmetic reveal systematic, dynamic spatial processing. *Quarterly Journal of Experimental Psychology*, 67, 1579-1596.

Marghetis, T., & Youngstrom, K. (2014). Pierced by the number line: Integers are associated with back-to-front sagittal space. *Proceedings of the 36th Annual Conference of the Cognitive Science Society.* Austin, TX: Cognitive Science Society.

McNeill, D. (1992). *Hand and Mind.* Chicago: University of Chicago Press.

Núñez, R., & Cooperrider, K. (2013). The tangle of space and time in human cognition. *Trends in Cognitive Science,* 17, 220-229.

Núñez, R., Cooperrider, K., & Wassman, J. (2012). Number concepts without number lines in an indigenous group of Papua New Guinea. *PLoS One,* 7, e35662.

Núñez, R, & Marghetis, T. (in press) Cognitive linguistics and the concept(s) of number. In R. C. Kadosh & A. Dowker (eds.), *Oxford Handbook of Numerical Cognition*. New York: Oxford University Press.

Opfer, J. E., Thompson, C. A., & Furlong, E. E. (2010). Early development of spatial-numeric associations: Evidence from spatial and quantitative performance of preschoolers. *Developmental Science,* 13, 761-771.

Rowe, M. L., Goldin-Meadow, S. (2009). Differences in early gesture explain SES disparities in child vocabulary size at school entry. *Science,* 323, 951-953.

Rugani, R., Vallortigara, G., Priftis, K., & Regolin, L. (2015). Number-space mapping in the newborn chick resembles humans' mental number line. *Science*, 347, 534-536.

Saxe, G. (2014) *Cultural Development of Mathematical Ideas.* New York: Cambridge University Press.

Shaki, S., Fischer, M. H., Pretrusic, W. M. (2009). Reading habits for both words and numbers contribute to the SNARC effect. *Psychonomics Bulletin & Review,* 16, 328-331.

Siegler, R. S., Ramani, G. B. (2009). Playing linear number board games—but not circular ones—improves low-income preschoolers' numerical understanding. *Journal of Educational Psychology*, 101, 545-560.

Simmons, J., Nelson, L., & Simonsohn, U. (2012) A 21 word solution. http://ssrn.com/abstract=2160588

Sperber, D. (1996) *Explaining Culture.* Oxford: Blackwell.

Tomasello, M. (2014). *A Natural History of Human Thinking.* Cambridge, MA: Harvard University Press.

Winter, B., Marghetis, T., & Matlock, T. (2015). Of magnitudes and metaphors: Explaining cognitive interactions between space, time, and number. *Cortex* 64, 209-224.

Zorzi, M., Priftis, K., Umiltà, C. (2002). Neglect disrupts the mental number line. *Nature,* 417, 138-139.

5.7 Supplementary Information

5.7.1 Mathematical facts stated in the Path and Collection videos

Table 5.S1. Mathematical facts stated in the Path and Collection videos

Operation	Mathematical Fact
Addition	"Four plus five equals nine." "Two plus four equals six."
Subtraction	"Eight minus six equals two." "Seven minus four equals three." "Nine minus two equals seven."
Inequality	"Seven is greater than three." "Four is greater than one." "Two is less than six."

5.7.2 Full analyses of reaction times in Studies 1 and 3

In Study 1, we analyzed reaction times using 2 x 2 x 4 mixed-design ANOVAs, with Content (*Path* vs. *Collection*) as a between-subjects factor, and Response (*left* vs. *right*) and Numerical Magnitude (*1-2, 3-4, 6-7,* or *8-9*) as within-subjects factors. We analyzed reaction times in Study 3 using the same mixed-design ANOVA, crossed with the additional between-subjects factor of Gesture Reproduction (*Reproduce* vs. *Observe*). Additional regression analyses in Studies 1 and 3 used a standard approach for analyzing number-space associations (Fias et al, 1996).

In Study 1, there was a main effect of Numerical Magnitude, $F_{3,138} = 24.0$, $P << 0.0001$, $\eta_p^2 = 0.34$, driven by the Distance Effect (Moyer and Landauer, 1967): reaction times for numbers closer to five (i.e. 3-4, 6-7) were significantly slower than for numbers farther from five (i.e. 1-2, 8-9), all *t*s ≥ 5, all *P*s < 0.01. The only other significant effects were the interactions described in the main text: the SNARC effect, and its modulation by gesture.

