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Authors

Bucur, Alina
Kedlaya, Kiran

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AN APPLICATION OF THE EFFECTIVE SATO-TATE CONJECTURE

ALINA BUCUR AND KIRAN S. KEDLAYA

Abstract. Based on the Lagarias-Odlyzko effectivization of the Chebotarev density theorem, Kumar Murty gave an effective version of the Sato-Tate conjecture for an elliptic curve conditional on analytic continuation and Riemann hypothesis for the symmetric power L-functions. We use Murty's analysis to give a similar conditional effectivization of the generalized Sato-Tate conjecture for an arbitrary motive. As an application, we give a conditional upper bound of the form $O((\log N)^2(\log\log 2N)^2)$ for the smallest prime at which two given rational elliptic curves with conductor at most N have Frobenius traces of opposite sign.

Let $\pi(x)$ denote the number of prime numbers less than or equal to x. Hadamard and de la Vallée-Poussin proved the prime number theorem

$$\pi(x) = (1 + o(1))\frac{x}{\log x}$$

by exploiting the relationship between prime numbers and the zeroes of the Riemann zeta function. Assuming Riemann's hypothesis that the zeroes in the critical strip $0 \leq \mathrm{Re}(s) \leq 1$ all lie on the line $\mathrm{Re}(s) = 1/2$, one gets a much more precise estimate for $\pi(x)$:

$$\pi(x) = \mathrm{Li}(x) + O(x^{1/2}\log x) \qquad \left(\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}\right).$$

Here and throughout this paper, the implied constant in the big-O notation is absolute and effectively computable, and the assertion applies for all x exceeding some other effectively computable absolute constant. (For details, see any introductory text on analytic number theory, e.g., [2].)

A similar paradigm applies to the distribution of values of various other functions of prime numbers, or more generally of prime ideals in a number field K: one gets an asymptotic result using some limited analytic information about L-functions, but under the analogue of the Riemann hypothesis one gets an estimate with a small effective error term. For example, for the Chebotarev density theorem (describing the distribution of Frobenius classes for a fixed Galois extension of K), this effectivization process was described by Lagarias and Odlyzko [10]. More recently, the Sato-Tate conjecture (describing the distribution of Frobenius traces for a fixed elliptic curve over K) has been established for K totally real through the efforts of Taylor et al. (see [1] for a definitive result); in this case, the effectivization process had been described previously by Kumar Murty [12].

The previous two examples can both be subsumed into a generalized Sato-Tate conjecture for an arbitrary motive, taking an Artin motive in the case of Chebotarev and the 1-motive

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of an elliptic curve in the case of Sato-Tate. The first purpose of this paper is to explain, under suitable analytic hypotheses on motivic L-functions (Conjecture 1.1), how to obtain effective error bounds for the generalized Sato-Tate conjecture. The technique is essentially an application of Weyl-type explicit formulas as in [10] and [12]; in fact, Murty's treatment of the analytic arguments in [12] is already general enough to apply to arbitrary motives, so it is not necessary to redo any of the complex analysis. (Murty was practically forced to work at this level of generality to handle the usual Sato-Tate conjecture, because he needed his arguments to apply uniformly over symmetric powers. Here we apply them uniformly over representations of a compact Lie group.)

The second purpose of this paper is to indicate an application of the effective form of the generalized Sato-Tate conjecture to a classical question about the arithmetic of elliptic curves. Let E_1 and E_2 be nonisogenous elliptic curves over K, neither having complex multiplication. The isogeny theorem of Faltings [3] implies that there exists a prime ideal $\mathfrak{p}$ of K at which E_1, E_2 both have good reduction and have distinct Frobenius traces. In particular, for any fixed prime ℓ, there exists a prime ideal $\mathfrak{p}$ of K at which the Frobenius traces of E_1, E_2 differ modulo ℓ. Assuming the generalized Riemann hypothesis for Artin L-functions, one can use the effective form of the Chebotarev density theorem (as suggested by Serre in [17]; see also Corollary 4.8) to show the least norm of such a prime ideal is

$$O((\log N)^2(\log\log 2N)^b)$$

for some fixed $b \geq 0$. Assuming the generalized Riemann hypothesis for L-functions of the form $L(s, \mathrm{Sym}^m E_1 \otimes \mathrm{Sym}^n E_2)$, we use the effective form of the generalized Sato-Tate conjecture for the abelian surface $E_1 \times_K E_2$ to obtain a similar bound for the least norm of a prime ideal at which the Frobenius traces of E_1, E_2 have opposite sign (Theorem 4.1). In both cases, the optimal bound is most likely closer to $O(\log N)$, but by analogy with the problem of finding the least quadratic nonresidue modulo N, it is unlikely that one can do better than $O((\log N)^2)$ using L-function methods.

