Title
E.E. REVIEW COURSE - LECTURE II

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ELECTRICAL ENGINEERING REVIEW COURSE

Lecture II
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VECTOR FIELDS

GRADIENT, grad

If a scalar quantity, temperature for example, is given as a function of a point in a region, we may then speak of a scalar field.

If a vector quantity, velocity for example, is given as a function of a point in a region, we may then speak of a vector field.

The mathematical shorthand for the above can be expressed:

$$T = T\,(x, y, z)$$

$$\vec{V} = \vec{V}\,(x, y, z)$$

These above expressions can be read:

Temperature, T, is equal to a scalar function T of x, y, and z, and velocity at any point is equal to a velocity vector function of x, y, and z. The above statements could of course be translated to cylindrical, spherical, oblique, or other coordinates by means of suitable operations, but for the purpose of discussion, let us consider only the rectangular cartesian coordinate system.

The change in value of the function corresponding to a displacement equal to dr, will depend on the direction of the displacement. If we now define $\vec{dr}$, the infinitesimal displacement vector, then:

$$\vec{dr} = dx\hat{i} + dy\hat{j} + dz\hat{k} \; *$$

and since:

$$dT = \frac{\partial T(x,y,z)}{\partial x}dx + \frac{\partial T(x,y,z)}{\partial y}dy + \frac{\partial T(x,y,z)}{\partial z}dz$$

then:

$$dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dx,$$ which appears to be a scalar product of two vectors.

*See Notes on Lecture I.

DISCLAIMER

This can be verified by the following procedure in which $\vec{F}$ is some vector field function:

$$dT = \vec{F} \cdot \vec{dr}$$

$$\vec{F} = \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}$$

$$\vec{dr} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

since:

$$\hat{i}\hat{i} = \hat{j}\hat{j} = \hat{k}\hat{k} = 1$$

$$\hat{i}\hat{j} = \hat{j}\hat{k} = \hat{k}\hat{i} = 0$$

$$dT = \vec{F} \cdot \vec{dr} = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz = (\text{grad } T) \cdot (\vec{dr})$$

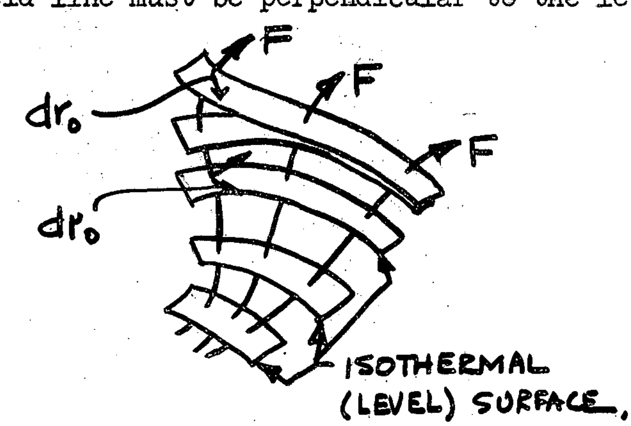
this can be expressed:

$$\left[\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right] T = \text{grad } T = \nabla T,$$ it being understood that the function T will be operated upon by the operator ∇ (read nabla or del or atled) to produce the field function as described above.

The meaning of the gradient may be expressed as follows: we can connect all points in the region having the same value of $T_{(x,y,z)}$ by means of surfaces. Each of these surfaces is characterized by the fact that displacements made along the surface do not alter the value of the function $T_{(x,y,z)}$. If, then we let dr lie in one of the surfaces, then:

$$dr_o \cdot \text{grad } (T) = 0$$

Since neither dr_o nor grad T is to vanish, then the vector, gradient T, is perpendicular to the level surface. Since the scalar product is equal to zero, the gradient field line must be perpendicular to the level surface.



If, in going from position A, where the value of $T_{(x,y,z)}$ is T_A, to position B, where $T_{(x,y,z)}$ is T_B, the difference in temperature between the two points is $T_B - T_A$, or:

