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NATURAL CONSTRAINTS FOR EXTENDED SUPERSPACE *

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Abstract

We present a simple systematic method to derive superspace constraints. We give constraints for extended supergravity with one-, two-, and three-form gauge potentials in four spacetime dimensions. The natural constraints lead to equations of motion for $N > 4$ (supergravity), resp. $N > 2$ (gauge potentials). We discuss modifications for higher N. We also discuss modifications of the three- and four-form field strengths and observe an interesting similarity between four- and ten-dimensional supergravity.

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1. Introduction

The geometry of superspace is determined by a structure group $SL(2,\mathbb{C}) \times \mathcal{G}$ and by covariant constraints. Since the choice of $\mathcal{G}$ is in some sense arbitrary [1], the question is: Which are the proper constraints? For several years the only way to find superspace constraints was by try and error. Later some more systematic methods were developped [1-4]. We present here what we consider as the simplest method to derive constraints for conventional extended superspace. We do not deal with central charges, which require the introduction of additional bosonic coordinates [5]. The reason is that central charges do not allow to construct unconstrained prepotentials, which (at present) are necessary for superfield perturbation theory [6].

To derive the natural constraints we use, on the one hand, the restrictions following from the Bianchi identities. On the other hand we employ the simple fact that some components of the field strengths can be absorbed by a redefinition of the gauge potentials. In other words, higher dimensional parts of the gauge potentials are expressed in terms of lower dimensional ones. By this method, only the constraints at the lowest dimension involve some guesswork. The remaining ones follow then automatically.

We use the conventions of the book of Wess and Bagger [7].

2. Geometry of U(N) superspace

The coordinates of extended superspace are $z^{\mathcal{M}} \sim (x^m, \theta^\mu_M, \bar{\theta}^M_{\dot{\mu}})$, where the index M = 1,...,N refers to an internal space. We choose the structure group of superspace to be $SL(2,\mathbb{C}) \times U(N)$. This is the maximal automorphism group of the supersymmetry algebra [8], resp. of the trivial constraints (9). The reason for the inclusion of the U(N) is that the natural constraints take their simplest form in this kind of superspace.

The Lie algebra valued parameters of the structure group are

$$\begin{aligned} &L_{\mathcal{B}}{}^{\mathcal{A}} \sim (L_b{}^a, L^{\beta\alpha}_{\beta A}, L^{\dot{\beta} A}_{B\dot{\alpha}}), \\ &L^{B\alpha}_{\beta A} = \delta^B_A L_\beta{}^\alpha + \delta^\alpha_\beta L^B{}_A, \\ &L^{\dot{\beta} A}_{B\dot{\alpha}} = \delta^A_B L^{\dot{\beta}}{}_{\dot{\alpha}} + \delta^{\dot{\beta}}_{\dot{\alpha}} L^+{}_B{}^A . \end{aligned} \tag{1}$$

The Lorentz parameters have the properties

$$\begin{aligned} &L_a{}^a = L_\alpha{}^\alpha = L^{\dot{\alpha}}{}_{\dot{\alpha}} = 0, \\ &L_{\beta\dot{\beta}\alpha\dot{\alpha}} = 2\varepsilon_{\dot{\beta}\dot{\alpha}} L_{\beta\alpha} - 2\varepsilon_{\beta\alpha} L_{\dot{\beta}\dot{\alpha}}, \end{aligned} \tag{2}$$

and the U(N) parameters are antihermitian:

$$L^B{}_A = -L^+{}_A{}^B . \tag{3}$$

The differential geometry of U(N) superspace is an immediate generalization of the case N = 1 [9]. Therefore we give here only the formulas which we will need later and refer for a more detailed description to [7] and [9].

The basic geometric objects are the vielbein forms $E^{\mathcal{A}}$ and the connection forms $\phi_{\mathcal{B}}{}^{\mathcal{A}}$. The torsion and curvature two-forms are defined through the structure equations

$$\begin{aligned} &T^{\mathcal{A}} = \mathcal{D}E^{\mathcal{A}}, \\ &R_{\mathcal{B}}{}^{\mathcal{A}} = d\phi_{\mathcal{B}}{}^{\mathcal{A}} + \phi_{\mathcal{B}}{}^{\mathcal{C}}\phi_{\mathcal{C}}{}^{\mathcal{A}} \end{aligned} \tag{4}$$

and satisfy the Bianchi identities

$$\begin{aligned} &\mathcal{D}T^{\mathcal{A}} = E^{\mathcal{B}} R_{\mathcal{B}}{}^{\mathcal{A}}, \\ &\mathcal{D}R_{\mathcal{B}}{}^{\mathcal{A}} = 0 . \end{aligned} \tag{5}$$

$\phi_{\mathcal{B}}{}^{\mathcal{A}}$ and $R_{\mathcal{B}}{}^{\mathcal{A}}$ are Lie algebra valued, i. e. they have the properties (1-3).