Although we will not do so here, we mention that the framework of the generalized Sato-Tate conjecture includes many additional questions about distinguishing L-functions, a number of which have been considered previously. For instance, Goldfeld and Hoffstein [8] established an upper bound on the first distinguishing coefficient for a pair of holomorphic Hecke newforms, by an argument similar to ours but with a milder analytic hypothesis (the Riemann hypothesis for the Rankin-Selberg convolutions of the two forms with themselves and each other). Sengupta [14] carried out the analogous analysis with the Fourier coefficients replaced by normalized Hecke eigenvalues (this only makes a difference when the weights are distinct). The analogue of Serre's argument for modular forms was given by Ram Murty [11] and subsequently extended to Siegel modular forms by Ghitza [6] for Fourier coefficients and Ghitza and Sayer [7] for Hecke eigenvalues.

1. Motivic L-functions and motivic Galois groups

We begin by recalling the conjectural properties of motivic L-functions, as in [16].

Fix two number fields K, L. Let $\mathcal{M}$ be a pure motive of weight w over K with coefficients in L. For each prime ideal $\mathfrak{p}$ of K, let $G_\mathfrak{p}$ be a decomposition subgroup of $\mathfrak{p}$ inside the absolute Galois group G_K, let $I_\mathfrak{p}$ be the inertia subgroup of $G_\mathfrak{p}$, and let $\mathrm{Frob}_\mathfrak{p} \in G_\mathfrak{p}/I_\mathfrak{p}$ be the Frobenius element. The *Euler factor* of $\mathcal{M}$ at $\mathfrak{p}$ (for the automorphic normalization) is

the function

$$L_{\mathfrak{p}}(s, \mathcal{M}) = \det(1 - \mathrm{Norm}(\mathfrak{p})^{-s-w/2} \mathrm{Frob}_{\mathfrak{p}}, V_v(\mathcal{M})^{I_{\mathfrak{p}}} \otimes_{L_v} \mathbb{C})^{-1}$$

for v a finite place of L equipped with an embedding $L_v \hookrightarrow \mathbb{C}$ and $V_v(\mathcal{M})$ the v-adic étale realization of $\mathcal{M}$ equipped with its action of $G_{\mathfrak{p}}$. It is clear that this definition does not depend on the choice of $G_{\mathfrak{p}}$; it is conjectured also not to depend on v or the embedding $L_v \hookrightarrow \mathbb{C}$, and this is known when $\mathcal{M}$ has good reduction at $\mathfrak{p}$ (which excludes only finitely many primes).

The ordinary L-function of $\mathcal{M}$ is the the Euler product

$$L(s, \mathcal{M}) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(s, \mathcal{M}).$$

For each infinite place ∞ of K, there is also an archimedean Euler factor defined as follows. Put

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2), \qquad \Gamma_{\mathbb{C}}(s) = 2^{-s}\pi^{-s}\Gamma(s).$$

Form the Betti realization of $\mathcal{M}$ at ∞ and the spaces $H^{p,q}$ for $p + q = w$, and put $h^{p,q} = \dim H^{p,q}$. Note that complex conjugation takes $H^{p,q}$ to $H^{q,p}$ and thus acts on $H^{w/2,w/2}$; let h^+ and h^- be the dimensions of the positive and negative eigenspaces (both taken to be 0 if w is odd). Then put

$$L_\infty(s, \mathcal{M}) = \Gamma_{\mathbb{R}}(s)^{h^+}\Gamma_{\mathbb{R}}(s+1)^{h^-} \prod_{p+q=w, p<q} \Gamma_{\mathbb{C}}(s + w/2 - p)^{h^{p,q}}.$$

The completed L-function is then defined as

$$\Lambda(s, \mathcal{M}) = N^{s/2} L(s, \mathcal{M}) \prod_{\infty} L_\infty(s, \mathcal{M}),$$

for N the absolute conductor of $\mathcal{M}$ (i.e., the norm from K to $\mathbb{Q}$ of the conductor ideal of $\mathcal{M}$).

Conjecture 1.1. *Let d be the dimension of the fixed subspace of the motivic Galois group of $\mathcal{M}(-w/2)$ (taken to be 0 if w is odd).*

(a) *The function $s^d(1-s)^d\Lambda(s, \mathcal{M})$ (which is defined* a priori *for $\mathrm{Re}(s) > 1$) extends to an entire function on $\mathbb{C}$ of order 1 which does not vanish at $s = 0, 1$. (Recall that an entire function $f : \mathbb{C} \to \mathbb{C}$ is of* order 1 *if $f(z)e^{-\mu|z|}$ is bounded for each $\mu > 1$.)*

(b) *Let $\mathcal{M}^*$ denote the Cartier dual of $\mathcal{M}$. Then there exists $\epsilon \in \mathbb{C}$ with $|\epsilon| = 1$ such that $\Lambda(1 - s, \mathcal{M}) = \epsilon\Lambda(s, \mathcal{M}^*)$ for all $s \in \mathbb{C}$.*

(c) *The zeroes of $\Lambda(s, \mathcal{M})$ all lie on the line $\mathrm{Re}(s) = 1/2$.*

Remark 1.2. At present, the most promising approach to proving parts (a) and (b) of Conjecture 1.1 for a given $\mathcal{M}$ is to show that $\Lambda(s, \mathcal{M})$ coincides with a potentially automorphic L-function. For example, this is known for the symmetric power L-functions of an elliptic curve over a totally real number field [1] and for the Rankin-Selberg product of two such L-functions [9]. This implies (a) and (b) for such L-functions, using the work of Gelbart and Shahidi [5] to verify the order 1 condition. See [13] for an overview of how to use potential automorphy to deduce the Sato-Tate conjecture. Part (c), the analogue of the Riemann hypothesis, is unknown in all cases.

