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Constrained dynamics: generalized Lie symmetries, singular Lagrangians, and the passage to Hamiltonian mechanics

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Abstract

Guided by the symmetries of the Euler–Lagrange equations of motion, a study of the constrained dynamics of singular Lagrangians is presented. We find that these equations of motion admit a generalized Lie symmetry, and on the Lagrangian phase space the generators of this symmetry lie in the kernel of the Lagrangian two-form. Solutions of the energy equation—called second-order, Euler–Lagrange vector fields (SOELVFs)—with integral flows that have this symmetry are determined. Importantly, while second-order, Lagrangian vector fields are not such a solution, it is always possible to construct from them a SOELVF that is. We find that all SOELVFs are projectable to the Hamiltonian phase space, as are all the dynamical structures in the Lagrangian phase space needed for their evolution. In particular, the primary Hamiltonian constraints can be constructed from vectors that lie in the kernel of the Lagrangian two-form, and with this construction, we show that the Lagrangian constraint algorithm for the SOELVF is equivalent to the stability analysis of the total Hamiltonian. Importantly, the end result of this stability analysis gives a Hamiltonian vector field that is the projection of the SOELVF obtained from the Lagrangian constraint algorithm. The Lagrangian and Hamiltonian formulations of mechanics for singular Lagrangians are in this way equivalent.

1. Introduction

The Lagrangian phase space formulation of mechanics [1–4], with its roots in differential geometry, provides an especially fruitful framework with which to analyze dynamical systems of singular Lagrangians L . Instead of trajectories $q(t) = (q^1(t), \dots, q^D(t))$ on a D -dimensional configuration space $\mathbb{Q}$ that are solutions of the Euler–Lagrange equations of motion, trajectories in the Lagrangian phase space formulation

$$u(t) = (q^1(t), \dots, q^D(t), v^1(t), \dots, v^D(t)), \quad (1)$$

are on a $2D$ -dimensional Lagrangian (or velocity) phase space $\mathbb{P}_L = T\mathbb{Q}$ embodied with a Lagrangian two-form Ω_L . They are determined by vector fields $X_E \in T_u\mathbb{P}_L$ that are solutions of the energy equation

$$i_{X_E}\Omega_L = dE, \quad (2)$$

with E being the energy of the system. For regular Lagrangians, Ω_L is symplectic. The solution to equation (2) is unique, and is a second-order, Lagrangian vector field (SOLVF)[1] (also called a second-order dynamical equation in the literature). For singular Lagrangians, on the other hand, Ω_L is presymplectic². The solution to equation (2) is not unique, need not be a SOLVF, nor need it even exist [5]. Nevertheless, with few exceptions in the literature [6], focus has been placed on solutions of equation (2) that are SOLVFs. This is done for physical

² There are two definitions of ‘presymplectic’—a broader one in which $\Omega_L(u)$ need only be closed, and a more narrow one that also requires the rank of $\Omega_L(u)$ to be constant on $\mathbb{P}_L$ —used in the literature. Although in this paper our focus is on systems where the rank of $\Omega_L(u)$ is constant, a few of our results still hold even when it is not, and we follow the broader definition of the word.

reasons: the condition $\dot{q}^a = v^a$, for $a = 1, \dots, D$, immediately follows for trajectories determined by such fields. This focus on SOLVFs has consequences, however.

The presence of a singular Lagrangian often predicates the existence of a Lagrangian constraint submanifold of $\mathbb{P}_L$, and for solutions of equation (2) to exist, trajectories of these dynamical systems must lie on this submanifold. Algorithms—called a constraint algorithm or a stability condition—for constructing such solutions have been developed [5, 7–13]. However, irrespective of the one used, the end result of these algorithms is a vector field that has a number of troubling attributes. First, this vector field need not be a SOLVF, even though physical arguments were used to restrict the starting point of these algorithms to such fields. This is called the second order problem, first noted within a different context by Künzle [14], and emphasized by Gotay and Nester [8]. Currently, it is known that requiring the end result of these algorithms be a SOLVF is very restrictive, and additional conditions may need to be imposed [15]. Second, the fibre derivative (Legendre transform) $\mathcal{L}$ for singular Lagrangians is singular, and thus the rank of the Hessian of L is not maximal. Because of this the passage from Lagrangian to Hamiltonian mechanics is problematic. The ability to map dynamical structures from the Lagrangian to the Hamiltonian phase space has long been studied for SOLVFs [4, 7, 10, 11, 15–23]. It is found that a general SOLVF—even after the application of the constraint algorithm, and even under the weak projectability condition [10]—is not projectable [10, 15]. (Examples of systems for which a SOLVF is projectable, and for which it is not are given in [10]). One immediate consequence of this non-projectability is dynamical systems for which the Hamiltonian flow field determined through constrained Hamiltonian mechanics as described in [24, 25] (see [26, 27] for more modern approaches)—even after its restriction to the primary constraint submanifold—need have little relation to the SOLVF obtained from the Lagrangian constraint algorithm. Third, it is known that the Lagrangian constraints obtained while a constraint algorithm is being imposed on a SOLVF also need not be projectable [4, 10, 15, 17, 18, 21, 28]. Combined, this means that dynamics on the Lagrangian phase space and dynamics on the Hamiltonian phase space can take place on two inequivalent submanifolds, be determined by two inequivalent vector fields, resulting in two *different* families of trajectories on the configuration space $\mathbb{Q}$ for the *same* dynamical system with the *same* initial data.

We take a different starting point in our analysis of singular Lagrangians, one that is rooted in the generalized Lie symmetries of the Euler–Lagrange equations of motion. The analysis is guided by the observation that if a symmetry of the dynamical system has been determined through Lagrangian mechanics, it must be present in the Lagrangian and Hamiltonian phase space descriptions of motion as well. We emphasize, however, that while these symmetries play an important role, this role is nevertheless supportive. One of the main goals of this work is the construction of algebraic-geometric structures within differential geometry that will then be used to implement these symmetries; to characterize the relevant geometry of the Lagrangian phase space; to determine the structures needed to describe dynamics on this phase space; and to show the equivalence of the Lagrangian and Hamiltonian phase space formulation of mechanics. In doing so, we are led to construct the second-order, Euler–Lagrange vector field (SOELVF). These fields avoid the second-order problem, are projectable to the Hamiltonian phase space, and lie on Lagrangian constraint submanifolds that also are projectable. Importantly, the projection of the SOELVF is the Hamiltonian flow field of the total Hamiltonian obtained from constrained Hamiltonian mechanics. (We follow the terminology in [25], and call the result of augmenting the canonical Hamiltonian with the primary constraint functions the total Hamiltonian).

The generalized Lie symmetry [29] of the Euler–Lagrange equations of motion is generated by second-order prolongation vectors in the tangent space $\mathbf{TM}^{(2)}$ of the second-order jet space $\mathbb{M}^{(2)}$. This symmetry is reflected in the Lagrangian phase space description of motion, and the projection of $\mathbf{TM}^{(2)}$ to $\mathbf{TP}_L$ maps these prolongation vectors into the kernel of Ω_L . Surprisingly, it is not the vertical vector fields of the kernel that generates this symmetry, as may have been expected. Also surprisingly, the corresponding symmetry group Gr_{Sym} is not a symmetry group for SOLVFs; action on a SOLVF by Gr_{Sym} results in a vector field that is no longer a SOLVF, nor need it even be a solution of equation (2). It is, however, always possible to construct from a SOLVF vector fields that do have Gr_{Sym} as a symmetry group, and are solutions to equation (2). These vector fields are the SOELVFs, and they resolve the issues listed above for the SOLVF.

That a SOELVF is projectable is a natural consequence of having Gr_{Sym} as a symmetry group. Moreover, all the Lagrangian constraints—both those due to the energy equation and those introduced through the application of the constraint algorithm to the SOELVF—are projectable as well. We find also that there is a choice of a basis for the kernel of Ω_L that is projectable, and that the primary Hamiltonian constraints can be constructed from their image. Indeed, all the dynamical structures needed to describe evolution on the Lagrangian phase space are projectable, and their image corresponds to the dynamical structures needed to describe evolution on the Hamiltonian phase space obtained through constrained Hamiltonian mechanics [13, 24–27, 30–32]. In this way the Lagrangian and Hamiltonian formulations of mechanics are equivalent even for singular Lagrangians.

Analysis of the symmetries of Lagrangian systems (both regular and singular) have been done before. However, such analyses have been focused on time-dependent Lagrangians (and in particular their Noether

symmetries) [9, 33–39]; on systems of first-order evolution equations on either the Lagrangian or Hamiltonian phase space [40–44]; or on general solutions of equation (2) [6] (see also [45] for an analysis of particle motion on curved spacetimes). Importantly, the great majority of these analyses have been done using first-order prolongations on first-order jet bundles with a focus on the Lie symmetries of first-order evolution equations. Our interest is in the symmetries of the Euler–Lagrange equations of motion—a system of second-order differential equations—that come from singular Lagrangians. This naturally leads us to consider generalized Lie symmetries and second-order prolongations. Such symmetry analysis of the Euler–Lagrange equations of motion has not been done before. (Although the framework for k th-order prolongations on k th-order jet bundles have been introduced before [9, 39, 46, 47], they have not been applied to the Euler–Lagrange equations of motion).

In this paper we only consider autonomous Lagrangians for which the rank of Ω_L is constant on $\mathbb{P}_L$. We also require such Lagrangians to have a fibre derivative that is a submersive map of $\mathbb{P}_L$ to the Hamiltonian phase space $\mathbb{P}_C$; the rank of the Hessian of such Lagrangians is necessarily constant on $\mathbb{P}_L$. In addition, the preimage of $(q, p) = \mathcal{L}(u)$, $u \in \mathbb{P}_L$, must be a connected submanifold of $\mathbb{P}_L$ (see also [32] where this condition is relaxed). These Lagrangians are called almost regular Lagrangians in the literature [2, 4, 7, 15, 23], and we also use this terminology.

Some of the results on the equivalence of singular Lagrangian and Hamiltonian mechanics presented in this paper have been presented elsewhere. However, the approaches used previously often rely on such dynamical structures from constrained Hamiltonian mechanics as the primary Hamiltonian constraints; pullbacks of their derivatives to the Lagrangian phase space are used to construct such mappings as the time-evolution operator K [11, 16, 18, 19, 21, 37], for example. We take a different approach, one that starts with the Lagrangian phase space formulation, and, with restrictions imposed by Gr_{Sym} , is one which shows that the dynamical structures on the Hamiltonian phase space necessary to describe the dynamical system can be obtained directly from those on the Lagrangian phase space.

The importance of establishing the equivalence between the Lagrangian and Hamiltonian formulations of mechanics for singular Lagrangians can be seen from the starting point of any physically relevant system: the action, and through it, its Lagrangian. For systems with local gauge or diffeomorphism symmetry, the Lagrangian is singular, and dynamics have traditionally been analysed using Hamiltonian constraints [25]. This is done by first using the fibre derivative to construct the Hamiltonian from the Lagrangian, and then using Hamiltonian stability analysis to determine both the total Hamiltonian and the Hamiltonian constraint surfaces. However, as the Lagrangian was the starting point, and as the fibre derivative is not invertible for such Lagrangians, a natural question to ask is whether the dynamics described by the total Hamiltonian has any relation to the original dynamics given by the Lagrangian. With the equivalence between the Lagrangian and the Hamiltonian phase space formulations of dynamics demonstrated, we have shown that they are. This equivalence is even more important for the path integral formulation of quantum mechanics and quantum field theory. Both are based on the action, and integration is over paths on the configuration space. For systems with local gauge or diffeomorphism symmetries these integrals must be restricted, which for non-abelian gauge theories leads to the use of BRST symmetries. These symmetries have traditionally been constructed using the Hamiltonian and Hamiltonian constraint analysis (see [25]).

Although we freely use the tools and language of differential geometry, we are aware that interest in constrained dynamics is often due to its application to quantum field theories. In these applications, the ability to calculate and determine symmetries is paramount. To ensure that the tools and methodologies given in this paper can be so applied to the analysis of quantum field-theoretic systems, we have also written a number of the expressions given in this paper in terms of local coordinates using a notation that is both familiar and useful for calculations. In particular, the general solution to $i_K \Omega_L = 0$ given in section 3.2 is given in terms of local coordinates as are the construction of second-order Lagrangian and Euler–Lagrangian vector fields.

The rest of the paper is arranged as follows. In section 2 we show that the Euler–Lagrange equations of motion have a generalized Lie symmetry, and determine the existence conditions for the generators of this symmetry. In section 3 the vectors that lie in the kernel of Ω_L are found, and the role they play in generating Gr_{Sym} is determined. Physically relevant solutions of the energy equation are characterized, and the SOELVF is defined and constructed. First-order Lagrangian constraints are also constructed, and a constraint algorithm for SOELVFs is presented. In section 4 focus is on the passage from the Lagrangian to the Hamiltonian phase space. The projectability of functions on $\mathbb{P}_L$ is reviewed, and a new result on the projectability of vector fields in $\mathbb{T}\mathbb{P}_L$ is presented. The dynamical structures needed to describe evolution with SOELVFs are shown to be projectable, and the primary Hamiltonian constraints are constructed. The equivalence of the constraint algorithm presented in section 3 with the usual Hamiltonian stability analysis is shown. In section 4.3 application of the analysis given here to three different dynamical systems with singular Lagrangians is presented. Concluding remarks can be found in section 4.4, with the crucial role that the vertical vector fields in the kernel of Ω_L play summarized.

2. Generalized Lie symmetries and Lagrangian mechanics

We begin with Lagrangian mechanics, and an analysis of the generalized Lie symmetry [29] of the Euler–Lagrange equations of motion. The existence conditions for the generators of this symmetry will be established.

The Euler–Lagrange equations of motion are the system of D second-order differential equations

$$M_{ab}\ddot{q}^b = -\frac{\partial E}{\partial q^a} - F_{ab}\dot{q}^b, \quad (3)$$

where

$$E(q, \dot{q}) := \dot{q}^b \frac{\partial L(q, \dot{q})}{\partial \dot{q}^b} - L(q, \dot{q}), \quad (4)$$

is the energy, while

$$M_{ab}(q, \dot{q}) := \frac{\partial^2 L(q, \dot{q})}{\partial \dot{q}^a \partial \dot{q}^b}, \quad \text{and} \quad F_{ab}(q, \dot{q}) := \frac{\partial^2 L(q, \dot{q})}{\partial \dot{q}^a \partial q^b} - \frac{\partial^2 L(q, \dot{q})}{\partial \dot{q}^b \partial q^a}. \quad (5)$$

Here, Einstein's summation convention is used.

For almost regular Lagrangians the rank of the Hessian $M_{ab}(u)$, where $u = (q, \dot{q})^3$, is constant on $\mathbb{P}_L$. However, as the rank of $M_{ab}(u) = D - N_0$, with $N_0 = \dim(\ker M_{ab}(u))$, this rank is not maximal, and thus equation (3) cannot be solved for a unique $\ddot{q}$. Instead, a chosen set of initial data $u_0 = (q_0, \dot{q}_0)$ given at $t = t_0$ determines a family of solutions to equation (3) that evolve from the same u_0 . These solutions are related to one another through a generalized Lie symmetry [29].

Following Olver [29], we define

$$\Delta_a(q, \dot{q}, \ddot{q}) := \frac{\partial E(q, \dot{q})}{\partial q^a} + F_{ab}(q, \dot{q})\dot{q}^b + M_{ab}\ddot{q}^b, \quad (6)$$

which reduces to equation (3) on the surfaces $\Delta_a(q, \dot{q}, \ddot{q}) = 0$. The set

$$\mathcal{O}(u_0) := \{q(t) \mid \Delta_a(q, \dot{q}, \ddot{q}) = 0 \text{ with } q(t_0) = q_0, \dot{q}(t_0) = \dot{q}_0\}, \quad (7)$$

is the family of solutions to equation (3) that evolve from u_0 . Consider two such solutions $q^a(t)$ and $Q^a(t)$. From equation (3) there exists a $z^a(u) \in \ker M(u)$ such that $\ddot{Q}^a - \ddot{q}^a = z^a(u)$. As z^a depends on both q^a and $\dot{q}^a$, we are led to consider generalized Lie symmetry groups generated by

$$\mathbf{g} := \rho(u) \cdot \frac{\partial}{\partial q}, \quad (8)$$

with a $\rho(u)$ that does not depend explicitly on time. This gives the total time derivative

$$\frac{d}{dt} := \dot{q} \cdot \frac{\partial}{\partial q} + \ddot{q} \cdot \frac{\partial}{\partial \dot{q}}, \quad (9)$$

with $\dot{\rho} := d\rho/dt$.

With this $\mathbf{g}$, the second-order prolongation vector,

$$\mathbf{pr} \mathbf{g} := \rho \cdot \frac{\partial}{\partial q} + \dot{\rho} \cdot \frac{\partial}{\partial \dot{q}} + \ddot{\rho} \cdot \frac{\partial}{\partial \ddot{q}}, \quad (10)$$

on the second-order jet space $\mathbb{M}^{(2)} = \{(t, q, \dot{q}, \ddot{q})\}$ with $\mathbf{pr} \mathbf{g} \in \mathbf{T}\mathbb{M}^{(2)}$ can be constructed. Its action on the Euler–Lagrange equations of motion gives

$$\mathbf{pr} \mathbf{g}[\Delta_a(q, \dot{q}, \ddot{q})] = -\frac{\partial \ddot{q}^b}{\partial q^a} M_{bc} \rho^c + \frac{d}{dt} [F_{ab} \rho^b + M_{ab} \dot{\rho}^b], \quad (11)$$

on the $\Delta_a = 0$ surface. However, because the rank of $M_{ab}(u)$ is not maximal, the solution for $\ddot{q}$ on this surface is not unique. For $\mathbf{g}$ to generate the same symmetry group for all the trajectories in $\mathcal{O}(u_0)$, we must require $\rho(u) \in \ker M_{ab}(u)$. It then follows that $\mathbf{pr} \mathbf{g}[\Delta_a(q, \dot{q}, \ddot{q})] = 0$ if and only if (iff) $b_a = F_{ab} \rho^b + M_{ab} \dot{\rho}^b$ for some function $b_a(u)$ where $\dot{b}_a = 0$. However, because all the solutions in $\mathcal{O}(u_0)$ have the same initial data, $\rho^a(u_0) = 0$ and $\dot{\rho}^a(u_0) = 0$, and thus $b_a = 0$. This leads to the following new result.