In the vielbein basis, the structure equations (4) read explicitly

$$T_{\mathcal{C}\mathcal{B}}{}^{\mathcal{A}} = (-)^{c(b+m)} E_{\mathcal{B}}{}^{\mathcal{M}} E_{\mathcal{C}}{}^{\mathcal{N}} (\partial_{\mathcal{N}} E_{\mathcal{M}}{}^{\mathcal{A}} - (-)^{nm} \partial_{\mathcal{M}} E_{\mathcal{N}}{}^{\mathcal{A}}), \tag{6}$$

$$\begin{aligned} R_{\mathcal{D}\mathcal{C}\mathcal{B}}{}^{\mathcal{A}} = {} & \mathcal{D}_{\mathcal{D}}\phi_{\mathcal{C}\mathcal{B}}{}^{\mathcal{A}} - (-)^{dc}\mathcal{D}_{\mathcal{C}}\phi_{\mathcal{D}\mathcal{B}}{}^{\mathcal{A}} + T_{\mathcal{D}\mathcal{C}}{}^{\mathcal{E}}\phi_{\mathcal{E}\mathcal{B}}{}^{\mathcal{A}} \\ & + \phi_{\mathcal{D}\mathcal{B}}{}^{\mathcal{E}}\phi_{\mathcal{C}\mathcal{E}}{}^{\mathcal{A}} - (-)^{dc}\phi_{\mathcal{C}\mathcal{B}}{}^{\mathcal{E}}\phi_{\mathcal{D}\mathcal{E}}{}^{\mathcal{A}}, \end{aligned} \tag{7}$$

and the first Bianchi identity (5) can be written as

$$\oint_{\mathcal{D}\mathcal{C}\mathcal{B}} (R_{\mathcal{D}\mathcal{C}\mathcal{B}}{}^{\mathcal{A}} - \mathcal{D}_{\mathcal{D}} T_{\mathcal{C}\mathcal{B}}{}^{\mathcal{A}} - T_{\mathcal{D}\mathcal{C}}{}^{\mathcal{E}} T_{\mathcal{E}\mathcal{B}}{}^{\mathcal{A}}) = 0 . \tag{8}$$

Here $\oint$ denotes the graded cyclic sum. (The sign changes according to $\mathcal{A}\mathcal{B} = -(-)^{ab}\mathcal{B}\mathcal{A}$.)

3. Constraints for supergravity

The purpose of supergravity constraints is to reduce the huge number of component fields contained in vielbein and connection to an irreducible multiplet. The restrictions must be covariant, i. e. constraints on torsion and curvature, and they should not lead to equations of motion in x-space.

We start at dimension 0 with the "trivial" constraints

$$T_{\gamma\beta}^{c\,b\,a} = T_{c\,b}^{\dot\gamma\,\dot\beta\,a} = 0\,, \quad T_{\gamma\,b}^{c\,\dot\beta\,a} = 2i\,\delta_b^c\,(\sigma^a)_\gamma{}^{\dot\beta}\,. \tag{9}$$

No further restrictions are possible at this dimension since (9) holds also in flat superspace. For N = 1 and N = 2, the constraint $T_{\gamma\beta}^{CBa} = 0$ has a geometrical meaning. Together with $T_{\gamma\beta\dot\alpha}^{CBA} = 0$ it allows to define chiral superfields which are scalars under the structure group.

In the next step we analyze the consequences of the trivial constraints for the dim 1/2 components of the torsion using the Bianchi identities (8). From the identity with indices $\binom{DC\,B\,a}{\delta\gamma\,\beta}$ follows

$$T_{\gamma\beta\,\dot\alpha}^{CB\,A} = \varepsilon_{\gamma\beta}\,T_{\dot\alpha}^{[CBA]}\,, \tag{10}$$

and from $\binom{DC\,\dot\beta\,a}{\delta\gamma\,B}$ we find

$$\begin{aligned} T_{\gamma\,\beta\dot\beta\,\alpha\dot\alpha}^{C} = 2\,\big(&\varepsilon_{\beta\alpha}\,\varepsilon_{\dot\beta\dot\alpha}\,T_\gamma^C + \varepsilon_{\dot\beta\dot\alpha}\,T_{\gamma\,(\beta\alpha)}^C + \varepsilon_{\beta\alpha}\,T_{\gamma\,(\dot\beta\dot\alpha)}^C \\ &+ \varepsilon_{\gamma\beta}\,S_{\alpha\,(\dot\beta\dot\alpha)}^C + \varepsilon_{\gamma\alpha}\,S_{\beta\,(\dot\beta\dot\alpha)}^C\big)\,, \end{aligned} \tag{11}$$

$$T_{\gamma\,\dot\beta B\,\dot\alpha}^{C\quad\;\, A} = \varepsilon_{\dot\beta\dot\alpha}\,T_{\gamma\,B}^{C\;\;A} + \delta_B^A\,(S-T)_{\gamma\,(\dot\beta\dot\alpha)}^C - 2\,\delta_B^C\,S_{\gamma\,(\dot\beta\dot\alpha)}^A\,,$$

$$T_{\gamma\beta\,\alpha A}^{CB} = \varepsilon_{\beta\alpha}\,\big(T_{\gamma\,A}^{C\;\;B} + \delta_A^B\,T_\gamma^C\big) + \delta_A^B\,T_{\gamma\,(\beta\alpha)}^C + \big(\begin{smallmatrix}C\\ \gamma\end{smallmatrix} \leftrightarrow \begin{smallmatrix}B\\ \beta\end{smallmatrix}\big)\,.$$

$T_{\dot\alpha}^{[CBA]}$ is antisymmetric in CBA and contains the physical spin 1/2 fermions of N ≥ 3 supergravity.