2. Equidistribution and motivic L-functions

We next recall how to use the analytic information about motivic L-functions provided by Conjecture 1.1 to obtain equidistribution statements with small effective error bounds. This combines the general approach to equidistribution described in [15, Appendix to Chapter 1] with the extraction of effective bounds from L-functions described in [10, 12].

Take $\mathcal{M}$ as before. The *motivic Sato-Tate group* of $\mathcal{M}$ is the kernel of the map from the motivic Galois group of $\mathcal{M} \oplus L(1)$ to the motivic Galois group of the Tate motive $L(1)$; this is a subgroup of the usual motivic Galois group of $\mathcal{M}$. Taking a compact form of the motivic Sato-Tate group yields the *Sato-Tate group* G. From the construction, one obtains a sequence $\{g_{\mathfrak{p}}\}$ in the space $\mathrm{Conj}(G)$ of conjugacy classes of G corresponding to prime ideals of good reduction, such that for any motive $\mathcal{N}$ pure of weight k in the Tannakian category generated by $\mathcal{M}$, the characteristic polynomial of $\mathrm{Norm}(\mathfrak{p})^{-k/2} \mathrm{Frob}_{\mathfrak{p}}$ on any étale realization of $\mathcal{N}$ equals the characteristic polynomial of $g_{\mathfrak{p}}$ on the corresponding representation of G.

Topologize $\mathrm{Conj}(G)$ as a quotient of G, and equip it with the measure μ with the property that for any continuous function $F : \mathrm{Conj}(G) \to \mathbb{C}$, $\mu(F)$ is the Haar measure of the pullback of F to G. The statement that the $g_{\mathfrak{p}}$ are equidistributed in $\mathrm{Conj}(G)$ would mean that for any F, if we write $F(\mathfrak{p})$ as shorthand for $F(g_{\mathfrak{p}})$, then

$$\sum_{\mathrm{Norm}(\mathfrak{p}) \leq x} F(\mathfrak{p}) = (\mu(F) + o(1)) \sum_{\mathrm{Norm}(\mathfrak{p}) \leq x} 1. \tag{2.1}$$

By the Peter-Weyl theorem, we have

$$F = \sum_{\chi} \mu(F\overline{\chi})\chi \tag{2.2}$$

where the sum runs over irreducible characters χ of G, so it suffices to check (2.1) for these characters. For such a character χ, let $L(s, \chi)$ be the L-function of the motive corresponding to χ in the Tannakian category generated by $\mathcal{M}$. One then shows that if[1] parts (a) and (b) of Conjecture 1.1 hold for each $L(s, \chi)$, then (2.1) holds.

Assume now that Conjecture 1.1, including part (c), holds for each $L(s, \chi)$. One can obtain information about the average behavior of $\chi(\mathfrak{p})$ by computing a suitable contour integral of the logarithmic derivative of $L(s, \chi)$, as in [10]. By keeping careful track of the dependence on various factors, as in [12, Proposition 4.1], one obtains the following estimate: for $d_\chi = \dim(\chi)[K : \mathbb{Q}]$,

$$\sum_{\mathrm{Norm}(\mathfrak{p}) \leq x} \chi(\mathfrak{p}) \log \mathrm{Norm}(\mathfrak{p}) = \mu(\chi)x + O(d_\chi x^{1/2} \log x \log(N(x + d_\chi))). \tag{2.3}$$

Using Abel partial summation, it then follows that

$$\sum_{\mathrm{Norm}(\mathfrak{p}) \leq x} \chi(\mathfrak{p}) = \mu(\chi) \,\mathrm{Li}(x) + O(d_\chi x^{1/2} \log(N(x + d_\chi))). \tag{2.4}$$

[1] In fact, somewhat less analytic information is needed; one only needs $L(s, \chi)$ to extend to a meromorphic function on $\mathrm{Re}(s) \geq 1$ with no zeroes or poles except for a simple pole at $s = 1$ in case χ is trivial. This resembles the standard proof of the prime number theorem; see [15, Theorem 1, Appendix to Chapter 1].