Lemma 1. *If $\mathbf{g}$ is a generalized infinitesimal symmetry of Δ_a , then $\rho^a(u) \in \ker M_{ab}(u)$ and $\dot{\rho}^a(u)$ is a solution of*

$$0 = F_{ab}(u) \rho^b(u) + M_{ab}(u) \dot{\rho}^a(u). \quad (12)$$

³ We use u to represent both $(q, \dot{q})$ for trajectories on $\mathbb{Q}$ and $(q, \dot{q}, \ddot{q})$ for trajectories on $\mathbb{P}_L$ to avoid introducing additional notation.

We denote by $\mathfrak{g}$ the set of all vector fields $\mathbf{g}$ that satisfy lemma 1, and by $\mathbf{pr} \mathfrak{g} := \{\mathbf{pr} \mathbf{g} \mid \mathbf{g} \in \mathfrak{g}\}$ the set of their prolongations. This $\mathbf{pr} \mathfrak{g}$ is involutive [29].

The conditions under which $\mathbf{pr} \mathfrak{g}$ generates a generalized Lie symmetry group are well known [29]. However, because our Lagrangians are singular, three additional conditions must be imposed:

1. While $\rho^a = 0$ and $\dot{\rho}^a = \dot{z}^a$ for any $z \in \ker M_{ab}(u)$ is a solution of equation (12), we require that $\dot{\rho}^a = d\rho^a/dt$, and they must be removed.
2. If $\dot{\rho}^a$ is a solution of equation (12), then $\dot{\rho}^a + \dot{z}^a$ is a solution of equation (12) as well. The $\dot{\rho}^a$ are not unique, and this, along with the first condition, leads us to generators that are constructed from equivalence classes of prolongations.
3. For any $\dot{z}^a \in \ker M_{ab}(u)$, equation (6) gives,

$$0 = \dot{z}^a \left(\frac{\partial E}{\partial q^a} + F_{ab}(q, \dot{q}) \dot{q}^b \right), \quad (13)$$

on the solution surface $\Delta_a(q, \dot{q}, \ddot{q}) = 0$. If equation (13) does not hold identically, it must be imposed, leading to the well-known, first-order Lagrangian constraints. As each $q(t) \in \mathcal{O}(u_0)$ must lie on this constraint submanifold, any symmetry transformation of $q(t)$ generated by $\mathbf{pr} \mathfrak{g}$ must give a path $Q(t)$ that also lies on the constraint submanifold.

Not all vectors in $\mathbf{pr} \mathfrak{g}$ will be generators of the generalized Lie symmetry group for $\mathcal{O}(u_0)$. Determining which vectors are is best done within the Lagrangian phase space framework. This will be done in the next section. For now, we note the following.

For any $\mathbf{g} \in \mathfrak{g}$, decompose

$$\mathbf{pr} \mathbf{g} = \mathbf{k} + \dot{\rho} \cdot \frac{\partial}{\partial \dot{q}}. \quad (14)$$

Then for $\mathbf{pr} \mathbf{g}_A, \mathbf{pr} \mathbf{g}_B \in \mathbf{pr} \mathfrak{g}$,

$$[\mathbf{pr} \mathbf{g}_A, \mathbf{pr} \mathbf{g}_B] = [\mathbf{k}_A, \mathbf{k}_B] + [\mathbf{pr} \mathbf{g}_A(\dot{\rho}_B^a) - \mathbf{pr} \mathbf{g}_B(\dot{\rho}_A^a)] \frac{\partial}{\partial \dot{q}^a}. \quad (15)$$

Because $\mathbf{pr} \mathfrak{g}$ is involutive, there exists a $\mathbf{pr} \mathbf{g}_C \in \mathbf{pr} \mathfrak{g}$ such that $\mathbf{pr} \mathbf{g}_C = [\mathbf{pr} \mathbf{g}_A, \mathbf{pr} \mathbf{g}_B]$. There is then a $\mathbf{k}_C$ such that $\mathbf{k}_C = [\mathbf{k}_A, \mathbf{k}_B]$, and the collection of vectors

$$\mathcal{K} = \left\{ \mathbf{k} = \rho(u) \cdot \frac{\partial}{\partial q} + \dot{\rho}(u) \cdot \frac{\partial}{\partial \dot{q}} \mid \rho^a(u) \in \ker M_{ab}(u), 0 = F_{ab}(u) \rho^b(u) + M_{ab}(u) \dot{\rho}^b(u) \right\}, \quad (16)$$

is involutive. Importantly, $\dim \mathbf{pr} \mathfrak{g} = \dim \mathcal{K} = 2N_0$.

3. The Lagrangian phase space

In this section we determine the generators of the generalized Lie symmetry found in section 2. This is done on the Lagrangian phase space using the tools of differential geometry. These generators are then used to determine the physically relevant solutions of the energy equation, and with them, the constraint algorithm and the Lagrangian constraint submanifold. Much of the content in sections 3.1 to 3.4 have been established elsewhere. They are gathered here for clarity and coherence of argument, and to establish notation and terminology. On the other hand, much of the construction and the results presented in sections 3.5 to 3.7 are new.

3.1. Passage from Lagrangian mechanics to the Lagrangian phase space

To treat singular Lagrangian dynamics using the methods of differential geometry, trajectories $t \rightarrow q(t) \in \mathbb{Q}$ are replaced by integral flows $t \rightarrow u(t) \in \mathbb{P}_L[1]$, which for the initial data $u_0 = (q_0, v_0)$ are given by solutions to

$$\frac{du}{dt} := \mathbf{X}(u), \quad (17)$$

where $\mathbf{X}$ is a smooth vector field in the tangent space $\mathbb{T}\mathbb{P}_L$. As $\mathbb{P}_L = \mathbb{T}\mathbb{Q}$, we have the bundle projections $\tau_{\mathbb{Q}} : \mathbb{T}\mathbb{Q} \rightarrow \mathbb{Q}$ and $\tau_{\mathbb{T}\mathbb{Q}} : \mathbb{T}(\mathbb{T}\mathbb{Q}) \rightarrow \mathbb{T}\mathbb{Q}$ with $\tau_{\mathbb{Q}} \circ \tau_{\mathbb{T}\mathbb{Q}} : \mathbb{T}(\mathbb{T}\mathbb{Q}) \rightarrow \mathbb{Q}$ (see [5] and [1]). It is also possible to construct $\mathbf{T}\tau_{\mathbb{Q}}$, the prolongation of $\tau_{\mathbb{Q}}$ to $\mathbb{T}(\mathbb{T}\mathbb{Q})$, which is a second projection map $\mathbf{T}\tau_{\mathbb{Q}} : \mathbb{T}(\mathbb{T}\mathbb{Q}) \rightarrow \mathbb{T}\mathbb{Q}$ which is determined by requiring that $\tau_{\mathbb{Q}} \circ \tau_{\mathbb{T}\mathbb{Q}}$ and $\tau_{\mathbb{Q}} \circ \mathbf{T}\tau_{\mathbb{Q}}$ map a point in $\mathbb{T}(\mathbb{T}\mathbb{Q})$ to the same point in $\mathbb{Q}$. The vertical subbundle $[\mathbb{T}\mathbb{P}_L]^v$ of $\mathbb{T}(\mathbb{T}\mathbb{Q})$ is defined by $[\mathbb{T}\mathbb{P}_L]^v = \ker \mathbf{T}\tau_{\mathbb{Q}}$ [5]; a vector $\mathbf{X}^v \in [\mathbb{T}_u \mathbb{P}_L]^v$ above a point $u \in \mathbb{P}_L$ is

called a vertical vector field. The horizontal subbundle $[\mathbb{T}\mathbb{P}_L]^q$ of $\mathbb{T}(\mathbb{T}\mathbb{Q})$ is defined by $[\mathbb{T}\mathbb{P}_L]^q = \text{Image } \mathbb{T}\tau_{\mathbb{Q}}$; a vector $\mathbf{X}^q \in [\mathbb{T}_u\mathbb{P}_L]^q$ is called a horizontal vector field. Each $\mathbf{X} \in \mathbb{T}_u\mathbb{P}_L$ can be expressed as $\mathbf{X} = \mathbf{X}^q + \mathbf{X}^v$ with $\mathbf{X}^q \in [\mathbb{T}_u\mathbb{P}_L]^q$ and $\mathbf{X}^v \in [\mathbb{T}_u\mathbb{P}_L]^v$, which in terms of local coordinates are,

$$\mathbf{X}^q := X^{qa} \frac{\partial}{\partial q^a}, \quad \text{and} \quad \mathbf{X}^v := X^{va} \frac{\partial}{\partial v^a}. \quad (18)$$

In particular, a second order Lagrangian vector field $\mathbf{X}_L$ is the solution of equation (2) such that $\mathbb{T}\tau_{\mathbb{Q}} \circ \mathbf{X}_L$ is the identity on $\mathbb{T}\mathbb{Q}$ (see [1]). In terms of local coordinates

$$\mathbf{X}_L = v^a \frac{\partial}{\partial q^a} + X^{va} \frac{\partial}{\partial v^a}, \quad (19)$$

where for singular Lagrangians X^{va} is not unique.

For a one-form $\alpha \in \mathbb{T}_u^*\mathbb{P}_L$, where $\mathbb{T}^*\mathbb{P}_L$ is the cotangent space of one-forms on $\mathbb{P}_L$, and a vector field $\mathbf{X} \in \mathbb{T}_u\mathbb{P}_L$, we define the dual prolongation map $\mathbb{T}^*\tau_{\mathbb{Q}}$ by

$$\langle \alpha | \mathbb{T}\tau_{\mathbb{Q}} \mathbf{X} \rangle = \langle \mathbb{T}^*\tau_{\mathbb{Q}} \alpha | \mathbf{X} \rangle. \quad (20)$$

Here, we adapt Dirac's bra and ket notation to denote the action of a k -form $\omega(x)$ by

$$\omega(x): \mathbf{Y}_1 \otimes \cdots \otimes \mathbf{Y}_k \rightarrow \langle \omega(x) | \mathbf{Y}_1 \otimes \cdots \otimes \mathbf{Y}_k \rangle \in \mathbb{R}, \quad (21)$$

where $\mathbf{Y}_j \in \mathbb{T}_u\mathbb{P}_L$. The k -form bundle is $\Lambda^k(\mathbb{P}_L)$, while the exterior algebra of forms is denoted by $\Lambda(\mathbb{P}_L)$. Then the vertical one-form subbundle $[\mathbb{T}^*\mathbb{P}_L]^v$ of $\mathbb{T}^*\mathbb{P}_L$ is defined by $[\mathbb{T}^*\mathbb{P}_L]^v := \ker \mathbb{T}^*\tau_{\mathbb{Q}}$; a one-form $\alpha_v \in [\mathbb{T}_u^*\mathbb{P}_L]^v$ is called a vertical one-form. The horizontal one-form subbundle $[\mathbb{T}^*\mathbb{P}_L]^q$ of $\mathbb{T}^*\mathbb{P}_L$ is defined by $[\mathbb{T}^*\mathbb{P}_L]^q = \text{Image } \mathbb{T}^*\tau_{\mathbb{Q}}$; a one-form $\alpha_q \in [\mathbb{T}_u^*\mathbb{P}_L]^q$ is called a horizontal one-form. Each one-form $\varphi \in \mathbb{T}_u^*\mathbb{P}_L$ can be expressed as $\varphi = \varphi_q + \varphi_v$ with $\varphi_q \in [\mathbb{T}_u^*\mathbb{P}_L]^q$ and $\varphi_v \in [\mathbb{T}_u^*\mathbb{P}_L]^v$. In terms of local coordinates, $\varphi_q := \varphi_{qa} dq^a$ and $\varphi_v := \varphi_{va} dv^a$.

There are a number approaches [1] used in the literature to obtain the Lagrange two-form Ω_L . We follow [5, 8], and define $\Omega_L = -\mathbf{d}\mathbf{d}_j L$, where $\mathbf{d}_j$ is the vertical derivative (see [5]). This two-form can be expressed as $\Omega_L := \Omega_F + \Omega_M$ where

$$\Omega_F(\mathbf{X}, \mathbf{Y}) := \Omega_L(\mathbb{T}\tau_{\mathbb{Q}}\mathbf{X}, \mathbb{T}\tau_{\mathbb{Q}}\mathbf{Y}), \quad (22)$$

for all $\mathbf{X}, \mathbf{Y} \in \mathbb{T}_u\mathbb{P}_L$; this is a horizontal two-form of Ω_L . Then $\Omega_M(\mathbf{X}, \mathbf{Y}) = \Omega_L(\mathbf{X}, \mathbf{Y}) - \Omega_F(\mathbf{X}, \mathbf{Y})$; this is a mixed two-form of Ω_L . In terms of local coordinates,

$$\Omega_L = -\mathbf{d}\theta_L, \quad \text{where } \theta_L = \frac{\partial L}{\partial v^a} dv^a, \quad (23)$$

while

$$\Omega_F := \frac{1}{2} F_{ab} dq^a \wedge dq^b, \quad \text{and} \quad \Omega_M := M_{ab} dq^a \wedge dv^b. \quad (24)$$

If $u(t)$ is to describe the evolution of the dynamical system given by L , then the vector field $\mathbf{X}$ must be chosen so that its integral flows on $\mathbb{P}_L$ faithfully represent their trajectories on $\mathbb{Q}$. For regular Lagrangians this is guaranteed by setting $\mathbf{X} = \mathbf{X}_L$, and is a unique solution of the energy equation equation (2) [1].

We adopt the general assumption that even for singular Lagrangians there are solutions of the energy equation that faithfully represent trajectories on $\mathbb{Q}$. For general singular Lagrangians neither the existence nor the uniqueness of solutions to equation (2) is assured. For almost regular Lagrangians, however, a number of general results are available. These results depend on the structure of the family of solutions evolving from u_0 , and in this the kernel of Ω_L ,

$$\ker \Omega_L(u) := \{\mathbf{K} \in \mathbb{T}_u\mathbb{P}_L \mid i_{\mathbf{K}} \Omega_L = 0\}, \quad (25)$$

plays a defining role.

3.2. Properties of $\ker \Omega_L(u)$

In this section we characterize the structure of $\ker \Omega_L(u)$, and determine the vectors that lie in it.

The two-form Ω_L gives the lowering map $\Omega_L^b: \mathbb{T}_u\mathbb{P}_L \rightarrow \mathbb{T}_u^*\mathbb{P}_L$, with $\Omega_L^b \mathbf{X} := i_{\mathbf{X}} \Omega_L$. As $\Omega_L^b = \Omega_F^b + \Omega_M^b$, the action of Ω_L^b on a vector $\mathbf{X}$ is given by $\Omega_F^b: \mathbf{X} \in \mathbb{T}_u\mathbb{P}_L \rightarrow [\mathbb{T}_u^*\mathbb{P}_L]^q$, and, since Ω_M is a mixed two-form, by $\Omega_M^b = \Omega_M^{vb} + \Omega_M^{qb}$, where $\Omega_M^{qb}: \mathbf{X} \in \mathbb{T}_u\mathbb{P}_L \rightarrow [\mathbb{T}_u^*\mathbb{P}_L]^q$ and $\Omega_M^{vb}: \mathbf{X} \in \mathbb{T}_u\mathbb{P}_L \rightarrow [\mathbb{T}_u^*\mathbb{P}_L]^v$. In terms of local coordinates, $\Omega_F^b \mathbf{X} = F_{ab} X^{qa} dq^b$, $\Omega_M^{qb} \mathbf{X} = -M_{ab} X^{va} dv^b$ and $\Omega_M^{vb} \mathbf{X} = M_{ab} X^{qa} dv^b$.

For almost regular Lagrangians $\ker \Omega_M^{vb} = \mathcal{C} \oplus [\mathbb{T}_u\mathbb{P}_L]^v$ while $\ker \Omega_M^{qb} = [\mathbb{T}_u\mathbb{P}_L]^q \oplus \mathcal{G}$, where

$$\mathcal{C} = \{\mathbf{C} \in [\mathbb{T}_u\mathbb{P}_L]^q \mid i_{\mathbf{C}} \Omega_M = 0\}, \quad (26)$$

while

$$\mathcal{G} = \{\mathbf{G} \in [\mathbf{T}_q \mathbb{P}_L]^\nu \setminus i_{\mathbf{G}} \Omega_M = 0\}. \quad (27)$$

Moreover, because the rank of $M_{ab}(\mathbf{u})$ is constant on $\mathbb{P}_L$ there exists a basis,

$$\{\mathfrak{z}_{(n)}(\mathbf{u}) = (\mathfrak{z}_{(n)}^1(\mathbf{u}), \dots, \mathfrak{z}_{(n)}^D(\mathbf{u})) \setminus M_{ab}(\mathbf{u}) \mathfrak{z}_{(n)}^b(\mathbf{u}) = 0, n = 1, \dots, N_0\}, \quad (28)$$

for $\ker M_{ab}(\mathbf{u})$ at each $\mathbf{u} \in \mathbb{P}_L$. This in turn gives the bases

$$\mathcal{C} = \text{span} \left\{ \mathbf{U}_{(n)}^q = \mathfrak{z}_{(n)} \cdot \frac{\partial}{\partial q}, n = 1, \dots, N_0 \right\}, \quad \mathcal{G} = \text{span} \left\{ \mathbf{U}_{(n)}^\nu = \mathfrak{z}_{(n)} \cdot \frac{\partial}{\partial \nu}, n = 1, \dots, N_0 \right\}, \quad (29)$$

for $\mathcal{C}$ and $\mathcal{G}$. It is well known [2] that for almost regular Lagrangians $\mathcal{G}$ is involutive. Furthermore, when the rank of Ω_L is constant on $\mathbb{P}_L$, $\ker \Omega_L(\mathbf{u})$ is involutive as well.