Follow up analyses confirmed that the left-to-right MNL was highly pronounced after reproducing Path gestures ($F_{(3,161)}$ = 11.5, P << 0.0001), but only trending after reproducing *Collection* gestures ($F_{(3,161)}$ = 2.1, P = 0.11). Indeed, participants in the Path condition responded significantly faster on the left for numbers less than five ($t_{(23)}$ = 2.34, P = 0.028), and significantly faster on the right for numbers greater than five ($t_{(23)}$ = 2.98, P < 0.01).

In Study 3, there was again a main effect of Numerical Magnitude, $F_{3,339}$ = 63.1, P << 0.0001, η_p^2 = 0.36, as well as a main effect of Response, $F_{3,113}$ = 13.3, P < 0.0001, η_p^2 = 0.11, with responses slightly faster on the right side (M = 445 vs. 453). The only other significant effects were the interactions described in the main text: the SNARC effect, and its modulation by gesture. Follow-up analyses revealed that participants in the Path condition showed a significant left-to-right MNL ($F_{(3,183)}$ = 14.0, P << 0.001), regardless of whether they had reproduced or observed the gestures ($F_{(3,180)}$ < 2.0, P > 0.10), while participants in the Collection condition showed only a marginal MNL ($F_{(3,162)}$ = 2.6, P = 0.06).

5.7.3 Analysis of accuracy in Studies 1 and 3

Before running any inferential statistical tests, we arcsine-square-root transformed all accuracy scores in order to account for heterogeneity of variance. The pattern of results was unchanged with untransformed values. For ease of interpretation, descriptive statistics report untransformed accuracy scores.

To analyze accuracy in Study 1, we used a 2 x 2 x 4 mixed-design ANOVAs, with Gesture (*Path* vs. *Collection*) as a between-subjects factor, and Response (*left* vs. *right*) and Numerical Magnitude (*1-2, 3-4. 6-7,* or *8-9*) as within-subjects factors. There was a highly significant main effect of Numerical Magnitude $F_{3,144}$ = 15.3, P << 0.001, η_p^2 = 0.24, due to the Distance Effect (Moyer and Landauer, 1967): reaction times were significantly slower for numbers closer to 5, the point of comparison, than for numbers farther away (1 or 2 *vs* 3 or 4: 3.2 percentage points more accurate, $t_{49} = 5.3$, $P << 0.001$; 6 or 7 *vs* 8 or 9: 3.3 percentage points more accurate, $t_{49} = 4.2$, $P = 0.0001$). The only other effect that approached significance was a marginal interaction between Response and Numerical Magnitude, $F_{3,144} = 2.537$, $P = 0.059$, $\eta_p^2 = 0.052$. The pattern of results suggested a canonical left-to-right mental number-line: For numbers less than 5, participants were more accurate when responding on the left than the right ($M = 93.7$ vs 93.0%), and, conversely, for numbers greater than 5, more accurate on the right than the left ($M = 95.2$ vs 94.0%)—although neither of these differences reached significance (both $t_s < 1.66$, both *Ps* > 0.1).

To analyze accuracy in Study 3, we performed the same 2 (Gesture) x 2 (Response) x 4 (Numerical magnitude) mixed-design ANOVA, crossed with Gesture Reproduction (*Reproduce* vs. *Observe*) as an additional between-subjects factor. There was, once again, a highly significant main effect of Numerical Magnitude $F_{3,345} = 46.6$, $P << o.001$, $\eta_p^2 = 0.29$. There was also a difficult-to-interpret two-way interaction between Response and Gesture condition, $F_{3,345} = 4.7$, $P = 0.03$, $\eta_p^2 = 0.04$, with slightly more accurate left-hand responses in the *Reproduce* condition but slightly more accurate right-hand responses in the *Observe* condition, although neither of these differences approached statistical significance (both *P*s > 0.25). The only other significant effects were a significant interaction between Response and Numerical Magnitude, $F_{3,345} = 4.71$, $P = 0.003$, $\eta_p^2 = 0.04$, complicated by a three-way interaction with the type of gesture (*Path* vs. *Collection*), $F_{3,345} = 1.9$, $P = 0.02$, $\eta_p^2 = 0.03$. Participants in the Path condition showed a highly significant interaction between Numerical Magnitude and Response, evidence of a canonical left-to-right mental number-line, $F_{3,186} = 6.7$, $P < 0.001$, $\eta_p^2 = 0.10$, with more accurate responses for smaller numbers on the left, and more accurate responses for larger numbers on the right. For participants in the Collection condition, by contrast, there was no hint of an interaction between Numerical Magnitude and Response, $F_{3,165} = 1.3$, $P > 0.27$, $\eta_p^2 = 0.02$.