All the other fields at dim 1/2 can be absorbed by a redefinition of the vielbein and the connection:

$$\begin{aligned} E'_{\mathcal{M}}{}^{\mathcal{A}} &= E_{\mathcal{M}}{}^{\mathcal{B}}\,X_{\mathcal{B}}{}^{\mathcal{A}}\,, \\ \phi'_{\mathcal{C}\mathcal{B}}{}^{\mathcal{A}} &= \phi_{\mathcal{C}\mathcal{B}}{}^{\mathcal{A}} + \chi_{\mathcal{C}\mathcal{B}}{}^{\mathcal{A}}\,. \end{aligned} \tag{12}$$

This redefinition changes torsion and curvature, too. From the first structure equation (6) one obtains [4]

$$\begin{aligned} T'_{\mathcal{C}\mathcal{B}}{}^{\mathcal{A}} = (-)^{c(b+d)}\,&X^{-1}{}_{\mathcal{B}}{}^{\mathcal{D}}\,X^{-1}{}_{\mathcal{C}}{}^{\mathcal{E}}\,\big(T_{\mathcal{E}\mathcal{D}}{}^{\mathcal{F}}\,X_{\mathcal{F}}{}^{\mathcal{A}} \\ &+ \mathcal{D}_{\mathcal{E}}\,X_{\mathcal{D}}{}^{\mathcal{A}} - (-)^{ed}\,\mathcal{D}_{\mathcal{D}}\,X_{\mathcal{E}}{}^{\mathcal{A}}\big) \\ &+ X^{-1}{}_{\mathcal{C}}{}^{\mathcal{D}}\,\chi_{\mathcal{D}\mathcal{B}}{}^{\mathcal{A}} - (-)^{cb}\,X^{-1}{}_{\mathcal{B}}{}^{\mathcal{D}}\,\chi_{\mathcal{D}\mathcal{C}}{}^{\mathcal{A}}\,. \end{aligned} \tag{13}$$

We choose

$$X_{\mathcal{B}}{}^{\mathcal{A}} \sim \begin{pmatrix} \delta_b^a & X_b{}^{\underline{\alpha}} \\ 0 & \delta_{\underline{\beta}}^{\underline{\alpha}} \end{pmatrix}\,, \quad \chi_{\mathcal{C}\mathcal{B}}{}^{\mathcal{A}} \sim \big(0, \chi_{\underline{\gamma}\mathcal{B}}{}^{\mathcal{A}}\big)\,,$$

$$\begin{aligned} X_{\beta\dot\beta\,\dot\alpha}^{\quad\;\;A} &= -i\,\big(S_{\beta\,(\dot\beta\dot\alpha)}^A + \varepsilon_{\dot\beta\dot\alpha}\,T_\beta^A\big)\,, \\ \chi_{\gamma\,\beta\alpha}^{C} &= -T_{\gamma\,(\beta\alpha)}^C + \varepsilon_{\gamma\beta}\,T_\alpha^C + \varepsilon_{\gamma\alpha}\,T_\beta^C\,, \\ \chi_{\gamma\,\dot\beta\dot\alpha}^{C} &= (T-S)_{\gamma\,(\dot\beta\dot\alpha)}^C\,, \\ \chi_{\gamma\;\;A}^{CB} &= T_{\gamma A}^{C\;\;B} + 2\,\delta_A^C\,T_\gamma^B\,. \end{aligned} \tag{14}$$

($\underline{\alpha}$ denotes either $\overset{A}{\alpha}$ or $\overset{\dot{\alpha}}{A}$.) Inserting this into (13) gives

$$T'^{c}_{\gamma b}{}^{a} = T'^{c\dot{\beta}A}_{\gamma B\dot{\alpha}} = T'^{cB\alpha}_{\gamma\beta A} = 0 , \tag{15}$$

while $T^{CBA}_{\gamma\beta\dot{\alpha}}$ remains unchanged.

At dimension 1 we have the freedom to shift the connection $\phi_{cB}{}^{A}$. Choosing

$$\chi_{cba} = -\frac{1}{2}\left(T_{cba} - T_{bac} + T_{acb}\right) , \tag{16}$$

we get from (13) $T'_{cb}{}^{a} = 0$, which is the standard constraint in x-space. The second structure equation (7) shows that $R^{b\dot{\gamma}B}_{\delta c}{}_{A} \sim 2i\,\delta^{b}_{c}\,(\sigma^{e})_{\delta}{}^{\dot{\gamma}} \cdot \phi_{e}{}^{B}{}_{A}$. Thus a suitable redefinition of the U(N) connection gives $R^{c\dot{\gamma}B}_{\gamma c}{}_{A} = 0$.