Corresponding to $\mathbf{U}_{(n)}^q$ and $\mathbf{U}_{(n)}^\nu$ we have the one-forms $\Theta_q^{(n)}$ and $\Theta_\nu^{(n)}$ where $\langle \Theta_q^{(n)} | \mathbf{U}_{(m)}^q \rangle = \delta_m^n$ and $\langle \Theta_\nu^{(n)} | \mathbf{U}_{(m)}^\nu \rangle = \delta_m^n$. Then $[\mathbf{T}_u \mathbb{P}_L]^q = \mathcal{C} \oplus \mathcal{C}_\perp$ and $[\mathbf{T}_u \mathbb{P}_L]^\nu = \mathcal{G} \oplus \mathcal{G}_\perp$, where

$$\begin{aligned} \mathcal{C}_\perp &:= \{\mathbf{X} \in [\mathbf{T}_u \mathbb{P}_L]^q \setminus \langle \Theta_q^{(n)} | \mathbf{X} \rangle = 0, n = 1, \dots, N_0\}, \text{ and} \\ \mathcal{G}_\perp &:= \{\mathbf{X} \in [\mathbf{T}_u \mathbb{P}_L]^\nu \setminus \langle \Theta_\nu^{(n)} | \mathbf{X} \rangle = 0, n = 1, \dots, N_0\}. \end{aligned} \quad (30)$$

To determine the vectors in $\ker \Omega_L(\mathbf{u})$, choose a $\mathbf{K} \in \ker \Omega_L(\mathbf{u})$. Then

$$\Omega_M^{\nu b} \mathbf{K}^q = 0, \quad \text{and} \quad \Omega_M^{qb} \mathbf{K}^\nu = -\Omega_F^b \mathbf{K}^q. \quad (31)$$

Solutions of these equations are found with the use of the following theorem from linear algebra stated without proof (see also [4] where a special case of this theorem was proved).

Theorem 2. For linear spaces $\mathbf{E}$ and $\mathbf{F}$ of dimension D and a linear map $\mathcal{A} : \mathbf{F} \rightarrow \mathbf{E}^*$ of rank r , the inhomogeneous linear equation,

$$\mathcal{A}\mathbf{f} = \boldsymbol{\varphi}, \quad (32)$$

has solutions if and only if

$$\langle \boldsymbol{\varphi} | \mathbf{e} \rangle = 0 \quad \forall \quad \mathbf{e} \in \mathbf{A}, \quad (33)$$

where

$$\mathbf{A} := \{\mathbf{e} \in \mathbf{E} \setminus \langle \mathcal{A}\mathbf{f} | \mathbf{e} \rangle = 0 \quad \forall \quad \mathbf{f} \in \mathbf{F}\}. \quad (34)$$

This theorem is applied to equation (31) by setting $\mathbf{F} = [\mathbf{T}_u \mathbb{P}_L]^\nu$, $\mathbf{E} = [\mathbf{T}_u \mathbb{P}_L]^q$, $\mathcal{A} = \Omega_M^{qb}$, and $\boldsymbol{\varphi} = -\Omega_F^b \mathbf{K}^q$. To make the connection with the results of section 2 clear, this application is done locally. The condition that $\mathbf{X}^q \in \mathbf{A}$ is $\langle \Omega_M^{qb} \mathbf{K} | \mathbf{X}^q \rangle = -K^{qa} M_{ab} X^{qb} = 0 \quad \forall K^{qa}$, which requires $M_{ab} X^{qb} = 0$. This establishes $\mathbf{A} = \mathcal{C}$. Equation (33) reduces to $\langle \Omega_F^b \mathbf{K}^q | \mathbf{C} \rangle = F_{ab} K^{qa} C^b = 0 \quad \forall \quad \mathbf{C} \in \mathcal{C}$, or equivalently $\mathfrak{z}_{(n)}^a F_{ab} K^{qb} = 0$. Using the action of $\Omega_M^{\nu b}$ and Ω_M^{qb} in equation (31), we find that $M_{ab} K^{qa} = 0$, and thus $\mathbf{K}^q \in \mathcal{C}$. The existence condition for solutions of equation (31) is then,

$$\mathfrak{z}_{(n)}^a F_{ab} K^{qb} = 0, n = 1, \dots, N_0. \quad (35)$$

With the basis given in equation (29), we may express

$$K^{qa} = \sum_{m=1}^{N_0} K^{q(m)} \mathfrak{z}_{(m)}^a, \quad (36)$$

and equation (35) becomes,

$$\sum_{m=1}^{N_0} \bar{F}_{nm} K^{q(m)} = 0, \text{ where } \bar{F}_{nm} := \mathfrak{z}_{(n)}^a F_{ab} \mathfrak{z}_{(m)}^b, \quad (37)$$

is the reduced matrix of F_{ab} . Then $\mathbf{K}^q$ is restricted to the subspace,

$$\bar{\mathcal{C}} := \left\{ \bar{\mathbf{C}} \in \mathcal{C} \setminus \sum_{m=1}^{N_0} \bar{F}_{nm} \bar{\mathbf{C}}^{(m)} = 0 \right\} \subset \mathcal{C}. \quad (38)$$

Theorem 3. The vectors $\mathbf{K} = \mathbf{K}^q + \mathbf{K}^\nu \in \ker \Omega_L$ are given by,

$$\mathbf{K}^q = \bar{\mathbf{C}}, \quad \mathbf{K}^\nu = \mathbf{G} + \hat{\mathbf{C}}, \quad (39)$$

where, $\bar{\mathbf{C}} \in \bar{\mathcal{C}}$, $\mathbf{G} \in \mathcal{G}$, and $\hat{\mathbf{C}} \in \mathcal{G}_\perp$ is the unique solution of $M_{ab} \hat{\mathbf{C}}^b = -F_{ab} \bar{\mathbf{C}}^b$.

Proof. The horizontal component, $\mathbf{K}^q = \bar{\mathbf{C}}$, of $\mathbf{K}$ satisfies equation (35), and is a solution of equation (31). As $\mathbf{G} \in \ker \Omega_M^{qb}$, the general solution of equation (31) is $\mathbf{K}^v = \mathbf{G} + \bar{\mathbf{C}}$, where $\bar{\mathbf{C}} \in \mathcal{G}_\perp$. With $M_{(\alpha)(\beta)} := \mathfrak{z}_{(\alpha)}^a M_{ab} \mathfrak{z}_{(\beta)}^a$ for $\alpha, \beta = 1, \dots, D$, and with the choice that $\mathfrak{z}_{(\alpha)} \in \ker M_{ab}(\mathbf{u})$ for $\alpha = 1, \dots, N_0$, the components of $\bar{\mathbf{C}}$ and M satisfy $\hat{C}^{(n)} = 0$, and $M_{(n)(e)} = M_{(f)(m)} = 0$, when $n, m = 1, \dots, N_0$. Thus

$$M = \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{M} \end{bmatrix}, \quad \mathcal{M}_{(f)(e)} := M_{(f)(e)}, \quad (40)$$

for $e, f = N_0 + 1, \dots, D$, and $\mathcal{M}$ is nonsingular. In this basis equation (31) becomes $M\hat{\mathbf{C}} = R$, where $R_{(f)} := -U_{(f)}^{va} F_{ab} \bar{C}^b$, and $R_{(n)} = 0$. The nonvanishing components of $\hat{\mathbf{C}}$ and R are vectors with $(D - N_0)$ -components that satisfy

$$\mathcal{M}\hat{\mathbf{C}} = R. \quad (41)$$

Thus there is a unique solution for $\hat{\mathbf{C}}$ that belongs to $\mathcal{G}_\perp$. ■

The constant rank assumption for $M_{ab}(\mathbf{u})$ together with the definition of $\mathcal{G}$ shows that there are N_0 free choices for $\mathbf{G}$ at each $\mathbf{u} \in \mathbb{P}_L$. According to theorem 3 the $\hat{\mathbf{C}}$ -term is uniquely specified by the choice of $\bar{\mathbf{C}} \in \bar{\mathcal{C}}$. In general, $\bar{F}$ is not determined by M_{ab} , and thus the constant rank of M_{ab} does not guarantee that the rank of $\bar{F}$ is constant. We then find

$$\dim(\ker \Omega_L(\mathbf{u})) = N_0 + \bar{D}, \quad \text{where } \bar{D} := \dim \bar{\mathcal{C}} \leq N_0. \quad (42)$$

The results of theorem 3 are more general than we need. Because its proof is local and algebraic, even though our focus is on Ω_L with constant rank the theorem would nevertheless still hold if the rank was not. It would only have to be applied to each region of $\mathbb{P}_L$ on which the rank of Ω_L is constant, resulting in a $\bar{D}$ that takes on different values on the Lagrangian phase space.

3.3. Projection of $\mathcal{K}$ to $\ker \Omega_L(\mathbf{u})$

Consider a region $U_{Sol} \in \mathbb{M}^{(2)}$ on which solutions of the Euler–Lagrange equations of motion exist, and a point $(t, q, \dot{q}, \ddot{q}) \in U_{Sol}$. Then under the isomorphism $(t, q, \dot{q}, \ddot{q}) \rightarrow (t, q, v, X_L^{va})$, $\mathcal{K} \rightarrow \mathcal{K}' \subset \mathbf{T}_u \mathbb{P}_L$, with a $\mathbf{k} \in \mathcal{K}$ mapped into a $\mathbf{k}' \in \mathcal{K}'$ where

$$\mathbf{k}' = \rho \cdot \frac{\partial}{\partial q} + \dot{\rho} \cdot \frac{\partial}{\partial v}. \quad (43)$$

Now $\dot{\rho} = \mathcal{L}_{X_L} \rho$, and $\mathcal{L}$ is the Lie derivative. From lemma 1, $\mathcal{K}' \subseteq \ker \Omega_L(\mathbf{u})$. But since $\dim \mathcal{K} = 2N_0$, while $\dim \ker \Omega_L(\mathbf{u}) = N_0 + \bar{D} \leq 2N_0$, it follows that $\mathcal{K}' = \ker \Omega_L(\mathbf{u})$. Although this conclusion can only be reached on U_{Sol} , the rank of Ω_L is constant on $\mathbb{P}_L$, and thus $\bar{D} = N_0$ on all of $\mathbb{P}_L$. As such $\bar{\mathcal{C}} = \mathcal{C}$, and for any $\mathbf{K} \in \ker \Omega_L(\mathbf{u})$, $\mathbf{K} = \mathbf{C} + \hat{\mathbf{C}} + \mathbf{G}$ where now $\hat{\mathbf{C}} = \hat{C}^a \partial / \partial v^a$. (This expression for $\mathbf{K}$ can also be established directly using equation (31)). This result is also new.

3.4. First-order Lagrangian constraints

For singular Lagrangians solutions of the energy equation $\mathbf{X}_E$ are not unique. They also do not, in general, exist throughout $\mathbb{P}_L$, but are rather confined to a submanifold of the space given by Lagrangian constraints. The first-order constraints, those that come directly from the energy equation, are the focus of this section. Although most of this analysis is done for a SOLVE, we show later that our results do not depend on this choice.

With $\mathbf{X}_L = \mathbf{X}_L^q + \mathbf{X}_L^v$, the energy equation can be expressed in terms of a one-form Ψ as

$$\Omega_M^{qb} \mathbf{X}_L^v = \Psi. \quad (44)$$

The existence condition for solutions to equation (44) is given again by theorem 2 by identifying $\mathcal{A} = \Omega_M^{qb}$, $\mathbf{F} = [\mathbf{T}_u \mathbb{P}_L]^v$, $\mathbf{E}^* = [\mathbf{T}_u^* \mathbb{P}_L]_q$, and $\mathbf{E} = [\mathbf{T}_u \mathbb{P}_L]^q$. Combining equation (34) with the action of Ω_M^{qb} on vector fields yields $\mathbf{A} = \mathcal{C}$; consequently equation (33) requires that $\langle \Psi | \mathbf{C} \rangle = 0 \quad \forall \quad \mathbf{C} \in \mathcal{C}$ or, after using the basis $\{\mathbf{U}_{(n)}^q\}$ of $\mathcal{C}$, that $\gamma_n^{[1]} := \langle \Psi | \mathbf{U}_{(n)}^q \rangle = 0$ for $n = 1, \dots, N_0$. In terms of local coordinates,

$$\gamma_n^{[1]} = U_{(n)}^{qa} \left(\frac{\partial E}{\partial q^a} + F_{ab} v^b \right). \quad (45)$$

The condition $\gamma_n^{[1]} = 0$ imposes relations on the coordinates q and v , and defines a set of submanifolds of $\mathbb{P}_L$. These $\gamma_n^{[1]}$ are called the first-order constraint functions. (Because they are obtained through the energy equation, they are also called dynamical constraints in the literature [10, 15]).

While the number of first-order constraint functions in $\mathcal{C}_L^{[1]} := \{\gamma_1^{[1]}, \dots, \gamma_{N_0}^{[1]}\}$ is equal to the dimension of $\mathcal{C}$, these functions need not be mutually independent. Let $I_{[1]}$ be the number of independent functions in $\mathcal{C}_L^{[1]}$. Then $I_{[1]} = \text{rank}\{\mathbf{d}\gamma_n^{[1]}\} \leq N_0$, and $\mathbb{P}_L^{[1]} := \{\mathbf{u} \in \mathbb{P}_L \setminus \gamma_n^{[1]}(\mathbf{u}) = 0, n = 1, \dots, N_0\}$ —called the first-order

Lagrangian constraint submanifold—has $\dim \mathbb{P}_L^{[1]} = 2D - I_{[1]}$. We will assume that $\mathbb{P}_L^{[1]}$ is not empty, i.e. that the first-order constraint functions are consistent. Otherwise, there is no SOLVF, and the integral flows that give the evolution of the dynamical system would not exist.

While the energy equation is usually written as $\mathbf{d}E = i_{\mathbf{X}_E} \Omega_L$, in doing so we have implicitly restricted ourselves to $\mathbb{P}_L^{[1]}$. This is too restrictive for our purposes, and in this paper we introduce the constraint one-form

$$\beta[\mathbf{X}_E] := \mathbf{d}E - i_{\mathbf{X}_E} \Omega_L, \quad (46)$$

(see also the approach in [15]). The condition $\beta[\mathbf{X}_E] = 0$ then gives both solutions of the energy equation and the submanifold $\mathbb{P}_L^{[1]}$. Furthermore, as $i_{\mathbf{U}_{(n)}^q} \beta = \gamma_n^{[1]}$,

$$\left\langle \left(\beta[\mathbf{X}_E] - \sum_{m=1}^{N_0} \gamma_m^{[1]} \Theta_q^{(m)} \right) \middle| \mathbf{U}_{(n)}^q \right\rangle = 0, \quad n = 1, \dots, N_0, \quad (47)$$

and

$$\beta[\mathbf{X}_E] = \sum_{n=1}^{N_0} \gamma_n^{[1]} \Theta_q^{(n)} + \vartheta, \quad (48)$$

where $\vartheta \in \mathbf{T}_u^* \mathbb{P}_L$ such that $\langle \vartheta | \mathbf{C} \rangle = 0$ for all $\mathbf{C} \in \mathcal{C}$. But as $\beta[\mathbf{X}_E] = 0$ on $\mathbb{P}_L^{[1]}$, we may choose $\vartheta = 0$.

From equation (46), $\beta[\mathbf{X}_E + \mathbf{K}] = \beta[\mathbf{X}_E]$, and the construction of $\gamma_n^{[1]}$ does not depend on our use of $\mathbf{X}_L$.

3.5. The generalized lie symmetry group

Our construction of the generalized Lie symmetry group for $\mathcal{O}(u_0)$ is guided by the three conditions listed in section 2, and makes use of the projection of $\mathcal{K}$ to $\ker \Omega_L(u)$ in section 3.3. It begins with the vector space,

$$\overline{\ker \Omega_L(u)} := \{ \mathbf{P} \in \ker \Omega_L(u) \setminus [\mathbf{G}, \mathbf{P}] \in [\mathbf{T}_u \mathbb{P}_L]^\vee \ \forall \ \mathbf{G} \in \mathcal{G} \}, \quad (49)$$

along with the following collection of functions on $\mathbb{P}_L$,

$$\overline{\mathcal{F}} := \{ f \in C^\infty \text{ on } \mathbb{P}_L \setminus \mathbf{G}f = 0 \ \forall \ \mathbf{G} \in \mathcal{G} \}. \quad (50)$$

The following result will be used a number of times in our analysis.

Lemma 4. *Let $\mathbf{X} \in \mathbf{T}_u \mathbb{P}_L$ and $\mathbf{G} \in \mathcal{G}$ such that $[\mathbf{G}, \mathbf{X}] \in \ker \Omega_L(u)$. Then $[\mathbf{G}, \mathbf{X}] \in [\mathbf{T}_u \mathbb{P}_L]^\vee$ iff $[\mathbf{G}, \mathbf{X}] \in \mathcal{G}$.*

Proof. Since $[\mathbf{G}, \mathbf{X}] \in \ker \Omega_L(u)$, from theorem 3 there exists a $\mathbf{C} \in \mathcal{C}$ and $\mathbf{G}' \in \mathcal{G}$ such that $[\mathbf{G}, \mathbf{X}] = \mathbf{C} + \widehat{\mathbf{C}} + \mathbf{G}'$, and we see that $[\mathbf{G}, \mathbf{X}] \in [\mathbf{T}_u \mathbb{P}_L]^\vee$ iff $\mathbf{C} = 0$. Then $\widehat{\mathbf{C}} = 0$, and $[\mathbf{G}, \mathbf{X}] \in \mathcal{G}$. ■

It then follows that $[\mathbf{G}, \mathbf{P}] \in \mathcal{G}$ for all $\mathbf{P} \in \overline{\ker \Omega_L(u)}$. As $\mathcal{G}$ is involutive and as $\mathcal{G} \subset \ker \Omega_L(u)$, $\mathcal{G} \subset \overline{\ker \Omega_L(u)}$ as well, and thus $\mathcal{G}$ is an ideal of $\overline{\ker \Omega_L(u)}$.