Since the implied constant in the big-O notation is absolute (and in particular independent of G) and effectively computable, one can obtain an effective bound on

$$\sum_{\mathrm{Norm}(\mathfrak{p}) \leq x} F(\mathfrak{p}) - \mu(F) \operatorname{Li}(x)$$

for a general continuous function $F : \mathrm{Conj}(G) \to \mathbb{C}$ by summing (2.4) over the terms of the expansion (2.2). In practice, one gets a slightly better result by applying (2.4) only to characters of small dimension, lumping the large characters directly into the error term. Even with this refinement, though, to obtain reasonable results one must impose enough regularity on f to get sufficient decay for the coefficients in (2.2). This was demonstrated explicitly for the case of SU(2) by Murty [12], whose arguments we recall in the next section.

3. The case of an elliptic curve

In this section, we recall the treatment of the effective Sato-Tate conjecture for elliptic curves by Murty [12] and indicate how it arises as a specialization of the preceding discussion. Our exposition is somewhat complementary to that of [12], where the explicit formula (2.3) and the application to the Lang-Trotter conjecture are treated in detail; we instead take (2.3) as a black box and discuss how effective Sato-Tate emerges from it in detail. (Concretely, this means that we apply Lemma 3.5 with slightly different parameters than in [12].)

For E an elliptic curve over a number field K and $\mathfrak{p}$ a prime ideal of K at which E has good reduction, let $a_\mathfrak{p} = a_\mathfrak{p}(E)$ be the Frobenius trace of E at $\mathfrak{p}$, so that $\mathrm{Norm}(\mathfrak{p}) + 1 - a_\mathfrak{p}$ is the number of rational points on the reduction of E modulo $\mathfrak{p}$. Then define the Frobenius angle $\theta_\mathfrak{p} = \theta_\mathfrak{p}(E) \in [0, \pi]$ by the formula

$$1 - a_\mathfrak{p}(E)T + \mathrm{Norm}(\mathfrak{p})T^2 = (1 - \mathrm{Norm}(\mathfrak{p})^{1/2} e^{i\theta} T)(1 - \mathrm{Norm}(\mathfrak{p})^{1/2} e^{-i\theta} T).$$

Let μ_{ST} denote the Sato-Tate measure, so that

$$\mu_{\mathrm{ST}}(f) = \int_0^\pi \frac{2}{\pi} \sin^2 \theta f(\theta)\, d\theta.$$

For I an interval, let χ_I denote the characteristic function. We prove the following theorem. (As usual, the implied constant in the big-O notation is absolute and effectively computable.)

Theorem 3.1 (after Murty)**.** *Let E be an elliptic curve over a number field K without complex multiplication. Let N denote the absolute conductor of E. Assume that $L(s, \mathrm{Sym}^k E)$ satisfies Conjecture 1.1 for all $k \geq 0$. Then for any closed subinterval I of $[0, \pi]$,*

$$\sum_{\mathrm{Norm}(\mathfrak{p}) \leq x, \mathfrak{p} \nmid N} \chi_I(\theta_\mathfrak{p}) = \mu_{\mathrm{ST}}(I) \operatorname{Li}(x) + O([K : \mathbb{Q}]^{1/2} x^{3/4} (\log(Nx))^{1/2}).$$

The weaker statement that $\sum_{\mathrm{Norm}(\mathfrak{p}) \leq x, \mathfrak{p} \nmid N} \chi_I(\theta_\mathfrak{p}) \sim \mu_{\mathrm{ST}}(I) \operatorname{Li}(x)$ is the Sato-Tate conjecture, which is known unconditionally when K is totally real. See [13] for more discussion.

To prove Theorem 3.1, we first note that the number of primes dividing N (which includes all primes of bad reduction) is $O(\log N)$, which is subsumed by our error term. We can thus safely neglect bad primes in what follows.

We next introduce a family of functions F for which we have control over the coefficients appearing in (2.2), which we will use to approximate the characteristic function χ_I. Since E

has no complex multiplication, its Sato-Tate group is SU(2), whose characters are

$$\chi_k(\theta) = \sum_{j=0}^{k} e^{(k-2j)i\theta} \qquad (k = 0, 1, \dots). \tag{3.2}$$

Thus expanding F in terms of the χ_k amounts to ordinary Fourier analysis: if we formally extend F to an even function on $[-\pi, \pi]$ and form the ordinary Fourier decomposition

$$F(\theta) = c_0 + \sum_{k=1}^{\infty} 2c_k \cos(k\theta), \tag{3.3}$$

we can then write

$$F(\theta) = \sum_{k=0}^{\infty} (c_k - c_{k+2})\chi_k(\theta). \tag{3.4}$$

We can thus avail ourselves of a construction of Vinogradov [19, Lemma 12]. For now, we take the construction as a black box; we will recall the method of proof in the context of the generalized Sato-Tate conjecture in §5.

Lemma 3.5. *Let r be a positive integer, and let A, B, Δ be real numbers satisfying*

$$0 < \Delta < \frac{1}{2}, \qquad \Delta \leq B - A \leq 1 - \Delta.$$

Then there exists a continuous periodic function $D_{A,B} = D : \mathbb{R} \to \mathbb{R}$ with period 1 that satisfies the following conditions.