Summarizing, the natural constraints in U(N) superspace are

$$\begin{aligned}
&T^{CB}_{\gamma\beta}{}^{a} = T^{\dot{\gamma}\dot{\beta}}_{CB}{}^{a} = 0 , \quad T^{C\dot{\beta}}_{\gamma B}{}^{a} = 2i\,\delta^{C}_{B}\,(\sigma^{a})_{\gamma}{}^{\dot{\beta}} ,\\
&T^{CBA}_{\gamma\beta\dot{\alpha}} = \varepsilon_{\gamma\beta}\,T^{[CBA]}_{\dot{\alpha}} , \quad T^{\dot{\gamma}\dot{\beta}\alpha}_{CBA} = \varepsilon^{\dot{\gamma}\dot{\beta}}\,T^{\alpha}_{[CBA]} ,\\
&T_{\underline{\gamma\beta}}{}^{\underline{\alpha}} = 0 \ \text{else} , \quad T_{\underline{\gamma}b}{}^{a} = 0 ,\\
&T_{cb}{}^{a} = 0 , \quad R^{c\dot{\gamma}B}_{\gamma c}{}_{A} = 0 .
\end{aligned} \tag{17}$$

They eliminate the connection and the higher dimensional parts of the vielbein as independent variables. The above constraints correspond for N = 1 with those given in [9] and, for general N, with those of Howe [1] (except for the curvature constraint). However, it has been shown by Gates and Grimm that they lead to equations of motion for the graviton and the gravitinos if $N > 4$ [10].

Thus the natural constraints have to be modified for $N > 4$. Since the trivial constraints (9) have been the only assumption in our derivation, it is clear that they should be relaxed. In order to analyze which modifications are possible, we use the linearized version of (13) and redefine the dim 0 components of the vielbein. Those parts of $T^{CB}_{\gamma\beta}{}^{a}$ and $T^{C\dot{\beta}}_{\gamma B}{}^{a}$ which cannot be transformed away are

$$\begin{aligned}
&T^{CB}_{\gamma\beta\,\alpha\dot{\alpha}} = T^{(CB)}_{(\gamma\beta\alpha)\dot{\alpha}} ,\\
&T^{C}_{\gamma\dot{\beta}B\,\alpha\dot{\alpha}} = -4i\,\delta^{C}_{B}\,\varepsilon_{\gamma\alpha}\,\varepsilon_{\dot{\beta}\dot{\alpha}} + \tilde{T}^{C}_{(\gamma\alpha)(\dot{\beta}\dot{\alpha})B} ,
\end{aligned} \tag{18}$$

where $\tilde{T}$ is traceless in the internal indices. These are therefore the most general dim 0 constraints. However, they cannot be called "natural" since they do not reduce naturally to the trivial constraints (9) for N = 1 and N = 2. Besides, $T^{(CB)}_{(\gamma\beta\alpha)\dot{\alpha}}$ and $\tilde{T}^{C}_{(\gamma\alpha)(\dot{\beta}\dot{\alpha})B}$ contain auxiliary fields with high spins. Therefore we do not expect that the constraints (18) lead to a consistent supergravity theory.

Apart from that, it seems that even N = 3 and N = 4 supergravity cannot be extended off-shell [11]. In superspace, this is due to problems in the Yang-Mills sector, which will be discussed in the next section.

4. Yang-Mills constraints

A one-form gauge potential A is the connection of a compact Lie group. The Yang-Mills field strength

$$F = dA + AA \tag{19}$$

satisfies the Bianchi identity

$$\mathcal{D}F = 0 \,, \qquad (20)$$

$$\oint_{\mathcal{E}\mathcal{B}\mathcal{A}} (\mathcal{D}_{\mathcal{E}} F_{\mathcal{B}\mathcal{A}} + T_{\mathcal{E}\mathcal{B}}{}^{\mathcal{D}} F_{\mathcal{D}\mathcal{A}}) = 0 \,,$$

where $\mathcal{D}$ is the gauge covariant derivative.

The dim 0 components of the field strength F can be decomposed into

$$F_{\beta\alpha}^{BA} = F_{(\beta\alpha)}^{(BA)} + \varepsilon_{\beta\alpha} W^{[BA]} \,, \qquad (21)$$

$$F_{\beta\dot{\alpha}A}^{B} = \tilde{F}_{\beta\dot{\alpha}A}^{B} + \delta_A^B F_{\beta\dot{\alpha}} \,, \quad \tilde{F}_{\beta\dot{\alpha}A}^{A} = 0 \,.$$

Analogously to supergravity, $F_{\beta\dot{\alpha}}$ can be absorbed by a redefinition of A_a. However, F has still too many components. We define the natural constraints as those which reduce to the correct Yang-Mills constraints for N = 1 and N = 2. They are obtained from (21) by eliminating the parts with the highest spin:

$$F_{\beta\alpha}^{BA} = \varepsilon_{\beta\alpha} W^{[BA]} \,, \quad F_{\beta A}^{B\dot{\alpha}} = 0 \,. \qquad (22)$$

For N = 1 this becomes [7]

$$F_{\beta\dot{\alpha}} = 0 \,. \qquad (23)$$

The constraint $F_{\beta\dot{\alpha}} = 0$ allows to define scalar superfields ϕ which satisfy the condition $\bar{\mathcal{D}}_{\dot{\alpha}} \phi = 0$.