Lemma 5. *There exists a choice of basis for $\ker \Omega_L(u)$ that is also a basis of $\overline{\ker \Omega_L(u)}$.*

Proof. Given a basis $\{ \mathbf{G}_{(n)} \}$ of $\mathcal{G}$, choose a set $\{ \mathbf{K}_{(n)} \}$ such that $\{ \mathbf{K}_{(n)}, \mathbf{G}_{(n)}, n = 1, \dots, N_0 \}$ form a basis of $\ker \Omega_L(u)$. These $\{ \mathbf{G}_{(n)} \}$ are also a basis of $\overline{\ker \Omega_L(u)}$. To show that $\{ \mathbf{K}_{(n)} \}$ can be chosen to complete this basis, express $\mathbf{K}_{(n)} = \mathbf{C}_{(n)} + \widehat{\mathbf{C}}_{(n)} + \mathbf{G}_{(n)}$. Then as $[\mathbf{G}_{(m)}, \mathbf{K}_{(n)}] = [\mathbf{G}_{(m)}, \mathbf{C}_{(n)}] + [\mathbf{G}_{(m)}, \widehat{\mathbf{C}}_{(n)} + \mathbf{G}_{(n)}]$, we need only show that there exists a choice of $\{ \mathbf{C}_{(n)} \}$ such that $[\mathbf{G}_{(m)}, \mathbf{C}_{(n)}] \in [\mathbf{T}_u \mathbb{P}_L]^\vee$. This we do by construction.

Because $\ker \Omega_L(u)$ is involutive, $[\mathbf{G}_{(m)}, \mathbf{K}_{(n)}] \in \ker \Omega_L(u)$, and there exist functions λ_{mn}^l on $\mathbb{P}_L$ such that

$$\mathbf{T}_{\mathbb{Q}}[\mathbf{G}_{(m)}, \mathbf{C}_{(n)}] = \sum_{l=0} \lambda_{mn}^l \mathbf{C}_{(l)}. \quad (51)$$

Let $\{ \underline{\mathbf{C}}_{(n)}, n = 1, \dots, N_0 \}$ be another choice of basis of $\mathcal{C}$ where

$$\underline{\mathbf{C}}_{(n)} = \sum_{m=1}^{N_0} \omega_n^m \mathbf{C}_{(m)}. \quad (52)$$

Requiring $\mathbf{T}_{\mathbb{Q}}[\mathbf{G}_{(m)}, \underline{\mathbf{C}}_{(n)}] = 0$ in turn requires that ω_n^m be a solution of

$$\mathbf{G}_{(l)} \omega_n^m + \sum_{k=1}^{N_0} \omega_n^k \lambda_{lk}^m = 0. \quad (53)$$

This is a linear, first-order Cauchy problem [48]. A solution exists for a given set of boundary conditions for ω_n^m given on a surface $\mathcal{S}$ as long as $\mathbf{G}_{(l)}$ is nowhere tangent to $\mathcal{S}$ [48]. As $N_0 < D$, and as we have complete freedom to choose both the boundary conditions and $\mathcal{S}$, a solution of equation (53) can always be found. The collection of vector fields $\{ \mathbf{P}_{(n)} = \underline{\mathbf{C}}_{(n)} + \widehat{\mathbf{C}}_{(n)}, \mathbf{G}_{(n)} \}$ is then a basis of both $\ker \Omega_L(u)$ and $\overline{\ker \Omega_L(u)}$. ■

For the rest of this paper we will assume that this choice of basis for $\mathcal{C}$ and $\overline{\ker \Omega_L(u)}$ has been made.

It follows from lemma 5 that $\dim(\overline{\ker \Omega_L(u)}) = 2N_0$. Next, choose two vectors $\mathbf{P}_{1,2} \in \overline{\ker \Omega_L(u)}$. Then $\mathbf{P}_{1,2} \in \ker \Omega_L(u)$ as well, and as $\ker \Omega_L(u)$ is involutive, $[\mathbf{P}_1, \mathbf{P}_2] \in \ker \Omega_L(u)$. Choose now a $\mathbf{G} \in \mathcal{G}$. From the Jacobi identity, $[\mathbf{G}, [\mathbf{P}_1, \mathbf{P}_2]] = -[\mathbf{P}_1, [\mathbf{P}_2, \mathbf{G}]] - [\mathbf{P}_2, [\mathbf{G}, \mathbf{P}_1]]$. From Lemma 4, there exists $\mathbf{G}_{1,2} \in \mathcal{G}$ such that $\mathbf{G}_{1,2} = [\mathbf{G}, \mathbf{P}_{1,2}]$. Then $[\mathbf{G}, [\mathbf{P}_1, \mathbf{P}_2]] = [\mathbf{P}_1, \mathbf{G}_2] - [\mathbf{P}_2, \mathbf{G}_1]$, and thus $\overline{\ker \Omega_L(u)}$ is involutive.

As $\mathcal{G}$ is an ideal of $\overline{\ker \Omega_L(u)}$, we may define for any $\mathbf{P}_1, \mathbf{P}_2 \in \overline{\ker \Omega_L(u)}$ the equivalence relation: $\mathbf{P}_1 \sim \mathbf{P}_2$ iff $\mathbf{P}_1 - \mathbf{P}_2 \in \mathcal{G}$. The equivalence class,

$$[\mathbf{P}] := \{\mathbf{Y} \in \overline{\ker \Omega_L(u)} \mid \mathbf{Y} \sim \mathbf{P}\}, \quad (54)$$

then follows, along with the quotient space $\overline{\ker \Omega_L(u)}/\mathcal{G}$. This results in a collection of vectors that lie in the kernel of Ω_L , but with the vectors in $\mathcal{G}$ removed. This $\overline{\ker \Omega_L(u)}/\mathcal{G}$ addresses the first two conditions listed at the end of section 2. We now turn our attention to the third condition.

Because the integral flow $u_X(t)$ of any solution $\mathbf{X}$ of the energy equation must lie on $\mathbb{P}_L^{[1]}$, a symmetry transformation of $u_X(t)$ must result in an integral flow $u_Y(t)$ of another solution $\mathbf{Y}$ of the energy equation. This flow must also lie on $\mathbb{P}_L^{[1]}$. Implementing this condition is done through $\beta[\mathbf{X}_E]$, and it is this one-form that singles out the vectors in $\ker \Omega_L(u)$ that generate the generalized Lie symmetry. We do this by looking at the action of a vector $\mathbf{G} \in \mathcal{G}$ on $\beta[\mathbf{X}_E]$.

As $i_G \beta[\mathbf{X}_E] = 0$ for all $\mathbf{G} \in \mathcal{G}$,

$$\mathcal{L}_G \beta[\mathbf{X}_E] = \sum_{n=1}^{N_0} (\mathbf{G} \gamma_n^{[1]}) \Theta_q^{(n)}, \quad (55)$$

since $\gamma_n^{[1]} = 0$ on $\mathbb{P}_L^{[1]}$. Given a $\mathbf{P}_{(n)} \in \overline{\ker \Omega_L(u)}$ such that $\mathbf{P}_{(n)} = \mathbf{U}_{(n)}^q + \widehat{\mathbf{U}}_{(n)} + \mathbf{G}'$ with $\mathbf{G}' \in \mathcal{G}$, it has been shown in [5, 8] that $\gamma_n^{[1]} = i_{\mathbf{P}_{(n)}} dE$. Then $\mathbf{G} \gamma_n^{[1]} = [\mathbf{G}, \mathbf{P}_{(n)}]E + \mathbf{P}_{(n)} \mathbf{G}E$. But $\mathcal{G}$ is an ideal of $\overline{\ker \Omega_L(u)}$, while a straightforward calculation shows that $\mathbf{G}E = 0$. Thus $\mathbf{G} \gamma_n^{[1]} = 0$, and it follows that $\mathcal{L}_G \beta = 0$. The subspace

$$\text{Sym} := \{\mathbf{P} \in \overline{\ker \Omega_L(u)}/\mathcal{G} \mid \mathcal{L}_P \beta[\mathbf{X}_E] = d(i_P \beta[\mathbf{X}_E]) \text{ on } \mathbb{P}_L^{[1]}\}, \quad (56)$$

is then well defined, and we find that $\mathbf{P} \in \text{Sym}$ iff $i_P d\beta[\mathbf{X}_E] = 0$. A simple calculation shows that Sym is involutive. There is then a corresponding set of one-parameter subgroups $\sigma_P(\epsilon, x)$ for any $\mathbf{P} \in \text{Sym}$ given by

$$\frac{d\sigma_P}{d\epsilon} := \mathbf{P}(\sigma_P), \quad (57)$$

with $\sigma_P(0, u) = u$ for $u \in \mathbb{P}_L$. The collection of such subgroups gives the Lie group Gr_{Sym} . This Gr_{Sym} is the generalized symmetry group we are looking for, as we see below.

3.6. Euler-lagrange solutions of the energy equation

The set of general solutions to the energy equation is

$$\text{Sol} := \{\mathbf{X}_E \in \mathbf{T}_u \mathbb{P}_L \mid i_{\mathbf{X}_E} \Omega_L = dE \text{ on } \mathbb{P}_L^{[1]}\}. \quad (58)$$

Importantly, while a SOLVF $\mathbf{X}_L \in \text{Sol}$, the majority of vectors in Sol are *not* SOLVFs. This is the root cause of the ‘second order problem’ first raised by Künzle [14] (see also [5, 8], and [4]).

If $u(t)$ is the integral flow of a vector in Sol whose projection onto $\mathbb{Q}$ corresponds to a trajectory $q(t)$ that is a solution of the Euler–Lagrange equations of motion, then Gr_{Sym} must map one of such flows into another. However, while $\mathcal{L}_G \mathbf{X}_L = [\mathbf{G}, \mathbf{X}_L] \in \ker \Omega_L(u)$, in general $\mathcal{L}_G \mathbf{X}_L \notin \mathcal{G}$. The action of σ_P on the flow $u_{\mathbf{X}_L}$ will in general result in a flow u_Y generated by a $\mathbf{Y}$ that is *not* a SOLVF. It need not even be a solution of the energy equation. By necessity, general solutions of the energy equation must be considered. Only a specific subset of such solutions are physically relevant, however.

As

$$i_{[\mathbf{X}_E, \mathbf{P}]} \Omega_L = i_P d\beta[\mathbf{X}_E], \quad (59)$$

for $\mathbf{P} \in \ker \Omega_L(u)$ and $\mathbf{X}_E \in \text{Sol}$, in general $\mathcal{L}_P \mathbf{X}_E \notin \ker \Omega_L(u)$. The exception is when $\mathbf{P} \in \text{Sym}$ as well, which leads us to the subset of solutions

$$\overline{\text{Sol}} := \{\overline{\mathbf{X}}_{EL} \in \text{Sol} \mid [\mathbf{G}, \overline{\mathbf{X}}_{EL}] \in [\mathbf{T}_u \mathbb{P}_L]^\vee \quad \forall \mathbf{G} \in \mathcal{G}\}. \quad (60)$$

Moreover, as $i_{[\mathbf{G}, \overline{\mathbf{X}}_{EL}]} \Omega_L = -\mathcal{L}_G \beta = 0$, from lemma 4 $[\mathbf{G}, \overline{\mathbf{X}}_{EL}] \in \mathcal{G}$.

Lemma 6. $[\overline{\mathbf{X}}_{EL}, \mathbf{P}] \in \overline{\ker \Omega_L(u)}$ for all $\mathbf{P} \in \text{Sym}$.

Proof. As $\mathbf{P} \in \text{Sym}$, from equation (59) $[\overline{\mathbf{X}}_{EL}, \mathbf{P}] \in \ker \Omega_L(u)$. Next, for each $\mathbf{G} \in \mathcal{G}$, there is a $\mathbf{G}_X \in \mathcal{G}$ such that $\mathbf{G}_X = [\mathbf{G}, \overline{\mathbf{X}}_{EL}]$. There is also a $\mathbf{G}_P \in \mathcal{G}$ such that $\mathbf{G}_P = [\mathbf{G}, \mathbf{P}]$. It then follows from the Jacobi identity that $[[\overline{\mathbf{X}}_{EL}, \mathbf{P}], \mathbf{G}] \in \mathcal{G}$, and $[\overline{\mathbf{X}}_{EL}, \mathbf{P}] \in \overline{\ker \Omega_L(u)}$. ■

The vector fields in $\overline{\text{Sol}}$ generate the family of integral flows

$$\mathcal{O}_{EL}(u_0) := \left\{ u(t) \mid \frac{du}{dt} = \overline{X}_{EL}(u), \overline{X}_{EL} \in \overline{\text{Sol}}, \text{ and } u(t_0) = u_0 \right\}. \quad (61)$$

The physical significance of these flows can be seen from the following theorem.

Theorem 7. Gr_{Sym} forms a group of symmetry transformations of $\mathcal{O}_{EL}(u_0)$.

Proof. Let $u_{\overline{X}_{EL}}(t, u_0) \in \mathcal{O}_{EL}(u_0)$ be an integral flow generated by $\overline{X}_{EL}$, and let $\sigma_P(\epsilon, u) \in \text{Gr}_{\text{Sym}}$ be a one-parameter subgroup of Gr_{Sym} generated by $P \in \text{Sym}$ with $\sigma_P(0, u) = u$. The action of σ_P on $u_{\overline{X}_{EL}}$ gives $u_Y(t, u_0) := \sigma_P(\epsilon, u_{\overline{X}_{EL}}(t, u_0))$, while the choice $\epsilon = 0$ when $t = t_0$ ensures that $u_{\overline{X}_{EL}}$ and u_Y have the same initial data. The tangent to this path is

$$Y = \sigma_P^* \circ \overline{X}_{EL}(\sigma_P^{-1}(\epsilon, u_{\overline{X}_{EL}})), \quad (62)$$

where σ_P^* is the pullback map of σ_P . As σ_P is also a mapping of $\mathbb{P}_L$ into itself, for a suitably small neighborhood about $u_{\overline{X}_{EL}}$ we may expand Y about $\epsilon = 0$ in the Lie series,

$$Y(\sigma(\epsilon, u_{\overline{X}_{EL}}(t, u_0))) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \mathcal{L}_P^{(n)} \overline{X}_{EL} \Big|_{u_{\overline{X}_{EL}}(t, u_0)}. \quad (63)$$

However, from lemma 6, $\mathcal{L}_P \overline{X}_{EL} \in \overline{\ker \Omega_L(u)}$, and $\overline{\ker \Omega_L(u)}$ is involutive. Then $Y(\sigma_P(\epsilon, u_{\overline{X}_{EL}}(t, u_0))) = \overline{X}_{EL}(u_{\overline{X}_{EL}}(t, u_0)) + \epsilon \overline{Z}(\epsilon, u_{\overline{X}_{EL}}(t, u_0))$, where $\overline{Z} \in \overline{\ker \Omega_L(u)}$. It then follows that $Y \in \overline{\text{Sol}}$, and $u_Y(t, u_0) \in \mathcal{O}_{EL}(u_0)$. ■

While $X_L \notin \overline{\text{Sol}}$, it is possible to construct from X_L a vector field $\overline{X}_L$ that is. Choose a basis $\{P_{(n)}, G_{(n)}, n = 1, \dots, N_0\}$ of $\overline{\ker \Omega_L(u)}$, and consider a vector field $\overline{X}_L$ such that

$$\overline{X}_L = X_L + \sum_{m=1}^{N_0} f^m(u) P_{(m)} + G, \quad (64)$$

where $G \in \mathcal{G}$, and $f^m(u)$ are functions on $\mathbb{P}_L$. For $\overline{X}_L \in \overline{\text{Sol}}$ as well, we must have $[\overline{X}_L, G_{(n)}] \in \mathcal{G}$, and thus these functions must be solutions of

$$G_{(n)} f^m(u) = -\langle \Theta_q^{(m)} | [X_L, G_{(n)}] \rangle. \quad (65)$$

Once again, this is a linear Cauchy problem, and a solution exists with the appropriate choice of boundary conditions and surfaces.

If $f^m(u)$ is a solution to equation (65), then $f^m(u) + u^m(u)$ is as well as long as $u^m(u) \in \overline{\mathcal{F}}$. This leads us to the second-order, Euler–Lagrange vector field (SOELVF),

$$\overline{X}_{EL} = \overline{X}_L + \sum_{m=1}^{N_0} u^m(u) [P_{(m)}], \quad (66)$$

where $\{[P_{(n)}], n = 1, \dots, N_0\}$ is a choice of basis for $\overline{\ker \Omega_L(u)}/\mathcal{G}$. By construction, $\overline{X}_{EL} \in \overline{\text{Sol}}$. Conversely, if $\overline{Y}_{EL} \in \overline{\text{Sol}}$, then $\overline{Y}_{EL} - \overline{X}_L \in \overline{\ker \Omega_L(u)}$, and $\overline{Y}_{EL}$ is a SOELVF. Thus, $\overline{Y}_{EL} \in \overline{\text{Sol}}$ iff $\overline{Y}_{EL}$ is a SOELVF.

3.7. A constraint algorithm for second-order, Euler–Lagrange vector fields

For most dynamical systems the flow fields in $\mathcal{O}_{EL}(u_0)$ will not be confined to $\mathbb{P}_L^{[1]}$, and yet this is the submanifold on which the solutions $\overline{X}_{EL} \in \overline{\text{Sol}}$ of the energy equations exist. In these cases it is necessary to jointly choose a SOELVF $\overline{X}_{EL}$ and a submanifold of $\mathbb{P}_L^{[1]}$ on which $u_{\overline{X}_{EL}}$ will be confined. Doing so requires that

$$\mathcal{L}_{\overline{X}_{EL}} \beta = 0, \quad (67)$$

which is called the constraint condition. Implementing it involves imposing successive conditions on $\overline{X}_{EL}$. At each step additional constraints may be introduced, giving a succession of submanifolds of $\mathbb{P}_L^{[1]}$. It is an iterative process that terminates either when $u_{\overline{X}_{EL}}$ is confined to the current submanifold under the current generator of time evolution, or when the possibility of dynamics on $\mathbb{P}_L$ is exhausted. This process is called a constraint algorithm, and has been introduced often in the literature. While such an algorithm will also be presented here, its purpose is to show that the end result $\overline{X}_{EL}$ of the algorithm is once again a SOELVF, and a second-order problem is avoided. Later, it will also be used to show that both this $\overline{X}_{EL}$ and the Lagrangian constraints—whether first-order or introduced by the algorithm—are projectable.