1. *For $A + \frac{1}{2}\Delta \leq x \leq B - \frac{1}{2}\Delta$, $D(x) = 1$.*
2. *For $B + \frac{1}{2}\Delta \leq x \leq 1 + A - \frac{1}{2}\Delta$, $D(x) = 0$.*
3. *For x in the remainder of the interval $\left[A - \frac{1}{2}\Delta, 1 + A - \frac{1}{2}\Delta\right]$, $0 \leq D(x) \leq 1$.*
4. *$D(x)$ has a Fourier series expansion of the form*

$$D(x) = \sum_{m \geq 0} \left(a_m \cos(2m\pi x) + b_m \sin(2m\pi x)\right)$$

in which $a_0 = B - A$ and for all $m \geq 1$,

$$|a_m|, |b_m| \leq \min\left\{2(B - A), \frac{2}{\pi m}, \frac{2}{\pi m}\left(\frac{r}{\pi m \Delta}\right)^r\right\}. \tag{3.6}$$

Leaving the choices of A, B, Δ, r unspecified for the moment, let us define D as in Lemma 3.5, then define the function $F_{A,B} : \mathbb{R} \to \mathbb{R}$ by $F_{A,B}(\theta) = D\left(\frac{\theta}{2\pi}\right) + D\left(-\frac{\theta}{2\pi}\right)$.

Its Fourier series has the form

$$F_{A,B}(\theta) = \sum_{m \in \mathbb{Z}} c_{m,A,B} e^{im\theta}, \tag{3.7}$$

where

$$c_{0,A,B} = 2a_0 = 2(B - A)$$

and

$$c_{m,A,B} = c_{-m,A,B} = a_m \text{ for all } m \geq 1.$$

Let M be a positive integer (to be specified later). By truncating the Fourier series (3.7) and using the bounds for the Fourier coefficients (3.6), we see that

$$F_{A,B}(\theta) = \sum_{|m| \leq M} c_{m,A,B} e^{im\theta} + O\left(\frac{(r/\pi)^r}{M^r \Delta^r}\right).$$

Rewriting in terms of the characters of SU(2) gives

$$F_{A,B}(\theta) = (c_{0,A,B} - c_{2,A,B}) + \sum_{m=1}^{M-2} (c_{m,A,B} - c_{m+2,A,B}) \chi_m(\theta) + O\left(\frac{(r/\pi)^r}{M^r \Delta^{r+1}}\right). \tag{3.8}$$

If we take $r = 1$, then (3.6) implies

$$\sum_{m=1}^{M-2} m \left| c_{m,A,B} - c_{m+2,A,B} \right| = O\left(\frac{\log M}{\Delta}\right).$$

Applying (2.4) then yields

(3.9)

$$\begin{aligned} &\sum_{\mathrm{Norm}(\mathfrak{p}) \leq x, \mathfrak{p} \nmid N} F_{A,B}(\theta_{\mathfrak{p}}) \\ &= (c_{0,A,B} - c_{2,A,B}) \operatorname{Li}(x) + O\left(\frac{[K:\mathbb{Q}] x^{1/2} \log(N(x+M)) \log M}{\Delta}\right) + O\left(\frac{x}{M \Delta \log x}\right). \end{aligned}$$

To deduce Theorem 3.1, note that on one hand, the characteristic function of the interval $I = [2\pi\alpha, 2\pi\beta]$ is bounded from above by $F_{\alpha - \Delta/2, \beta + \Delta/2}$ and from below by $F_{\alpha + \Delta/2, \beta - \Delta/2}$. On the other hand, the quantities $c_{0,\alpha-\Delta/2,\beta+\Delta/2} - c_{2,\alpha-\Delta/2,\beta+\Delta/2}$ and $c_{0,\alpha+\Delta/2,\beta-\Delta/2} - c_{2,\alpha+\Delta/2,\beta-\Delta/2}$ each differ from $\mu_{ST}(I)$ by $O(\Delta)$. We obtain the theorem by balancing the error terms in (2.4) with each other and with $O(\Delta \operatorname{Li}(x))$ by setting

$$\Delta = x^{-1/4} [K:\mathbb{Q}]^{1/2} (\log x)(\log(Nx))^{1/2}, \qquad M = \lceil \Delta^{-2} \rceil.$$

4. The case of two elliptic curves

We now consider a variant of the previous situation.