The N = 2 Yang-Mills constraints are [12]

$$F_{\beta\alpha}^{BA} = \varepsilon_{\beta\alpha} \varepsilon^{BA} W \,, \quad F_{\beta A}^{B\dot{\alpha}} = 0 \,. \qquad (24)$$

Here the constraints $F_{(\beta\alpha)}^{(BA)} = 0$ and $\tilde{F}_{\beta A}^{B\dot{\alpha}} = 0$ are required for the consistency of the conditions $\mathcal{D}_{\alpha}^{(A} \phi^{B)} = \bar{\mathcal{D}}_{\dot{\alpha}}^{(A} \phi^{B)} = 0$ which define a hypermultiplet with a central charge. They also allow to define real linear multiplets $L^{(AB)}$ satisfying $\mathcal{D}_{\alpha}^{(A} L^{BC)} = 0$.

For $N > 2$ the constraints (22) lead to an equation of motion for $W^{[BA]}$ [13] and it seems to be hopeless to analyze possible modifications. Namely, simple counting arguments show that the N = 4 super Yang-Mills theory cannot be extended off-shell [14,6]. This means that the vector fields have to be described off-shell within a different multiplet, e. g. within the supergravity multiplet.

In the following we consider N = 3 supergravity. The three vector fields are described by field strengths $F_{[BA]}$ and $\bar{F}^{[BA]}$ which satisfy the Bianchi identity (20). In order to avoid the field equation for W, we impose the constraints

$$F_{\delta\gamma[BA]}^{DC} = \varepsilon_{\delta\gamma} (\delta_B^D \delta_A^C - \delta_A^D \delta_B^C) \,, \qquad (25)$$

$$F_{\delta C[BA]}^{D\dot{\gamma}} = F_{DC[BA]}^{\dot{\delta}\dot{\gamma}} = 0 \,,$$

which are also valid on-shell [15]. The Bianchi identities (20) subject to the constraints (17) and (25) yield at dim 1 [1]

$$\mathcal{D}_{\delta}^{D} T_{\dot{\gamma}}^{[CBA]} = 0 \,, \quad \bar{\mathcal{D}}_{D}^{\dot{\alpha}} T_{\dot{\alpha}}^{[CBA]} = 0 \,. \qquad (26)$$

The first equation is also a consequence of the Bianchi identities for supergravity. Both equations together give in the linearized approximation

$$\partial_{\alpha}{}^{\dot{\alpha}} T_{\dot{\alpha}}^{[CBA]} = 0 \,. \qquad (27)$$

This is the Dirac equation for the spin 1/2 fermion. Again, it seems to be hopeless to look for modifications of the constraints (25) [11].

5. Two-form gauge potential

We introduce a two-form gauge potential B = (1/2) $E^{\mathcal{A}} E^{\mathcal{B}} B_{\mathcal{BA}}$ with the transformation law

$$\delta B = d\omega , \tag{28}$$

where ω is a one-form gauge parameter. The field strength

$$G = dB = \frac{1}{3!} E^{\mathcal{A}} E^{\mathcal{B}} E^{\mathcal{C}} G_{\mathcal{CBA}} ,$$
$$G_{\mathcal{CBA}} = \oint_{\mathcal{CBA}} \left(\mathcal{D}_{\mathcal{C}} B_{\mathcal{BA}} + T_{\mathcal{CB}}{}^{\mathcal{D}} B_{\mathcal{DA}} \right) \tag{29}$$

satisfies the Bianchi identity

$$dG = 0 ,$$
$$E^{\mathcal{A}} E^{\mathcal{B}} E^{\mathcal{C}} E^{\mathcal{D}} \left(\mathcal{D}_{\mathcal{D}} G_{\mathcal{CBA}} + \tfrac{3}{2} T_{\mathcal{DC}}{}^{\mathcal{F}} G_{\mathcal{FBA}} \right) = 0 . \tag{30}$$