To present the constraint algorithm we introduce the following notation used in conjunction with the constraint analysis

$$\bar{\mathbf{X}}_{EL}^{[1]} := \bar{\mathbf{X}}_{EL}, \quad \bar{\mathbf{X}}_L^{[1]} := \bar{\mathbf{X}}_L, \quad \mathbf{P}_{(n)}^{[1]} := \mathbf{P}_{(n)}, \quad u_{[1]}^m := u^m, \quad N_0^{[1]} := N_0. \quad (68)$$

As both $u_{[1]}^n, \gamma_n^{[1]} \in \bar{\mathcal{F}}, [\mathbf{P}_{(n)}^{[1]}] \gamma_n^{[1]} = \mathbf{P}_{(n)} \gamma_n^{[1]}$. The constraint condition equation (67) requires $\mathcal{L}_{\bar{\mathbf{X}}_E} \gamma_n^{[1]} = 0$, which, after using equation (66) for a general SOELVF, reduces to

$$\sum_{m=1}^{N_0} \Gamma_{nm}^{[1]} u_{[1]}^m = -\langle \mathbf{d}\gamma_n^{[1]} | \bar{\mathbf{X}}_L^{[1]} \rangle, \quad \text{with } \Gamma_{nm}^{[1]} := \langle \mathbf{d}\gamma_n^{[1]} | \mathbf{P}_{(m)}^{[1]} \rangle. \quad (69)$$

Then $r^{[1]} := \text{rank} \Gamma_{nm}^{[1]}$ of the $u_{[1]}^m$ is determined by equation (69), while $N_0^{[2]} := N_0^{[1]} - r^{[1]}$ are not. Moreover, $N_0^{[2]}$ second-order Lagrangian constraint functions

$$\gamma_{n_{[2]}}^{[2]} := \langle \mathbf{d}\gamma_{n_{[2]}}^{[1]} | \bar{\mathbf{X}}_L^{[1]} \rangle, \quad n_{[2]} = 1, \dots, N_0^{[2]}. \quad (70)$$

are introduced with the conditions $\gamma_{n_{[2]}}^{[2]} = 0$ imposed. In general there will be $I_{[2]} := \text{rank} \{ \mathbf{d}\gamma_{n_{[1]}}^{[1]}, \mathbf{d}\gamma_{n_{[2]}}^{[2]} \}$ independent functions in $C_L^{[2]} := C_L^{[1]} \cup \{ \gamma_{n_{[2]}}^{[2]} \mid n_{[2]} = 1, \dots, N_0^{[2]} \}$, and $\mathbb{P}_L^{[1]}$ is reduced to the second-order constraint submanifold,

$$\mathbb{P}_L^{[2]} := \{ \mathbf{u} \in \mathbb{P}_L^{[1]} \mid \gamma_{n_{[2]}}^{[2]} = 0, \quad n_{[2]} = 1, \dots, N_0^{[2]} \}, \quad (71)$$

where $\dim \mathbb{P}_L^{[2]} = 2D - I_{[2]}$. At this point, there are two possibilities. If $I_{[2]} = I_{[1]}$ or $I_{[2]} = 2D$, the iterative process stops, and no new Lagrangian constraints are introduced. If not, the process continues.

For the second step in the iterative process, we choose a basis $\{ [\mathbf{P}_{(n)}^{[2]}] \}$ for $\ker \bar{\Omega}_L(\mathbf{u})/\mathcal{G}$ and arbitrary functions $\{ u_{[2]}^m \}$ such that for $m = 1, \dots, N_0^{[2]}$, $u_{[2]}^m$ are linear combinations of $u_{[1]}^m$ that lie in the kernel $\Gamma_{nm}^{[1]}$. Then

$$\bar{\mathbf{X}}_{EL}^{[2]} = \bar{\mathbf{X}}_L^{[2]} + \sum_{m=1}^{N_0^{[2]}} u_{[2]}^m [\mathbf{P}_{(m)}^{[2]}], \quad (72)$$

with

$$\bar{\mathbf{X}}_L^{[2]} = \bar{\mathbf{X}}_L^{[1]} + \sum_{m=N_0^{[2]}+1}^{N_0^{[1]}} u_{[2]}^m [\mathbf{P}_{(m)}^{[2]}]. \quad (73)$$

Here, the functions $u_{[2]}^m$ for $m = N_0^{[2]} + 1, \dots, N_0^{[1]}$ have been determined through the constraint analysis of $\gamma_n^{[1]}$.

Because for any $\mathbf{G} \in \mathcal{G}$, $\mathbf{G} i_{\bar{\mathbf{X}}_E^{[1]}} \mathbf{d}\gamma_n^{[1]} = \mathcal{L}_{[\mathbf{G}, \bar{\mathbf{X}}_E^{[1]}]} \gamma_n^{[1]} = 0$ and $\mathbf{G} \Gamma_{nm}^{[1]} = \mathcal{L}_{[\mathbf{G}, \mathbf{P}_{(m)}^{[1]}]} \gamma_n^{[1]} = 0$, it follows that $\mathbf{G} u_{[1]}^m = 0$, as required. Similarly, $\mathbf{G} \gamma_n^{[2]} = \mathcal{L}_{[\mathbf{G}, \bar{\mathbf{X}}_{EL}]} \mathbf{d}\gamma_n^{[2]} = 0$. Clearly $\gamma_n^{[2]} \in \bar{\mathcal{F}}$ and we may require $u_{[2]}^m \in \bar{\mathcal{F}}$ as well. It then follows that $[\mathbf{P}_{(n)}^{[2]}] \gamma_m^{[2]} = \mathbf{P}_{(n)}^{[2]} \gamma_m^{[2]}$ and imposing equation (67) on $\gamma_n^{[2]}$, gives

$$\sum_{m=1}^{N_0^{[2]}} \Gamma_{nm}^{[2]} u_{[2]}^m = -\langle \mathbf{d}\gamma_n^{[2]} | \bar{\mathbf{X}}_L^{[2]} \rangle, \quad \text{where } \Gamma_{nm}^{[2]} := \langle \mathbf{d}\gamma_n^{[2]} | \mathbf{P}_{(m)}^{[2]} \rangle, \quad n = 1, \dots, N_0^{[2]}. \quad (74)$$

Then $r^{[2]} := \text{rank} \Gamma_{nm}^{[2]}$ of the remaining $u_{[2]}^m$ functions are determined, up to $N_0^{[3]} = N_0^{[2]} - r^{[2]}$ third-order Lagrangian constraint functions,

$$\gamma_{n_{[3]}}^{[3]} = \langle \mathbf{d}\gamma_{n_{[3]}}^{[2]} | \bar{\mathbf{X}}_L^{[2]} \rangle, \quad n_{[3]} = 1, \dots, N_0^{[3]}, \quad (75)$$

are introduced with the conditions $\gamma_{n_{[3]}}^{[3]} = 0$ imposed. With

$$I_{[3]} := \text{rank} \{ \mathbf{d}\gamma_{n_{[1]}}^{[1]}, \mathbf{d}\gamma_{n_{[2]}}^{[2]}, \mathbf{d}\gamma_{n_{[3]}}^{[3]} \}, \quad (76)$$

independent functions in $C_L^{[3]} := C_L^{[2]} \cup \{ \gamma_{n_{[3]}}^{[3]} \mid n_{[3]} = 1, \dots, N_0^{[3]} \}$, we now have the third-order constraint submanifold,

$$\mathbb{P}_L^{[3]} := \{ \mathbf{u} \in \mathbb{P}_L^{[2]} \mid \gamma_{n_{[3]}}^{[3]}(\mathbf{u}) = 0, \quad n_{[3]} = 1, \dots, N_0^{[3]} \}. \quad (77)$$

Once again, the process stops when $I_{[3]} = I_{[2]}$ or $I_{[3]} = 2D$. However, if $I_{[2]} < I_{[3]} < 2D$, the process continues until at the n_F -step either $I_{[n_F]} = I_{[n_F]-1}$ or $I_{[n_F]} = 2D$.

The end result of this algorithm is

1. A submanifold $\mathbb{P}_L^{[n_F]} \subset \mathbb{P}_L$ on which dynamics takes place.
2. A collection $C_L^{[n_F]} \subset \bar{\mathcal{F}}$ of constraint functions of order 1 to n_F .
3. A second-order, Euler–Lagrange vector field

$$\bar{\mathbf{X}}_{EL}^{[n_F]} = \bar{\mathbf{X}}_L^{[n_F]} + \sum_{m=1}^{N_0^{[n_F]}} u_{[n_F]}^m(\mathbf{u}) [\mathbf{P}_{(m)}^{[n_F]}], \quad (78)$$

with $N_0^{[n_F]}$ arbitrary functions $u_{[n_F]}^m(\mathbf{u}) \in \bar{\mathcal{F}}$ for $m = 1, \dots, N_0^{[n_F]}$, and

$$\mathbf{X}_L^{[n_F]} = \mathbf{X}_L^{[1]} + \sum_{m=N_0^{[n_F]}+1}^{N_0^{[1]}} u_{[n_F]}^m(u) [\mathbf{P}_{(m)}^{[n_F]}], \quad (79)$$

where the $N_0^{[1]} - N_0^{[n_F]}$ functions $u_{[n_F]}^m(u) \in \mathcal{F}$, $m = N_0^{[n_F]} + 1, \dots, N_0^{[1]}$, have been uniquely determined through the constraint algorithm.

Importantly, the end result of the constraint algorithm $\mathbf{X}_{EL}^{[n_F]}$ is still a SOELVF.

As with the first-order constraint manifold $\mathbb{P}_L^{[1]}$, we assume that $\mathbb{P}_L^{[n_F]}$ is non-empty. In addition, we assume that the rank of $\Gamma_{nm}^{[l]}$ is constant on $\mathbb{P}_L$ for each $l = 1, \dots, n_F$.

4. The passage to Hamiltonian mechanics

The question of whether and how dynamical structures on the Lagrangian phase space are equivalent to such structures on the Hamiltonian phase space has had a long history [8, 28] (see also [11, 16, 18, 19, 21, 37]). These analyses have focused solely on SOLVFs, and often make use of pullbacks of structures on the Hamiltonian phase space in the construction of such operators as the evolution operator K and the vector field operator R (see [11, 16, 18, 19, 21, 37]) which are used to determine the projectability of Lagrangian constraints and vector fields on $\mathbb{T}\mathbb{P}_L$, respectively. However, while primary Hamiltonian constraints play a central role in the construction of both operators, the existence of such constraints is presumed. Moreover, because of the reliance on primary constraints, a number of subtleties involving first- and second-class Hamiltonian constraints must be dealt with.

These subtleties and their conclusions, present for SOLVFs, are not present for SOELVFs. The approach used here focuses on the symmetry group Gr_{Sym} , and the geometric structures inherent to almost regular Lagrangians. The passage from Lagrangian to Hamiltonian mechanics follows naturally. Much of the content of this section is new.

4.1. Projectability of functions and vector fields on $\mathbb{P}_L$

The canonical phase space $\mathbb{P}_C := \mathbf{T}^*\mathbb{Q}$ has the cotangent bundle coordinates $\mathfrak{s} = (q, p)$ with $q \in \mathbb{Q}$ and $p \in \mathbf{T}_q^*\mathbb{Q}$. The fiber derivative is the map $\mathcal{L} : (q, v) \in \mathbb{P}_L \rightarrow (q, p = \partial L / \partial v) \in \mathbb{P}_C$. For regular Lagrangians, its action on a function $f(u) \in C^\infty$ on $\mathbb{P}_L$ gives the function $f_C(\mathfrak{s}) := (f \circ \mathcal{L}^{-1})(\mathfrak{s}) = f(\mathcal{L}^{-1}(\mathfrak{s})) = f(u)|_{\mathcal{L}(u)=\mathfrak{s}}$, on $\mathbb{P}_C$. The action of $\mathcal{L}$ on a vector field $\mathbf{X} \in \mathbf{T}_u\mathbb{P}_L$ is given by the pushforward map $\mathcal{L}_* : \mathbf{T}_u\mathbb{P}_L \rightarrow \mathbf{T}_{\mathcal{L}(u)}\mathbb{P}_C$, while its action on a one-form $\sigma \in \mathbf{T}_\mathfrak{s}^*\mathbb{P}_C$ is given by the pullback map $\mathcal{L}^* : \mathbf{T}_{\mathcal{L}(u)}^*\mathbb{P}_C \rightarrow \mathbf{T}_u^*\mathbb{P}_L$.

The situation changes for singular Lagrangians. While the pullback of one-forms simply involves replacing $\mathfrak{s}$ by $\mathcal{L}(u)$, the action of both $\mathcal{L}$ and $\mathcal{L}_*$ involve solving for u in $\mathfrak{s} = \mathcal{L}(u)$. For singular Lagrangians solutions of this equation are not single valued, but instead gives the preimage of $\mathcal{L}$,

$$\mathcal{L}^{-1}(\mathfrak{s}) := \{u \in \mathbb{P}_L \mid \mathcal{L}(u) = \mathfrak{s}\}, \quad (80)$$

which is a submanifold of $\mathbb{P}_L$. As such, the pullback of a function now results in the collection of functions

$$(f \circ \mathcal{L}^{-1})(\mathfrak{s}) = \{f(u) \mid u \in \mathcal{L}^{-1}(\mathfrak{s})\}, \quad (81)$$

while the pushforward $\mathcal{L}_*\mathbf{X}$ of $\mathbf{X}$ gives the collection of vectors

$$\mathcal{L}_*\mathbf{X}(\mathfrak{s}) = \{\mathbf{X}_C(u) \mid u \in \mathcal{L}^{-1}(\mathfrak{s})\}. \quad (82)$$

This ambiguity for equation (81) can be avoided by focusing on functions that are constant on $\mathcal{L}^{-1}(\mathfrak{s})$. Then $f(u) = f_C(\mathfrak{s}) \forall u \in \mathcal{L}^{-1}(\mathfrak{s})$, so that

$$(f \circ \mathcal{L}^{-1})(\mathfrak{s}) = \{f(u) \mid u \in \mathcal{L}^{-1}(\mathfrak{s})\} = f_C(\mathfrak{s}), \quad (83)$$

and is thus single valued. Following the literature, we say that a function f on $\mathbb{P}_L$ is projectable when equation (83) holds. It is well known [2, 4, 5, 20, 23, 30] that the condition for a function f to be projectable is

$$\mathbf{G}f(u) = 0, \quad (84)$$

for all $\mathbf{G} \in \mathcal{G}$.

For vector fields, the ambiguity equation (82) is avoided when the components of $\mathbf{X}_C$ are constant on the preimage. Then

$$\mathcal{L}_*\mathbf{X}(\mathfrak{s}) = \{\mathbf{X}_C(u) \mid u \in \mathcal{L}^{-1}(\mathfrak{s})\} = \mathbf{X}_C(\mathfrak{s}), \quad (85)$$

and we say a vector field on $\mathbf{T}_u\mathbb{P}_L$ is projectable when equation (85) holds. To determine which vectors are projectable, consider first the collection of vectors in $\mathbf{T}_u\mathbb{P}_L$ for which $\mathcal{G}$ is an ideal,

$$\overline{\mathbf{T}_u \mathbb{P}_L} := \{\mathbf{X} \in \mathbf{T}_u \mathbb{P}_L \mid [\mathbf{X}, \mathbf{G}] \in \mathcal{G} \ \forall \ \mathbf{G} \in \mathcal{G}\}, \quad (86)$$

Applying the same arguments using lemma 5 to $\overline{\mathbf{T}_u \mathbb{P}_L}$ as was applied to $\overline{\ker \Omega_L(u)}$ gives similar results: $\dim \overline{\mathbf{T}_u \mathbb{P}_L} = 2D$, and $\overline{\mathbf{T}_u \mathbb{P}_L}$ is involutive. The equivalence relation $\overline{\mathbf{X}}_1 \sim \overline{\mathbf{X}}_2$ iff $\overline{\mathbf{X}}_1 - \overline{\mathbf{X}}_2 \in \mathcal{G}$ then follows along with the quotient space $\overline{\mathbf{T}_u \mathbb{P}_L}/\mathcal{G}$.

Theorem 8. $\overline{\mathbf{T}_u \mathbb{P}_L}/\mathcal{G}$ is projectable.

Proof. Choose an open covering $\mathcal{U}$ of $\mathbb{P}_L$, and a point u in an open neighborhood $U_u \in \mathcal{U}$ such that $\mathbf{G}u^A = 0$, $A = 1, \dots, 2D$, for all $\mathbf{G} \in \mathcal{G}$. Choose also a $\mathbf{X} \in \overline{\mathbf{T}_u \mathbb{P}_L}/\mathcal{G}$, and consider the path $u_{\mathbf{X}}(t, u)$ given by

$$\frac{du_{\mathbf{X}}}{dt} = \mathbf{X}(u_{\mathbf{X}}), \quad (87)$$

with $u_{\mathbf{X}}(0, u) = u$. This open neighborhood can always be chosen small enough such that,

$$u_{\mathbf{X}}^A(t, u) = e^{t\mathbf{X}}u^A, \quad (88)$$

on U_u . Then as $\mathcal{G}$ is an ideal of $\overline{\mathbf{T}_u \mathbb{P}_L}$, $e^{-t\mathbf{X}}\mathbf{G}e^{t\mathbf{X}} \in \mathcal{G}$, and $\mathbf{G}u_{\mathbf{X}}^A(t, u) = 0$ in U_u . By applying equation (88) to a sequence of such open neighborhoods, we can extend this result to any connected region $\mathcal{R}$ of $\mathbb{P}_L$. Importantly, as the path $u_{\mathbf{X}}(t, u)$ is projectable on $\mathcal{R}$, there is the path $s_{\mathbf{X}}(t, \mathcal{L}(u)) = \mathcal{L}(u_{\mathbf{X}}(t, u))$ on $\mathbb{P}_C$ with tangent vector $\mathfrak{X}$ and initial data $s_{\mathbf{X}}(0) = \mathcal{L}(u)$. The integral flow $u_{\mathbf{X}}(t, u)$ is unique for a given $\mathbf{X}$ and u . Similarly, the integral flow $s_{\mathbf{X}}(t, \mathcal{L}(u))$ is unique for a given $\mathfrak{X}$ and initial data $s = \mathcal{L}(u)$. As the projection of $u_{\mathbf{X}}(t)$ to $s_{\mathbf{X}}(t, \mathcal{L}(u))$ is also unique, we conclude that each $\mathbf{X} \in \overline{\mathbf{T}_u \mathbb{P}_L}/\mathcal{G}$ is projectable with $\mathfrak{X} = \mathcal{L}_*\mathbf{X}$. ■

(A coordinate-based proof using equation (84) can also be given). The converse is also true, as we show in the next section.