Theorem 4.1. *Let E_1, E_2 be two $\overline{\mathbb{Q}}$-nonisogenous elliptic curves over a number field K, neither having complex multiplication. Let N be the product of the absolute conductors of E_1 and E_2. For each prime ideal $\mathfrak{p}$ of K not dividing N, let $\theta_{1,\mathfrak{p}}, \theta_{2,\mathfrak{p}}$ be the Frobenius angles of E_1, E_2 at $\mathfrak{p}$. Assume that the L-functions $L(s, \mathrm{Sym}^i E_1 \otimes \mathrm{Sym}^j E_2)$ for $i, j = 0, 1, \ldots$ all satisfy Conjecture 1.1. Then for any closed subintervals I_1, I_2 of $[0, \pi]$,*

$$\sum_{\mathrm{Norm}(\mathfrak{p}) \leq x, \mathfrak{p} \nmid N} \chi_{I_1}(\theta_{1,\mathfrak{p}}) \chi_{I_2}(\theta_{2,\mathfrak{p}}) = \mu_{\mathrm{ST}}(I_1) \mu_{\mathrm{ST}}(I_2) \operatorname{Li}(x) + O([K:\mathbb{Q}]^{1/3} x^{5/6} (\log(Nx))^{1/3}).$$

To prove Theorem 4.1, note that by [4, §4.2] the Sato-Tate group of the 1-motive associated to $E_1 \times_K E_2$ is $\mathrm{SU}(2) \times \mathrm{SU}(2)$, whose conjugacy classes we identify with $[0, \pi] \times [0, \pi]$. For real numbers $A_1, B_1, A_2, B_2, \Delta$ with

$$0 < \Delta < \frac{1}{2}, \qquad \Delta \leq B_1 - A_1, B_2 - A_2 \leq 1 - \Delta,$$

put $d_{m_i,A_i,B_i} = c_{m_i,A_i,B_i} - c_{m_i+2,A_i,B_i}$; then

$$F_{A_1,B_1}(\theta_1)F_{A_2,B_2}(\theta_2) = \sum_{m_1,m_2=0}^{M} d_{m_1,A_1,B_1}d_{m_2,A_2,B_2}\chi_{m_1}(\theta_1)\chi_{m_2}(\theta_2) + O\left(\frac{(r/\pi)^r}{M^r\Delta^{2r}}\right).$$

If we take $r = 1$, then (3.6) implies

$$\sum_{m_1,m_2=0}^{M-2} m_1m_2\,|d_{m_1,A_1,B_1}d_{m_2,A_2,B_2}| = O\left(\frac{(\log M)^2}{\Delta^2}\right).$$

We thus have

(4.2)

$$\sum_{\mathrm{Norm}(\mathfrak{p})\le x,\mathfrak{p}\nmid N} F_{A_1,B_1}(\theta_{1,\mathfrak{p}})F_{A_2,B_2}(\theta_{2,\mathfrak{p}})$$
$$= d_{0,A_1,B_1}d_{0,A_2,B_2}\,\mathrm{Li}(x) + O\left(\frac{[K:\mathbb{Q}]x^{1/2}\log(N(x+M))(\log M)^2}{\Delta^2}\right) + O\left(\frac{x}{M\Delta^2\log x}\right).$$

As in the proof of Theorem 3.1, all that remains is to balance the error terms in (4.2) with each other and with $O(\Delta\,\mathrm{Li}(x))$ by taking

$$\Delta = x^{-1/6}[K:\mathbb{Q}]^{1/3}(\log x)(\log(Nx))^{1/3}, \qquad M = \lceil\Delta^{-3}\rceil.$$

This yields Theorem 4.1 as desired.

Theorem 4.1 immediately implies that there exists a prime ideal $\mathfrak{p}$ of K with $\mathrm{Norm}(\mathfrak{p}) = O([K:\mathbb{Q}]^2(\log N)^2(\log\log N)^6)$ for which E_1 and E_2 have good reduction and $a_\mathfrak{p}(E_1) \neq a_\mathfrak{p}(E_2)$. Namely, if we fix two disjoint intervals $[A_1, B_1]$ and $[A_2, B_2]$, then for Δ as above, the count of prime ideals is at least $c_1\,\mathrm{Li}(x)(1 - c_2\Delta)$ for some absolute constants c_1, c_2. This count is forced to be positive as soon as $c_2\Delta < 1$, proving the claim.

Note however that applying Theorem 4.1 is not the correct optimization for this problem, as the parameters of the proof were tuned to optimize for large x rather than for small x. Changing this optimization leads to a sharper result.

Theorem 4.3. *With hypotheses and notation as in Theorem 4.1, there exists a prime ideal $\mathfrak{p}$ not dividing N with $\mathrm{Norm}(\mathfrak{p}) = O([K:\mathbb{Q}]^2(\log N)^2(\log\log 2N)^2)$ such that $a_\mathfrak{p}(E_1)$ and $a_\mathfrak{p}(E_2)$ are nonzero and of opposite sign.*

Proof. Fix once and for all some $\epsilon > 0$, then put

$$A_1 = \epsilon, \quad B_1 = 1/4 - \epsilon, \quad A_2 = 1/4 + \epsilon, \quad B_2 = 1/2 - \epsilon.$$