Thus, while accuracy is not typically used to measure the SNARC effect, analyses of accuracy in Studies 1 and 3 ruled out the presence of a speed-accuracy trade-off, confirmed the presence of an overall SNARC effect (Study 1 and 3), and confirmed the central finding that gesture shapes the SNARC effect (Study 3).

5.7.4 Latent Dirichlet Allocation topic model of Study 2

To get an intuitive sense of the latent topics extracted by the LDA model, we determined the twenty most likely words for each topic, and examined those words that were unique to each topic (Table 5.S2).

Table 5.S2. Twenty most likely unique terms for latent topics in LDA model from Study 2

Topic	Most likely unique terms
Path-based	arithmetic, basic, right, left, math, think, simple, terms, using, understands, less
Collection-based	hands, adding, subtracting, one, two, together, another, part, video, whole, like

5.7.5 Is the effect of gesture due to associations between spoken numerals and visual locations?

Could the effect of gesture of the MNL be explained by perceived associations between spoken numerals (e.g. "four") and the visual location on the computer monitor of the accompanying gesture? On this deflationary account, the amplification of the MNL is due not to meaningful gesture *per se,* but to low-level number-space associations in visual experience (i.e. on the computer monitor). We can rule out this alternative, however, by considering the visual perspective of the video clips. In every video, the speaker was recorded from the front and slightly to his left (see Fig. 5.1A in the main text). As a result, when the speaker produced gestures that were oriented left-to-right from his perspective, the gesture's trajectory across the screen was actually reversed, right-to-left. Thus, when the video-recorded Path gestures represented numbers on a left-to-right axis *from the speaker's perspective*, participants actually saw a *reversed* pattern of associations between spoken numbers and movement. Similarly, whenever one gesture was to the left of another from the speaker's perspective, the first gesture was actually *right* of the second from the observer's perspective. In the Path videos, as a result, the visual association between number and spatial location was actually reversed from the observer's perspective—exactly as it would be in a canonical communicative encounter. Simple visual associations between spoken numbers and space, therefore, cannot account for the amplification of the left-to-right MNL in the Path condition (compared to the Collection condition). Instead, we think the main finding of Study 2—that these gestures are meaningful for naïve observers, who make use spontaneously of Path gestures to infer a path-like understanding of number—suggests that participants in Studies 1 and 3 were influenced by the inferred meaning of the Path and Collection gestures.

5.7.6 Is the effect of gesture due to a suppressed or reversed MNL in the Collection condition?

If, as we claim, the MNL spreads by gestural contagion, then the effect of gesture on the MNL should be driven primarily by an amplification of the canonical left-to-right MNL by Path gestures. An alternative, however, is that the observed difference between Path and Collection conditions was due to extinguished or even reversed number-space associations among some participants in the Collection condition, rather than a selective impact of left-to-right Path gestures on the left-to-right MNL. To adjudicate between these possibilities, we conducted supplementary analyses of individual differences in Studies 1 and 3.

If Collection gestures reverse the MNL, then there should be a greater proportion of participants with a reversed right-to-left MNL in the Collection condition. However, gesture did not have a significant effect on the proportion of participants who exhibited a left-to-right rather than right-to-left MNL, as indexed by the sign of their SNARC coefficients ($P > 0.3$, Fisher's exact test; Fig. 5.S1A). Thus, there is no evidence that the effect of gesture was due to a reversal of the MNL in the Collection condition.