4.2. Projection of dynamical structures

By construction, $\overline{\mathcal{F}}$ is projectable, and as both $u^m(u)$ and $\gamma \in \overline{\mathcal{F}}$ for any $\gamma \in C_L^{[n_f]}$, they also are projectable. In addition, $\mathbf{G}E = 0$, and E is projectable with its image $H_C = (E \circ \mathcal{L}^{-1})(s)$ being the canonical Hamiltonian. With the exception of the energy, we avoid introducing new notation, and will represent the projection of any function $f(u) \in \overline{\mathcal{F}}$ through its argument: $f(s)$.

Both $\overline{\text{Sol}}$ and $\overline{\ker \Omega_L(u)}$ are subsets of $\overline{\mathbf{T}_u \mathbb{P}_L}/\mathcal{G}$, and are projectable. Of particular interest are

$$\text{Prim} := \mathcal{L}_*(\overline{\ker \Omega_L(u)}) = \{\mathfrak{P} \in \mathbf{T}_s \mathbb{P}_C \mid \mathfrak{P} = \mathcal{L}_*[\mathbf{P}], \ \forall \ [\mathbf{P}] \in \overline{\ker \Omega_L(u)}/\mathcal{G}\}, \quad (89)$$

and

$$\text{Flow}_{H_T} := \mathcal{L}_*(\overline{\text{Sol}}) = \{\mathfrak{X}_{H_T} \in \mathbf{T}_{s=\mathcal{L}(u)} \mathbb{P}_C \mid \mathfrak{X}_{H_T} = \mathcal{L}_*\overline{\mathbf{X}_{EL}} \ \forall \ \overline{\mathbf{X}_{EL}} \in \overline{\text{Sol}}\}. \quad (90)$$

In particular, the general $\overline{\mathbf{X}_{EL}}$ in equation (66) gives the general vector field

$$\mathfrak{X}_{H_T} = \overline{X}_L^{qa}(s) \frac{\partial}{\partial q^a} + [\overline{X}_L^{qa} N_{ab}]|_s \frac{\partial}{\partial p_b} - \frac{\partial H_C}{\partial q^b} \frac{\partial}{\partial p_b} + \sum_{m=1}^{N_0} u^m(s) \mathfrak{P}_{(m)}, \quad (91)$$

in Flow_{H_T} when expressed in terms of local coordinates. Here,

$$N_{ab} = \frac{\partial^2 L}{\partial v^a \partial q^b}, \quad (92)$$

and $\mathfrak{P}_{(m)} = \mathcal{L}_*[\mathbf{P}_{(m)}]$ for a choice $\{[\mathbf{P}_{(m)}], m = 1, \dots, N_0\}$ of basis for $\overline{\ker \Omega_L(u)}/\mathcal{G}$.

With the canonical two-form $\omega = dq^a \wedge dp_a$ on $\mathbf{T}_s^* \mathbb{P}_C$, we have the collection of one-forms,

$$\text{Prim}^b := \{\pi \in \Lambda^1(\mathcal{L}(\mathbb{P}_L)) \mid \pi = \omega^b \mathfrak{P} \ \forall \ \mathfrak{P} \in \text{Prim}\}. \quad (93)$$

which gives the primary constraints, and

$$\text{Flow}_{H_T}^b := \{\alpha \in \Lambda^1(\mathcal{L}(\mathbb{P}_L)) \mid \alpha = \omega^b \mathfrak{X}_{H_T} \ \forall \ \mathfrak{X}_{H_T} \in \text{Flow}_{H_T}\}, \quad (94)$$

which gives the set of total Hamiltonians.

4.2.1. Prim and the Primary Hamiltonian Constraints

Using the kernel of the pullback map,

$$\ker \mathcal{L}^* := \{\phi \in \Lambda(\mathbb{P}_C) \mid \mathcal{L}^*\phi = 0\}, \quad (95)$$

in this section we construct from $\overline{\ker \Omega_L(u)}/\mathcal{G}$ the primary Hamiltonian constraints.

Lemma 9. For any one-form $\sigma \in \Lambda^1(\mathcal{L}(\mathbb{P}_L))$, $\sigma \in \ker \mathcal{L}^*$ iff $\sigma \in \text{Prim}^b$.

Proof. Suppose first that $\sigma \in \text{Prim}^b$. Then there exists a $[\mathbf{P}] \in \overline{\ker \Omega_L(u)}/\mathcal{G}$ such that $\sigma = i_{\mathcal{L}^*[\mathbf{P}]} \omega$. As $\mathcal{L}^* \omega = \Omega_L$, $\mathcal{L}^* \sigma = i_{\mathbf{P}} \Omega_L = 0$, and it follows that $\sigma \in \ker \mathcal{L}^*$.

Next suppose that $\sigma \in \ker \mathcal{L}^*$. Let $\mathfrak{X}$ be the unique vector in $\mathbf{T}_s \mathbb{P}_C$ such that $i_{\mathfrak{X}} \omega = \sigma$. Then $\mathcal{L}^*[i_{\mathfrak{X}} \omega] = 0$. But as both $i_{\mathfrak{X}} \omega$ and ω are differential forms, their pullbacks are well-defined and there must then be a $\mathbf{X} \in \mathbf{T}_u \mathbb{P}_L$ such that $\mathcal{L}_* \mathbf{X} = \mathfrak{X}$. It then follows that $i_{\mathbf{X}} \Omega_L = 0$, and thus $\sigma \in \text{Prim}^b$. ■

(A coordinate-based proof of this lemma can also be given).

Consider now the Pfaff system of exterior equations,

$$Pf(\text{Prim}^b) := \{\pi_{(n)} = 0, n = 1, \dots, N_0\}. \quad (96)$$

and the integral manifold $(\mathbb{P}_L, \mathcal{L})$ of $Pf(\text{Prim}^b)$ [49]. As $N_0 = \dim \overline{(\ker \Omega_L(u) \mathcal{G})} = \dim \text{Prim} = \dim \text{Prim}^b$, $\text{rank} Pf(\text{Prim}^b) = N_0$. Of particular interest is the ideal [49] of $Pf(\text{Prim}^b)$

$$I[Pf(\text{Prim}^b)] := \left\{ \sum_{n=1}^{N_0} \xi^n \wedge \pi_{(n)} \mid \xi^n \in \Lambda(\mathbb{P}_C), \pi_{(n)} \in Pf(\text{Prim}^b) \right\}. \quad (97)$$

Lemma 10. $I[Pf(\text{Prim}^b)] = \ker \mathcal{L}^*$.

Proof. If $\sigma \in I[Pf(\text{Prim}^b)]$, then

$$\sigma = \sum_{n=1}^{N_0} \xi^n \wedge \pi_{(n)}. \quad (98)$$

From lemma 9,

$$\mathcal{L}^* \sigma = \sum_{n=1}^{N_0} \mathcal{L}^* \xi^n \wedge \mathcal{L}^* \pi_{(n)} = 0, \quad (99)$$

so that $I[Pf(\text{Prim}^b)] \subseteq \ker \mathcal{L}^*$.

Next, choose a basis $\theta_{(n)}$, $n = 1, \dots, 2D$ of $\Lambda^1(\mathcal{L}(\mathbb{P}_L))$ such that $\theta_{(n)} = \pi_{(n)}$ for $n = 1, \dots, N_0$. Let $\sigma \in \ker \mathcal{L}^*$ be the p -form,

$$\sigma(\mathfrak{s}) := \frac{1}{p!} \sum_{n_1, \dots, n_p=1}^D \sigma_{n_1 \dots n_p}(\mathfrak{s}) \theta_{(n_1)} \wedge \dots \wedge \theta_{(n_p)}. \quad (100)$$

Then as $\mathcal{L}^* \sigma = 0$,

$$0 = \frac{1}{p!} \sum_{n_1, \dots, n_p=1}^D \sigma_{n_1 \dots n_p}(\mathcal{L}(u)) \mathcal{L}^* \theta_{(n_1)} \wedge \dots \wedge \mathcal{L}^* \theta_{(n_p)}, \quad (101)$$

and from lemma 9 we conclude that $\sigma_{n_1 \dots n_p}(\mathcal{L}(u)) = 0$ for $n_s > N_0$, $s = 1, \dots, p$. Thus there exists forms $\xi^{(n)}$ such that

$$\sigma = \sum_{n=1}^{N_0} \xi^{(n)} \wedge \pi_{(n)}, \quad (102)$$

so that $\ker \mathcal{L}^* \subseteq I[Pf(\text{Prim}^b)]$ as well. ■

The construction of the primary Hamiltonian constraints is now trivial.

Consider a $\pi \in Pf(\text{Prim}^b)$. As $\mathcal{L}^* d\pi = d\mathcal{L}^* \pi = 0$, $d\pi \in \ker \mathcal{L}_*$, and from lemma 10, $d\pi \in I[Pf(\text{Prim}^b)]$. There are then one-forms $\xi^{(n)}$, $n = 1, \dots, N_0$ such that

$$d\pi = \sum_{n=1}^{N_0} \xi^{(n)} \wedge \pi_{(n)}. \quad (103)$$

Then $d\pi \wedge \pi_{(1)} \wedge \dots \wedge \pi_{(N_0)} = 0$, and thus $Pf(\text{Prim}^b)$ is closed [49]. It follows from the Frobenius theorem that $Pf(\text{Prim}^b)$ is completely integrable. There are then N_0 first integrals $\gamma_n^{[0]}$ of $Pf(\text{Prim}^b)$ such that in a neighborhood about each generic point $u \in \mathbb{P}_C$, $\{\pi_{(n)} = 0\} \sim \{d\gamma_n^{[0]} = 0\}$; these forms may be chosen such that $\pi_{(n)} = f_n(\mathfrak{s}) d\gamma_n^{[0]}$ where f_n is a C^∞ function on $\mathbb{P}_C$. The functions $\gamma_n^{[0]}$ are the primary Hamiltonian constraints while

$$\mathbb{P}_C^{[0]} := \{\mathfrak{s} \in \mathbb{P}_C \mid \gamma_n^{[0]}(\mathfrak{s}) = 0, n = 1, \dots, N_0\}, \quad (104)$$

is the primary constraint submanifold. Connections between the primary constraints and vectors in $\ker \Omega_L(u)$ have been found previously by using the time-evolution operator K [23]. Such analyses make use of pullbacks of the primary Hamiltonian constraints, however, while the approach here is constructive.

Lemma 11. $\mathcal{L}_*(\overline{\mathbf{T}_u \mathbb{P}_L}/\mathcal{G}) = \mathbf{T}_{\mathfrak{s}=\mathcal{L}(u)} \mathbb{P}_C^{[0]}$.

Proof. Let $[\overline{\mathbf{X}}] \in \overline{\mathbf{T}_u \mathbb{P}_L}/\mathcal{G}$. As $[\overline{\mathbf{X}}]$ is projectable, $\langle d\gamma_n^{[0]} | \mathcal{L}_*[\overline{\mathbf{X}}] \rangle = \langle \mathcal{L}^* d\gamma_n^{[0]} | [\overline{\mathbf{X}}] \rangle = 0$ since $d\gamma_n^{[0]} \in \ker \mathcal{L}^*$, and it follows that $\mathcal{L}_*(\overline{\mathbf{T}_u \mathbb{P}_L}/\mathcal{G}) \subseteq \mathbf{T}_{\mathfrak{s}=\mathcal{L}(u)} \mathbb{P}_C^{[0]}$. But as $\dim \overline{\mathbf{T}_u \mathbb{P}_L}/\mathcal{G} = 2D - N_0 = \dim \mathbf{T}_{\mathfrak{s}=\mathcal{L}(u)} \mathbb{P}_C^{[0]}$, $\mathcal{L}_*(\overline{\mathbf{T}_u \mathbb{P}_L}/\mathcal{G}) = \mathbf{T}_{\mathfrak{s}=\mathcal{L}(u)} \mathbb{P}_C^{[0]}$ follows. ■

The converse of theorem 8 then follows. Importantly, because $\mathcal{L}_*(\overline{\mathbf{Sol}}) \subset \mathbf{T}_{\mathfrak{s}=\mathcal{L}(u)} \mathbb{P}_C^{[0]}$, the integral flow fields of SOELVFs lie on $\mathbb{P}_C^{[0]}$.

4.2.2. $\overline{\mathbf{Sol}}$ and the total Hamiltonian

On $\mathbb{P}_L^{[1]}$, $\beta[\mathbf{X}_E] = 0$, and the energy equation may be written as $0 = dE - i_{\overline{\mathbf{X}}_L} \mathcal{L}^* \omega$. It follows that

$$i_{\mathcal{L}_* \overline{\mathbf{X}}_L} \omega = dH_C, \quad (105)$$

from which we conclude that if $\mathfrak{X}_C$ is the Hamiltonian flow field for H_C , then $\mathfrak{X}_C = \mathcal{L}_* \overline{\mathbf{X}}_L$. The image of the pushforward of equation (66) gives the vector field $\mathfrak{X}_{H_T} := \mathcal{L}_* \overline{\mathbf{X}}_{EL} \in \text{Flow}_{H_T}$,

$$\mathfrak{X}_{H_T} = \mathfrak{X}_C + \sum_{m=1}^{N_0} u^m(\mathfrak{s}) \mathfrak{P}_{(m)}, \quad (106)$$

that is everywhere tangent to $\mathbb{P}_C^{[0]}$. Correspondingly, a general one form in $\text{Flow}_{H_T}^b$ is

$$i_{\mathfrak{X}_{H_T}} \omega = dH_C + \sum_{m=1}^{N_0} u^m(\mathfrak{s}) f_m(\mathfrak{s}) d\gamma_m^{[0]}, \quad (107)$$

which gives the total Hamiltonian,

$$H_T = H_C + \sum_{m=1}^{N_0} u^m(\mathfrak{s}) f_m(\mathfrak{s}) \gamma_m^{[0]}, \quad (108)$$

for the dynamical system. This leads to the sequence of maps:

$$\overline{\mathbf{Sol}} \xrightarrow{\mathcal{L}_*} \text{Flow}_{H_T} \xrightarrow{\omega^b} \text{Flow}_{H_T}^b \xrightarrow{\mathcal{L}^*} E, \quad (109)$$

and to each $\overline{\mathbf{X}}_{EL} \in \overline{\mathbf{Sol}}$ there is a corresponding total Hamiltonian $H_T \in \text{Flow}_{H_T}^b$.

4.2.3. The equivalence of the constraint algorithm for lagrangians and the stability analysis of canonical Hamiltonians

It is well known that the integral flow generated by $\mathfrak{X}_{H_T}$ need not be confined to $\mathbb{P}_C^{[0]}$ even though its initial data is chosen to be on this submanifold. This difficulty is resolved through a stability analysis [25] where $\{H_T, \gamma_n^{[0]}\} = 0$ is imposed on the primary constraints, and when necessary, successively on the secondary, tertiary, and higher-level Hamiltonian constraints. While this process is traditionally applied to the canonical Hamiltonian, section 3.7 describes a constraint algorithm for SOELVFs. We show here that this constraint algorithm is equivalent to the stability analysis of the canonical Hamiltonian.

Choose a $\overline{\mathbf{X}}_{EL}^{[1]} \in \overline{\mathbf{Sol}}$, where we follow the notation established in equation (68). There is then a corresponding $\mathfrak{X}_{H_T}^{[1]} = \mathcal{L}_* \overline{\mathbf{X}}_{EL}^{[1]}$, and total Hamiltonian $H_T^{[1]}$. The stability analysis of the primary constraints under $H_T^{[1]}$ then results in

$$f_n \frac{d\gamma_n^{[0]}}{dt} = \langle \pi_{(n)} | \mathfrak{X}_C \rangle + \sum_{m=1}^{N_0} u^m \langle \pi_{(n)} | \mathfrak{P}_{(m)} \rangle, \quad (110)$$

after using equation (106). But $\langle \pi_{(n)} | \mathfrak{P}_{(m)} \rangle = \langle \omega | \mathcal{L}_*(\mathbf{P}_{(n)}) \otimes \mathcal{L}_*(\mathbf{P}_{(m)}) \rangle = \langle \Omega_L | \mathbf{P}_{(n)} \otimes \mathbf{P}_{(m)} \rangle = 0$, while $\langle \pi_{(n)} | \mathfrak{X}_C \rangle = \langle \omega | \mathcal{L}_*(\mathbf{P}_{(n)}) \otimes \mathcal{L}_*(\overline{\mathbf{X}}_L) \rangle = \langle \Omega_L | \mathbf{P}_{(n)} \otimes \overline{\mathbf{X}}_L \rangle = -\langle dE | \mathbf{P}_{(n)} \rangle$ since we are on the $\beta[\mathbf{X}_E] = 0$ surface. As $\langle dE | \mathbf{P}_{(n)} \rangle = \gamma_n^{[1]}$,

$$f_n \frac{d\gamma_n^{[0]}}{dt} = -\gamma_n^{[1]}(\mathfrak{s}). \quad (111)$$

The projection of first-order constraints then automatically gives the secondary Hamiltonian constraints. It follows that $\{H_T, \gamma_n^{[0]}\} = 0$ is automatically satisfied through the Lagrangian constraint condition $\gamma_n^{[1]}(u) = 0$.