We set notation as in the proof of Theorem 4.1 except that now we take $r = 2$. In this case, we have

$$\sum_{m_1,m_2=0}^{M-2} m_1m_2\,|d_{m_1,A_1,B_1}d_{m_2,A_2,B_2}| = O\left(\frac{1}{\Delta^4}\right)$$

and

$$\sum_{\mathrm{Norm}(\mathfrak{p}) \leq x, \mathfrak{p} \nmid N} F_{A_1,B_1}(\theta_{1,\mathfrak{p}}) F_{A_2,B_2}(\theta_{2,\mathfrak{p}})$$
$$= d_{0,A_1,B_1} d_{0,A_2,B_2} \operatorname{Li}(x) + O\left(\frac{[K:\mathbb{Q}] x^{1/2} \log(N(x+M))}{\Delta^4}\right) + O\left(\frac{x}{M^2 \Delta^4 \log x}\right).$$

This time, balancing the error terms with $O(\Delta \operatorname{Li}(x))$ yields

$$\Delta = x^{-1/10} [K:\mathbb{Q}]^{1/5} (\log x)^{1/5} (\log(Nx))^{1/5}, \qquad M = \lceil \Delta^{-5/2} \rceil.$$

The proof of Theorem 4.1 implies that there exists an absolute constant c such that there is a prime ideal of the desired form whenever $c\Delta < 1$. This is the same as $x > c^5 [K : \mathbb{Q}]^2 (\log x)^2 (\log(Nx))^2$, from which the claim follows. □

Remark 4.4. In the proof of Theorem 4.3, it is not really necessary to take Δ decreasing to 0. It would suffice to fix $A_1, B_1, A_2, B_2, \Delta$ so that the functions F_{A_1,B_1} and F_{A_2,B_2} have disjoint support, then balance the error terms in (4.2) against the main term.

Remark 4.5. The conclusion of Theorem 4.3 remains true if E_1, E_2 are isogenous over $\overline{\mathbb{Q}}$ but not over K: in this case they differ by a twist, so the claim reduces directly to effective Chebotarev [17, Théorème 5].

For the remainder of §4, retain notation as in Theorem 4.3 but assume for simplicity that $K = \mathbb{Q}$.

Remark 4.6. Theorem 4.3, which distinguishes two Frobenius traces using their archimedean behavior, should be compared with similar results which distinguish the traces using their mod-ℓ behavior for some prime ℓ. For example, in [17, §8.3, Théorème 21], Serre shows that there exists a prime number p not dividing N with $p = O((\log N)^2 (\log \log 2N)^{12})$ such that $a_p(E_1)$ and $a_p(E_2)$ differ modulo some auxiliary prime ℓ.

Both this argument and Theorem 4.3 give upper bounds on the norm of a prime ideal $\mathfrak{p}$ for which $a_{\mathfrak{p}}(E_1)$ and $a_{\mathfrak{p}}(E_2)$ differ. However, Serre has subsequently remarked [18, p. 715, note 632.6] that by replacing the mod-ℓ argument with an ℓ-adic argument, one can improve these bounds to $O((\log N)^2)$; since the details have not appeared in print elsewhere, Serre has kindly provided them and permitted us to reproduce them as Theorem 4.7 and Corollary 4.8 below. This suggests the possibility of a similar improvement to Theorem 4.3; see Remark 4.9.

Theorem 4.7 (Serre)**.** *Let Γ be a group, let ℓ be a prime number, let r be a positive integer, and let $\rho_1, \rho_2 : \Gamma \to \mathrm{GL}_r(\mathbb{Z}_\ell)$ be two homomorphisms with distinct traces. Then there exist a finite quotient G of Γ and a nonempty subset C of G with the following properties.*

(a) *The order of G is at most $\ell^{2r^2} - 1$.*
(b) *For any $\gamma \in \Gamma$ whose image in G belongs to C, $\mathrm{Trace}(\rho_1(\gamma)) \neq \mathrm{Trace}(\rho_2(\gamma))$.*

Proof. The argument is based on the proof of [3, Satz 6]. Let M be the (noncommutative) ring of $r \times r$ matrices over $\mathbb{Z}_\ell$, and let A be the $\mathbb{Z}_\ell$-subalgebra of $M \times M$ generated by the image of $\rho_1 \times \rho_2 : \Gamma \to \mathrm{GL}_r(\mathbb{Z}_\ell) \times \mathrm{GL}_r(\mathbb{Z}_\ell)$. Let G be the image of Γ in $A/\ell A$; since $\mathrm{rank}(A) \leq 2r^2$, the order of G is at most $\ell^{2r^2} - 1$. To define C, let m be the largest nonnegative integer

such that $\mathrm{Trace}(\rho_1(\gamma)) \equiv \mathrm{Trace}(\rho_2(\gamma)) \pmod{\ell^m}$ for all $\gamma \in \Gamma$; this integer exists because we assumed that ρ_1, ρ_2 have distinct traces. Define the linear form $\lambda : A \to \mathbb{Z}_\ell$ by

$$\lambda(x_1, x_2) = \ell^{-m}(\mathrm{Trace}(x_1) - \mathrm{Trace}(x_2));$$

by reduction modulo ℓ, λ defines a linear form $\overline{\lambda} : A/\ell A \to \mathbb{F}_\ell$. Let C be the set of $g \in G$ for which $\overline{\lambda}(g) \neq 0$; this set is nonempty by the choice of m. For any $\gamma \in \Gamma$ whose image in G belongs to C, $\mathrm{Trace}(\rho_1(\gamma)) \not\equiv \mathrm{Trace}(\rho_2(\gamma)) \pmod{\ell^{m+1}}$. □