At dim -1/2 we impose the constraint

$$G_{\underline{\gamma\beta\alpha}} = 0 \tag{31}$$

and analyze the consequences for the dim 0 components of G using the Bianchi identities (30). The identity with indices $\begin{pmatrix} D\,C\,B\,A \\ \delta\,\gamma\,\beta\,\alpha \end{pmatrix}$ is empty. From $\begin{pmatrix} D\,C\,B\,\dot{\alpha} \\ \delta\,\gamma\,\beta\,A \end{pmatrix}$ follows

$$G_{\gamma\beta\alpha\dot{\alpha}} = \sum_{\gamma\beta} \varepsilon_{\gamma\alpha} G_{\beta\dot{\alpha}} \qquad (N = 1) , \; {}^{*}$$
$$G^{\;C\,B}_{\gamma\,\beta\;a} = 0 \qquad\qquad (N > 1) , \tag{32}$$

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* $\sum_{\alpha\beta} X_{\alpha\beta}$ means $X_{\alpha\beta} + X_{\beta\alpha}$.

and $\begin{pmatrix} D\,C\,\dot{\beta}\,\dot{\alpha} \\ \delta\,\gamma\,B\,A \end{pmatrix}$ gives

$$G^{C}_{\gamma\dot{\beta}B\alpha\dot{\alpha}} = -4i\, \varepsilon_{\gamma\alpha} \varepsilon_{\dot{\beta}\dot{\alpha}} G^{C}{}_{B} + \delta^{C}_{B} \left(\varepsilon_{\dot{\beta}\dot{\alpha}} G_{(\gamma\alpha)} + \varepsilon_{\gamma\alpha} G_{(\dot{\beta}\dot{\alpha})} \right) ,$$
$$\text{where} \quad G^{A}{}_{A} = 0 \;\; (N = 2) , \quad G^{C}{}_{B} = 0 \;\; (N > 2) . \tag{33}$$

Now we observe in (29) that $G^{C\dot{\beta}}_{\gamma B a} \sim 2i\, \delta^{C}_{B} (\sigma^{d})_{\gamma}{}^{\dot{\beta}} B_{da}$. Thus a suitable redefinition of B_{ba} gives $G_{(\gamma\alpha)} = G_{(\dot{\beta}\dot{\alpha})} = 0$. In addition, we require $G_{\beta\dot{\alpha}} = 0$ for N = 1 and end up with the constraints

$$G_{\underline{\gamma\beta\alpha}} = 0 , \quad G^{\;C\,B}_{\gamma\,\beta\;a} = 0 ,$$
$$G_{\gamma\dot{\beta}a} = 2i\, (\sigma_{a})_{\gamma\dot{\beta}}\, G \qquad (N = 1) ,$$
$$G^{C\dot{\beta}}_{\gamma B a} = 2i\, (\sigma_{a})_{\gamma}{}^{\dot{\beta}} G^{C}{}_{B} , \quad G^{A}{}_{A} = 0 \qquad (N = 2) ,$$
$$G^{C\dot{\beta}}_{\gamma B a} = 0 \qquad (N > 2) . \tag{34}$$

For N = 1 and N = 2 these constraints were first given in [16] and [17]. They lead to the N = 1 tensor multiplet [18], resp. to the N = 2 tensor multiplet [19].

For N > 2 we find from the remaining Bianchi identities $G_{\underline{\gamma} ba} = G_{cba} = 0$, i. e. G vanishes identically. Therefore the constraint (31) has to be relaxed for N > 2 since it has been the only assumption in our derivation. The only modification of this constraint which is not affected by redefinitions of $B_{\underline{\beta} a}$ and reduces to (31) for N = 1 and N = 2, is

$$G^{\;C\,B\,A}_{\gamma\,\beta\,\alpha} = 0 ,$$
$$G^{\;C\,B\,\dot{\alpha}}_{\gamma\,\beta\;A} = \varepsilon_{\gamma\beta}\, \tilde{G}^{[CB]\dot{\alpha}}{}_{A} , \quad \tilde{G}^{[CA]\dot{\alpha}}{}_{A} = 0 . \tag{35}$$

In curved superspace, however, the above constraints are incompatible with the supergravity constraints (17). This can be seen from the Bianchi identity $\binom{DC\,\beta\,A}{\delta\gamma\,\beta\,\alpha}$. Therefore we stop our analysis at this point.

6. Three-form gauge potential

A three-form gauge potential $C = (1/3!)\,E^{A}E^{B}E^{C}C_{CBA}$ transforms as

$$\delta C = d\chi \,, \tag{36}$$

where χ is a two-form gauge parameter. The field strength

$$H = dC \,,$$
$$H = \frac{1}{4!}\,E^{A}E^{B}E^{C}E^{D}H_{DCBA} \tag{37}$$

satisfies the Bianchi identity

$$dH = 0 \,,$$
$$E^{A}E^{B}E^{C}E^{D}E^{E}\left(\mathcal{D}_{E}H_{DCBA} + 2\,T_{ED}{}^{F}H_{FCBA}\right) = 0 \,. \tag{38}$$