The stability analysis must now be applied to the secondary constraints: $\mathcal{L}_{\mathfrak{X}_{H_T}^{[1]}\gamma_n^{[1]}}(\mathfrak{s}) = 0$. But as $\mathfrak{X}_{H_T}^{[1]} = \mathfrak{L}_*\overline{\mathfrak{X}}_{EL}^{[1]}$, this requirement is equivalent to imposing the constraint condition: $\mathcal{L}_{\overline{\mathfrak{X}}_{EL}^{[1]}\gamma_n^{[1]}}(\mathfrak{u}) = 0$. From section 3.7 doing so results in the SOELVF $\overline{\mathfrak{X}}_{EL}^{[2]}$, and thus gives a corresponding Hamiltonian flow field $\mathfrak{X}_{H_T}^{[2]}$ and total Hamiltonian $H_T^{[2]}$. If second-order Lagrangian constraints are introduced at this step, their projection will give the tertiary Hamiltonian constraints.

This procession continues with the stability analysis of the n th-level Hamiltonian constraints giving a $\overline{\mathfrak{X}}_{EL}^{[n]}$, and thus a corresponding Hamiltonian flow field $\mathfrak{X}_{H_T}^{[n]} = \mathfrak{L}_*\overline{\mathfrak{X}}_{EL}^{[n]}$ and total Hamiltonian $H_T^{[n]}$. If $(n+1)^{th}$ -level Hamiltonian constraints are introduced, they are the projection of the n^{th} -order Lagrangian constraints. The analysis stops when the Lagrangian constraint algorithm ends: at the n_F -step. The end result $\overline{\mathfrak{X}}_{EL}^{[n_F]}$ of the constraint algorithm gives a $\mathfrak{X}_{H_T}^{[n_F]}$ with integral flows that lie on the Hamiltonian constraint submanifolds. Correspondingly, there is a $H_T^{[n_F]}$ that agrees with the end result of the stability analysis of the total Hamiltonian. The Lagrangian constraint algorithm applied to $\overline{\mathfrak{X}}_{EL}$ is thus equivalent to the stability analysis of the canonical Hamiltonian.

5. Examples of almost regular Lagrangians

In this section we present three examples of dynamical systems with almost regular Lagrangians. The first example describes a single particle interacting with an external potential. It illustrates the role $\mathcal{G}$ plays, and the tight relationship between Gr_{Sym} , the symmetries of the Euler–Lagrange equations of motion, and the gauge symmetries of the Lagrangian. Moreover, it explicitly shows that $\mathcal{G}$ is not the generator of the local gauge symmetry, as is sometimes asserted in the literature. The second example consists of two interacting particles with a Lagrangian that has a local conformal symmetry. It is an example of a dynamical system for which only a subset of vectors in $\ker \overline{\Omega}_L(\mathfrak{u})$ generate the symmetry group. The third example consists of a particle with both local conformal symmetry and time-reparametrization invariance. It is an example of a fully constrained dynamical system—as such, $\mathcal{S}ol = \ker \overline{\Omega}_L(\mathfrak{u})$ —that has two gauge symmetries. The analysis of all three systems are done using the techniques and tools presented above.

5.1. A Lagrangian With and Without a Local Gauge Symmetry

Whether the action

$$S_1 := \int \left[\frac{1}{2} m \left(\frac{d\hat{q}}{dt} \right)^2 - V(q^a) \right] dt, \quad (112)$$

with $|q| = \sqrt{q^a q_a}$ and $\hat{q}^a := q^a/|q|$, $a = 1, \dots, D$, has a local gauge symmetry depends on the choice of potential $V(q)$. With one choice both the Lagrangian and the equations of motion have a local gauge symmetry; with another choice the equations of motion has a symmetry while the Lagrangian does not have a local gauge symmetry; and with a third choice, neither has a symmetry. Interestingly, L is singular irrespective the choice of $V(q)$, showing that not all singular Lagrangians need have a symmetry.

With $\Pi_{ab}(q) := \delta_{ab} - \hat{q}_a \hat{q}_b$,

$$\Omega_M = \frac{m}{|q|^2} \Pi_{ab}(q) dq^a \wedge dv^b, \quad \Omega_F = \frac{m}{|q|^3} (\hat{q} \cdot dq) \wedge (v \cdot \Pi(q) \cdot dq), \quad (113)$$

and $\mathcal{C}$ and $\mathcal{G}$ are spanned by $\mathbf{U}_{(1)}^q = \hat{q} \cdot \partial/\partial q$ and $\mathbf{U}_{(1)}^v = \hat{q} \cdot \partial/\partial v$, respectively, while $\overline{\ker \Omega_L(\mathfrak{u})}$ is spanned by $\mathbf{U}_{(1)}^q$ and

$$P_{(1)} = \hat{q} \cdot \frac{\partial}{\partial q} + \frac{1}{|q|} v \cdot \frac{\partial}{\partial v}. \quad (114)$$

The energy is

$$E = \frac{1}{2} \frac{m}{|q|^2} v \cdot \Pi(q) \cdot v + V(q), \quad (115)$$

and there is only one first-order, Lagrangian constraint,

$$\gamma^{[1]} = \mathbf{U}_{(1)}^q V, \quad (116)$$

with $\beta[\mathbf{X}_E] = \gamma^{[1]} \Theta_q^{(1)}$, $\Theta_q^{(1)} = \hat{q} \cdot dq$. As expected, $\mathcal{L}_G \gamma^{[1]} = 0$.

We may choose

$$\mathbf{X}_L = v \cdot \frac{\partial}{\partial q} + 2 \frac{(\hat{q} \cdot v)}{|q|} v \cdot \frac{\partial}{\partial v} - \frac{|q|^2}{m} \frac{\partial V}{\partial q} \cdot \frac{\partial}{\partial v}. \quad (117)$$

As $[\mathbf{X}_L, \mathbf{U}_{(1)}^\nu] \sim -\mathbf{P}_{(1)}$, a symmetry transformation of $\mathbf{X}_L$ does not result in a SOLVF. Instead,

$$\bar{\mathbf{X}}_L = \nu \cdot \Pi(q) \cdot \frac{\partial}{\partial q} + \frac{(\hat{q} \cdot \nu)}{|q|} \nu \cdot \frac{\partial}{\partial \nu} - \frac{|q|^2}{m} \frac{\partial V}{\partial q} \cdot \Pi(q) \cdot \frac{\partial}{\partial \nu}, \quad (118)$$

is constructed, and a general SOELVF is $\bar{\mathbf{X}}_{EL} = \bar{\mathbf{X}}_L + u(u)[\mathbf{P}_{(1)}]$, where $u(u) \in \mathcal{F}$. Because

$$\mathcal{L}_{\mathbf{P}_{(1)}} \beta = \mathbf{d}[\mathbf{U}_{(1)}^q V] - \frac{1}{|q|^2} \hat{q} \cdot \frac{\partial}{\partial q} \left(\Pi_a^b(q) \frac{\partial V}{\partial \hat{q}^b} \right) \mathbf{d}q^a, \quad (119)$$

whether or not Sym is empty depends on the symmetries of $V(q)$. As the constraint algorithm gives

$$\mathcal{L}_{\bar{\mathbf{X}}_{EL}} \gamma^{[1]} = \nu \cdot \Pi \cdot \frac{\partial \gamma^{[1]}}{\partial q} + u(u) \mathbf{U}_{(1)}^q \gamma^{[1]}, \quad (120)$$

whether or not $u(u)$ is determined also depends on the symmetries of $V(q)$. There are three possibilities, none of which will require the introduction of higher-order Lagrangian constraints.

5.1.1. The symmetric potential

For $\mathbf{P}_{(1)}$ to generate a symmetry,

$$0 = \frac{1}{|q|^2} \hat{q} \cdot \frac{\partial}{\partial q} \left(\Pi_a^b(q) \frac{\partial V}{\partial \hat{q}^b} \right), \quad (121)$$

and it follows that

$$\frac{\partial V}{\partial \hat{q}^a} = \frac{\partial V_{AS}(\hat{q}^a)}{\partial \hat{q}^a}, \quad (122)$$

where V_{AS} is a function of $\hat{q}^a$ only. Then $\mathbf{P}_{(1)}$ generates a symmetry iff $V(q, \hat{q}^a) = V_{Sph}(q) + V_{AS}(\hat{q}^a)$, where V_{Sph} is a function of q only. The group Sym is one-dimensional, and spanned by $\mathbf{P}_{(1)}$.

The constraint condition equation (120) for this potential reduces to

$$0 = u(u) \frac{d^2 V_{Sph}(q)}{d|q|^2}, \quad (123)$$

which must be satisfied on $\mathbb{P}_L$. There are two cases:

Case 1: $\frac{d^2 V_{Sph}}{d|q|^2} = 0$.

Then $V_{Sph} = aq + b$, but since

$$\gamma^{[1]} = \frac{dV_{Sph}}{d|q|} = a, \quad (124)$$

the condition $\gamma^{[1]} = 0$ requires $a = 0$. As we may choose $b = 0$, $V(q) = V_{AS}(\hat{q}^a)$ only. The Lagrangian is invariant under the local conformal transformation $q^a \rightarrow \alpha q^a$, where α is an arbitrary, nonvanishing function on $\mathbb{P}_L$. The function $u(u)$ is not determined, and correspondingly, the dynamics of the particle is determined only up to an arbitrary function.

Case 2: $\frac{d^2 V_{Sph}}{d|q|^2} \neq 0$.

In this case $u(u) = 0$, and the dynamics of the particle is completely determined by its initial data. The Lagrangian does not have a local gauge symmetry. The first-order, Lagrangian constraint $\gamma^{[1]} = 0$ defines a surface on $\mathbb{P}_L$, and for dynamics to be possible the set of solutions

$$\left\{ R_i \in \mathbb{R} \setminus \frac{dV_{Sph}}{d|q|} \Big|_{R_i} = 0 \right\}, \quad (125)$$

must be non-empty. Dynamics are on the surfaces $|q| - R_i = 0$, and on them the potential reduces to $V(q) = V_{Sph}(R_i) + V_{AS}(\hat{q}^a)$. This reduced potential has the same symmetry as the potential $V_{AS}(\hat{q}^a)$ in *Case 1*, leading to equations of motion that have the same generalized Lie symmetry. The Lagrangian for the two cases, however, do not have the same invariances, resulting in one case to dynamics that are determined up to an arbitrary $u(u)$ while in the other case to $u(u) = 0$ and dynamics that are completely determined by the choice of initial data.

A specific example of this type of potential is the Mexican hat potential: $V(q) = -\lambda|q|^2/2 + \beta|q|^4/4$. Then

$$\gamma^{[1]} = -\lambda|q| + \beta|q|^3. \quad (126)$$

As $|q| \neq 0$, dynamics are thus on the surface $|q| = (\beta/\lambda)^{1/2}$ for $\beta/\lambda > 0$. This breaks the local conformal symmetry while preserving rotational symmetry.

The asymmetric potential

For a general V , the second term in equation (119) does not vanish, $\mathbf{P}_{(1)}$ does not generate a symmetry of the equations of motion, $\text{Sym} = \{\emptyset\}$, and equation (120) gives

$$u = -\frac{\nu \cdot \Pi \cdot \frac{\partial \gamma^{[1]}}{\partial q}}{\mathbf{U}_{(1)}^q \gamma^{[1]}}. \quad (127)$$

The dynamics of the particle is uniquely determined by its initial data.

The passage to Hamiltonian mechanics is straightforward. With the canonical momentum, $p_a = m\Pi_{ab}(q)v^b/q^2$, equation (115) gives $H_C = q^2 p^2 / 2m + V(q)$, while $\gamma^{[1]}$ does not change under $\mathcal{L}$. The projection of $\mathbf{P}_{(1)}$ is

$$\mathfrak{P}_{(1)} = \frac{1}{|q|} \left(q \cdot \frac{\partial}{\partial q} - p \cdot \frac{\partial}{\partial p} \right), \quad (128)$$

giving $\pi = \mathbf{d}(q \cdot p)/|q|$, and the primary constraint $\gamma^{[0]} = q \cdot p$.

The projection of $\bar{X}_L$ gives the Hamiltonian flow

$$\mathfrak{X}_C = \frac{|q|^2}{m} p \cdot \frac{\partial}{\partial q} - \frac{|q|p^2}{m} \hat{q} \cdot \frac{\partial}{\partial p} - \frac{\partial V}{\partial q} \cdot \Pi \cdot \frac{\partial}{\partial p}, \quad (129)$$

and the total Hamiltonian $H_T = H_C + u\gamma^{[0]}(\mathfrak{s})$. The projection of equation (120) is

$$0 = \frac{|q|^2}{m} p \cdot \frac{\partial \gamma^{[1]}}{\partial q} + u \mathbf{U}_{(1)}^q \gamma^{[1]}. \quad (130)$$

For each of the three possible choices of $V(q)$ outlined above the total Hamiltonian obtained here agrees with the one obtained using constrained Hamiltonian mechanics.

5.2. A Lagrangian with local conformal symmetry

The action,

$$S_2 := \int \left\{ \frac{1}{2} m \left(\frac{d\hat{q}_1}{dt} \right)^2 + \frac{1}{2} m \left(\frac{d\hat{q}_2}{dt} \right)^2 + \frac{\lambda}{2} \left[\frac{q_1^a}{q_2} \frac{d}{dt} \left(\frac{q_{2a}}{|q_1|} \right) - \frac{q_2^a}{q_1} \frac{d}{dt} \left(\frac{q_{1a}}{|q_2|} \right) \right] \right\} dt, \quad (131)$$

where $a = 1, \dots, d$, $D = 2d$, describes an interacting, two particle system that is invariant under the local conformal transformation $q_1^a \rightarrow \alpha(u)q_1^a$ and $q_2^a \rightarrow \alpha(u)q_2^a$.

With

$$\begin{aligned} \Omega_M &= \frac{m}{|q_1|^2} \Pi_{ab}(q_1) \mathbf{d}q_1^a \wedge \mathbf{d}v_1^b + \frac{m}{|q_2|^2} \Pi_{ab}(q_2) \mathbf{d}q_2^a \wedge \mathbf{d}v_2^b, \text{ and} \\ \Omega_F &= \frac{m}{|q_1|^3} (\hat{q}_1 \cdot \mathbf{d}q_1) \wedge (\nu_1 \cdot \Pi(q_1) \cdot \mathbf{d}q_1) + \frac{m}{|q_2|^3} (\hat{q}_2 \cdot \mathbf{d}q_2) \wedge (\nu_2 \cdot \Pi(q_2) \cdot \mathbf{d}q_2) \\ &\quad - \frac{\lambda}{|q_1||q_2|} [\mathbf{d}q_1^a \wedge (\Pi(q_2) \cdot \mathbf{d}q_2)_a + (\Pi(q_1) \cdot \mathbf{d}q_1)_a \wedge \mathbf{d}q_2^a - (\Pi(q_1) \cdot \mathbf{d}q_1)^a \wedge (\Pi(q_2) \cdot \mathbf{d}q_2)_a] \\ &\quad - \frac{\lambda}{|q_1|^2} (\hat{q}_1 \cdot \mathbf{d}q_1) \wedge (\hat{q}_2 \cdot \Pi(q_1) \cdot \mathbf{d}q_1) + \frac{\lambda}{|q_2|^2} (\hat{q}_2 \cdot \mathbf{d}q_2) \wedge (\hat{q}_1 \cdot \Pi(q_2) \cdot \mathbf{d}q_2), \end{aligned} \quad (132)$$

$\mathcal{C}$ and $\mathcal{G}$ are two-dimensional and are spanned by

$$\mathbf{U}_{(1)}^q = \hat{q}_1 \cdot \frac{\partial}{\partial q_1}, \quad \mathbf{U}_{(2)}^q = \hat{q}_2 \cdot \frac{\partial}{\partial q_2}, \quad \text{and} \quad \mathbf{U}_{(1)}^\nu = \hat{q}_1 \cdot \frac{\partial}{\partial \nu_1}, \quad \mathbf{U}_{(2)}^\nu = \hat{q}_2 \cdot \frac{\partial}{\partial \nu_2}, \quad (133)$$

respectively. The reduced $\bar{F} = 0$, and $\overline{\ker \Omega_L(u)}$ is spanned by $\mathbf{U}_{(1)}^\nu$, $\mathbf{U}_{(2)}^\nu$, and

$$\begin{aligned} \mathbf{P}_{(1,2)} &= \hat{q}_{1,2} \cdot \frac{\partial}{\partial q_{1,2}} + \frac{\nu_{1,2}}{|q_{1,2}|} \cdot \Pi(q_{1,2}) \cdot \frac{\partial}{\partial \nu_{1,2}} \\ &\quad + (-1)^{1,2} \frac{\lambda}{m} \left[\hat{q}_{2,1} \cdot \Pi(q_{1,2}) \cdot \frac{\partial}{\partial \nu_{1,2}} + \frac{|q_{2,1}|}{|q_{1,2}|} \hat{q}_{1,2} \cdot \Pi(q_{2,1}) \cdot \frac{\partial}{\partial \nu_{2,1}} \right], \end{aligned} \quad (134)$$

The energy is

$$E = \frac{1}{2} \frac{m}{|q_1|^2} \nu_1 \cdot \Pi(q_1) \cdot \nu_1 + \frac{1}{2} \frac{m}{|q_2|^2} \nu_2 \cdot \Pi(q_2) \cdot \nu_2. \quad (135)$$

Although $\mathcal{C}$ is two-dimensional, $\gamma_{(1)}^{[1]} = -\lambda\gamma^{[1]}/|q_1|$ and $\gamma_{(2)}^{[1]} = \lambda\gamma^{[1]}/|q_2|$, and the two first-order Lagrangian constraints reduce to one

$$\gamma^{[1]} = \hat{q}_2 \cdot \Pi(q_1) \cdot \frac{v_1}{|q_1|} + \hat{q}_1 \cdot \Pi(q_2) \cdot \frac{v_2}{|q_2|}, \quad (136)$$

with

$$\beta[\mathbf{X}_E] = -\lambda\gamma^{[1]} \left(\frac{\Theta_q^{(1)}}{|q_1|} - \frac{\Theta_q^{(2)}}{|q_2|} \right). \quad (137)$$