Corollary 4.8. *Assume the Riemann hypothesis for Artin L-functions. Then there exists a prime number p not dividing N with $p = O((\log N)^2)$ such that $a_p(E_1) \neq a_p(E_2)$.*

Proof. Put $\ell = 2$, $r = 2$, and $\Gamma = G_{\mathbb{Q}}$, and apply Theorem 4.7 to the ℓ-adic representations associated to E_1, E_2. The resulting group G may be viewed as the Galois group of a finite extension of $\mathbb{Q}$ of absolutely bounded degree unramified away from the primes dividing $N_1 N_2$. The claim then follows from the effective Chebotarev theorem as stated in [17, Théorème 6]. □

Remark 4.9. The absence of a factor of $\log\log 2N$ in the bound appearing in Corollary 4.8 is a consequence of [17, Théorème 5], which is a refinement to effective Chebotarev described at the end of [10]. Without such a refinement, the bound would have the form $O((\log N)^2(\log\log 2N)^4)$ as indicated in the remarks following [17, Théorème 5].

This analysis suggests that it should also be possible to obtain a bound of the form $O((\log N)^2)$ in Theorem 4.3 by refining the analysis of [12] in the style of [10, pp. 461–462]. We have not attempted to do this.

5. Notes on the general case

We conclude by returning to the case of a general motive M and sketching how to derive effective equidistribution under the assumption of Conjecture 1.1.

Given a function $F : \mathrm{Conj}(G) \to \mathbb{C}$, one would like to approximate $\sum_{\mathrm{Norm}(\mathfrak{p}) \leq x, \mathfrak{p} \nmid N} F(\mathfrak{p})$ by writing F in terms of characters using (2.2), truncating the approximation by discarding the characters of large dimension, then applying (2.4) to the characters of small dimension. To get a meaningful result, it may be necessary to first replace F by a close approximation for which the coefficients in (2.2) decay sufficiently rapidly.

In case G is connected, it is not hard to explain how to explicitly carry out these steps in terms of classical Lie theory. Let H be a Cartan subgroup of G, and identify $\mathrm{Conj}(G)$ with the quotient of H by the action of the Weyl group W. The $\mathbb{Z}$-module of characters of G may then be identified with the W-invariant part of the $\mathbb{Z}$-module of characters of H. If we fix a Weyl chamber in the lattice of characters of H, then each element of the Weyl chamber is the highest weight of a unique irreducible representation of G, whose full character is computed by the Weyl character formula. Any function $F : \mathrm{Conj}(G) \to \mathbb{C}$ corresponds naturally to a W-invariant function $H \to \mathbb{C}$; the coefficients computed by the Weyl character formula then provide the change-of-basis matrix converting the expansion of F in (2.2) into the usual Fourier expansion of F. But this change-of-basis matrix is triangular, so we can invert it to convert the Fourier expansion into the expansion in terms of characters. In the case $G = \mathrm{SU}(2)$, the Weyl character formula produces (3.2), so this construction specializes to the conversion of (3.3) into (3.4).

Suppose however that the coefficients of F in (2.2) do not themselves converge sufficiently rapidly to obtain the desired estimates. We then take a small H-invariant neighborhood U of the identity in H and define $g : H \to \mathbb{C}$ to be the characteristic function of U rescaled so that its integral equals 1. The ordinary Fourier coefficients of g do decay: if we fix a basis $\chi_1, \dots, \chi_n$ of characters of H, where $n = \operatorname{rank}(G)$, then the coefficient of $\chi_1^{m_1} \cdots \chi_n^{m_n}$ is on the order of $\prod_{i=1}^n (|m_i| + 1)^{-1}$. Taking the convolution $F * g^r$ over H corresponds to pointwise multiplication of Fourier coefficients, so we may achieve any desired polynomial decay of Fourier coefficients without changing F too much. For r sufficiently large, this decay persists when we convert Fourier coefficients into character coefficients. In the case $G = \mathrm{SU}(2)$, this process with F taken to be the characteristic function of an interval gives precisely the function $F_{A,B}$ considered by Vinogradov.

In general, the group G need not be connected; for example, for E an elliptic curve over K with complex multiplication not defined over K, G is the normalizer of $\mathrm{SO}(2)$ in $\mathrm{SU}(2)$, which has two connected components. In this case, one can carry out the above analysis after restricting the representations involved to the connected part of G.

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Department of Mathematics, University of California at San Diego, 9500 Gilman Dr #0112, La Jolla, CA 92093
E-mail address: alina@math.ucsd.edu

Department of Mathematics, University of California at San Diego, 9500 Gilman Dr #0112, La Jolla, CA 92093
E-mail address: kedlaya@ucsd.edu
URL: http://kskedlaya.org