If Collection gestures extinguish participants' number-space associations, then the absolute value of SNARC coefficients should be smaller in the Collection condition—or, stated otherwise, both negative and positive SNARC coefficients in the Collection condition should be closer to zero. If, on the other hand, the effect of gesture is the result of Path gestures amplifying the canonical left-to-right MNL, as we have argued, then SNARC coefficients should be systematically more negative in the Path condition, especially among participants with a canonical left-to-right MNL. We thus examined separately the effect of gesture on the MNL among participants who had a canonical or a reversed MNL (i.e. SNARC coefficients less than vs. greater than zero). Among participants who had a reversed MNL, there was no difference between gesture conditions (M_{Path} = 8.2, $M_{Collection}$ = 11.7, $t_{(114)}$ = 1.1, P = 0.29). By contrast, among participants with a canonical left-to-right MNL, SNARC coefficients were significantly more negative in the Path condition (M_{Path} = -19.6, $M_{Collection}$ = -12.2, $t_{(114)}$ = 2.1, P = 0.035; Fig. 5.S1B). Thus, the effect of gesture (Path vs. Collection) on the MNL appears to have been driven, as predicted, by a systematic amplification of the left-to-right MNL by Path gestures.

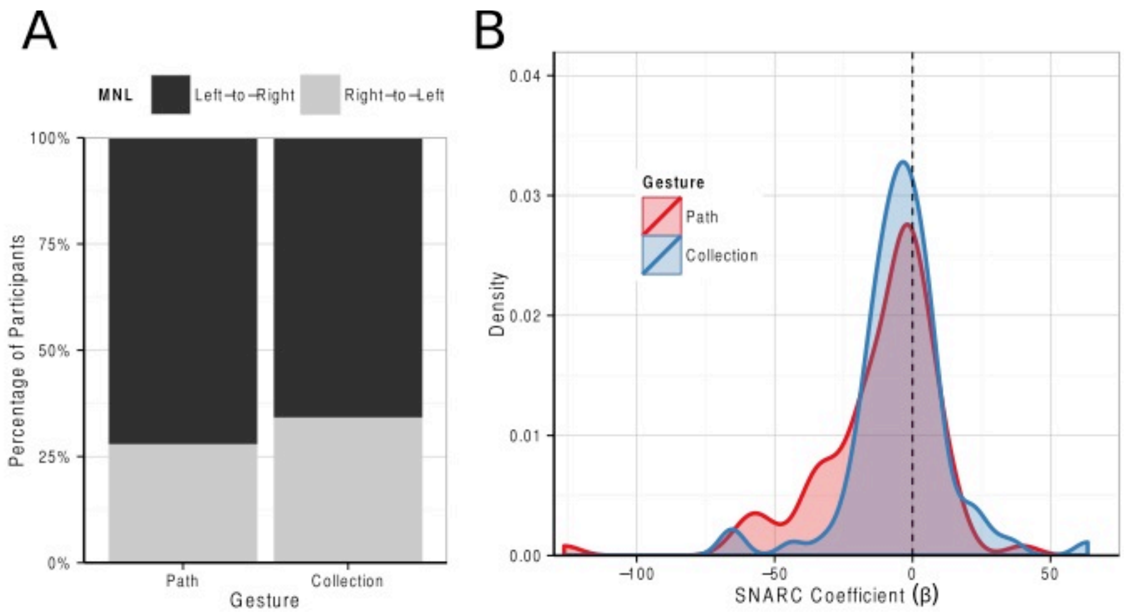


Figure 5.S1. (A) Orientation of participants' MNL in Studies 1 and 3, as indexed by the sign of SNARC coefficients (left-to-right = negative coefficient; right-to-left = positive coefficient). Gesture had no effect on the proportion of participants with a left-to-right rather than right-to-left MNL ($P > 0.3$). (B) Density plot of SNARC coefficients from Studies 1 and 3. Negative values indicate a canonical left-to-right MNL. The effect of gesture on the MNL was driven by participants who exhibited a canonical MNL (i.e. negative SNARC coefficients), whose SNARC coefficients were significantly more negative in the Path condition ($P = 0.035$). By contrast, among participants with a reversed MNL (i.e. positive SNARC coefficients), SNARC coefficients did not differ between conditions ($P > 0.3$)

5.7.7 Supplementary References

Moyer, R. S., & Landauer, T. K. (1967). Time required for judgments of numerical inequality. *Nature,* 215, 1519-1520.