At dim -1 we restrict H through

$$H_{\underline{\delta\gamma\beta\alpha}} = 0 \,. \tag{39}$$

The Bianchi identities (38) subject to this constraint give at dim -1/2

$$H_{\delta\gamma\beta\alpha\dot\alpha} = \oint_{\delta\gamma\beta} \varepsilon_{\delta\alpha}\, H_{(\gamma\beta)\dot\alpha} \qquad (N=1)\,,$$
$$H^{DCB}_{\delta\gamma\beta a} = 0 \qquad (N>1) \tag{40}$$

and, after a suitable redefinition of $C^{C}_{\gamma ba}$,

$$H^{DCB}_{\delta\gamma\dot\beta\,\alpha\dot\alpha} = 4i\,\varepsilon_{\dot\beta\dot\alpha}\sum_{\delta\gamma}\varepsilon_{\delta\alpha}\,\Lambda^{(DCB)}_{\gamma} \qquad (N=2)\,,$$
$$H^{DC\dot\beta}_{\delta\gamma B a} = 0 \qquad (N\neq 2)\,. \tag{41}$$

For N = 1 we set $H_{(\gamma\beta)\dot\alpha} = 0$. At dim 0 we find

$$H_{\delta\gamma\,\beta\dot\beta\,\alpha\dot\alpha} = -\,\varepsilon_{\dot\beta\dot\alpha}\left(\varepsilon_{\delta\beta}\varepsilon_{\gamma\alpha} + \varepsilon_{\delta\alpha}\varepsilon_{\gamma\beta}\right)H \qquad (N=1)\,,$$
$$H^{DC}_{\delta\gamma\,\beta\dot\beta\,\alpha\dot\alpha} = \varepsilon_{\dot\beta\dot\alpha}\sum_{\delta\gamma}\sum_{\beta\alpha}\varepsilon_{\delta\beta}\,H^{(DC)}_{\gamma\alpha} \qquad (N=2)\,, \tag{42}$$
$$H^{DC}_{\delta\gamma ba} = 0 \qquad (N>2)$$

and, after a redefinition of C_{cba},

$$H^{DC}_{\delta\dot\gamma\,\beta\dot\beta\,\alpha\dot\alpha} = \varepsilon_{\dot\beta\dot\alpha}\sum_{\beta\alpha}\varepsilon_{\delta\beta}\,H^{(DC)}_{\alpha\dot\gamma} - \varepsilon_{\beta\alpha}\sum_{\dot\beta\dot\alpha}\varepsilon_{\dot\gamma\dot\beta}\,\bar H^{(DC)}_{\delta\dot\alpha} \qquad (N=2)\,,$$
$$H^{D\dot\gamma}_{\delta C ba} = 0 \qquad (N\neq 2)\,. \tag{43}$$

$H^{(DC)}_{\gamma\alpha}$ and $H^{(DC)}_{\alpha\dot\gamma}$ can be expressed in terms of the covariant derivatives of $\Lambda^{(DCB)}_{\gamma}$.

Summarizing, the constraints for N = 1 are

$$H_{\underline{\delta\gamma\beta\alpha}} = 0 \,,\quad H_{\underline{\delta\gamma\beta}a} = 0 \,,$$
$$H_{\delta\gamma ba} = (\sigma_{ba})_{\delta\gamma}\,H \,,\quad H_{\delta\dot\gamma ba} = 0 \,. \tag{44}$$

They lead to the N = 1 three-form multiplet [16]. For N = 2 we have found the constraints

$$H_{\underline{\delta\gamma}\,\underline{\beta\alpha}} = 0 \ , \quad H^{DCB}_{\delta\gamma\beta\,a} = 0 \ ,$$

$$H^{DCB}_{\delta\gamma\dot\beta\,\alpha\dot\alpha} = 4i\,\varepsilon_{\dot\beta\dot\alpha} \sum_{\delta\gamma} \varepsilon_{\delta\alpha}\,\Lambda^{(DCB)}_{\gamma} \ , \tag{45}$$

$$H^{c\dot\gamma}_{\gamma c\,ba} = 0 \ .$$

We expect that they lead to a N = 2 three-form multiplet.

For N > 2 the remaining Bianchi identities give $H_{\underline{\delta}cba} = H_{dcba} = 0$, i. e. H vanishes identically. We conclude that the constraint (39) has to be relaxed for N > 2. Those modifications which are invariant under shifts of $C_{\underline{\gamma\beta}a}$ and reduce to (39) for N = 1 and N = 2, are

$$H^{DCBA}_{\delta\gamma\beta\alpha} = 0 \ , \quad H^{DCB\dot\alpha}_{\delta\gamma\beta A} = \oint_{DCB} \varepsilon_{\delta\gamma}\,\tilde H^{[DC]B\dot\alpha}{}_{\beta A} \ ,$$

$$H^{DC\dot\beta\dot\alpha}_{\delta\gamma BA} = \varepsilon_{\delta\gamma}\,\varepsilon^{\dot\beta\dot\alpha}\,\tilde H^{[DC]}{}_{[BA]} + \varepsilon^{\dot\beta\dot\alpha}\,\tilde H^{(DC)}{}_{(\delta\gamma)[BA]} + \varepsilon_{\delta\gamma}\,\tilde H^{[DC](\dot\beta\dot\alpha)}{}_{(BA)} \ , \tag{46}$$

where all the traces of $\tilde H$ vanish. Again, however, the above constraints are inconsistent in curved superspace. This follows from the Bianchi identities $\binom{EDCBA}{\varepsilon\delta\gamma\beta\alpha}$, $\binom{EDCB\dot\alpha}{\varepsilon\delta\gamma\beta A}$, and $\binom{EDCB}{\varepsilon\delta\gamma\beta a}$. Therefore we do not analyze them further.