$$\int_A^B \nabla T \cdot d\vec{r} = \int_A^B dT = T_B - T_A$$

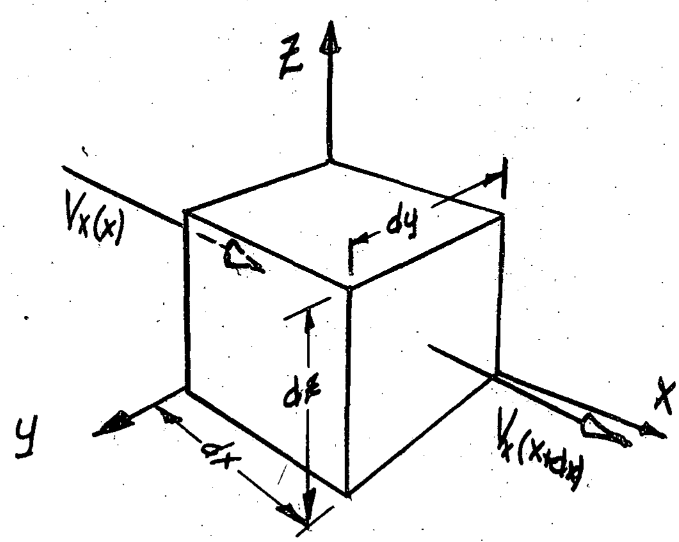
this means that the values of T at any points are independent of the path used to get from one point to any other point. If we return to our starting point, then the value is the same as it was before we left the point:

$$\oint \nabla T\, d\vec{r} = 0$$

Let us now consider the case where the function for which the field is defined as a vector quantity. If:

$\oint \vec{V}\, d\,\vec{r} = 0$, then the field is particularly defined as being irrotational, and as having no sources or sinks.

DIVERGENCE, div



$\left[V_x\ (x+dx)\ dy\ dz\right]$ is read V sub x parenthesis x plus dx parenthesis, dy dz, and means the x component of the function $\vec{V}$ at position $x+dx$ is to be multiplied by the area, dy dz

$V_x\ (x+dx)\ dy\ dz =$ the amount of fluid flowing from the box in the x direction at $(x+dx)$.

$V_x\ (x)\ dy\ dz =$ the amount of fluid flowing into the box in the x direction at (x).

$\therefore \left(V_x\ (x+dx) - V_x\ (x)\right) dy\ dz =$ the net fluid flowing out of the box in the x direction.

$V_x\ (x+dx) = V_x\ (x) + \frac{\partial V_x}{\partial x} dx.$ Substituting the last equation into the previous one, we get:

$$\left(V_x\ (x) + \frac{\partial V_x}{\partial x} dx - V_x\ (x)\right) dy\ dz = \frac{\partial V_x}{\partial x} dx\ dy\ dz =$$ the net fluid flowing or the net rate of fluid flow out of the box in the x direction. Note that the fluid can be leaving the box or can be piling up inside the box. That is, at certain points in the box, fluid may be continually generated and at other points destroyed. This allows us to consider the fluid as incompressible.

Following the same reasoning as above for the y and z components of the vector $\vec{V}$, finally we get:

The total rate at which fluid flows out of the box or the total flux of fluid through the surfaces of the box $= \iiint_{Volume} \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) dx\ dy\ dz = \int_{surface} \vec{V} \cdot \hat{n}\ ds,$

where $\hat{n} =$ unit vector normal to the surface s.

$\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)$ is called the divergence of the vector field $\vec{V}\ (x,y,z)$ and is represented by div $\vec{V}$. The divergence of a vector field at any point may be defined as the outward flow per unit volume from a volume element including the point in question.

Using the notation just defined: $$\iiint_{Volume} \text{div}\ \vec{V}\ dx\ dy\ dz = \int_{SURFACE} \vec{V} \cdot \hat{n}\ ds$$

This expression for converting a volume integral into a surface integral is true for any vector field and is called the divergence theorem or Gauss's theorem.

If $\vec{F} = \vec{V}$, where $\vec{F}$ is any vector field function, and $\vec{V}$ is the particular function used for illustrative purposes, then:

$$\text{div}\ \vec{F} = \nabla \cdot \vec{F} = \left[\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right] \cdot \vec{F}$$ (continued)

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

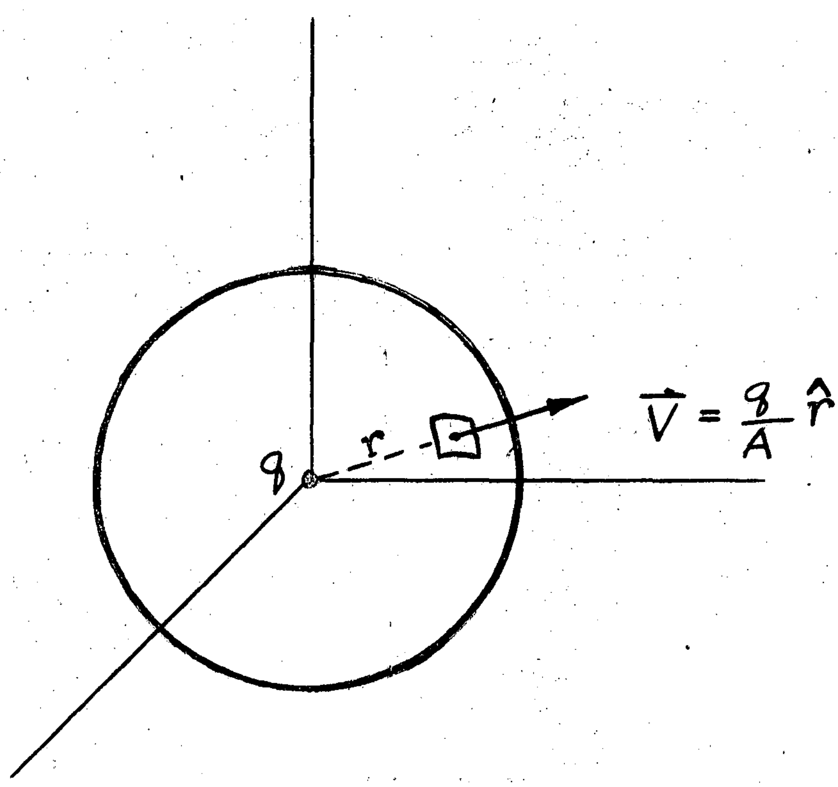
$$\nabla \cdot \vec{F} = \left[\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right] \cdot \left[F_x\hat{i} + F_y\hat{j} + F_z\hat{k}\right]$$