Chapter 6

Conclusion: Autonomy, entwining, and self-reproducing systems

Here, then, is the central claim of this essay: Abstract thinking is constituted, in part, by the situated activity of assemblages of spatialization. These assemblages consist of the more-or-less stable coordination of diverse sites of spatialization. These sites are autonomous of each other, both mechanistically and semiotically. These assemblages, moreover, are not pre-given, but dynamically produced and reproduced. This is true even for mathematics. Mathematics is marked by certainty, stability, interpersonal alignment, and precision; in a sense, it is "perfect" abstraction. And yet even this is the accomplishment of an assemblage of spatialization. In short, it is this distributed practice that *makes* perfect.

6.1. Autonomy of spatialization

We began this essay, in Chapter 1, by suggesting that the entwined sites in which number and arithmetic are spatialized retain a degree of autonomy. But in what ways can coordinated sites remain autonomous? In two ways: in their mechanism and in the features of space with which they associate number. Let's call these "Simon" and "semiotic" autonomy, respectively.

A site of spatialization is *Simon autonomous* whenever its mechanisms are unchanged by the activity of the sites with which it is coordinated (cf. Simon, 1965). For this to be true, a site need not be entirely insulated from the activity of other sites. Quite the contrary. The most Simon-autonomous site may be acutely sensitive to the activities of surrounding sites without changing the nature of its mechanistic underpinnings. The mechanisms responsible for speech production, for instance, are sensitive to but not transformed by the material context. Indeed, it's an open question—an empirical question—whether people are more likely to describe a greater number as "higher" when interacting with an artifact which adopts a more-is-up graphical convention. But certainly the basic computational processes involved in speech production are not changed, qualitatively, by their coupling with a material artifact. Speech production and graphical conventions, therefore, are Simon autonomous, at least on the timescales involved in situated mathematical activity.

A site is *semiotically autonomous* whenever its spatialization—the way it reliably associates features of space with features of other domains (e.g., number)—dissociates from the spatialization accomplished by the domains with which it is coupled. Again, a site might be tightly coupled with others while retaining semiotic autonomy. In speech, for instance, you might describe a number as being "higher" or "bigger" than another, while simultaneously spatializing the number in gesture as more *rightwards*. Thus, while speech and gesture interact during the production of multimodal utterances (McNeill, 1992), the particular spatialization that is accomplished in these sites is often distinct—that is, speech and gesture often exhibit *semiotic autonomy*. Whether speech and gesture are entirely autonomous rather than just autonomous in principle, of course, is an empirical question—and an open one, at that. Or, prompted by a graphical representation of two numerical intervals, in which the trace corresponding to one interval is *to the left* of the trace corresponding to the other, you might describe one interval as "lower" than the other. Here, the content of the graph has a causal impact on the content of speech—you are talking about the numbers because you saw how they were graphed—and yet graph and speech are semiotically autonomous because their spatialization of these intervals deploys distinct aspects of space. In this case, the speaker bound by the conventions of English is prohibited, in principle, from describing numbers in a way that aligns with the graphical spatialization; in

English, a greater number can be bigger or higher, but never *rightward*. Other sites, however, may exhibit a degree of semiotic dependence. Producing and reproducing metaphorical gestures, for instance, shapes the brain-internal spatialization of arithmetic (Ch. 5). Therefore, an assemblage may include sites that, despite each being coupled causally to each other, nevertheless persist in deploying distinct aspects of space, while other sites within the same assemblage are mutually dependent in their semiotic properties.

6.2. Circulation of spatialization

Assemblages of spatialization can be ad-hoc and idiosyncratic, put together in the moment in response to the particular demands and affordances of the situation. But they are also bound by convention, which sometimes become normative and thus binding. In English, greater numbers are larger or higher, never rightward; in a graph, greater numbers are higher or rightward, never bigger. But assemblages are marked by their *entwining*—the coordination between diverse sites that may be subject to different conventions. And communities of practice exhibit regularities in the assemblages of spatialization they contain.

This raises questions of how these constraints on spatialization propagate and perpetuate, both within an assemblage but also between assemblages, individuals, and communities. A set of spatial dispositions will remain forever idiosyncractic and isolated if it is restricted to an organism ensconced within their lifeworld. The flexible conceptualization of an individual are a shadow of a shared, normative conceptual system. In contrast, a conceptual system is distributed across time and space, produced and reproduced by relations of power (Giddens, 1984). These three kinds of relations —space, time, power— require ongoing maintenance.