As expected, $\mathbf{G}\gamma^{[1]} = 0$ for any $\mathbf{G} \in \mathcal{G}$. We may choose

$$\begin{aligned} \mathbf{X}_L = & v_1 \cdot \frac{\partial}{\partial q_1} + v_1 \cdot \frac{\partial}{\partial q_1} + \left[2 \frac{(\hat{q}_1 \cdot v_1)}{|q_1|} v_1 + \frac{\lambda}{m} \left(\frac{|q_1|}{|q_2|} v_2 - (\hat{q}_1 \cdot v_1) \hat{q}_2 \right) \right] \cdot \frac{\partial}{\partial v_1} \\ & + \left[2 \frac{(\hat{q}_2 \cdot v_2)}{|q_2|} v_2 - \frac{\lambda}{m} \left(\frac{|q_2|}{|q_1|} v_1 - (\hat{q}_2 \cdot v_2) \hat{q}_1 \right) \right] \cdot \frac{\partial}{\partial v_2}. \end{aligned} \quad (138)$$

As $[\mathbf{X}_L, \mathbf{U}_{(1,2)}^\vee] \sim -\mathbf{P}_{(1,2)}/q_1$, the action on $\mathbf{X}_L$ by Gr_{Sym} does not give a SOLVE. Instead, we construct

$$\begin{aligned} \bar{\mathbf{X}}_L = & v_1 \cdot \Pi(q_1) \cdot \frac{\partial}{\partial q_1} + v_2 \cdot \Pi(q_2) \cdot \frac{\partial}{\partial q_2} \\ & + \left(\frac{\hat{q}_1 \cdot v_1}{|q_1|} \right) v_1 \cdot \Pi(q_1) \cdot \frac{\partial}{\partial v_1} + \left(\frac{\hat{q}_2 \cdot v_2}{|q_2|} \right) v_2 \cdot \Pi(q_2) \cdot \frac{\partial}{\partial v_2} \\ & + \frac{\lambda}{m} \left(\frac{|q_1|}{|q_2|} v_2 \cdot \Pi(q_2) \cdot \Pi(q_1) \cdot \frac{\partial}{\partial v_1} - \frac{|q_2|}{|q_1|} v_1 \cdot \Pi(q_1) \cdot \Pi(q_2) \cdot \frac{\partial}{\partial v_2} \right). \end{aligned} \quad (139)$$

A general SOELVF is then $\bar{\mathbf{X}}_{EL} = \bar{\mathbf{X}}_L + u^{(-)}(\mathbf{u})[\mathbf{P}_{(-)}] + u^{(+)}(\mathbf{u})[\mathbf{P}_{(+)}]$, where $u^{(\pm)}(\mathbf{u}) \in \bar{\mathcal{F}}$ and $\mathbf{P}_{(\pm)} = |q_1|\mathbf{P}_{(1)} \pm |q_2|\mathbf{P}_{(2)}$. One of the arbitrary functions

$$u^{(-)}(\mathbf{u}) = \frac{m}{4\lambda} \frac{i\bar{\mathbf{X}}_L \mathbf{d}\gamma^{[1]}}{[1 - (\hat{q}_1 \cdot \hat{q}_2)]}, \quad (140)$$

is determined through the constraint algorithm with

$$i\bar{\mathbf{X}}_L \mathbf{d}\gamma^{[1]} = -2(\hat{q}_1 \cdot \hat{q}_2) \frac{E}{m} + \frac{2}{|q_1||q_2|} v_1 \cdot \Pi(q_1) \cdot \Pi(q_2) \cdot v_2 - \frac{\lambda}{m} (\hat{q}_1 \cdot \hat{q}_2) [v_2 \cdot \Pi(q_2) \cdot \hat{q}_1 - v_1 \cdot \Pi(q_1) \cdot \hat{q}_2]. \quad (141)$$

The other one, $u^{(+)}(\mathbf{u})$, is not.

We find that $\mathcal{L}_{\mathbf{P}_{(+)}}\beta = 0$, while

$$\mathcal{L}_{\mathbf{P}_{(-)}}\beta = -\frac{4\lambda}{m} [1 - (\hat{q}_1 \cdot \hat{q}_2)^2]. \quad (142)$$

Then Sym is one-dimensional, and spanned by $\mathbf{P}_{(+)}$.

With the canonical momenta,

$$p_{1a} := \frac{m}{|q_1|^2} \Pi_{ab}(q_1) v_1^b - \frac{\lambda}{2} \frac{\tau_{ab}(q_1)}{|q_1|} \hat{q}_2^b, \quad p_{2a} := \frac{m}{|q_2|^2} \Pi_{ab}(q_2) v_2^b + \frac{\lambda}{2} \frac{\tau_{ab}(q_2)}{|q_2|} \hat{q}_1^b, \quad (143)$$

where $\tau_{ab} := \delta_{ab} + \hat{q}_a \hat{q}_b$, the passage to Hamiltonian mechanics is straightforward. Equation (135) gives $H_C = |q_1|^2 L_1^2 / 2m + |q_2|^2 L_2^2 / 2m$ with

$$L_{1a} := |q_1| p_{1a} + \frac{\lambda}{2} \tau_{ac}(q_1) \hat{q}_2^c, \quad L_{2a} := |q_2| p_{2a} - \frac{\lambda}{2} \tau_{ac}(q_2) \hat{q}_1^c, \quad (144)$$

and the projection of the first-order Lagrangian constraint is $\gamma^{[1]} = [\hat{q}_1 \cdot L_2 + \hat{q}_2 \cdot L_1] / m$. The projection of $\mathbf{P}_{(\pm)}$ is

$$\begin{aligned}
\mathfrak{P}_{(+)} &= \hat{q}_1 \cdot \frac{\partial}{\partial q_1} + \hat{q}_2 \cdot \frac{\partial}{\partial q_2} - p_1 \cdot \frac{\partial}{\partial p_1} - p_2 \cdot \frac{\partial}{\partial p_2}, \\
\mathfrak{P}_{(-)} &= \hat{q}_1 \cdot \frac{\partial}{\partial q_1} - \hat{q}_2 \cdot \frac{\partial}{\partial q_2} - p_1 \cdot \frac{\partial}{\partial p_1} + p_2 \cdot \frac{\partial}{\partial p_2} \\
&\quad - 2\lambda \left[\frac{\hat{q}_2}{|q_1|} \cdot \Pi(q_1) \cdot \frac{\partial}{\partial p_1} + \frac{\hat{q}_1}{|q_2|} \cdot \Pi(q_2) \cdot \frac{\partial}{\partial p_2} \right],
\end{aligned} \tag{145}$$

giving $\pi_{(+)} = \mathbf{d}(q_1 \cdot p_1 + q_2 \cdot p_2)$, and $\pi_{(-)} = \mathbf{d}(q_1 \cdot p_1 - q_2 \cdot p_2 + \lambda \hat{q}_1 \cdot \hat{q}_2)$. The primary Hamiltonian constraints are $\gamma_{(+)}^{[0]} := q_1 \cdot p_1 + q_2 \cdot p_2$ and $\gamma_{(-)}^{[0]} := q_1 \cdot p_1 - q_2 \cdot p_2 + \lambda \hat{q}_1 \cdot \hat{q}_2$.

The projection of $\bar{\mathbf{X}}_L$ gives

$$\begin{aligned}
\mathfrak{X}_{H_C} &= \frac{|q_1|}{m} L_1 \cdot \frac{\partial}{\partial q_1} + \frac{|q_2|}{m} L_2 \cdot \frac{\partial}{\partial q_2} \\
&\quad - \frac{\lambda}{2m|q_1|} [(\hat{q}_1 \cdot \hat{q}_2) L_1 - L_2] \cdot \frac{\partial}{\partial p_1} - \frac{[L_1^2 + \lambda(\hat{q}_1 \cdot L_2)]}{m|q_1|} \hat{q}_1 \cdot \frac{\partial}{\partial p_1} \\
&\quad + \frac{\lambda}{2m|q_2|} [(\hat{q}_1 \cdot \hat{q}_2) L_2 - L_1] \cdot \frac{\partial}{\partial p_2} - \frac{[L_2^2 + \lambda(\hat{q}_2 \cdot L_1)]}{m|q_2|} \hat{q}_2 \cdot \frac{\partial}{\partial p_2}.
\end{aligned} \tag{146}$$

Then, $H_T = H_C + u^{(-)} \gamma_{(-)}^{[0]}(\mathfrak{s}) + u^{(+)} \gamma_{(+)}^{[0]}(\mathfrak{s})$, where after using equation (143) in equations (140) and (141),

$$u^{(-)} = \frac{1}{2\lambda} \left[\frac{L_1 \cdot L_2 / m + \lambda(\hat{q}_1 \cdot \hat{q}_2) \hat{q}_1 \cdot L_2 - (\hat{q}_1 \cdot \hat{q}_2) H_C}{1 - (\hat{q}_1 \cdot \hat{q}_2)^2} \right], \tag{147}$$

while $u^{(+)}$ remains undetermined.

5.3. A Lagrangian with local conformal and time-reparametrization invariance

The action

$$S_3 := sm \int \left[s \left(\frac{d\hat{q}}{dt} \right)^2 \right]^{1/2} dt, \tag{148}$$

where $s = \pm 1$, is invariant under both the local conformal transformations, $q^a \rightarrow \alpha(u)q^a$, and the reparametrization $t \rightarrow \tau(t)$, where τ is a monotonically increasing function of t (see also [50] and [51, 52] for systems with Lagrangians linear in the velocities). This action is a generalization of that for the relativistic particle, with the additional requirement that it have a local conformal invariance.

For this action,

$$\Omega_L = \frac{m}{|q|} \frac{P_{ab}(u)}{\sqrt{sv \cdot \Pi(q)} \cdot v} \mathbf{d}q^a \wedge \mathbf{d}v^b, \tag{149}$$

and $\Omega_F = 0$. Here, $a = 1, \dots, D$,

$$u_a = \frac{\Pi_{ab}(q)v^b}{\sqrt{sv \cdot \Pi(q)} \cdot v}, \tag{150}$$

so that $u^2 = s$, while $P_{ab}(u) = \Pi_{ab}(q) - su_a u_b$. Then $\ker \Omega_L(u) = \ker \Omega_M(u)$, and both $\mathcal{C}$ and $\mathcal{G}$ are two-dimensional. They are spanned by

$$\mathbf{U}_{(1)}^q = \hat{q} \cdot \frac{\partial}{\partial q}, \quad \mathbf{U}_{(2)}^q = u \cdot \frac{\partial}{\partial q}, \quad \text{and} \quad \mathbf{U}_{(1)}^v = \hat{q} \cdot \frac{\partial}{\partial v}, \quad \mathbf{U}_{(2)}^v = u \cdot \frac{\partial}{\partial v}, \tag{151}$$

respectively.

Because this system is fully constrained, $E = 0$. As $\Omega_F = 0$ as well, there are no Lagrangian constraints. We may choose $\mathbf{X}_L = v \cdot \partial / \partial q$. As $[\mathbf{X}_L, \mathbf{U}_{(1,2)}^v] \sim -\mathbf{U}_{(1,2)}^q$, the action of $\mathbf{X}_L$ by Gr_{Sym} does not give a SOLVF. The vector field $\bar{\mathbf{X}}_L$ can be constructed, and as expected for a fully constrained system, $\bar{\mathbf{X}}_L = 0$. A general SOELVF is then $\bar{\mathbf{X}}_{EL} = u^1(u)[\mathbf{U}_{(1)}^q] + u^2(u)[\mathbf{U}_{(2)}^q]$, with $u^n(u) \in \bar{\mathcal{F}}$ for $n = 1, 2$. The Lie algebra Sym itself is two dimensional, and spanned by $\mathbf{U}_{(1)}^q$ and $\mathbf{U}_{(2)}^q$.

For the passage to Hamiltonian mechanics, $H_C = 0$ as $E = 0$. With the canonical momentum $p_a = mu_a/|q|$, the projection of $\mathbf{P}_{(1,2)}$ is

$$\mathfrak{P}_{(1)} = \hat{q} \cdot \frac{\partial}{\partial q} - \frac{1}{|q|} p \cdot \frac{\partial}{\partial p}, \quad \mathfrak{P}_{(2)} = \frac{q}{m} p \cdot \frac{\partial}{\partial q} - \frac{p^2}{m} |\hat{q}| \cdot \frac{\partial}{\partial p}, \tag{152}$$

and $\pi_{(1)} = \mathbf{d}(q \cdot p)/|q|$, $\pi_{(2)} = \mathbf{d}(|q|^2 p^2)/2m|q|$. The primary Hamiltonian constraints are $\gamma_1^{[0]} := q \cdot p$ and $\gamma_2^{[0]} := |q|^2 p^2 - sm^2$. As $\bar{\mathbf{X}}_L = 0$, $\bar{\mathbf{X}}_C = 0$, and we find that

$$H_T = \frac{u_{(1)}}{|q|} \gamma_1^{[0]}(\mathbf{s}) + \frac{u_{(2)}}{2m|q|} \gamma_2^{[0]}(\mathbf{s}). \quad (153)$$

6. Concluding remarks

With the benefit of hindsight, the many roles that $\mathcal{G}$ plays in determining both the geometric structure of $\mathbb{P}_L$ for singular Lagrangians, and the connection between these structures and dynamics become readily apparent. What also becomes clear are the reasons why SOELVs and their dynamical structures are projectable.

Because $\mathcal{G}$ is involutive, it gives a foliation of $\mathbb{P}_L$. There is then a neighborhood U about each point $\mathbf{u} \in \mathbb{P}_L$ on which we can define the equivalence relation $\mathbf{u}_1 \sim \mathbf{u}_2$ iff $\mathbf{u}_1 - \mathbf{u}_2 = \mathbf{g}$, where $\mathbf{g}$ is a point on the leaves $\mathfrak{Fol}(\mathcal{G})$ of the foliation. This leads to the quotient space $\mathbb{P}_L/\mathfrak{Fol}(\mathcal{G})$, which has dimension $2D - N_0$. Importantly, $\mathbb{P}_L/\mathfrak{Fol}(\mathcal{G})$ is projectable, and $\mathcal{L}(\mathbb{P}_L/\mathfrak{Fol}(\mathcal{G})) = \mathcal{L}(\mathbb{P}_L) = \mathbb{P}_C^{[0]}$. This structure, and the role that $\mathcal{G}$ plays in its construction, is well known in the literature [4, 5].

Next, because $\mathcal{G} \subset \mathbf{T}_u \mathbb{P}_L$ and is involutive, it is natural to follow the construction of $\mathbb{P}_L/\mathfrak{Fol}(\mathcal{G})$ and consider the set of vector fields in $\mathbf{T}_u \mathbb{P}_L$ for which $\mathcal{G}$ is an ideal. This leads us to $\overline{\mathbf{T}}_u \mathbb{P}_L$, and the quotient space $\overline{\mathbf{T}}_u \mathbb{P}_L/\mathcal{G}$. As $\dim \overline{\mathbf{T}}_u \mathbb{P}_L/\mathcal{G} = 2D - N_0$, it is expected that $\overline{\mathbf{T}}_u \mathbb{P}_L/\mathcal{G} = \mathbf{T}_p[\mathbb{P}_L/\mathfrak{Fol}(\mathcal{G})]$ for $\mathbf{p} \in \mathbb{P}_L/\mathfrak{Fol}(\mathcal{G})$. That $\overline{\mathbf{T}}_u \mathbb{P}_L/\mathcal{G}$ is projectable is then readily apparent.

Finally, for singular Lagrangians the acceleration is not determined uniquely by the Euler–Lagrange equations of motion, an ambiguity due to the generalized Lie symmetry. This symmetry is generated by vectors that must lie in the kernel of $\bar{\Omega}_L$, and yet cannot be in $\mathcal{G}$, leading naturally first to the construction of $\ker \bar{\Omega}(\mathbf{u})/\mathcal{G}$, and then to the construction of $\overline{\mathbf{Sol}}$. Both $\ker \bar{\Omega}(\mathbf{u})/\mathcal{G} \subset \overline{\mathbf{T}}_u \mathbb{P}_L/\mathcal{G}$ and $\overline{\mathbf{Sol}} \subset \overline{\mathbf{T}}_u \mathbb{P}_L/\mathcal{G}$, and thus the evolution of the dynamical system is confined to the tangent bundle $\mathbf{T}[\mathbb{P}_L/\mathfrak{Fol}(\mathcal{G})]$. The projectability of $\mathbf{T}[\mathbb{P}_L/\mathfrak{Fol}(\mathcal{G})]$ ensures that all of the dynamical structures needed to describe the evolution of dynamical systems on the Lagrangian phase space is projectable, and agrees with those obtained through constrained Hamiltonian mechanics.

While $\mathcal{G}$ does play an important role in determining the general Lie symmetry group, it itself is not the generator of this group. This can be readily seen in the first example in section 4.3 where the Lagrangian may or may not have a local gauge symmetry depending on the choice of potential. Nevertheless, $\mathcal{G}$ is present and plays its usual role in determining $\mathbb{P}_L/\mathfrak{Fol}(\mathcal{G})$. It is instead vectors in $\ker \bar{\Omega}_L(\mathbf{u})/\mathcal{G}$ —with $\mathcal{G}$ removed—that generate the generalized Lie symmetry. We emphasize here that while this symmetry plays an important and guiding role, this role is nevertheless supportive in the construction of the algebraic-geometric structures on $\mathbb{P}_L$ needed in determining both the geometric structure of $\mathbb{P}_L$, and the connection between these structures and the evolution of the dynamical system.

The application of these algebraic-geometric structures go beyond showing the equivalence of the Lagrangian and Hamiltonian formulations of mechanics for singular Lagrangians, however. While the primary Hamiltonian constraints play a critical role in the Hamiltonian constraint analysis, the constraints themselves have traditionally been found by inspection; the expectation is that this inspection is able to both determine their form and to ensure that all of the constraints has been found for the system at hand. As a result of the Lagrangian phase space analysis presented here we are able to determine the number of primary constraints for any dynamical system, and the constraints themselves can be calculated by solving a first-order, quasi-linear differential equation. In addition, while the end result of the Lagrangian constraint algorithm is a SOELVF defined in terms of a certain number of arbitrary functions, and the end result of the Hamiltonian constraint analysis is a total Hamiltonian with the same number of arbitrary functions, how many arbitrary functions are needed, and their relationship to the original symmetries of the action is not known. With direct access to the Lagrangian and its symmetries, these questions can now be addressed in the Lagrangian phase space formulation.

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