7. Modified field strengths

In the preceding sections we have introduced three kinds of field strengths X ~ F,G,H satisfying the Bianchi identity $\mathcal{D}X = 0$ ($R_{\mathcal{B}}{}^{\mathcal{A}}$ is here included in F). The two-form field strength F is given by the Ricci identity $\mathcal{D}\mathcal{D}\Omega = \Omega F$ for any n-form Ω. This is not the case for G and H. Therefore these field strengths can be modified. In the following we consider modifications of the form X = dY + non-linear terms. The Bianchi identity for X then becomes dX = Z, where Z is covariant, non-linear, and satisfies dZ = 0.

We start with the field strength of the two-form potential and extend the Bianchi identity dG = 0 to

$$dG = k\,\mathrm{tr}\,(FF) \ , \tag{47}$$

where k is a real parameter. From this we conclude

$$G = dB + k\,\mathrm{tr}\,(AF - \tfrac{1}{3}A^3) \ . \tag{48}$$

The field strength G thus contains a superspace generalization of the Chern-Simons three-form. In order to cancel the gauge variation of this three-form $(\delta A = -\mathcal{D}\Lambda)$, the transformation law $\delta B = d\omega$ has to be extended to

$$\delta B = d\omega + k\,\mathrm{tr}\,(\Lambda\, dA) \ . \tag{49}$$

The above modifications are consistent in N = 1 superspace. They describe the coupling of Yang-Mills theories to 16+16 supergravity [20,9]. In N = 2 superspace the modifications (47-49) are not possible since the constraints on G are incompatible with the Yang-Mills constraints on F. This can be seen from the Bianchi identity (47) with indices $\binom{DCBA}{\delta\gamma\beta\alpha}$. – We remark that we could also choose $F = R_{\mathcal{B}}{}^{\mathcal{A}}$, $A = \phi_{\mathcal{B}}{}^{\mathcal{A}}$, and $\Lambda = L_{\mathcal{B}}{}^{\mathcal{A}}$ in (47-49). We expect that we will obtain this way a superspace generalization of the curvature squared terms which appear in the field theory limit of string theories [21]. Again, this is not possible in N = 2 superspace.

Now we consider the field strength of the three-form potential and extend the Bianchi identity dH = 0 to

$$dH = k\,F\,G \ , \tag{50}$$

where F is an abelian field strength. From this we find

$$H = dC - k\,AG \ . \tag{51}$$

H is invariant if the transformation law $\delta C = d\chi$ is modified to

$$\delta C = d\chi + k \wedge G \ . \tag{52}$$

If we assume (47-49) to be valid in N = 1 superspace, the above modifications are no longer possible since (47) and (50) are incompatible. In N = 2 superspace it would suggest itself to identify F with the field strength of the vector in the supergravity multiplet. We have checked that this is consistent at dim -1/2 and dim 0. To prove full consistency, of course, the Bianchi identities (50) have to be solved completely.

Finally we note an interesting observation. The modified three-form field strength (48) also describes the coupling of Yang-Mills theories to N = 1 supergravity in ten dimensions [22] (it was first introduced in [23]), whereas the modified four-form field strength (51) occurs in the non-chiral N = 2, D = 10 supergravity theory [24]. In fact, N = 1, 16+16 supergravity in four dimensions is an almost exact copy of N = 1, D = 10 supergravity. A similar relation would be possible between N = 2, D = 4 and N = 2, D = 10 supergravity.

8. Conclusions

The conclusions of this paper are the following. First, N = 1 and N = 2 supergravity with one-, two-, and three-form gauge potentials have a natural off-shell formulation in superspace. Secondly, there are a lot of indications that off-shell versions without central charges do not exist for $N > 2$. If such theories exist, conventional extended superspace is not an appropriate setting for their description. Thirdly, there seems to be a relation between N = 1, resp. N = 2 supergravity in four and in ten dimensions.

As an example, consider the N = 4 supergravity theory with an antisymmetric tensor gauge potential [23]. It can be obtained by dimensional reduction from N = 1 supergravity in ten dimensions [22]. A truncation of the N = 4 theory leads to a N = 1 supergravity theory with a two-form gauge potential [20,9]. Since auxiliary fields exist for the N = 1, D = 4 theory as well as for the linearized N = 1, D = 10 theory [25], one might conjecture that they also exist for the N = 4, D = 4 theory. However, all attempts to find N = 4 off-shell representations failed [14,11] and a formulation of the mentioned N = 4 supergravity theory in conventional superspace is even on-shell inconsistent [26].

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