To establish and maintain a norm of spatialization, spatial strategies must propagate between agents and assemblages. One way in which this can happen is when a particular site of spatialization is shared among disparate assemblages. If a site of spatialization contributes to multiple assemblages—sometimes entangled with one assemblage, other times entangled with another—then it can discipline and be disciplined by those disconnected sites. In this way, a strategy of spatialization can spread between assemblages, shaping individuals throughout a community. Cartesian graphs, for instance, are created and shared, thus serving as "immutable mobiles" (Latour, 1996) that can travel throughout a community, incorporated into distant and distinct assemblages and thus shaping the spatialization of other sites (gestural, neural) in those assemblages. Likewise, humans reproduce the gesture forms of their interlocutors via processes of social transmission (Sperber, 1996; Tomasello, 2014) and gestural alignment (Kimbara, 2005)

The spatiotemporal distribution of conceptualization is inflected by relations of power that determine *which* sites become regimented *when*. The conventions that govern practices of literacy, for instance, are contingent upon larger sociopolitical systems that maintain and constrain those practices. It is no accident that literate adults in Gaza have internalized spatial-numerical dispositions that reverse those of their neighbors, mere miles away, in Israel. State borders often limit the circulation of practices and artifacts, or they mark an inflection between areas in which a practice or artifact is ubiquitous and those in which it is sparse.

What about the emergence of entirely new strategies of spatialization, or strategies that subvert the existing conventions of spatialization? The studies of gestural contagion reported in Chapter 5 dealt with spatializations that were already prevalent with the community: a left-to-right mental number-line, and the system of left-to-right Path gestures. I suspect that similar mechanisms of gestural influence could, for instance, reverse the mental number-line or induce a vertical mental timeline.

6.3. Assemblages of spatialization

This approach resonates with a number of similar proposals. Building on Conversation Analysis, Goodwin (2000, 2013) argues that successful meaning-making and situated action are accomplished by and within contextual configurations, which consist of laminated and interrelated semiotic fields. A semiotic field is a cohesive set of sign phenomena in a particular medium: the constructional units of speech; meaningful movements of the hands; artifacts that are meaningful in virtue of their embedding within particular cultural practices and a nexus of other artifacts. Within talk-in-interaction, we deploy these semiotic fields in concert as *contextual configurations* in order to produce and interpret meaningful communication and action (Goodwin, 2000). In so doing, these semiotic fields are laminated, that is, brought into coordination during situated activity (Goodwin, 2013). Contextual configurations of semiotic fields, therefore, share many features with the assemblages of spatialization described in this essay. Indeed, Goodwin's analytic framework could be productive deployed to analyze the segment of situated activity with which we started this essay in Chapter 1.

The approach adopted here departs from Goodwin in two critical ways. The first is the focus on the precise mechanisms responsible for producing and reproducing the coordination of diverse sites of spatialization. Instead of taking for granted that actors are able to laminate semiotic fields into contextual configurations, the assemblage approach zooms in on processes of lamination and delamination to identify the mechanisms by which coordination is produced and reproduced. Second, the assemblage approach is tied less to interactional encounters between multiple agents in a public setting, and interested more in the accomplishment of abstraction more generally, whether performed alone or with others.

Two other approaches shift the focus from individual brains to the larger systems in which they are situated. Within the philosophy of mind, Clark has argued that the vehicles of mental content and the locus of mental activity is often the "ecological assembly" put together by a "canny cognizer" who "tends to recruit, on the spot, whatever mix of problem-solving resources will yield an acceptable result with a minimum of effort" (2006, p. 13). How this process of recruitment actually happens, however, is largely left unspecified. Like Goodwin, Clark (2000) has little to say about the details of how the resources are actually assembled; indeed, "accounting for the organization of ecological assemblies is the central and unsolved problem in the book" (Hutchins, 2011, p. 438). Hutchins (1995, 2010) responds to this by decentralizing the system even further, arguing that the functional systems responsible for cognitive activity are typically assembled by *cultural practices*, which recruit individual brains alongside a motley mix of other resources. Insofar as assemblages of spatialization are regimented by spatial practices and conventions of spatialization, one contribution of this essay is to spell out how, exactly, this process of cultural recruitment plays out during situated mathematical activity.