$$\nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

If div $\vec{F}$ is everywhere equal to zero and the fluid is incompressible, then we have no sources or sinks of fluid.

EXAMPLE

Let us consider a point source of strength q and investigate the flux through a bounding sphere.



$\vec{V}$ = velocity of the fluid in cm/sec

q = source strength, cu cm/sec

v = volume

A = area over which the velocity is to be measured, sq. cm., i.e. sphere surface area

r = radius at which A is taken

$\hat{r}$ = unit radius vector

$\vec{r}$ = radius vector = $|\vec{r}|\ \hat{r}$

s = surface area

$\hat{n}$ = unit vector normal to surface

The flux through the sphere is given by:

$$\oint_s \vec{V} \cdot \hat{n}\, ds$$

$$\vec{V} = \frac{q}{A}\hat{r} = \frac{q}{4\pi r^2}\hat{r} \qquad \text{and } \hat{n} = \hat{r} \text{ for a sphere}$$

$$\therefore \text{Flux} = \oint_s \frac{q}{4\pi r^2}\, \hat{r} \cdot \hat{r}\, ds$$

$$\text{since } \hat{r} \cdot \hat{r} = 1, \quad \text{flux} = \oint_s \frac{q}{4\pi r^2}\, ds$$

$$\text{flux} = \frac{q}{4\pi r^2} \oint_s ds = \frac{q}{4\pi r^2}\, 4\pi r^2 = q$$

Thus the flux through any closed surface surrounding the point source is equal to the source strength. We can get another equation by considering the fact that:

$$\oint_s \vec{V} \cdot \hat{n}\, ds = \int_{\text{Volume}} \text{div}\, \vec{V}\, dv$$

$$\text{therefore} \quad \int_{\text{Volume}} \text{div}\, \vec{V}\, dv = q \text{ for a point source.}$$

For a distributed source: $\text{Flux} = \int_{\text{Volume}} Q_{(x,y,z)}\, d\vec{V} = \iiint_{\text{Volume}} \text{div}\, \vec{V}\, dx\, dy\, dz$

(See next page for symbols)

where $Q_{(x,y,z)}$ = strength of source as a function of x,y, and z.

V = velocity

v = volume

$\operatorname{div} \vec{V} = Q_{(x,y,z)}$

CURL, curl

When $\oint \vec{V} \cdot \vec{dr} \neq 0$, we may define a new vector: $\oint_c \vec{V} \cdot \vec{dr} = \int_s \operatorname{curl} \vec{V} \cdot \vec{ds}$, which indicates there is something circulating in the closed field, i.e., we have a rotational field. In a rotational field, the integral of the normal component of the curl of the field over the area enclosed by a curve gives the circulation of the vector field about this closed curve. Or, the line integral of vector V, taken over a closed curve C, is equal to the surface integral of the curl of V, taken over any surface having C as a boundary.

$\int \vec{V} \cdot \vec{dr}$ A physical example of a field that has curl is a whirlpool, because there, the $\int \vec{V} \cdot \vec{dr}$ about the center of the whirlpool is different from zero.

$$\operatorname{curl} \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} \quad \text{(read del cross Vector } \vec{V}\text{)}$$

$$\nabla \times \vec{V} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{i} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{k}$$

SUMMARY

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\operatorname{grad} u = \nabla u = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} + \frac{\partial u}{\partial z}\hat{k}$$

$$\operatorname{div} \vec{V} = \nabla \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$\operatorname{curl} \vec{V} = \nabla \times \vec{V} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{i} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{k}$$

$$\operatorname{curl} \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

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