Neither Clark nor Hutchins, however, have much to say about how, when it comes to the assemblages that accomplish cognition, the parts and the whole are co-constitutive and mutually transforming. When diverse sites of spatialization are brought into coordination, they shape and transform each other in ways that are not accounted for by the metaphors of "recruitment." And if cultural practices are ways of seeing and doing that are constrained by the activities of others, then the putative mechanism of recruitment—the cultural practice—is going to be transformed as diverse sites are brought into coordination. Our ways of seeing and doing change as we engage in them, as the tension between entwining and autonomy creates new structure and destroys old, as the assemblage transforms as a whole and in its parts.

6.4. Last words

This was an essay on the regimentation of thought: the disciplining of insight, the development of dispositions, the structuring of reason. Nowhere is this more stark than in mathematics. The claims of mathematics impress themselves upon the individual as truths at once inherited and intuitive, received and discovered anew. For the contemporary student of Euclid, the experience is not one of authoritarian transmission of ancient dogma but of a gentle coaxing toward self-evident truths—at least, self-evident in hindsight. Having arrived at an insight, the mathematician finds herself part of a community of believers, a community that transcends gender, race, class, language, culture. We are compelled to believe the truths of mathematics, both individually and as a community, both now and, it seems, always.

Or so the story goes. I'm not so sure.

After all, mathematics is a messy, situated human practice, one kind of manual labor among many. It is this mundane, material character that nominates mathematics as a critical case study in the regimentation of thought. The foregoing chapters were written in this spirit. Each was a study in the contribution of space to the regimentation of mathematical thinking, intentionally cutting across timescales, from the microdynamics of individual reckoning to the propagation and perpetuation of understanding within communities of practice. We saw in Chapters 2 and 3 that mathematical reckoning on the timescale of milliseconds is regimented by systemic spatialization during numerical comparison and calculation. In Chapter 4, we considered activity on the timescale of seconds, analyzing reflexive, precise mathematical reasoning as enacted in multimodal utterances. We then turned, in Chapter 5, from the regimentation of the individual to the propagation and perpetuation of regimentation, between individuals and across time. Taken together, these four projects trace the outlines of an assemblage of autonomous but entwined mechanisms, responsible for spatializing—and thus regimenting—not only the simple whole numbers but also the negative integers, exact calculation, and reflexive reasoning. If mathematics is in some sense "perfect" knowledge, then it is this kind of distributed practice that makes perfect.

Despite the focus on number and arithmetic, the morals of this essay may apply widely. Given any abstract domain, I predict we'll find a cognitive ecosystem rife with spatialization, in which situated activity requires the production and reproduction of coordinated assemblages. Capitalist ideology may propagate via gestural contagion. Metaphorical gestures for social inequality may reflect the simulation of balance and motion. Careful, reflexive reasoning about structural racism may co-opt neural systems specialized for space and action. Intervening on these systems will require attention to the distributed assemblages in which they occur.

L'Homo Vista, Los Angeles, April 2015

6.5. References

Clark, A. (2008). *Supersizing the Mind.* Oxford: Oxford University Press.

Giddens, A. (1984). *The constitution of society: Outline of the theory of structuration*. Berkeley: University of California Press.

Goodwin, C. (2000). Action and embodiment within situated human interaction. *Journal of Pragmatics*, 32, 1489-1522.

Hutchins, E. (1995). *Cognition in the Wild*. Cambridge, MA: MIT Press.

Hutchins, E. (2011). Enculturating the Supersized Mind. *Philosophical Studies,* 152, 437-446.

Kimbara, I. (2006). On gestural mimicry. *Gesture*, 6, 39-61.

Latour, B. (1986). Visualization and cognition: Thinking with eyes and hands. *Knowledge and Society,* 6, 1–40.

McNeill, D. (1992). *Hand and Mind.* Chicago: University of Chicago Press.

Simon, H. A. (1965). The architecture of complexity. *General Systems*, 10, 63-76.

Sperber, D. (1996) *Explaining Culture.* Oxford: Blackwell.

Tomasello, M. (2014). *A Natural History of Human Thinking.* Cambridge, MA: Harvard University Press.